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Distribution in the Mean in Progressions of Numbers with a Large Number of Prime Factors

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We denote by $\Omega(n)$ the number of prime factors n with due account of their multiplicity, $\Omega(1) = 0$. Let $f_k(n)$ be a characteristic function of the set n for which $\Omega(n) = k$, i.e., $f_k(n) = 1$ if $\Omega(n) = k$ and $f_k(n) = 0$ otherwise. We set

$$\mathsf{P}(x,k) = \sum_{n\le x} f_k(n).$$

Before beginning the investigation of the distribution of numbers with a specified value of $\Omega(n)$ in progressions, we give the information concerning $\mathsf{P}(x,k)$ which is necessary for the sequel. If k is a fixed number, then the asymptotics of $\mathsf{P}(x,k)$ was found by Landau [1]. Sathe [2] and Selberg [3] proved that if $1 \le k \le (2-\varepsilon)\ln_2 x$, $0 \le \varepsilon \le 1$, $x \ge 3$, then

$$\mathsf{P}(x,k) = \frac{x}{\ln x}\, F\left(\frac{k}{\ln_2 x}\right)\frac{(\ln_2 x)^{k-1}}{(k-1)!}\left(1 + O_\varepsilon\left(\frac{1}{\ln_2 x}\right)\right), \tag{1}$$

where $\ln_2 x = \ln\ln x$,

$$F(z) = \frac{1}{\Gamma(z+1)}\prod_p\left(1-\frac{1}{p}\right)^z\left(1-\frac{z}{p}\right)^{-1},$$

$\Gamma(z)$ is Euler's gamma-function.

For $(2+\varepsilon)\ln_2 x \le k \le \frac{\ln x}{\ln 2}$ the asymptotics of $\mathsf{P}(x,k)$ was found in [4]. Nickolas showed that in this case

$$\mathsf{P}(x,k) = \frac{1}{4}\prod_{p\ge 2}\left(1+\frac{1}{p(p-2)}\right)\frac{x}{2^k}\,\ln\frac{x}{2^k} + O\left(\frac{x}{2^k}\,\ln^b\frac{x}{2^k}\right), \tag{2}$$

where b is a constant, $0 < b < 1$.

It is shown in Lemma 2 of this work that for $1.9\ln_2 x \le k \le 2.1\ln_2 x$ there exists a constant $c_1 > 0$ with which

$$\mathsf{P}(x,k) \ge c_1\frac{x}{\ln x}\left(\frac{e}{2.2}\right)^k. \tag{3}$$

We introduce the following notation: $f_k(n,t) = 1$ if $\Omega(n) = k$ and all prime factors n exceed t and $f_k(n,t) = 0$ otherwise, $f_k(n,1) = f_k(n)$,

$$\Delta_k(Q,x,t)=\sum_{d\le Q}\max_{y\le x}\max_{(a,d)=1}\left|\sum_{\substack{n\le y\\ n\equiv a \pmod d}} f_k(n,t)-\frac{1}{\varphi(d)}\sum_{\substack{n\le y\\ (n,d)=1}} f_k(n,t)\right|,$$

$$\Delta_k(Q,x,1)=\Delta_k(Q,x).$$

The following theorem was proved in [5].

Theorem A. *If* $\ln^{5B}x\le t\le x$, *then* $\Delta_k(Q,x,t)\ll Q\sqrt{x}\ln^{3+2B}x+x\ln^{4-B/2}x$.

This theorem implies that for $t\ge \ln^{10A+40}x$ we have

$$\Delta_k(\sqrt{x}\,\ln^{-5A-19}x,x,t)\ll x\ln^{-A}x, \tag{4}$$

where A is any positive constant. Theorem 2 in the work [5] also gives the estimates for $\Delta_k(Q,x)$. We shall give them here with better values of Q than in [5]. Note that the estimates of $\Delta_k(Q,x)$ in [5] become trivial if $k\ge b\ln_2 x$, where b is a sufficiently large constant. The main goal of our work is to obtain the estimates for $\Delta_k(Q,x)$ when $k\ge(2+\varepsilon\ln_2 x)$, $\varepsilon>0$. In the theorem that follows this is done for $k\le\frac{\ln x}{2\ln 13}$.

Theorem 1. *If* $k\le b\ln_2 x$, *where* b *is a constant*, *then*

$$\Delta_k(Q,x)\ll\left(\frac{Q}{\sqrt{x}}\ln^{6B+b\ln 2+2}x+\ln^{-B/2+b\ln 2+1}x\right)\mathsf{P}(x,k).$$

If $b\ln_2 x\le k\le\Theta(x)\frac{\ln x}{A\ln 13}$, *where* $\frac{A\ln 2}{\ln_3 x}\le\Theta(x)\le\frac{(1-\varepsilon)A}{2}$, $\ln_3 x=\ln\ln\ln x$, *then*, *with*

$$Q\le\sqrt{x}\,\exp\left(-\frac{\Theta(x)}{A}\ln x\right)\ln^{-5A-19}x,$$

we have $\Delta_k(Q,x)\ll\mathsf{P}(x,k)\ln^{-A/12+9}x$, *where* $\varepsilon>0$, A *is any positive constant.*

A similar result for numbers which have k different prime factors, i.e., when $\omega(n)=k$, was obtained in the work [6] in 1993. The theorem proved in [6] is valid for $k\le\frac{\eta\ln x}{\ln_2 x}$, $\eta>0$, with $Q\le x^{1/2-\varepsilon}$, where $0<\varepsilon<1/2$. This value of Q does not allow us to prove theorems of the type given below.

