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On the Construction of Probability Measures on Functional Classes

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1. The problem of introduction of measures on functional classes, besides being of interests in itself, has numerous applications, in particular, it is used for comparing, in the same terms, the determinate methods of numerical integration and Monte–Karlo methods (see, e.g., [1]–[7]).

Obviously, the most well-known of the measures on classes of functions is Wiener's measure (see, e.g., [8]) defined on the class of all functions $f(x)$, continuous on the interval $[0, 1]$, such that $f(0) = 0$.

Wiener's measure is concentrated on the set of nowhere differentiable continuous functions, in the scale of Hölder classes which have smoothness equal to $1/2$ (to be more precise, the measurable set $\operatorname{Lip}\alpha$ has measure 1 or 0 according as $0 < \alpha < 1/2$ or $1/2 < \alpha < 1$).

However, in different problems of the theory of functions (quadrature formulas, restoration operators, widths, etc.) classes are studied which reflect certain difference-differential properties of functions.

In this article Kolmogorov's general theorem on the extension of measure [9] (formulated as two theorems for convenience) is defined concretely to serve as the basis for the presentation of efficient methods of constructing probability measures on a number of functional classes which are important in the theory of functions and in other fields of mathematics.

Let us now pass to constructions.

2. Suppose that we are given a number s $(s = 1, 2, \dots)$ and that $\{\Gamma_k\}_{k=0}^{\infty}$ is a sequence of pairwise nonintersecting finite sets $\Gamma_k \subset \mathbb{Z}^s$ whose union is all $\mathbb{Z}^s$. We denote by d_k the number of points in Γ_k. Let $\nu_{-1} = 0$ and $\nu_k = d_0 + d_1 + \dots + d_k$, and let $j(m)$ be fixed ordering of Γ_k. Then every collection (the s-dimensional sequence)

$$Y = \{y_m\}_{m \in \mathbb{Z}^s} \tag{1}$$

of complex numbers, with due account of the relation $y_m = a_{j(m)} + ib_{j(m)}$, can be considered to be represented as the sequence $Y = (a_1, b_1, \dots, a_{\nu_k}, b_{\nu_k}, \dots)$. Finally, suppose that for every k on the $2d_k$-dimensional Euclidean space $\mathbb{R}^{2d_k}$ a nonnegative continuous function Ψ_k is defined such that $\Psi_k(0) = 0$. Let us define the classes $H(\Gamma_k, \Psi_k)$ as the set of all collections (1) such that the inequality

$$\Psi_k(a_{\nu_{k-1}+1}, b_{\nu_{k-1}+1}, \dots, a_{\nu_k}, b_{\nu_k}) \le 1$$

is satisfied for every k $(k = 0, 1, 2, \dots)$.

Let

$$D_k = \{(a_{\nu_{k-1}+1}, b_{\nu_{k-1}+1}, \dots, a_{\nu_k}, b_{\nu_k}) \in \mathbb{R}^{2d_k} : \Psi_k(a_{\nu_{k-1}+1}, b_{\nu_{k-1}+1}, \dots, a_{\nu_k}, b_{\nu_k}) \le 1\}.$$

We denote by $\mathcal{L}(D_k)$ the σ-algebra of all Lebesgue measurable subsets $D_k \subset \mathbb{R}^{2d_k}$.

We define the measurable space $\big(H, \mathcal{F}(H)\big)$ as the least σ-algebra containing the algebra $\mathcal{L}(H)$ of all cylindrical sets ($k = 0, 1, 2, \dots,\ E_k \in \mathcal{L}(D_k)$):

$$T_k(E_k) \equiv \{(a_1, b_1, \dots, a_{\nu_k}, b_{\nu_k}, \dots) \in H(\Gamma_k, \Psi_k) : (a_{\nu_{k-1}+1}, b_{\nu_{k-1}+1}, \dots, a_{\nu_k}, b_{\nu_k}) \in E_k\}.$$

The following theorem is valid.

Theorem 1. *Every one of the various probability measures μ on the measurable space $\big(H, \mathcal{F}(H)\big)$ is uniquely defined by the specification of the sequence of probability measures μ_k on $\big(D_k, \mathcal{L}(D_k)\big)$ such that the relation*

$$\mu\big(T_k(E_k)\big) = \mu_k(E_k)$$

holds for all k ($k = 0, 1, 2, \dots$) and E_k ($E_k \in \mathcal{L}(D_k)$).

