Proceedings of the Steklov Institute of Mathematics, Vol. 218, 1997, pp. 372–383.
Translated from Trudy Matematicheskogo Instituta imeni V. A. Steklova, Vol. 218, 1997, pp. 374–384.

Estimates of the Derivatives of Polynomials on Convex Bodies

V. I. Skalyga

Received February, 1997

In this paper, we obtain the best possible estimate in Markov's inequality for algebraic polynomials on convex bounded closed bodies in Banach spaces.

1. INTRODUCTION

Let $\mathfrak{P}_n$ be the set of all algebraic polynomials of degree not higher than $n \in \mathbb{Z}_+$, and let $[a,b] \subset \mathbb{R}$, $\|P_n\|_{[a,b]} = \max_{x\in[a,b]} |P_n(x)|$. The well-known Markov inequality (see [1]) states that for any $P_n \in \mathfrak{P}_n$ we have

$$\|P_n'\|_{[a,b]} \le \frac{2n^2}{b-a}\|P_n\|_{[a,b]}, \tag{1}$$

the equality being attained only for polynomials of the form

$$P_n(x) = cT_n\Big(\frac{2x-a-b}{b-a}\Big), \qquad \text{where} \quad T_n(x) = \cos(n \arccos x).$$

For interior points of the interval $[a,b]$, Bernstein's inequality holds (see [2]); according to which we have

$$|P_n'(x)| \le n\bigg(\frac{\|P_n\|^2_{[a,b]} - P_n^2(x)}{(x-a)(b-x)}\bigg)^{1/2}. \tag{2}$$

The inequality (2) is an identity for the polynomial $T_n\big((2x-a-b)/(b-a)\big)$. The Markov and Bernstein inequalities were extended to polynomials in several variables (see [3–9] and the references therein). Note the results of [3], where it is shown that inequality (1) can be extended to the case of a ball in a Hilbert space without a change in the constant. In [4], an analog of the Markov inequality is obtained for a convex compact body in $\mathbb{E}^m$ with the constant $4n^2/\omega$ (where ω is the width of the body), and a conjecture is stated that $2n^2/\omega$ can be the constant. Counterexamples to this conjecture are constructed in [8].

Let X and Y be two Banach spaces over $\mathbb{R}$. By $\mathfrak{P}_n(X,Y)$ we denote the space of all polynomials of degree not higher than n, i.e., the set of all maps of the form

$$P_n(x) = \sum_{j=0}^{n} u_j(x,\dots,x), \qquad x \in X,$$

where $u_j \colon X^j \to Y$ is a j-linear continuous mapping. In the sequel, $\|P_n'(x)\|_{\mathcal{L}(X,Y)}$ is the norm of the Frechét derivative of the polynomial P_n at the point x in the space $\mathcal{L}(X,Y)$ of all bounded linear operators from X into Y. If $Y=\mathbb{R}$, then we use the symbol ∇ to denote the derivative. The value of the linear mapping $P_n'(x)$ on the vector h is denoted by $P_n'(x)[h]$.

Everywhere in the sequel, V is a convex bounded closed body in the Banach space X. The set of all bodies of this kind is denoted by W. We set

$$\|P_n\|_V=\sup_{x\in V}\|P_n(x)\|_Y, \qquad \|P_n'\|_V=\sup_{x\in V}\|P_n'(x)\|_{\mathcal{L}(X,Y)}.$$

Let $h\in X$ and $\|h\|=1$. We denote by $V(h)$ the set

$$V(h)=\{[y_1,y_2]: y_1,y_2\in V \text{ and } \exists c\in\mathbb{R}: y_1-y_2=ch\},$$

and by $\omega_V(h)$ and ω_V the width of the body V in the direction of h and the width of this body V, respectively, where

$$\omega_V(h)=\sup_{[y_1,y_2]\in V(h)}\|y_1-y_2\|, \qquad \omega_V=\inf_{\|h\|=1}\omega_V(h).$$

The following estimate for $P_n\in\mathfrak{P}(X,Y)$ and $V\in W$ is proved in [6]:

$$\|P_n'\|_V\le\frac{4n^2}{r(V)}\|P_n\|_V,$$

where $r(x_0)=\sup\{r: B(x_0,r)\subset V\}$ and $r(V)=\sup\{r(x_0): x_0\in V\}$.

The stronger estimate

$$\|P_n'\|_V\le\frac{4n^2}{\omega_V}\|P_n\|_V$$

is proved in [9] for $P_n\in\mathfrak{P}_n(X,Y)$ and $V\in W$.

In the present paper, we obtain the unimprovable estimate

$$\|P_n'\|_V\le 2n\,\cot\Big(\frac{\pi}{4n}\Big)\frac{\|P_n\|_V}{\omega_V}$$

for $P_n\in\mathfrak{P}_n(X,Y)$ and $V\in W$.

All the results are formulated for $Y=\mathbb{R}$, but this does not lead to lose of generality for the majority of estimates as Lemma 1 shows, which follows from the Hahn–Banach theorem.

2. AUXILIARY LEMMAS

Lemma 1. *Let $V\subset X$, $P_n\in\mathfrak{P}_n(X,Y)$, $\|P_n\|_V\le 1$. If the inequality*

$$\|\widetilde{P}_n'(x)\|_{X^*}\le C(x), \qquad x\in V$$

holds for any $\widetilde{P}_n\in\mathfrak{P}_n(X,\mathbb{R})$ such that $\|\widetilde{P}_n\|_V\le 1$, then the inequality

$$\|P_n'(x)\|_{\mathcal{L}(X,Y)}\le C(x), \qquad x\in V,$$

also holds for P_n.