Theorem 2. *Let* $(2+\varepsilon)\ln_2 x\le k\le\eta(x)\ln x$, *where* $\varepsilon>0$, $\eta(x)\le\left(\frac{\ln_3 x}{\ln_2 x}\right)^{\rho}$, $\rho>2$, $A(x,k)=\{n:n\le x,\ \Omega(n)=k\}$. *Then, uniformly with respect to* k, *we have*

$$\sum_{n\in A(x,k)}\tau(n-1)=\mathsf{P}(x,k)\times\ln x\left(1+O\left(\frac{1}{\ln^{\varepsilon}x}\right)+O\left(\eta(x)\left(\frac{\ln_2 x}{\ln_3 x}\right)^{\rho}\right)\right).$$

Here $\tau(n)$ *is the number of factors of* n.

We shall need the following result proved in [7] (see Lemma 3 in [7]).

Lemma 1. *If $f(n) \ge 0$ and $Q < x$, then*

$$\sum_{d\le Q} \max_{(a,d)=1} \left(\sum_{\substack{n\le x \\ n\equiv a \pmod d}} f(n) + \frac{1}{\varphi(d)} \sum_{n\le x} f(n) \right) \ll \Big(\sum_{n\le x} f^2(n) \Big)^{1/2} \big(\sqrt{x}\, \ln^{3/2} x + Q\big).$$

For $(2-\varepsilon)\ln_2 x \le k \le (2+\varepsilon \ln_2 x)$ there is no asymptotic relation for $\mathsf{P}(x,k)$. In this case we shall use the following lower estimate.

Lemma 2. *There exists a constant $c_1 > 0$ with which, for $1.9\ln_2 x \le k \le 2.1\ln_2 x$, $x \ge 27$, the inequaltiy*

$$\mathsf{P}(x,k) \ge c_1 \frac{x}{\ln x} \Big(\frac{e}{2.2}\Big)^k$$

is satisfied.

Proof. We have

$$\mathsf{P}(x,k) \ge \frac{1}{\ln x} \sum_{n\le x} f_k(n) \sum_{p^r \| n} \ln p^r.$$

Here $p^r \parallel n$ means that p^r divides n whereas p^{r+1} does not divide n. Estimating the part of the sum in which the powers of the primes higher than the first participate, we obtain

$$\mathsf{P}(x,k) \ge \frac{1}{\ln x} \sum_{n\le x} f_{k-1}(n) \sum_{p\le x/n} \ln p + O\Big(\frac{x}{\ln x}\Big) \ge \frac{x}{\ln x} \sum_{n\le x} \frac{f_{k-1}(n)}{n} \mu^2(n) + O\Big(\frac{x}{\ln x}\Big). \tag{5}$$

We denote $\Pi(s,y) = \prod_{s\le p<y} p$, where $\ln s = \ln_2 x$, $\ln y = \frac{\ln x}{(\ln_2 x)^2}$. Then

$$\sum_{n\le x} \frac{f_{k-1}(n)}{n}\mu^2(n) \ge \sum_{n|\Pi(s,y)} \frac{f_{k-1}(n)}{n}\mu^2(n) - \sum_{\substack{n>x \\ n|\Pi(s,y)}} \frac{f_{k-1}(n)}{n}\mu^2(n). \tag{6}$$

We have

$$\sum_{n|\Pi(s,y)} \frac{f_m(n)}{n}\mu^2(n) = \frac{1}{2\pi i} \int_{|z|=1} \sum_{n|\Pi(s,y)} \frac{z^{\Omega(n)}}{n}\mu^2(n) \frac{dz}{z^{m+1}}.$$

The integrand sum is

$$\prod_{s\le p<y} \Big(1 + \frac{z}{p}\Big) = \exp\Big(z \ln \frac{\ln y}{\ln s}\Big) + O\Big(\frac{1}{\ln s}\Big).$$

Hence we get

$$\sum_{n|\Pi(s,y)} \frac{f_m(n)}{n}\mu^2(n) = \frac{\ln \frac{\ln y}{\ln s}}{m!} + O\Big(\frac{1}{\ln s}\Big).$$

Consequently, for our choice of s and y, we have

$$\sum_{n|\Pi(s,y)} \frac{f_m(n)}{n}\mu^2(n) = \frac{(\ln_2 x - 3\ln_3 x)^m}{m!} + O\Big(\frac{1}{\ln_2 x}\Big). \tag{7}$$

