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On the Algebraic Independence of the Values of E-Functions

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The theorem on the algebraic independence of values at algebraic points of the subcollection of E-functions is proved in the case where their principal collection constitutes a solution of a system of linear differential equations with coefficients from $\mathbb{C}(z)$ and is connected with the coefficients from $\mathbb{C}[z]$ by a unique algebraic equation.

Consider the collection of E-functions

$$f_1(z), \ldots, f_m(z), \tag{1}$$

which is a solution of the system of linear differential equations

$$y'_k = Q_{k,i} + \sum_{i=1}^{m} Q_{k,i} y_i, \qquad k = 1, \ldots, m, \quad Q_{k,i} \in \mathbb{C}(z), \tag{2}$$

or of the system of homogeneous differential equations

$$y'_k = \sum_{i=1}^{m} Q_{k,i} y_i, \qquad k = 1, \ldots, m, \quad Q_{k,i} \in \mathbb{C}(z). \tag{3}$$

Let $T(z) \in \mathbb{C}[z]$ and be the least common denominator of all functions $Q_{k,i}$ in system (2) or (3) and let $\xi \in \mathbb{A}$, $\xi T(\xi) \neq 0$, where $\mathbb{A}$ is the field of all algebraic numbers over $\mathbb{Q}$.

The problem on the algebraic independence of m values of functions (1) at algebraic points is completely solved. It has been proved (see [1]) that the numbers

$$f_1(\xi), \ldots, f_m(\xi) \tag{4}$$

are algebraically independent (homogeneously algebraically independent) if and only if functions (1) are algebraically independent (homogeneously algebraically independent) over $\mathbb{C}(z)$.

In the case where functions (1) are algebraically independent (homogeneously algebraically independent) over $\mathbb{C}(z)$, a number of results have been established for certain subclasses of E-functions concerning the algebraic independence of subcollection of numbers (4) (see, e.g., [1]). However, the investigation in this direction have not yet been completed.

The following statement has been proved which we shall need in the sequel (see [1, ch. 4, Section 4]).

Fundamental theorem. *The degree of transcendence* (*homogeneous transcendence*) *of numbers* (4) *is equal to* l, $0 \le l \le m$, *if and only if the degree of transcendence* (*homogeneous transcendence*) *of functions* (1) *over* $\mathbb{C}(z)$ *is equal to* l.

Let the degree of transcendence (homogeneous transcendence) of functions (1) over $\mathbb{C}(z)$ be equal to l, $1 \le l \le m-1$, and the functions $f_1(z), \dots, f_l(z)$ be algebraically independent (homogeneously algebraically independent) over $\mathbb{C}(z)$. It has been proved that the numbers

$$f_1(\xi), \dots, f_l(\xi) \tag{5}$$

are algebraically independent (homogeneously algebraically independent) for all numbers $\xi \in \mathbb{A}$, except for a finite amount of them (see [1, ch. 5, Section 6]). A problem naturally arises of how to find conditions under which these numbers are algebraically independent (homogeneously algebraically independent) and describe the set of exceptional points at which they are algebraically dependent.

It has been found out that numbers (5) are algebraically independent (homogeneously algebraically independent) in certain special cases. However, even in the case where $l = m-1$, the problem has not been completely solved (see, e.g., [1]).

The necessary and sufficient conditions for the algebraic dependence of numbers (5) were published in the articles [2] and [3], but they do not make it possible to apply these conditions freely to specific collections of functions since these conditions only assert the existence of a certain polynomial with the necessary properties but do not show how it can be found.

In this article we consider the case where $l = m-1$.

Suppose that $\mathbb{K}$ is an algebraic field over $\mathbb{Q}$ which contains all coefficients of the power series in the powers z of all functions (1) and the number ξ, and $[\mathbb{K} : \mathbb{Q}] = h$, $\mathbb{K}_i$, $i = 1, \dots, h$, are algebraic fields which are conjugates of the field $\mathbb{K}$. The numbers $\xi_i \in \mathbb{K}_i$, $i = 1, \dots, h$, denote the conjugates of ξ in the field $\mathbb{K}$.