The **proof** of this result see in [9].

Let $\{cZ_{n,\sigma}(t)\}$ be the set of Zolotarev polynomials, where $c, \sigma \in \mathbb{R}$, $Z_{n,\sigma}(t) \in \mathfrak{P}_n(\mathbb{R}, \mathbb{R})$, and let $Z_{n,\sigma}(t)$ be a Zolotarev polynomial of the form

$$t^n - n\sigma t^{n-1} + \sum_{i=0}^{n-2} \alpha_i t^i$$

which has the minimal norm in $C[-1,1]$. For $0 \le \sigma \le \tan^2(\pi/(2n))$, $Z_{n,\sigma}(t)$ is defined by the relation

$$Z_{n,\sigma}(t) = LT_n\Big(\frac{t-\sigma}{1+\sigma}\Big), \qquad \text{where} \quad L = (1+\sigma)^n 2^{-(n-1)}.$$

For $\sigma > \tan^2(\pi/(2n))$, $Z_{n,\sigma}(t)$ is expressed in terms of the well-known elliptic functions [10]. For $\sigma \le 0$, $Z_{n,\sigma}(t)$ is defined by the relation

$$Z_{n,\sigma}(t) = (-1)^n Z_{n,-\sigma}(-t). \tag{3}$$

Let $\varepsilon, \tau \in [-1,1]$ and let $M_{n,\varepsilon,\tau}$ be the subset of polynomials $Z_n(t)$ from the set $\{cZ_{n,\sigma}(t)\}$ such that

$$Z_n(\tau) = \varepsilon, \qquad \|Z_n(t)\|_{C[-1,1]} = 1, \qquad Z_n'(\tau) \le 0.$$

Let $Z_{n,\varepsilon,\tau}(t)$ be a polynomial from $M_{n,\varepsilon,\tau}$ such that $Z'_{n,\varepsilon,\tau}(\tau) \le Z_n'(\tau)$ for any $Z_n(t) \in M_{n,\varepsilon,\tau}$.

Lemma 2. *The polynomial $Z_{n,\varepsilon,\tau}(t)$ is a solution to the problem*

$$P_n'(\tau) \to \min, \qquad P_n(\tau) = \varepsilon, \qquad \tau \in [-1,1], \qquad \|P_n(t)\|_{C[-1,1]} \le 1, \tag{4}$$

for $P_n(t) \in \mathfrak{P}_n(\mathbb{R}, \mathbb{R})$.

Proof. Problem (4) is a convex programming problem. We replace it by the equivalent problem

$$\alpha_1 \to \min, \qquad \max_{-2+\beta \le t \le \beta} \Big|\varepsilon + \sum_{i=1}^n \alpha_i t^i\Big| \le 1, \qquad \beta = 1 - \tau \ge 0. \tag{5}$$

Let $|\varepsilon| < 1$. It follows from the Weierstrass compactness principle that there exists a solution

$$\widehat{P}_n(t) = \varepsilon + \sum_{i=1}^n \widehat{\alpha}_i t^i$$

to problem (5). It follows from the Kuhn–Tacker theorem (see [11]) that there exists a number $\lambda \ge 0$ such that

$$0 \in \partial_\alpha \bigg(\alpha_1 + \lambda \Big\|\varepsilon + \sum_{i=1}^n \alpha_i t^i\Big\|_{C[-2+\beta,\beta]}\bigg)\bigg|_{\alpha=\widehat{\alpha}}, \tag{6}$$

$$\lambda\bigg(\Big\|\varepsilon + \sum_{i=1}^n \widehat{\alpha}_i t^i\Big\|_{C[-2+\beta,\beta]} - 1\bigg) = 0, \tag{7}$$

where ∂ is the symbol of the Clarke generalized gradient (see [11]).

From (6)–(7) and the theorem of the pointwise maximum gradient [11] we have the system of equations

$$1 + \sum_{k=1}^{r} c_k \nu_k t_k = 0, \qquad \sum_{k=1}^{r} c_k \nu_k t_k^i = 0, \quad i = \overline{2, n}, \tag{8}$$

where $r \le n+1$, $|\widehat{P}_n(t_k)| = 1$, $t_k \in [-2+\beta, \beta]$, $\nu_k = \operatorname{sign} \widehat{P}_n(t_k)$, $k = \overline{1, r}$, $c_k \ge 0$, $\sum_{k=1}^{r} c_k = \lambda$.

The system of equations (8) implies the following equality for any $P_n(t)$:

$$P_n'(0) - P_n(0) \sum_{k=1}^{r} c_k \nu_k + \sum_{k=1}^{r} c_k \nu_k P_n(t_k) = 0. \tag{9}$$

It follows from (9) that $r \ge n$, since, otherwise, there would exist a polynomial $P_n(t)$ having $n-1$ zeros on the set of nonzero points which includes $\{t_k\}$ $(k = \overline{1, r})$ and the double zero at zero.