Here $\ln_2 x = \ln\ln x$, $\ln_3 x = \ln\ln_2 x$. Using Stirling's formula $n! = n^n\sqrt{2\pi n}\exp(-n+\theta/(12n))$, $0<\theta<1$, we find out that for $m=k-1$ satisfying the hypothesis of the theorem, the last sum is equal to

$$\exp\Big((k-1)\ln\frac{\ln_2 x - 3\ln_3 x}{k-1} + k - 1 + O(\ln k)\Big) + O\Big(\frac{1}{\ln_2 x}\Big)$$
$$\geq \exp\bigg((k-1)\Big(1-\ln 2.1 + \ln\Big(1-\frac{3\ln_3 x}{\ln_2 x}\Big) + O\Big(\frac{\ln k}{k}\Big)\Big)\bigg) + O\Big(\frac{1}{\ln_2 x}\Big) \geq c_2\Big(\frac{e}{2.2}\Big)^k,$$

where $c_2>0$. The second sum on the right-hand side of (6) is

$$\leq \frac{1}{\ln x}\sum_{n|\Pi(s,y)}\mu^2(n)\frac{f_{k-1}(n)}{n}\ln n \leq \frac{1}{\ln x}\sum_{s\leq p\leq y}\frac{\ln p}{p}\sum_{n|\Pi(s,y)}\frac{f_{k-2}(n)}{n}\mu^2(n)$$
$$\leq \frac{1}{(\ln_2 x)^2}\sum_{n|\Pi(s,y)}\frac{f_{k-2}(n)}{n}\mu^2(n).$$

Using relation (7) with $m=k-2$, we find that it is

$$\leq \frac{1}{(\ln_2 x)^2}\bigg(\frac{(\ln_2 x - 3\ln_3 x)^{k-2}}{(k-2)!} + O\Big(\frac{1}{\ln_2 x}\Big)\bigg) \leq \frac{2.1}{(\ln_2 x)^2}\sum_{n|\Pi(s,y)}\frac{f_{k-1}(n)}{n}\mu^2(n) + O\bigg(\frac{1}{(\ln_2 x)^3}\bigg).$$

Thus, the right-hand side of inequality (6) is $\geq c_3(e/2.2)^k$ with a certain constant $c_3>0$. From this and from inequality (5) follows the statement of Lemma 2.

Proof of Theorem 1. We represent n as $n=n_1n_2$, where n_1 are numbers all of whose prime factors are smaller than t whereas all prime factors of n_2 are larger than or equal to t; n_1 and n_2 may be equal to unity. Then we have

$$\Delta_k(Q,x) \leq \sum_{n_1\leq t}\Delta_{k-\Omega(n_1)}\Big(Q,\frac{x}{n_1},t\Big) + \Delta_{k,1}(Q,x) + \Delta_{k,2}(Q,x), \tag{8}$$

where

$$\Delta_{k,1}(Q,x) = \sum_{d\leq Q}\max_{y\leq x}\max_{(a,d)=1}\left|\sum_{\substack{t\leq n_1\leq T\\(n_1,d)=1}}\left(\sum_{\substack{n_2\leq y/n_1\\ n_1n_2\equiv a \ (\mathrm{mod}\ d)}} f_k(n_1n_2) - \frac{1}{\varphi(d)}\sum_{\substack{n_2\leq y/n_1\\(n_2,d)=1}} f_k(n_1n_2)\right)\right|,$$

$$\Delta_{k,2}(Q,x) \leq \sum_{d\leq Q}\max_{(a,d)=1}\left(\sum_{\substack{n_1>T\\(n_1,d)=1}}\left(\sum_{\substack{n_2\leq x/n_1\\ n_1n_2\equiv a \ (\mathrm{mod}\ d)}} f_k(n_1n_2) + \frac{1}{\varphi(d)}\sum_{\substack{n_2\leq x/n_1\\(n_2,d)=1}} f_k(n_1n_2)\right)\right).$$

For $\ln^{5B}x \leq t\leq T\leq \sqrt[4]{x}$ it follows from the theorem that the first sum

$$\leq \sum_{\substack{n_1\leq t\\ k-\frac{\ln x}{\ln t}\leq\Omega(n_1)\leq k}}\left(Q\sqrt{\frac{x}{n_1}}\ln^{3+2B}x + \frac{x}{n_1}\ln^{4-B/2}x\right) \ll Q\sqrt{t}\sqrt{x}\ln^{3+2B}x + x\ln^{5-B/2}x.$$

Let us first considr the case of $k \le b \ln_2 x$, where b is a positive constant. Taking into account that $T \le \sqrt[4]{x}$ and estimating the part of the sum $\Delta_{k,1}(Q,x)$ with $n_2 \le \sqrt{x}$, we have

$$\Delta_{k,1}(Q,x) \ll k \max_{l\le k} \sum_{d\le Q} \max_{y\le x} \max_{(a,d)=1} \left| \sum_{\substack{t\le n_1\le T\\ (n_1,d)=1}} f_l(n_1) \left(\sum_{\substack{\sqrt{x}<n_2\le y/n_1\\ n_1n_2\equiv 1 \pmod d}} f_{k-l}(n_2) - \frac{1}{\varphi(d)} \sum_{\substack{\sqrt{x}<n_2\le y/n_1\\ (n_2,d)=1}} f_{k-l}(n_2) \right) \right| + x^{3/4}\ln x.$$

We use Lemma 4 from [7] in order to estimate the last sum. Its special case can be formulated as follows.