We denote by $f_{1,i}(z), \dots, f_{m,i}(z)$, $i = 1, \dots, h$, the E-functions resulting from functions (1) upon the replacement of all coefficients of their power series in the powers of z by their conjugates from the field $\mathbb{K}_i$.

Let

$$f_1(z), \dots, f_{m-1}(z) \tag{6}$$

be any $l = m-1$ functions (1) which are algebraically independent (homogeneously algebraically independent) over $\mathbb{C}(z)$. Then the function $f_m(z)$ is connected with functions (6) (with an accuracy to within a constant factor) by a unique algebraic equation with coefficients from $\mathbb{C}[z]$ whose numerical coefficients can be chosen from the field $\mathbb{K}$ (see [1, ch. 4, Lemma 5]):

$$P\big(z, f_1(z), \dots, f_m(z)\big) = 0, \qquad P = P(z, z_1, \dots, z_m) \in \mathbb{K}[z, z_1, \dots, z_m], \tag{7}$$

where P is an irreducible (homogeneous with respect to $z_1, \dots, z_m$ irreducible) polymonial, containing z_m, of degree s with respect to $z_1, \dots, z_m$, $s \ge 1$.

We denote by $D(z) \in \mathbb{K}[z]$ the greatest common divisor of all coefficients-polynomials of z in the products of the degrees $z_1, \dots, z_m$ in the polynomial P including z_m, and by Λ the set of numbers $\alpha \in \mathbb{A}$ which are roots of the polynomial $D(z)$. It may turn out that $\Lambda = \varnothing$.

Let

$$P_i = P_i(z, z_1, \dots, z_m) \in \mathbb{K}_i[z, z_1, \dots, z_m], \quad D_i(z) \in \mathbb{K}_i[z], \qquad i = 1, \dots, h,$$

denote polynomials resulting from the polynomials P and D upon the replacement of all their numerical coefficients by conjugate numbers from the fields $\mathbb{K}_i$. Then it follows from equation (7) that

$$P_i\big(z, f_{1,i}(z), \dots, f_{m,i}(z)\big) = 0, \qquad i = 1, \dots, h.$$

Next we denote by Λ_i, $i = 1, \dots, h$, the set of roots of the polynomials $D_i(z)$.

Theorem 1. *Suppose that the collection of* m *E-functions* (1) *is a solution of the system of linear differential equations* (2) (*homogeneous differential equations* (3)), *the degree of transcendence* (*homogeneous transcendence*) *of these functions over* $\mathbb{C}(z)$ *is equal to* $m-1$, *functions* (6) *are algebraically independent* (*homogeneously algebraically independent*) *over* $\mathbb{C}(z)$, *the set* Λ *is defined by the polynomial* P (7), *and* $\xi \in \mathbb{A}$, $\xi T(\xi) \neq 0$, $\xi \notin \Lambda$.

Then, at least for one value of i, $1 \le i \le h$, *the numbers*

$$f_{1,i}(\xi_i), \dots, f_{m-1,i}(\xi_i) \tag{8}$$

are algebraically independent (*homogeneously algebraically independent*).

Proof. Let us consider the homogeneous case.

Assume that the numbers

$$f_1(\xi), \dots, f_{m-1}(\xi) \tag{9}$$

are homogeneously algebraically dependent.

Since $\xi \notin \Lambda$, it follows that in the polynomial P (7) at least one of the coefficients in the powers of z_m^t, $t = 1, \dots, s$, which are polynomials from $\mathbb{K}[z, z_1, \dots, z_{m-1}]$ is not divisible by $z - \xi$.