The equality $r = n+1$ can be attained only in the case where Chebyshev polynomial $cT_n(t+1-\beta)$ $(|c| = 1)$ is an extremal for the interval $[-2+\beta, \beta]$ satisfying the conditions

$$cT_n(1-\beta) = \varepsilon, \qquad cT_n'(1-\beta) \le 0$$

at zero.

Let $r = n$. Suppose that we have the numbers ν_k and t_k $(k = \overline{1, n})$. For system (8) to have a solution with positive c_k it is necessary that the elements of the first column $a_{k,1}$ of the matrix A, which is an inverse of the matrix $(\nu_k t_k^i)$ $(k, i = \overline{1, n})$, be negative. From the Cramer rule and the properties of the Vandermonde determinant we have

$$a_{k,1} = (-1)^{k+1} \prod_{i \ne k} \nu_i t_i^2 \prod_{\substack{n \ge j > i \ge 1 \\ j, i \ne k}} (t_j - t_i) \Big/ \biggl(\prod_{n \ge i \ge 1} \nu_i t_i \prod_{n \ge j > i \ge 1} (t_j - t_i) \biggr). \tag{10}$$

It follows from (10) that the condition

$$(-1)^{k+p} \nu_k < 0, \tag{11}$$

where p is the number of positive t_k, is necessary for ν_k.

It follows from (11) that the desired extremal should have an n-alternance. Hence, by the Chebyshev uniqueness theorem, the Zolotarev polynomial is a solution to problem $\widehat{P}_n(t)$.

The problem is solved by the use of a limiting process for $|\varepsilon| = 1$.

Remark 1. The set $M_{n,\varepsilon,1}$ consists of one polynomial for $\varepsilon \in (-1, 1)$ and the following relation holds:

$$Z_{n,\varepsilon,1}(t) = -T_n\Bigl(\frac{2t - y}{2 + y}\Bigr), \tag{12}$$

where $y = 2\tan^2\bigl(a(-\varepsilon)/(2n)\bigr)$, $a(\varepsilon) = \arccos(\varepsilon)$, $\varepsilon \in [-1, 1]$.

The relation

$$Z_{n,\varepsilon,1}'(1) = -(1-\varepsilon^2)^{1/2} \frac{n}{2} \cot\biggl(\frac{a(-\varepsilon)}{2n}\biggr)$$

follows from (12) and (2).

Remark 2. Note the following obvious equality for $\tau \in [-1,1]$ which follows from the symmetry considerations:

$$\max_{\substack{P_n(\tau)=\varepsilon \\ \|P_n\|_{C[-1,1]}\le 1}} |P_n'(\tau)| = \max\bigl\{|Z'_{n,\varepsilon,\tau}(\tau)|, |Z'_{n,-\varepsilon,\tau}(\tau)|\bigr\}. \tag{13}$$

Remark 3. Let $Q_{n,\varepsilon,\tau}(t) = Z_{n,\varepsilon,2\tau-1}(2t-1)$, where $\tau \in [0,1]$. Obviously, $Q_{n,\varepsilon,\tau}(t)$ is a Zolotarev polynomial for the interval $[0,1]$ which solves a problem similar to (4). This implies the inequality

$$|Q'_{n,\varepsilon,\tau_1}(\tau_1)|\tau_1 \le |Q'_{n,\varepsilon,\tau_2}(\tau_2)|\tau_2 \qquad \text{for} \quad \tau_1 < \tau_2. \tag{14}$$

Indeed, let $\tau_1/\tau_2 = \tau_3 < 1$. Consider a problem which is similar to (4) on the closed interval $[0,\tau_3]$ at the point τ_1 ($\tau_1 \le \tau_3$); assume that $P_n(t)$ is a solution to this problem. Then the inequality

$$|P_n'(\tau_1)| = \tau_3^{-1}\left|Q'_{n,\varepsilon,\tau_1/\tau_3}\left(\frac{\tau_1}{\tau_3}\right)\right| \ge |Q'_{n,\varepsilon,\tau_1}(\tau_1)|$$

holds; and this implies (14).

Note an obvious equality which follows from (3):

$$Q'_{n,-\varepsilon,1-\tau}(1-\tau) = Q'_{n,\varepsilon,\tau}(\tau). \tag{15}$$

Corollary. *Let $U_n(\theta)\colon \mathbb{R} \to [-1,1]$ be an even trigonometric polynomial of order n. Then the inequality*

$$|U_n'(\theta_0)| \le \max\bigl\{|Z'_{n,\varepsilon,c}(c)|, |Z'_{n,-\varepsilon,c}(c)|\bigr\}|\sin\theta_0|$$

holds, where $c = \cos\theta_0$, $\varepsilon = U_n(\theta_0)$.

Lemma 3. *The function*

$$C_n(\varepsilon) = n(1-\varepsilon^2)^{1/2}\left(\cot\left(\frac{a(\varepsilon)}{2n}\right) + \cot\left(\frac{a(-\varepsilon)}{2n}\right)\right),$$

where $a(\varepsilon) = \arccos\varepsilon$, $\varepsilon \in [-1,1]$, is an even and decreasing function on the interval $[0,1]$.

Proof. The evenness of $C_n(\varepsilon)$ is obvious; in order to prove the second assertion, we make the change of variables

$$x = a(\varepsilon), \qquad C_n(\varepsilon) = n\,\sin x\Bigl(\cot\Bigl(\frac{x}{2n}\Bigr) + \cot\Bigl(\frac{\pi - x}{2n}\Bigr)\Bigr).$$

We have $C_n'(\varepsilon) = -f'(x)(1-\varepsilon^2)^{-1/2}$; therefore, in order to prove the lemma, it suffices to show that $f'(x) \ge 0$ for $x \in [0,\pi/2]$.