Lemma 3. *Let $a(n)$, $b(n)$ be arbitrary sequences of complex numbers $a(n)=0$ for $n\le R$ and $b(n)=0$ for $n\le S$. Then, for $Q\le\sqrt{x}$, $\Delta \ll R_1+R_2+R_3$, where*

$$\Delta = \sum_{d\le Q} \max_{y\le x} \max_{(a,d)=1} \left| \sum_{\substack{nm\le y\\ nm\equiv a \pmod d}} a(n)b(m) - \sum_{\substack{nm\le y\\ (nm,d)=1}} a(n)b(m) \right|,$$

$$R_1 \ll (\ln x)^{2A_1+C_1+4} \max_{y\le x} \left(\sum_{n\le y} |a(n)| \max{}^* \max_{v\le 2x/y} \left| \sum_{S<m\le v} b(m)\chi_q^*(m) \right| \right).$$

Here A_1, C_1 are arbitrary positive constants and the symbol $\max^*$ means that the maximum is taken over all $q\le \ln^{C_1} x$, $d\le x$ and all primitive characters χ_q^*,

$$R_2 = \left(Q\ln x + \left(\sqrt{\frac{x}{R}} + \sqrt{\frac{x}{S}} \right) \ln^2 x + \sqrt{x}\,\ln^{2-C_1} x \right) \ln x \left(\max_{y\le x} \sum_{n\le y} |a(n)|^2 \sum_{m\le x/y} |b(m)|^2 \right)^{1/2},$$

$$R_3 = x(\ln x)^{-A_1/5+9} \left(\frac{1}{x} \sum_{n\le 2x} |a(n)|^4 \sum_{m\le 2x/n} |b(m)|^4 \right)^{1/4}.$$

In our case $R=t$, $S=\sqrt{x}$, $|a(n)|\le 1$, $|b(m)|\le 1$, and therefore

$$R_2 \le Q\sqrt{x}\,\ln^2 x + \frac{x}{\sqrt{t}} \ln^3 x + x\ln^{3-C_1} x, \qquad R_3 \ll x(\ln x)^{10-A_1/5}.$$

In order to estimate R_1, we note that

$$\sum_{\substack{\sqrt{x}\le n_2\le v\\ (n_2,d)=1}} f_m(n_2)\chi_q^*(n_2) = \sum_{\sqrt{x}\le n_2\le v} f_m(n_2)\chi_q^*(n_2) + O\left(\sum_{\substack{p>t\\ p\mid d}} \frac{x}{p} \right).$$

The second sum is equal to $O\left(\frac{x}{t}\ln x\right)$, and, in order to estimate the first sum, we use Lemma 1 [5], in which it is shown that for $q \le \left(\frac{1}{k}\ln x\right)^A$, $\chi_q \ne \chi_0$, we have

$$\sum_{n_2\le x}\chi_q(n_2) \ll (1+\varepsilon)^k x\exp\left(-c_4\sqrt{\frac{\ln x}{k}}\right)(\ln_2 x)^k,$$

where $c_4 > 0$.

Taking into account that $k \le b\ln_2 x$, we find that $R_1 \ll x\ln^{-B}x + xt^{-1}\ln^{2A_1+C_1+4}x$. Let us take $t=\ln^{8B}x$, $A_1 = 3B$, $C_1 = B$. Then

$$\Delta_{k,1}(Q,x) \ll Q\sqrt{x}\ln^2 x + x\ln^{10-B/2}x. \tag{9}$$

Let us estimate the third sum on the right-hand side of relation (8). It follows from Lemma 1 that

$$\Delta_{k,2}(Q,x) \ll \left(\sum_{n_1>T}\sum_{n_2\le x/n_1} 1\right)^{1/2}\left(\sqrt{x}\ln^{3/2}x + Q\right).$$

We shall need the following estimate.

Lemma 4. *Let n_1 be numbers none of whose prime factors exceed y. Then, for $y\ge 2$, $2 \le \ln\frac{\ln T}{\ln y} \le \frac{1}{3}\ln y$, $\varepsilon > 0$, we have*

$$\sum_{n_1\ge T}\frac{1}{n_1} \ll \exp\left(-(1-\varepsilon)\frac{\ln T}{\ln y}\,\ln\frac{\ln T}{\ln y}\right)\times(\ln y)^e.$$

Proof. We denote $\sigma = (\ln y)^{-1}\ln\frac{\ln T}{\ln y}$, and then we have

$$\sum_{n_1\ge T}\frac{1}{n_1} \le \frac{1}{T^\sigma}\sum_{n_1}\frac{n_1^\sigma}{n_1} \ll \exp\left(-\sigma\ln T + \sum_{p\le y}\frac{p^\sigma}{p}\right).$$

The second sum is

$$\le e\sum_{p\le\exp(1/\sigma)}\frac{1}{p} + y^\sigma\sum_{\exp(1/\sigma)<p\le y}\frac{1}{p} \le \ln\frac{1}{\sigma} + y^\sigma\ln(\sigma\ln y) + O(y^\sigma),$$

i.e., the sum being estimated is

$$\ll \exp\left(-\frac{\ln T}{\ln y}\,\ln\frac{\ln T}{\ln y} + e\ln\frac{\ln y}{\ln\frac{\ln T}{\ln y}} + \frac{\ln T}{\ln y}\,\ln_2\frac{\ln T}{\ln y} + O\left(\frac{\ln T}{\ln y}\right)\right).$$

Hence follows the statement of the lemma.