The coefficient in z_m^s, which is a polynomial of z, cannot be a coefficient of this kind since, otherwise, setting $z = \xi$ and $z_j = f_j(\xi)$, $j = 1, \dots, m$, in the polynomial P, we find, by virtue of (7), that the number $f_m(\xi)$ is homogeneously algebraic over the field $\mathbb{K}[f_1(\xi), \dots, f_{m-1}(\xi)]$. However, by the assumption, the degree of homogeneous transcendence of this field is smaller than $m-1$, and, consequently, the degree of homogeneous transcendence of the field $\mathbb{K}[f_1(\xi), \dots, f_m(\xi)]$ is also smaller than $m-1$, and this contradicts the fundamental theorem formulated above since the degree of homogeneous transcendence of function (1) over $\mathbb{C}(z)$ is equal to $m-1$.

The contradiction obtained proves that for $s = 1$ the statement of the theorem is valid (since in P the coefficient of z_m is not divisible by $z - \xi$) and for $s > 1$ we find that in the polynomial P (7) at least one of the coefficients of z_m^t, $1 \le t \le s-1$, is not divisible by $z - \xi$. Let

$$F = F(z, z_1, \dots, z_{m-1}) \in \mathbb{K}[z, z_1, \dots, z_m] \tag{10}$$

be a coefficient od this kind and suppose that its degree with respect to $z_1, \dots, z_{m-1}$ is equal to k, $1 \le k \le s-1$. Then

$$F(\xi, z_1, \dots, z_{m-1}) \not\equiv 0. \tag{11}$$

Let us set $z = \xi$ and $z_j = f_j(\xi)$, $j = 1, \dots, m$, in the polynomial P (7). Then, all coefficients in P of the different degrees of z_m^t, $t = 0, 1, \dots, s$, vanish (see [1, ch. 4, Section 6, Lemma 11]). In particular,

$$F\big(\xi, f_1(\xi), \dots, f_{m-1}(\xi)\big) = 0. \tag{12}$$

We denote by $F_i(z, z_1, \dots, z_{m-1})$, $i = 1, \dots, h$, the polynomials resulting from the polynomial F (10) after the replacement of all numerical coefficients by numbers from the field $\mathbb{K}_i$.

Let us now assume the contrary, i.e., that for every value of i, $1 \le i \le h$, numbers (8) are homogeneously algebraically dependent. Since $\xi \notin \Lambda$, it follows that $\xi_i \notin \Lambda_i$, $i = 1, \dots, h$. Therefore, repeating the preceding arguments, we find, just as in relations (12), that

$$F_i\bigl(\xi_i, f_{1,i}(\xi_i), \dots, f_{m-1,i}(\xi_i)\bigr) = 0, \qquad i = 1, \dots, h.$$

Consider the collection $\mu = \dfrac{(k+m-1)!}{k!(m-1)!}$ of the products of the powers of functions (1):

$$f_1^{k_1}(z) \cdots f_m^{k_m}(z), \qquad k_1 + \dots + k_m = k. \tag{13}$$

These functions constitute the solution of a system of μ linear homogeneous differential equations of form (3) in which the number m is replaced by μ and which does not have singular points different from the singular points of system (3) (see [1, ch. 3, Section 12, Lemma 18]). By the hypothesis, since $k < s$, functions (13) are linearly independent over $\mathbb{C}[z]$.

By virtue of condition (11), we can consider the polynomial $F\bigl(\xi, f_1(z), \dots, f_{m-1}(z)\bigr)$ as a linear form in the variables (13) with coefficients from $\mathbb{K}$ which is not identically zero. Therefore, using Theorem 1 from [1, ch. 11, Section 2], we find that the inequality

$$\max_{1 \le i \le h} \bigl|F_i\bigl(\xi_i, f_{1,i}(\xi_i), \dots, f_{m-1,i}(\xi_i)\bigr)\bigr| > 0 \tag{14}$$

is satisfied. However, this inequality contradicts relation (12).