Computing the derivative $f'(x)$, after simple transformations, we obtain

$$f'(x) = \frac{\sin\bigl(\pi/(2n)\bigr)\,\varphi(x)}{2\sin^2\bigl(x/(2n)\bigr)\sin^2\bigl((\pi-x)/(2n)\bigr)},$$

where

$$\varphi(x) = n\,\cos x\Bigl(\cos\Bigl(\frac{2x-\pi}{2n}\Bigr) - \cos\Bigl(\frac{\pi}{2n}\Bigr)\Bigr) + \sin x\,\sin\Bigl(\frac{2x-\pi}{2n}\Bigr).$$

Since $\varphi(0) = \varphi(\pi/2) = 0$, in order to prove that $\varphi(x) \geq 0$ on $[0, \pi/2]$, it is sufficient to justify the following assertion.

A. $\varphi(x)$ has one local extremum on $(0, \pi/2)$, and this extremum is positive.

We have

$$\varphi'(x) = \sin x\, \varphi_1(x),$$

where

$$\varphi_1(x) = -n\Big(\cos\Big(\frac{2x-\pi}{2n}\Big) - \cos\Big(\frac{\pi}{2n}\Big)\Big) + n^{-1}\cos\Big(\frac{2x-\pi}{2n}\Big).$$

Since $\varphi_1(0) = n^{-1}\cos\big(-\pi/(2n)\big) > 0$ and $\varphi_1'(x) = (1 - 1/n^2)\sin\big((2x-\pi)/(2n)\big) < 0$ for $0 \leq x < \pi/2$, asserion A is proved, and so is the lemma.

Lemma 4. *Let X be a two-dimensional normed space, ABC be a triangle belonging to X, CD be its median, $x \in [C, D]$, $B_1 \in [A, C]$, $x \in [B, B_1]$, $A_1 \in [B, C]$, $x \in [A, A_1]$, and let $\|D - x\|/\|D - C\| = \rho$. Then the relations*

$$\frac{\|x - A\|}{\|A_1 - A\|} = \frac{\|x - B\|}{\|B_1 - B\|} = \frac{1+\rho}{2} \tag{16}$$

hold.

Proof. The point x is the center of gravity of the system of particles $(A, 1)$,$(B, 1)$,$\big(C, 2\rho/(1-\rho)\big)$, and the points A_1 and B_1 are the centers of gravity of the system of particles $(B, 1)$, $\big(C, 2\rho/(1-\rho)\big)$ and $(A, 1)$, $\big(C, 2\rho/(1-\rho)\big)$ respectively. This implies relation (16).

3. ANALOGS OF MARKOV AND BERNSTEIN INEQUALITIES ON CONVEX BODIES IN BANACH SPACES

Let X be a Banach space, $x \in V \in W$, $h \in X$, $\|h\| = 1$. We introduce the following definitions: let $\varepsilon \in [-1, 1]$, $[y_1, y_2] \in V(h)$, $[y_1, x] \subset [u_{11}, u_{12}]$, $[y_2, x] \subset [u_{21}, u_{22}]$, $u_{ij} \in \partial V$ $(i, j = 1, 2)$,

$$\frac{\|x - u_{i2}\|}{\|u_{i1} - u_{i2}\|} = \rho_{1i}(x, y_i) = \rho_{1i}, \qquad \frac{\|x - y_i\|}{\|u_{i1} - u_{i2}\|} = \rho_{0i}(x, y_i) = \rho_{0i}, \qquad 1 - \rho_{1i} = \rho_i, \quad i = 1, 2, \tag{17}$$

$$b_1(X, V, h, x, \varepsilon, n) = \inf_{[y_1, y_2] \in V(h)} \|y_1 - y_2\|^{-1} \max_{i=1,2}\big\{|Q'_{n,\varepsilon_{i-1},\rho_1}(\rho_1)|\rho_{01} + |Q'_{n,\varepsilon_i,\rho_2}(\rho_2)|\rho_{02}\big\}, \tag{18}$$

where $\varepsilon_i = (-1)^i\varepsilon$, $Q_{n,\varepsilon,\rho}(t)$ is defined in Remark 3.

Let $x_0 \in V$, $\rho_0(x, x_0) = \|x - x_0\|/\|y - x_0\|$, where $y \in \partial V$, $x \in [x_0, y]$;

$$\begin{gathered}\overline{\rho}_1 = \frac{1}{2}\big(1 + \rho_0(x, x_0)\big), \qquad \omega_V(x_0, h) = 2\max\{t \geq 0 : [x_0 - th, x_0 + th] \subset V\}, \\ b_2(X, V, h, x, \varepsilon, n) = \inf_{x_0 \in V}\big\{|Q'_{n,\varepsilon,\overline{\rho}_1}(\overline{\rho}_1)| + |Q'_{n,-\varepsilon,\overline{\rho}_1}(\overline{\rho}_1)|\big\}\overline{\rho}_1/\omega_V(x_0, h), \\ B_n(V) = \{P_n : P_n \in \mathfrak{P}_n(X, \mathbb{R}),\ \|P_n\|_V \leq 1\}, \qquad \Phi_n(X, V, h, x, \varepsilon) = \sup_{\substack{P_n \in B_n(V) \\ P_n(x) = \varepsilon}} |\nabla P_n(x)[h]|.\end{gathered} \tag{19}$$