Using Lemma 4, we get $\Delta_{k,2}(Q,x) \ll x\ln^{-B}x + Q\sqrt{x}\,\ln^{-B}x$, where B is any positive constant. Thus it follows from here and from (8) and (9) that

$$\Delta_k(Q,x) \ll Q\sqrt{x}\,\ln^{6B+2}x + x\ln^{10-B/2}x$$

for $k\le b\ln_2 x$. If $k\le b\ln_2 x$, then it follows from (1), (2) and from Lemma 2 that $\mathsf{P}(x,k)\ge c_1x2^{-b\ln_2 x}$, where $c_1>0$.

Therefore the final result is

$$\Delta_k(Q,x)\ll\left(\frac{Q}{\sqrt{x}}\ln^{6B+b\ln 2+2}x+\ln^{-B/2+b\ln 2+10}x\right)\mathsf{P}(x,k).$$

Let $k\ge b\ln_2 x$, $b>3$. In this case, taking into account that all prime factors n_2 exceed t, we have

$$\Delta_k(Q,x)\le\sum_{\substack{n\le T\\ k-\frac{\ln x}{\ln t}\le\Omega(n_1)\le k}}\Delta_{k-\Omega(n_1)}\left(Q,\frac{x}{n_1},t\right)+\Delta_{k,2}(Q,x), \tag{10}$$

where $\Delta_{k,2}(Q,x)$ is the same as in relation (8).

Let

$$Q\le\frac{\sqrt{x}}{\sqrt{T}}\ln^{-5A-19}x,\qquad \ln^{10A+40}x\le t\le\sqrt{x},\qquad T\le x^{1-\varepsilon}.$$

Then, by virtue of (4), the first sum will be

$$\ll\sum_{k-\frac{\ln x}{\ln t}\le l\le k}\sum_{n_1\le T}\frac{f_l(n_1)}{n_1}x\ln^{-A}x.$$

Let us represent n_1 as $n_1=2^rm$, where $(m,2)=1$. Then we have

$$\sum_{n_1\le T}\frac{f_l(n_1)}{n_1}\le\frac{l}{2^l}\sum_{m\le T}\frac{2^{\Omega(m)}}{m}\ll\frac{l}{2^l}\ln^2 t.$$

Thus the first sum is

$$\ll\frac{x}{\ln^A x}k2^{\frac{\ln x}{\ln t}}\frac{\ln^2 t}{2^k}. \tag{11}$$

Let us estimate $\Delta_{k,2}(Q,x)$. We represent n_1 as $n_1=d_1m_1$, where all prime factors of d_1 are smaller than 17 and all prime factors of $m_1\ge 17$ but $<t$. We assume that $Q\le\sqrt{x/13^k}$. Using Lemma 1, we obtain

$$\Delta_{k,2}(Q,x)\ll\sum_{d_1\le 13^k}\left(\sum_{\substack{d_1m_1n_2\le x\\ d_1m_1\ge T}}f_k(d_1m_1n_2)\right)^{1/2}\sqrt{\frac{x}{d_1}}\ln^{3/2}x.$$

Let T be such that $13^k\le\sqrt{T}$. Then we have

$$\sum_{\substack{m_1n_2\le x/d_1\\ m_1>T/d_1}}f_k(d_1m_1n_2)\le\frac{4^{\Omega(d_1)}}{4^k}\sum_{\substack{m_1n_2\le x/d_1\\ m_1>\sqrt{T}}}\frac{4^{\Omega(m_1n_2)}}{m_1n_2}\frac{x}{d_1}\ll\frac{x}{4^k}\frac{4^{\Omega(d_1)}}{d_1}\ln^4 x\sum_{\sqrt{T}\le m_1\le x/d_1}\frac{4^{\Omega(m_1)}}{m_1}.$$

Thus we obtain

$$\Delta_{k,2}(Q,x)\ll\frac{x}{2^k}\ln^4 x\sum_{d_1\le x}\frac{2^{\Omega(d_1)}}{d_1}\left(\sum_{\sqrt{T}\le m_1\le x}\frac{4^{\Omega(m_1)}}{m_1}\right)^{1/2}.$$

The sum over d_1 is

$$\le k \prod_{3\le p\le 13}\left(1+\sum_{r=1}^{\infty}\left(\frac{2}{p}\right)^r\right) \ll \ln x.$$