The contradiction obtained completes the proof of the theorem in the homogeneous case.

The nonhomogeneous case follows from the homogeneous one if we replace the number m by $m+1$ and set $f_{m+1}(z) = 1$.

Theorem 2. *Suppose that the collection of E-functions* (1) *with coefficients of the power series in the powers of z from the imaginary quadratic field $\mathbb{I}$ over $\mathbb{Q}$ satisfies all conditions of Theorem* 1, *the set Λ is defined just as in Theorem* 1, *$\xi \in \mathbb{I}$ and $\xi T(\xi) \ne 0$.*

Then numbers (9) *are algebraically independent* (*homogeneously algebraically independent*) *if and only if $\xi \notin \Lambda$.*

Theorem 2 is a corollary of Theorem 1 and the consequence of the fact that at the point $\xi \in \Lambda$ numbers (3) are algebraically dependent (homogeneously algebraically dependent).

The statement of Theorem 2 was proved in 1987 in the article [4] and can be found in the book [1, ch. 4, Section 9, Theorem 11].

It is interesting to find out whether Theorem 2 is valid when the field $\mathbb{I}$ is replaced by the field $\mathbb{K}$.

Theorem 3. *Suppose that the collection of E-functions* (1) *and the number ξ satisfy all conditions of Theorem* 1, *the polynomial P* (7) *and the set Λ are defined as in Theorem* 1, *and the polynomial $P(\xi, z_1, \dots, z_m)$ is irreducible* (*or its factorization into irreducible factors does not include the factor from $\mathbb{A}[z_1, \dots, z_{m-1}]$*).

Then numbers (9) *are algebraically independent* (*homogeneously algebraically independent*).

Proof. Consider the homogeneous case. Let us assume the contrary, i.e., that numbers (9) are homogeneously algebraically dependent. Then they are connected by the algebraic equation

$$\varphi\bigl(f_1(\xi), \dots, f_{m-1}(\xi)\bigr) = 0, \qquad \varphi = \varphi(z_1, \dots, z_{m-1}) \in \mathbb{A}[z_1, \dots, z_{m-1}], \tag{15}$$

where φ is an irreducible homogeneous polynomial.

If the polynomial $P(\xi, z_1, \ldots, z_m)$ is irreducible, then, by virtue of (15), it is divisible by the irreducible polynomial $\varphi(z_1, \ldots, z_{m-1})$, but this is impossible since $P(\xi, z_1, \ldots, z_m)$ contains z_m by the hypothesis.

Now if $P(\xi, z_1, \ldots, z_m)$ is reducible, then, reasoning as in the proof of Theorem 1, we find that all coefficients of the polynomial $P(\xi, z_1, \ldots, z_m)$ in different powers of z_m are divisible by the polynomial $\varphi(z_1, \ldots, z_{m-1})$. Consequently, the polynomial $P(\xi, z_1, \ldots, z_m)$ is divisible by the polynomial $\varphi(z_1, \ldots, z_{m-1})$, and this contradicts the hypothesis of the theorem.

The contradictions obtained prove the statement of the theorem.

It is of interest to investigate the arithmetic properties of numbers (5) for $l < m - 1$. However, even under additional constraints imposed on the properties of functions (1) and the number ξ, these problems usually turn out to be very complicated.

In 1989, a theorem was proved in paper [5] which is an analog of Theorem 1 for $l \leq m - 1$, but this was done under the additional constraints imposed on the points ξ.

Also of interest are the incomplete investigations of the arithmetical properties of numbers (5) in the case where the coefficients of the power series of functions (1) and the number ξ belong to the field $\mathbb{I}$.

In 1989, in the paper [6], the necessary and sufficient condition for the algebraic independence of numbers (5) was established for this case. However, it has the same drawback as the conditions of the theorems from the papers [2] and [3] mentioned above.

It is easy to give examples of applications of the theorem that we have proved to specific functions (see, e.g., [7]).

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