Theorem 1. *Let X be a Banach space, $x \in V \in W$, $h \in X$, $\|h\| = 1$, and let $C_n(\varepsilon)$ be the function defined in Lemma 3. Then the following inequalities hold:*

$$\Phi_n(X, V, h, x, \varepsilon) \leq b_1(X, V, h, x, \varepsilon, n) \leq \frac{C_n(\varepsilon)}{\omega_V(h)}, \tag{20}$$

$$F_n(X, V, h, x) = \sup_{P_n \in B_n(V)} |\nabla P_n(x)[h]| \leq b_2(X, V, h, x, 0, n) \leq \frac{C_n(0)}{\omega_V(h)}, \tag{21}$$

$$\Psi_n(X, V) = \sup_{\substack{P_n \in B_n(V) \\ x \in V}} \|\nabla P_n(x)\|_{X^*} \leq \frac{C_n(0)}{\omega_V}. \tag{22}$$

Proof. Let us prove inequalities (20). Let $x \in V$, $[y_1, y_2] \in V(h)$, $P_n(x) = \varepsilon \in [-1, 1]$. Then the relation

$$|\nabla P_n(x)[y_1 - y_2]| = |\nabla P_n(x)[y_1 - x] + \nabla P_n(x)[x - y_2]| \tag{23}$$

holds. Let the terms on the right-hand side of (23) have different signs, and let the modulus of the sums is not exceeded, for example, $|\nabla P_n(x)[y_1 - x]|$. Then the inequality

$$|\nabla P_n(x)[h]| \leq \frac{|\nabla P_n(x)[y_1 - x]|}{\|y_1 - y_2\|}. \tag{24}$$

holds. This inequality, Lemma 2, and definitions (17) imply the inequality

$$|\nabla P_n(x)[h]| \leq \|y_1 - y_2\|^{-1} \max\big\{|Q'_{n,\varepsilon,\rho_1}(\rho_1)|, |Q'_{n,-\varepsilon,\rho_1}(\rho_1)|\big\}\rho_{01}. \tag{25}$$

If the terms on the right-hand side of (23) are of the same sign, then the values $\nabla P_n(x)[x - y_1]$ and $\nabla P_n(x)[x - y_2]$ have different signs, and we obtain from Lemma 2 and definition (17) the inequality

$$|\nabla P_n(x)[h]| \leq \|y_1 - y_2\|^{-1} \max_{i=1,2}\big\{|Q'_{n,\varepsilon_{i-1},\rho_1}(\rho_1)|\rho_{01} + |Q'_{n,\varepsilon_i,\rho_2}(\rho_2)|\rho_{02}\big\}, \tag{26}$$

where $\varepsilon_i = (-1)^{i-1}\varepsilon$.

The inequality

$$\Phi_n(X, V, h, x, \varepsilon) \leq b_1(X, V, h, x, \varepsilon, n)$$

follows from (25), (26), and the definition of $b_1(X, V, h, x, \varepsilon, n)$. Since

$$C_n(\varepsilon) = |Q'_{n,\varepsilon,1}(1)| + |Q'_{n,-\varepsilon,1}(1)|,$$

the inequality

$$b_1(X, V, h, x, \varepsilon, n) \leq \frac{C_n(\varepsilon)}{\omega_V(h)}$$

follows from (25), (26), and (14). Inequalities (20) are proved. Let us prove inequality (21).

Let $P_n(x) \in B_n(V)$, $x \in V$, $[y_1, y_2] \in V(h)$ and let x_0 be the midpoint of the interval $[y_1, y_2]$. Let $y \in \partial V$ and $y - x_0 = (x - x_0)/\rho_0(x, x_0)$ $(x \in [x_0, y])$. We introduce the change of variables

$$U_n(t_1, t_2) = P_n\big(x_0 + t_1(y - x_0) + t_2(y_1 - x_0)\big). \tag{27}$$

$U_n(t_1, t_2)$ is an nth degree polynomial in two variables, and we shall consider the vector $t = (t_1, t_2)$ in the rectangular coordinate system. Relation (27) implies the equality

$$U'_{n,t_2}(\rho_0(x, x_0), 0) = \nabla P_n(x)[y_1 - x_0]$$

and the condition

$$|U_n(t_1, t_2)| \le 1 \qquad \text{for} \quad t_1 + |t_2| \le 1 \quad \text{and} \quad t_1 \ge 0. \tag{28}$$

Let the polynomial

$$\widehat{U}_n(t) = \sum_{i+j \le n} \widehat{\alpha}_{ij} t_1^i t_2^j$$

has a maximum of the modulus of the derivative $U'_{n,t_2}(t)$ at the point $(\rho_0(x, x_0), 0)$ under condition (28). But then this maximum is also attained on the polynomial

$$\overline{U}_n(t) = \frac{\widehat{U}_n(t_1, t_2) - \widehat{U}_n(t_1, -t_2)}{2}$$

which satisfies condition (28) and has only odd degrees with respect to t_2. By Lemmas 2 and 4, this implies the inequality

$$|\widehat{U}'_{n,t_2}(\rho_0, 0)| = \left| \sum_{i=0}^{n-1} \widehat{\alpha}_{i1} \rho_0^i \right| \le 2|Q'_{n,0,\overline{\rho}_1}(\overline{\rho}_1)| \overline{\rho}_1, \tag{29}$$

where $\overline{\rho}_1 = \big(1 + \rho_0(x, x_0)\big)/2$. Now, (21) follows from (29) and (19). Inequality (22) follows from (21).