Applying the Cauchy inequality, we get

$$\sum_{\sqrt{T}\le m_1\le x}\frac{4^{\Omega(m_1)}}{m_1} \le \left(\sum_{m_1\le x}\frac{16^{\Omega(m_1)}}{m_1}\right)^{1/2}\left(\sum_{\sqrt{T}\le m_1\le x}\frac{1}{m_1}\right)^{1/2}.$$

The first sum is equal to $O(\ln^{16} x)$. We take

$$t=\exp\left(\frac{2\ln 2}{A\ln_2 x}\ln x\right),\quad T=t^{\Theta(x)\frac{\ln_2 x}{\ln 2}},\qquad \text{where}\quad \frac{A\ln 2}{\ln_3 x}\le \Theta(x)\le (1-\varepsilon)\frac{A}{2}.$$

In this case,

$$\frac{A\ln_2 x}{\ln_3 x}\le \frac{\ln T}{\ln t}\le (1-\varepsilon)\frac{A\ln_2 x}{2\ln 2}\le \ln t.$$

Using Lemma 4, we obtain

$$\sum_{\sqrt{T}\le m_1\le x}\frac{1}{m_1}\ll \ln x\exp\left(-\frac{2}{5}\frac{\ln T}{\ln t}\ln\frac{\ln T}{\ln t}\right)\ll \ln^{1-A/3} x,$$

and therefore, $\Delta_{k,2}(Q,x)\ll \frac{x}{2^k}\ln^{10-A/12} x$. From this and from (10), (11) it follows that

$$\Delta_k(Q,x)\ll \frac{x}{2^k}\left(k2^{\frac{\ln x}{\ln t}}(\ln t)^2(\ln x)^{-A}+\ln^{-A/12+10} x\right),$$

i.e., for our choice of t and T, we obtain the statement of Theorem 1 with

$$Q\le \sqrt{\frac{x}{T}}\ln^{-19-5A} x,\qquad k\le \Theta(x)\frac{\ln x}{A\ln 13}.$$

Proof of Theorem 2. By the hypothesis, $k\le \eta(x)\ln x$, where $\eta(x)\le \frac{1-\varepsilon}{2\ln 13}$. We take $Q=\sqrt{x}\exp(-\eta(x)\times\ln 13\times\ln x)\times\ln^{-19-5A} x$, i.e., in the notation of Theorem 1, $\Theta(x)=\eta(x)\ln 13$. After the summation under the hyperbola, we obtain

$$\sum_{n\in A(x,k)}\tau(n-1)=2\sum_{d\le Q}\sum_{\substack{n\in A(x,k)\\ n\equiv 1\ (\mathrm{mod}\, d)}}1+O\left(\sum_{Q\le d\le\sqrt{x+1}}\sum_{\substack{n\in A(x,k)\\ n\equiv 1\ (\mathrm{mod}\, d)}}1+\sum_{d\le Q}\sum_{\substack{n\in A(d\sqrt{x+1},k)\\ n\equiv 1\ (\mathrm{mod}\, d)}}1\right). \tag{12}$$

We apply Theorem 1 to the first sum. We find that it is equal to

$$2\sum_{n\in A(x,k)}\sum_{\substack{d\le Q\\ (d,n)=1}}\frac{1}{\varphi(d)}+O\big(\mathsf{P}(x,k)\ln^{14-A/6} x\big).$$

Let us investigate the principal term. Since (see, e.g., [8, Lemma 3]),

$$\sum_{\substack{d\le Q\\(d,n)=1}}\frac{1}{\varphi(d)}=F\prod_{p|n}\Bigl(1+\frac{1}{p(p-1)}\Bigr)^{-1}\Bigl(1-\frac{1}{p}\Bigr)\ln Q+O\Bigl(\sum_{\delta|n}\frac{\ln 2\delta}{\delta}\Bigr),$$

where

$$F=\prod_p\frac{p^2-p+1}{p(p-1)},$$

it is equal to

$$2F\ln Q\sum_{n\in A(x,k)}\Bigl(\sum_{\delta|n}\mu(\delta)f(\delta)+O\Bigl(\frac{1}{\ln x}\sum_{\delta|n}\frac{\ln 2\delta}{\delta}\Bigr)\Bigr).$$

Here $f(n)$ is a multiplicative function $f(p)=p/(p^2-p+1)<1/\varphi(p)$. Since

$$\begin{aligned}\sum_{\delta|n}\frac{\ln 2\delta}{\delta}&\ll\Bigl(1+\sum_{p^r\|n}\frac{\ln p^r}{p^r}\Bigr)\prod_{p|n}\Bigl(1+\frac{1}{p}\Bigr)^{-1}\\&\ll\Bigl(\sum_{p^r\le\ln 10n}\frac{\ln p^r}{p^r}+O(1)\Bigr)\exp\Bigl(\sum_{p\le\ln 10n}\frac{1}{p}+O(1)\Bigr)\ll(\ln_2 10n)^2,\end{aligned}$$

we can write the principal term as

$$2F\ln Q\Bigl(\sum_{\delta\le z}\mu(\delta)f(\delta)P_{k-\Omega(\delta)}\Bigl(\frac{x}{\delta}\Bigr)+O\Bigl(\sum_{n\in A(x,k)}\Bigl(\sum_{\delta|n,\ \delta>z}\frac{1}{\varphi(z)}+\frac{(ln_2 10x)^2}{\ln x}\Bigr)\Bigr)\Bigr).$$