As was pointed out above, this statement is a concretization of the theorem on the infinite product of probability measures $(D_k, \mathcal{L}(D_k), \mu_k)$ (see, e.g., [10, pp. 152–156]).

Now we denote by $B(\Gamma_k, \Psi_k)$ the set of all collections (1) such that

$$\sum_{n=0}^{\infty} \Psi_n(a_{\nu_{n-1}+1}, b_{\nu_{n-1}+1}, \dots, a_{\nu_n}, b_{\nu_n}) \le 1. \tag{2}$$

Let

$$D_k = \left\{(a_1, b_1, \dots, a_{\nu_k}, b_{\nu_k}) : \sum_{n=0}^{k} \Psi_n(a_{\nu_{n-1}+1}, b_{\nu_{n-1}+1}, \dots, a_{\nu_n}, b_{\nu_n}) \le 1\right\}$$

and $\mathcal{L}(D_k)$ be the σ-algebra of all subsets D_k measurable in the sense of $2\nu_k$-dimensional Lebesgue measure.

We define the measurable space $\big(B, \mathcal{F}(B)\big)$ as the least σ-algebra containing the algebra $\mathcal{L}(B)$ of cylindrical sets ($k = 0, 1, 2, \dots,\ E_k \in \mathcal{L}(D_k)$)

$$S(E_k) = \{(a_1, b_1, \dots, a_{\nu_k}, b_{\nu_k}, \dots) \in B(\Gamma_k, \Psi_k) : (a_1, b_1, \dots, a_{\nu_k}, b_{\nu_k}) \in E_k\}.$$

We have the following theorem.

Theorem 2. *Every one of the probability measures μ on the measurable space $\big(B, \mathcal{F}(B)\big)$ is uniquely defined by the specification of the sequence of probability measures μ_k on $\big(D_k, \mathcal{L}(D_k)\big)$ such that for all k ($k = 0, 1, 2, \dots$) and all $E_k \in \mathcal{L}(D_k)$, besides the relation*

$$\mu\big(S(E_k)\big) = \mu(E_k),$$

the following compatibility condition is fulfilled ($l = 1, 2, \dots$)

$$\mu_k(E_k) = \mu_{k+l}\Big\{(a_1, b_1, \dots, a_{\nu_{k+l}}, b_{\nu_{k+l}}) \in \mathbb{R}^{2\nu_{k+l}} : \\ (a_1, b_1, \dots, a_{\nu_k}, b_{\nu_k}) \in E_k, \quad \sum_{n=0}^{k+l} \Psi_n(a_{\nu_{n-1}+1}, b_{\nu_{n-1}+1}, \dots, a_{\nu_n}, b_{\nu_n}) \le 1\Big\}. \tag{3}$$

Since for every $E_k \in \mathcal{L}(D_k)$ the set $E_k \times \mathbb{R}^{2(\nu_{k+l}-\nu_k)}$ is measurable in the sense of $2\nu_{k+l}$-dimensional Lebesgue measure as the product of measurable sets and the function

$$\xi(a_1, b_1, \dots, a_{\nu_{k+l}}, b_{\nu_{k+l}}) \equiv \sum_{n=0}^{k+l} \Psi_n(a_{\nu_{n-1}+1}, b_{\nu_{n-1}+1}, \dots, a_{\nu_n}, b_{\nu_n})$$

is continuous on it, the set

$$\Big\{(a_1, b_1, \dots, a_{\nu_{k+l}}, b_{\nu_{k+l}}) \in \mathbb{R}^{2\nu_{k+l}} : (a_1, b_1, \dots, a_{\nu_k}, b_{\nu_k}) \in E_k,$$

$$\sum_{n=0}^{k+l} \Psi_n(a_{\nu_{n-1}+1}, b_{\nu_{n-1}+1}, \dots, a_{\nu_n}, b_{\nu_n}) \le 1\Big\} =$$

$$= \{(a_1, b_1, \dots, a_{\nu_{k+l}}, b_{\nu_{k+l}}) \in E_k \times \mathbb{R}^{2(\nu_{k+l}-\nu_k)} : \xi(a_1, b_1, \dots, a_{\nu_{k+l}}, b_{\nu_{k+l}}) \le 1\}$$

is measurable in the sense of $2\nu_{k+l}$-dimensional Lebesgue measure.