Theorem 2. *The following assertions hold*:

(a) *if $X = \mathbb{R}^2$ with a certain norm and if V is a triangle ABC with $x = A$, then the following equalities hold*:

$$\Phi_n(X, V, h, x, \varepsilon) = \begin{cases} b_1(X, V, h, x, \varepsilon, n) & \textit{for each } h \ (\|h\| = 1) \textit{ and } \varepsilon \in [-1, 1] \\ \dfrac{C_n(\varepsilon)}{\omega_V(h)} & \textit{for each } \varepsilon \in [-1, 1] \textit{ and } h = \dfrac{B - C}{\|B - C\|}; \end{cases}$$

(b) *for any Banach space $X \ne \mathbb{R}$ and two numbers $\delta \in (0, 1)$ and $\varepsilon \in [-1, 1]$, there exist $V_\delta \in W$, $x_0 \in V_\delta$, h ($\|h\| = 1$), and $P_n(x, \delta, \varepsilon) \in B_n(V_\delta)$ such that $P_n(x_0, \delta, \varepsilon) = \varepsilon$,*

$$\Phi_n(X, V_\delta, h, x_0, \varepsilon) = |\nabla P_n(x_0, \delta, \varepsilon)[h]| = \frac{C_n(\varepsilon)}{\omega_{V_\delta}(h)} \ge \frac{(1 - \delta) C_n(\varepsilon)}{\omega_{V_\delta}};$$

(c) *there exist X, $V \in W$, $h \in X$, $\|h\| = 1$, and $x \in V$ such that*

$$\Phi_n(X, V, h, x, \varepsilon) = \frac{C_n(\varepsilon)}{\omega_V}$$

for any $\varepsilon \in [-1, 1]$.

Proof. Let us prove assertion (a). First, we set $X = \mathbb{R}^2$ and assume that X is endowed with the Euclidean norm, i.e., $X = \ell_2^{(2)}$. Let V be the triangle ABC with the vertices $C = (0,1)$, $B = (0,-1)$, and $A = (1,0)$. Let the point $A_1 \in [B,C]$, the point $x_0 \in [A, A_1]$, and let $\rho_1 = \|x_0 - A\|/\|A_1 - A\|$, $h = (A_1 - A)/\|A_1 - A\|$.

Suppose that $\overline{x}$ is a projection of the point x onto the straight line containing the points A and A_1, which is parallel to the line containing the points B and C. Consider the polynomials

$$\overline{P}_n(x) = Q_{n,\varepsilon,\rho_1}\left(\frac{\overline{x}}{\|A_1 - A\|}\right), \qquad \widetilde{P}_n(x) = Q_{n,-\varepsilon,\rho_1}\left(\frac{\overline{x}}{\|A_1 - A\|}\right).$$

The definitions of Φ_n and b_1, and relations (13) and (15) imply the inequalities

$$\begin{aligned} D &= \max\{|Q'_{n,\varepsilon,\rho_1}(\rho_1)|, |Q'_{n,-\varepsilon,\rho_1}(\rho_1)|\} \\ &\le \|A_1 - A\|\Phi_n(X,V,h,x_0,\varepsilon) \le \|A_1 - A\| b_1(X,V,h,x_0,\varepsilon,n) \\ &\le \max\{|Q'_{n,\varepsilon,\rho_1}(\rho_1)|\rho_1 + |Q'_{n,-\varepsilon,(1-\rho_1)}(1-\rho_1)|(1-\rho_1), \\ &\qquad |Q'_{n,-\varepsilon,\rho_1}(\rho_1)|\rho_1 + |Q'_{n,\varepsilon,(1-\rho_1)}(1-\rho_1)|(1-\rho_1)\} = D. \end{aligned}$$

This implies the relation

$$\Phi_n(X,V,h,x_0,\varepsilon) = b_1(X,V,h,x_0,\varepsilon,n) = \frac{D}{\|A_1 - A\|}. \tag{30}$$

Let the point $B_1 \in [C, A]$, $h = (B_1 - B)/\|B_1 - B\|$, $\rho = \|A - B_1\|/\|A - B\|$, and let the inequality

$$|Q'_{n,\varepsilon,1}(1)|\rho + |Q'_{n,-\varepsilon,1}(1)| \ge |Q'_{n,-\varepsilon,1}(1)|\rho + |Q'_{n,\varepsilon,1}(1)|$$

hold for the given $\varepsilon \in [-1,1]$. (In the case of the inverse inequality, the proof is similar).