We have used here the fact that $\mu^2(n)f(n)\le 1/\varphi(n)$. Let us estimate the remainder. We choose $z=(\ln x)^{\varepsilon/2}$. We divide the sum over $\delta\mid n$ into two sums over δ_1 with $\Omega(\delta_1)\le\frac{\varepsilon}{2}\ln_2 x$ and over the remaining δ_2. With (2) taken into account, the first sum is equal to

$$\sum_{\delta_1>z}\frac{1}{\varphi(\delta_1)}P_{x-\Omega(\delta_1)}\Bigl(\frac{x}{\delta_1}\Bigr)\ll\sum_{\delta>z}\frac{2^{\Omega(\delta)}}{\delta\varphi(\delta)}\frac{x}{2^k}\ \ln\frac{x}{2^k}\ll\frac{\mathsf{P}(x,k)}{\sqrt{z}}\sum_{\delta}\frac{2^{\Omega(\delta)}}{\sqrt{\delta}\varphi(\delta)}.$$

The sum over δ_2 is

$$\ll q^{-\frac{\varepsilon}{2}\ln_2 x}\sum_{n\in A(x,k)}\sum_{\delta|n}\frac{q^{\Omega(\delta)}}{\varphi(\delta)}.$$

We take $q=\sqrt{e}$. Having in mind that

$$\sum_{\delta|n}\frac{q^{\Omega(\delta)}}{\varphi(\delta)}\ll\exp\Bigl(\sum_{p|n}\frac{q}{p}\Bigr)\ll(\ln_2 10n)^q,$$

we find that the remainder is equal to $O\bigl(P_k(x)\ln^{-\varepsilon/5}x\bigr)$. In the principal term we use relation (2). Since $\delta\le z$, it follows that $\Omega(\delta)\le\frac{\varepsilon}{2}\ln_2 x$, i.e., the constraints imposed on $k-\Omega(\delta)$ are fulfilled.

We obtain

$$\sum_{\delta\le z}\mu(\delta)f(\delta)P_{k-\Omega(\delta)}\Big(\frac{x}{\delta}\Big)=\frac14 c\frac{x}{2^k}\ln\frac{x}{2^k}\bigg(\sum_{\delta}\mu(\delta)\frac{f(\delta)}{\delta}+O\bigg(\sum_{\delta>z}\frac{1}{\delta\varphi(\delta)}+\ln^{b+1}x\bigg)\bigg)$$
$$=\frac{c}{4}\prod_p\bigg(1-\frac{1}{p^2-p+1}\bigg)\frac{x}{2^k}\ln\frac{x}{2^k}\big(1+O(\ln^{-\varepsilon/2}x)\big).$$

Thus the first sum on the right-hand side of relation (12) is equal to

$$2(\ln Q)\mathsf{P}(x,k)\big(1+O(\ln^{-\varepsilon/5}x)\big). \tag{13}$$

Let us estimate the third sum in relation (12). It is

$$\sum_{n\le 2\sqrt{x}Q}f_k(n)\tau(n-1)\le\bigg(\sum_{n\le 2\sqrt{x}Q}f_k(n)\bigg)^{1/2}\bigg(\sum_{n\le 2\sqrt{x}Q}\tau^2(n)\bigg)^{1/2}.$$

Taking into account that $Q\le\sqrt{x}13^{-k}\ln^{-9-5A}x$, we find that it is equal to $O(x/2^k)$.

Let us estimate the second sum in relation (12). We represent n as $n=d_1n_1n_2$, where all prime factors of d_1 are smaller than 17 and those of $n_1\ge 17$, but smaller than s, and the prime factors of n_2 are $\ge s$. Then $d_1\le 13^k$, and, since $k\le\eta(x)\ln x$, where $\eta(x)\to 0$ as $x\to\infty$, we can assume that $d_1\le x^{1/5}$. Applying the Cauchy inequality, we find that the part of the sum in which $n_1\ge\Lambda$ is

$$\le\sum_{d_1\le x}\Bigg(\sum_{\substack{d_1n_1n_2\le x\\ n_1\ge\Lambda}}\Bigg)^{1/2}\bigg(\sum_{d_1n_1n_2\le x}\tau^2(d_1n_1n_2-1)\bigg)^{1/2}.$$

Taking into account that the number of d, δ such that $[d,\delta]=n$, does not exceed $\tau_3(n)$, we obtain

$$\sum_{n_1n_2\le x/d_1}\tau^2(d_1n_1n_2-1)\le\sum_{n_1n_2\le x/d_1}\tau_4(d_1n_2n_1-1)\le\sum_{\substack{m\le x\\ m\equiv -1\ (\mathrm{mod}\ d_1)}}\tau_4(m)\ll\frac{x}{d_1}\ln^3 x.$$