Thus the formulation of the compatibility condition (3) is correct. In all other respects, the proof of Theorem 2 repeats the proof of Kolmogorov's theorem on the extension of the measure into $\big(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty)\big)$ from [11, pp. 178–181]) if the compatibility condition in it, where in $(x_1, \dots, x_n, \dots) \in \mathbb{R}^\infty$ the coordinates x_i and x_j for $i \ne j$ are independent of each other, is replaced by condition (3), where this independence no longer exists, which fact is reflected in condition (2).

In the case of measures μ_k which are absolutely continuous respective to the Lebesgue measure, conditions (3) become more visible which makes it possible to construct the corresponding measures. Namely, by virtue of the Radon–Nikodým theorem, for every k there exist nonnegative functions $\varphi(a_1, b_1, \dots, a_{\nu_k}, b_{\nu_k})$, defined on D_k, measurable in the sense of the $2\nu_k$-dimensionsl Lebesgue measure, such that

$$\mu_k(E_k) = \int_{E_k} \varphi_k. \tag{4}$$

It is easy to see that the compatibility conditions (3) become the conditions

$$\varphi_k(a_1, b_1, \dots, a_{\nu_k}, b_{\nu_k}) = \int_{A_k(a_1, b_1, \dots, a_{\nu_k}, b_{\nu_k})} \varphi_{k+1}(a_1, b_1, \dots, a_{\nu_{k+l}}, b_{\nu_{k+l}})\, da_{\nu_k+1}\, db_{\nu_k+1} \cdots da_{\nu_{k+1}}\, db_{\nu_{k+1}}, \tag{5}$$

where

$$A_k(a_1, b_1, \dots, a_{\nu_k}, b_{\nu_k}) = \Big\{a_{\nu_k+1}, b_{\nu_k+1}, \dots, a_{\nu_{k+1}}, b_{\nu_{k+1}}) :$$

$$\Psi_{k+1}(a_{\nu_k+1}, b_{\nu_k+1}, \dots, a_{\nu_{k+1}}, b_{\nu_{k+1}}) \le 1 - \sum_{n=0}^{k} \Psi_n(a_{\nu_{n-1}+1}, b_{\nu_{n-1}+1}, \dots, a_{\nu_n}, b_{\nu_n})\Big\}.$$

In turn, the functions φ_k, which satisfy conditions (4) and (5), can be derived by means of functions $\{t_k\}$ such that

$$\int_{A_k(a_1, b_1, \dots, a_{\nu_k}, b_{\nu_k})} t_{k+1}(a_1, b_1, \dots, a_{\nu_{k+1}}, b_{\nu_{k+1}})\, da_{\nu_k+1}\, db_{\nu_k+1} \cdots da_{\nu_{k+1}}\, db_{\nu_{k+1}} = 1 \tag{6}$$

for almost all $(a_1, b_1, \ldots, a_{\nu_k}, b_{\nu_k}) \in D_k$, and then

$$\varphi_k(a_1, b_1, \ldots, a_{\nu_k}, b_{\nu_k}) = \prod_{n=0}^{k} t_n(a_1, b_1, \ldots, a_{\nu_n}, b_{\nu_n}), \tag{7}$$

and the fulfillment of conditions (4) and (5) is equivalent to the fulfillment of conditions (6) and (7).

Note that the case, where t_k does not depend on $(a_{\nu_{k-1}+1}, b_{\nu_{k-1}+1}, \ldots, a_{\nu_k}, b_{\nu_k})$ was studied by Banach [12] (see also [4]).

3. The Nikolskii–Besov classes (see [13, pp. 75–76]). Let $s = 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$,

$$Q_k = \{m = (m_1, \ldots, m_s) \in \mathbb{Z}^s : -2^k \le m_j \le 2^k,\ j = 1, \ldots, s\}, \qquad Q_{-1} = \varnothing, \qquad \Gamma_k = Q_k \setminus Q_{k-1}.$$

Here and in the sequel we denote by $\widehat{f}(m)$ the Fourier trigonometric coefficients

$$\widehat{f}(m) = \int\limits_{[0,1]^s} f(x) e^{-2\pi i(m,x)}\, dx$$

for the functions $f(x) \in L^p(0,1)^s$.