Consider the polynomial $U_n(x_1, x_2)$, where

$$U_n(x_1,x_2) = J_n(x_1) + J_{n-1}(x_1)x_2, \tag{31}$$

$J_n \in \mathfrak{P}_n(\mathbb{R},\mathbb{R})$, $J_{n-1} \in \mathfrak{P}_{n-1}(\mathbb{R},\mathbb{R})$, and the relation

$$J_n(1 - x_2) + J_{n-1}(1 - x_2)x_2 = Q_{n,\varepsilon,1}(1 - x_2) = Z_{n,\varepsilon,1}(1 - 2x_2) \tag{32}$$

holds for any $x_2 \in [0,1]$ and the relation

$$J_n(1 + x_2) + J_{n-1}(1 + x_2)x_2 = -Q_{n,-\varepsilon,1}(1 + x_2) = -Z_{n,-\varepsilon,1}(2x_2 + 1) \tag{33}$$

holds for any $x_2 \in [-1,0]$.

Let a_i ($i = \overline{0,n}$) and b_i ($i = \overline{0,n-1}$) be coefficients of the polynomials J_n and J_{n-1} respectively. Then (32) and (33) imply

$$(-1)^i a_i + (-1)^{i-1} b_{i-1} = c_i,$$

$$a_i + b_{i-1} = d_i,$$

$$a_0 = \varepsilon - \sum_{i=1}^{n} a_i,$$

for a_i and b_i, where c_i (d_i) are a linear combination of coefficients of the polynomial $Q_{n,\varepsilon,1}$ ($Q_{n,-\varepsilon,1}$), a_j, and b_{j-1} ($j=\overline{i+1,n}$). Therefore the polynomial $U_n(x_1,x_2)$ of the form (31) satisfying identities (32) and (33) does exist. Let us show that $|U_n(x_1,x_2)|\le 1$ for $(x_1,x_2)\in V$. From conditions (31)–(33) we have

$$U_n(0,1)=J_n(0)+J_{n-1}(0)=(-1)^{n-1},\qquad U_n(0,-1)=J_n(0)-J_{n-1}(0)=(-1)^n.$$

This implies that $J_n(0)=0$, $J_{n-1}(0)=(-1)^{n-1}$, and $|U_n(x)|\le 1$ on the interval $[B,C]$. Hence, taking into account conditions (32) and (33), we find that the inequality

$$|U_n(x)|\le 1,\qquad x\in\partial V, \tag{34}$$

holds on the boundary of V.

Suppose that the inequality

$$|U_n(x_0)|>1$$

holds at a certain point $x_0\in V\setminus\partial V$.

Then (34) implies the existence of a point $x_*\in V\setminus\partial V$ such that

$$|U_n(x_*)|=\max_{x\in V}|U_n(x)|>1. \tag{35}$$

It follows from (35) that $U'_{n,x_2}(x_*)=J_{n-1}(x_{1*})=0$ and $|J_n(x_{1*})|>1$; but this contradicts (34). Consequently, $U_n(x)\in B_n(V)$.

The following relations hold:

$$\begin{aligned}|\nabla U_n(1,0)[B_1-B]|&=|\nabla U_n(1,0)[B_1-A]-\nabla U_n(1,0)[B-A]|\\&=|Q'_{n,\varepsilon,1}(1)|\rho+|Q'_{n,-\varepsilon,1}(1)|=d.\end{aligned} \tag{36}$$

The inequalities

$$d\le\omega_V(h)\Phi_n(X,V,h,A,\varepsilon)\le\omega_V(h)b_1(X,V,h,A,\varepsilon,n)\le d$$

follow from (36). This implies the relations

$$\Phi_n(X,V,h,A,\varepsilon)=b_1(X,V,h,A,\varepsilon,n)=\frac{d}{\omega_V(h)}. \tag{37}$$

Note that $d=C_n(\varepsilon)$ for $\rho=1$ (when $B_1=C$).

Let now $X=\mathbb{R}^2$ with a given norm, and let ABC be an arbitrary triangle in X. The relation corresponding to (30) can be proved by analogy in this case, and the relations corresponding to (37) follow from the change of coordinates

$$\begin{gathered}x=(x_1,x_2)=M+t_1(A-M)+t_2(C-M),\\ \widehat{P}_n(x_1,x_2)=\widehat{P}_n\bigl(M+t_1(A-M)+t_2(C-M)\bigr)=U_n(t_1,t_2),\end{gathered}$$

where M is the midpoint of the interval $[B,C]$ and $U_n(t_1,t_2)$ is the polynomial satisfying conditions (31)–(33).

Let us prove item (b). Let X be an arbitrary Banach space, V_δ be a convex body which is a triangular "prism" whose bases are the equal triangles ABC and $A_1B_1C_1$ belonging to the parallel planes Π_1 and Π_2 and three lateral faces Γ_3, Γ_4, Γ_5 are parts of the hyperplanes Π_3, Π_4, and Π_5, where $\Pi_i = H \times a_i$ $(i = 3,4,5)$, H is a subspace of codimension 2, and a_i $(i = 3,4,5)$ are the straight lines containing the intervals $[A,B]$, $[B,C]$, and $[C,A]$ respectively. Let the point C be chosen such that $\|C - B\| = 1$ and the inequality

$$d_1 = \inf_{x\in\Pi_3} \|C - x\| \geq 1 - \frac{\delta}{2} \tag{38}$$

holds. This can be done by the Riesz lemma on an "almost perpendicular".