Consequently, the part of the sum being estimated is

$$\ll\frac{x}{2^k}\sum_{d_1\le 13^k}\frac{2^{\Omega(d_1)}}{d_1}\Bigg(\sum_{\substack{n_1n_2\le x\\ n_1>\Lambda}}\frac{4^{\Omega(n_1n_2)}}{n_1n_2}\Bigg)^{1/2}.$$

We choose $s=\exp\big(100\ln_3 x\times\ln x\times(\ln_2 x)^{-1}\big)$, $\Lambda=x^{1/5}$. Just as in the proof of Theorem 1, we find that the sum over d_1 is equal to $O(k)=O(\eta(x)\ln x)$ and the sum over n_1n_2 is

$$\ll\ln^9 x\exp\bigg(-\frac25\frac{\ln\Lambda}{\ln s}\ln\frac{\ln\Lambda}{\ln s}\bigg)\ll\ln^9 x\exp(-8\ln_2 x)\ll\ln x,$$

i.e., the part of the sum being estimated is equal to $O\big(\frac{x}{2^k}\ln^{3/2}\frac{x}{2^k}\big)$.

Thus, it remains to estimate the part of the second sum of relation (12) in which $d_1n_1 \le \Lambda 13^k \le x^{1/3}$. We can assume that all prime factors of n_2 divide n_2 in the first degree. Indeed, the remaining part of the sum is

$$\ll \frac{x}{2^k} \sum_{\substack{d_1n_1p^2n_2 \le x \\ p>s}} \frac{2^{\Omega(d_1n_1p^2n_2)}}{d_1n_1p^2n_2} \tau(d_1n_1p^2n_2 - 1) \ll \frac{x}{2^k}.$$

Consequently, we have to estimate the sum which is

$$\le \sum_{Q \le d \le \sqrt{x}} \sum_{i=0}^{J} \frac{1}{\rho^i} \sum_{\substack{d_1n_1 \le x^{1/3} \\ \Omega(d_1n_1)=k-i}} \sum_{\substack{n_2 \le x/d_1n_1 \\ d_1n_1n_2 \equiv 1 \pmod d}} \rho^{\omega(n_2)},$$

where $J = \frac{\ln x}{\ln s}$, $\rho > 1$. In order to estimate this sum, we use Shiu's theorem [9].

Lemma 5. *Let $g(n)$ be a multiplicative function such that $O \le g(p^r) \le A_1^r$ for all prime numbers p and natural r, $g(n) \le A_2 n^{\varepsilon}$ for $n \ge 1$, $(a,d) = 1$, and $d \le x^{1-\varepsilon}$, $0 < \varepsilon < 1$. Then there exists a constant A_3, depending only on A_1, A_2, ε, with which*

$$\sum_{\substack{n \le x \\ n \equiv a \pmod d}} g(n) \le A_3 \frac{x}{\varphi(d) \ln x} \exp\left(\sum_{\substack{p \le x \\ p \nmid d}} \frac{g(p)}{p} \right).$$

Using Lemma 5, we find that the sum being estimated is

$$\ll \frac{x \ln \frac{\sqrt{x}}{Q}}{\ln x} \sum_{i=0}^{J} \frac{1}{\rho^i} \sum_{d_1n_1 \le x^{1/3}} \frac{f_{k-i}(d_1n_1)}{d_1n_1} \exp\Big(\rho \ln \frac{\ln x}{\ln s}\Big).$$

Since $J \le \frac{\ln x}{\ln s}$, it follows that $k - i \ge (2 + \varepsilon/2)\ln x$. Therefore, taking into account relation (2), after the Abel summation, we obtain

$$\sum_{n \le x} \frac{f_{k-i}(n)}{n} \ll \frac{x}{2^{k-i}} \ln^2 x.$$

Consequently, if we choose $\rho > 2$, then the sum being estimated is equal to

$$O\bigg(\frac{x}{2^k} \ln x \times \ln \frac{\sqrt{x}}{Q} \times \Big(\frac{\ln x}{\ln s}\Big)^{\rho}\bigg).$$

Substituting the obtained estimate into (12) and taking into account (13), we obtain

$$\sum_{n \in A(x,k)} \tau(n-1) = \mathsf{P}(x,k) \ln x \big(1 + O(\ln^{-\varepsilon} x)\big) + O\bigg(\frac{x}{2^k} \ln \frac{x}{2^k} \times \ln \frac{\sqrt{x}}{Q} \times \Big(\frac{\ln_2 x}{\ln_3 x}\Big)^{\rho}\bigg),$$

where $\rho > 2$. For $Q = \sqrt{x} \exp(-\eta(x) \ln x) \ln^{-19-5A} x$ the seconnd term is equal to

$$O\bigg(\frac{x}{2^k} \ln \frac{x}{2^k} \times \ln x \times \eta(x) \Big(\frac{\ln_2 x}{\ln_3 x}\Big)^{\rho}\bigg).$$

Hence follows the statement of Theorem 2.

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