For the measurable function $g(x)$ we set

$$\|g\|_p^p = \int\limits_{[0,1]^s} |g(x)|^p\, dx,$$

and for the collection of numbers $\{a(k)\}_{k \in A}$, where A is a given countable set, we denote by $\|a(k)\|_{l^\theta(A)}$ $\sup_{k\in A} |a(k)|$ for $\theta = \infty$ and $\left(\sum_{k\in A} |a(k)|^\theta\right)^{1/\theta}$ for $1 \le \theta < \infty$.

The Nikolskii–Besov class $B^r_{p,\theta}(0,1)^s$ ($s = 1, 2, \ldots,\ r > 0,\ 1 < p < \infty,\ 1 \le \theta \le \infty$) consists of all functions $f(x) \in L^p(0,1)^s$, which are 1-periodic with respect to every variable and are such that

$$\left\| 2^{kr} \left\| \sum_{m \in \Gamma_k} \widehat{f}(m) e^{2\pi i(m,x)} \right\|_p \right\|_{l^\theta(k \ge 0, k \in \mathbb{Z})} \le 1.$$

We set

$$\widehat{f}(m) = y_m = a_{j(m)} + i b_{j(m)},$$
$$g_k(a_{\nu_{k-1}+1}, b_{\nu_{k-1}+1}, \ldots, a_{\nu_k}, b_{\nu_k}) = 2^{kr} \left\| \sum_{m\in\Gamma_k} (a_{j(m)} + i b_{j(m)}) e^{2\pi i(m,x)} \right\|_p. \tag{8}$$

Let $\Psi_k = g_k$ or $\Psi_k = g_k^\theta$ irrespective of whether $\theta = \infty$ or $1 \le \theta < \infty$. Then the constructions described in Theorems 1 and 2 allow us to obtain various probability measures on the classes $B^r_{p,\infty} \equiv H^r_p$ and $B^r_{p,\theta}$ respectively.

4. Nikolskii–Besov classes with a dominant mixed derivative (see, e.g., [14]). Let $r = (r_1, \ldots, r_s)$, $r_j > 0$, $1 < p < \infty$, $1 \le \theta \le \infty$. The class $SB^{r_1,\ldots,r_s}_{p,\theta}$ consists of all functions $f(x) \in L^p(0,1)^s$, which are 1-periodic with respect to every variable and are such that

$$\left\| 2^{\tau_1 r_1 + \cdots + \tau_s r_s} \left\| \sum_{2^{\tau_j - 1} - 1 \le |m_j| < 2^{\tau_j} - 1} \widehat{f}(m) e^{2\pi i(m,x)} \right\|_p \right\|_{l^\theta(\tau_j = 1,2,\ldots,\ j=1,\ldots,s)} \le 1.$$

The introduction of measures on these classes fall into the pattern from Theorems 1 and 2 for the following choice of Γ_k and Ψ_k:

$$\Gamma_{\tau_1,\dots,\tau_s} = \{(m_1,\dots,m_s) \in \mathbb{Z}^s : 2^{\tau_j-1} - 1 \le |m_j| \le 2^{\tau_j} - 1\},$$

$\Psi_{\tau_1,\dots,\tau_s} = g$ for $\theta = \infty$ and $\Psi_{\tau_1,\dots,\tau_s} = g^\theta$ for $1 \le \theta < \infty$, where

$$g = 2^{\tau_1 r_1 + \dots + \tau_s r_s} \left\| \sum_{2^{\tau_j-1}-1 \le |m_j| < 2^{\tau_j}-1} (a_{j(m)} + i b_{j(m)}) e^{2\pi i (m,x)} \right\|_p . \tag{9}$$

Note that in the case of even $p = 2N$ functions (8) and (9) can be directly expressed via $\widehat{f}(m) = a_{j(m)} + i b_{j(m)}$:

$$\left\| \sum_{m\in\Gamma} \widehat{f}(m) e^{2\pi i (m,x)} \right\|_p^p = \int_{[0,1]^s} \left| \sum_{m\in\Gamma} \widehat{f}(m) e^{2\pi i(m,x)} \right|^{2N} dx = \left(\sum_{m\in\Gamma} |\widehat{f}(m)|^2 \right)^N .$$

The case of $p = 2$, $\theta = \infty$ was studied in [6].