Assume that the relation

$$\inf_{\substack{x\in\Pi_1\\ y\in\Pi_2}} \|x - y\| = 2 \tag{39}$$

holds and the point A is chosen such that the following inequalities hold true:

$$d_2 = \inf_{x\in\Gamma_4} \|A - x\| \geq 2, \tag{40}$$

$$d_3 = \inf_{x\in\Pi_5} \|B - x\| \geq 1 - \delta. \tag{41}$$

This can be done since $d_2 \to \infty$, and $d_3 \to d_1$ as $\|A\| \to \infty$. By (38)–(41) we have the inequality

$$\omega_V \geq 1 - \delta. \tag{42}$$

Let $\overline{x}$ be the projection of a point $x \in X$ onto the plane Π_1 parallel to the subspace H. Let $\widehat{P}_n(x_1,x_2)$ be a polynomial in two variables constructed in item (a) for the triangle ABC, and let $h = B - C$, $P_n(x) = \widehat{P}_n(\overline{x})$. Then, by (37) and (42), the inequality

$$|\nabla P_n(A)[h]| = \Phi_n(X, V_\delta, h, A, \varepsilon) = \frac{C_n(\varepsilon)}{\omega_V(h)} \geq \frac{(1-\delta)C_n(\varepsilon)}{\omega_V}$$

holds for any $\varepsilon \in [-1,1]$.

(c) We can consider by way of example $X = \mathbb{R}^2$ with a norm such that there exists a triangle V in which the length of one side is ω_V. Suppose that we have a triangle ABC whose vertices are $A = (1,0)$, $B = (0,1)$, $C = (0,-1)$.

Example 1. Let $X = \ell_\infty^{(2)}$, where $\|x\|_{\ell_\infty^{(2)}} = \max\{|x_1|,|x_2|\}$. Then $\|A - B\|_{\ell_\infty^{(2)}} = \omega_V = 1$ and the relation

$$\Phi_n(X, V, A - B, C, \varepsilon) = \frac{C_n(\varepsilon)}{\omega_V}$$

holds.

Example 2. Let $X = \overline{\ell}_1^{(2)}$, where $\|x\|_{\overline{\ell}_1^{(2)}} = 2|x_1| + |x_2|$. Then $\|B - C\|_{\overline{\ell}_1^{(2)}} = \omega_V = 2$ and the relation

$$\Phi_n\Big(X, V, \frac{1}{2}(B - C), A, \varepsilon\Big) = \frac{C_n(\varepsilon)}{\omega_V}$$

holds.

Remark 4. Let X and Y be two Banach spaces, and let $B_n(V) = \{P_n : P_n \in \mathfrak{P}_n(X,Y), \|P_n\|_V \leq 1\}$. Then the following inequalities for $x \in V \in W$ and the normed $h \in X$ follow from Theorem 1 and Lemma 1:

$$F_n(X,V,h,x) = \sup_{P_n \in B_n(V)} |P'_n(x)[h]| \leq b_2(X,V,h,x,0,n) \leq \frac{C_n(0)}{\omega_V(h)}.$$

Items (b) and (c) of Theorem 2 also hold for an arbitrary Y if we replace the function Φ_n by F_n and ε by 0. To this end, it suffices to consider the polynomial $U_n(t_1,t_2)y$, where y is a normed vector in Y, instead of the polynomial $U_n(t_1,t_2)$.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Fundamental Research (project 96-01-00767).

REFERENCES

1. Markov, A. A., Ob odnom voprose D. I. Mendeleeva (On a Certain Problem of D. I. Mendeleev), in: *Selected Works*, Moscow: GITTL, 1948.
2. Bernstein, S. N., *Extremal'nye svoistva polinomov i nailuchshee priblizhenie nepreryvnykh funktsii odnoi veshchestvennoi peremennoi* (Extremal Properties of Polynomials and the Best Approximation of Continuous Functions of one Real Variable), Leningrad: ONTI, 1931, part 1.
3. Kellogg, O. D., On Bounded Polynomials on Several Variables, *Math. Z.*, 1927, vol. 27, no. 1, pp. 55–66.
4. Wilhelmsen, Don R., A Markov Inequality in Several Dimensions, *J. Approx. Theory*, 1974, vol. 11, no. 3, pp. 216–220.
5. Pawlucki, W. and Plesniak, W., Markov's Inequality and C^{∞} Functions on Sets with Polynomial Cusps, *Math. Ann.*, 1986, vol. 275, pp. 467–480.
6. Andrianov, A. V., Analogs of the A. Markov and S. Bernstein Inequalities for Polynomials in Banach Spaces, *Mat. Zametki*, 1992, vol. 52, no. 5, pp. 13–21.
7. Baran, M., Markov Inequality on Sets with Polynomial Parametrization, *Ann. Polon. Math.*, 1994, vol. 60, no. 1, pp. 69–79.
8. Bialas-Ciez, L. and Goetgheluck, P., Constants in Markov's Inequality on Convex Sets, *East J. Approx.*, 1995, vol. 1, no. 3, pp. 379–389.
9. Skalyga, V. I., Analogs of the Markovs and Bernstein Inequalities for Polynomials in Banach Spaces, *Izv. Ross. Akad. Nauk, Ser. Mat.*, 1997, vol. 61, no. 1, pp. 141–156.
10. Akhiezer, N. I., *Lektsii po teorii approksimatsii* (Lectures on Approximation Theory), Moscow: OGIZ, 1947.
11. Clarcke, F., *Optimizatsiya i negladkii analiz* (Optimization and Nonsmooth Analysis), Moscow: Nauka, 1988.

Translated by S. Vakhromeev