5. The Korobov and Sobolev classes. Suppose that $\lambda(m) > 0$ ($m \in \mathbb{Z}^s$) and the class $E_s^\theta(\lambda)$ ($1 \le \theta \le \infty$) consists of summable functions $f(x_1,\dots,x_s)$, which are 1-periodic with respect to every variable and are such that

$$\sum_{m\in\mathbb{Z}^s} \lambda^{-1}(m) < \infty$$

and

$$\big\| \lambda(m) |\widehat{f}(m)| \big\|_{l^\theta(\mathbb{Z}^s)} \le 1.$$

Let

$$\lambda(m) = \left(\prod_{j=1}^s \max(|m_j|, 1) \right)^r, \qquad r > 1,$$

and then, for $\theta = \infty$, the set $E_s^\infty(\lambda) \equiv E_s^r$ is Korobov's class [15], and for $\theta = 2$ the set $E_s^2(\lambda) = W_2^r$ is Sobolev's class with a dominant mixed derivative.

It is clear that in order to specify measures on the classes $E_s^\theta(\lambda)$, we must use Theorems 1 and 2 for $\Gamma_m = \{m\}$, $\Psi_m(y) = |y|\lambda(m)$, if $\theta = \infty$ and $\Psi_m(y) = \big(|y|\lambda(m)\big)^\theta$ if $1 \le \theta < \infty$, respectively. Note that on the class E_s^r the probability measure was introduced in [3], and on the class W_2^r in [12] (see also [4]).

6. In all cases of measure introduction on classes of functions presented above, we made an essential use of the specification of a function by means of the definition of a complete collection of its trigonometric Fourier coefficients.

We can approach this problem in a different way, namely, via the values of the defining trigonometric polynomials at the nodes of lattices as it was proposed in [7] for the classes $H_p^r \equiv B_{p,\infty}^r$. Namely, (we restrict the consideration to the case $p = 2$, $s = 1$) for the 1-periodic function $f(x) \in L^2(0,1)$ we set ($k = 0, \pm 1, \pm 2, \dots$)

$$f_k(x) = \sum_{m\in\Gamma_k} \widehat{f}(m) e^{2\pi i (m,x)},$$

where $\Gamma_0 = \{-1, 0, 1\}$ and $\Gamma_k = \{\operatorname{sign} k(2^{|k|-1}+1), \dots, \operatorname{sign} k(2^{|k|})\}$, for all $k \neq 0$.

We refer to the class $B^r_{2,\theta}$ all functions $f(x)$ such that

$$\left\| 2^{|k|r} \|f_k\|_2 \right\|_{l^\theta(\mathbb{Z})} \leq 1.$$

It is easy to see that

$$\|f_k\|_2^2 = \sum_{m \in \Gamma_k} |\widehat{f}(m)|^2 = \frac{1}{2^{|k|-1}} \sum_{\nu=1}^{2^{|k|-1}} \left| f_k\left(\frac{\nu}{2^{|k|-1}}\right)\right|^2.$$

Next, for the given collection of complex numbers $\{y_\nu\}_{\nu \in \Gamma_k}$ there exists a uniquely defined by it collection of numbers $\{c_\nu\}_{\nu\in\Gamma_k}$ such that

$$y_\nu = \sum_{m \in \Gamma_k} c_m \exp 2\pi i m \frac{\nu}{2^{|k|-1}}$$

for all $\nu \in \Gamma_k$.

Therefore, setting $\Psi_k = g_k$ for $\theta = \infty$ and $\Psi_k = g_k^\theta$ for $1 \leq \theta < \infty$, where

$$g_k(x_1, \dots, x_{2^{|k|}}) = 2^{|k|r} \left(\frac{1}{2^{|k|-1}} \sum_{\nu=1}^{2^{|k|-1}} (x_{2\nu-1}^2 + x_{2\nu}^2) \right)^{1/2},$$

we obtain, by virtue of Theorems 1 and 2, the probability measure in the classes $B^r_{2,\infty}$ and $B^r_{2,\theta}$ respectively.

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