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On the Number of Representations of an Odd Integer as a Sum of Three Primes, One of Which Belongs to an Arithmetic Progression

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1. INTRODUCTION AND STATEMENT OF THE RESULT

In 1937 I. M. Vinogradov [8] proved that for every sufficiently large integer N the equation

$$p_1 + p_2 + p_3 = N \tag{1}$$

had solutions in prime numbers p_1, p_2, p_3. After the publication of Vinogradov's remarkable work many mathematicians investigated the solvability of equation (1) in primes satisfying some additional conditions. Zulauf [9, 10] considered (1) with prime numbers lying in arithmetic progressions. Suppose that k, l are integers, $(k, l) = 1$, and denote

$$J_{k,l}(N) = \sum_{\substack{p_1+p_2+p_3=N \\ p_1 \equiv l; (\mathrm{mod}\, k)}} \log p_1 \log p_2 \log p_3. \tag{2}$$

If $k \leq (\log N)^D$ for some fixed $D > 0$, then the following asymptotic formula holds as N tends to infinity:

$$J_{k,l}(N) = \frac{N^2}{2\varphi(k)} \mathfrak{S}_{k,l}(N) + O\left(\frac{N^2}{(\log N)^A} \right).$$

Here $A > 0$ is arbitrarily large,

$$\mathfrak{S}_{k,l}(N) = \prod_{p \nmid kN} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{\substack{p \nmid k \\ p \mid N}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid k \\ p \nmid N-l}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid k \\ p \mid N-l}} \frac{p}{p-1}, \tag{3}$$

and the constant in the O-term depends only on D and A.

In 1939 Van der Corput [5] used the method of I. M. Vinogradov and proved that there existed infinitely many arithmetic progressions consisting of three different primes. This result has not been proved yet for progressions of four or more terms. In 1981, however, Heath-Brown [3] proved that there existed infinitely many arithmetic progressions of four different terms, three of which are primes and the fourth is an integer with no more than two prime factors (counted according to multiplicity).

The main point in [3] is a result of Bombieri–Vinogradov type for the quantity

$$w(n) = \sum_{\substack{x<p_2,\ p_3\le 2x\\ p_1+p_3=2p_2\\ p_2+n=2p_3}} u(p_1)u(p_2)u(p_3),$$

where $u(n) = \log n/\log 3x$ for $n \ge 5$ and $u(n) = 0$ otherwise. Heath-Brown proved that for each $\varepsilon > 0$

$$\sum_{\substack{d\le x^{1/2-\varepsilon}\\ (d,6)=1}} \mu^2(d)\left|\sum_{n:d|n} w(n) - X\frac{\omega(d)}{d}\right| \ll \frac{X}{(\log x)^{4/3}},$$

where

$$X = 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\times\frac{x^2}{(\log 3x)^3} \qquad\text{and}\qquad \omega(d) = \prod_{p|d}\frac{p}{p-2}.$$

It is interesting to find a similar result for the quantity $J_{k,l}(N)$. Let us denote

$$\mathcal{E} = \mathcal{E}(N,\theta) = \sum_{k\le N^\theta}\max_{(l,k)=1}\left|J_{k,l}(N) - \frac{N^2}{2\varphi(k)}\mathfrak{S}_{k,l}(N)\right|. \tag{4}$$

It is natural to expect that using the method of Heath-Brown one can prove that

$$\mathcal{E} \ll \frac{N^2}{(\log N)^A} \tag{5}$$

for each $\theta < 1/2$ and for an arbitrarily large A. However, at present we can verify (5) only for $\theta < 1/3$. We shall prove the following

Theorem. *Suppose that $J_{k,l}(N)$, $\mathfrak{S}_{k,l}(N)$, and $\mathcal{E}$ are defined by (2), (3) and (4), respectively. Then, for an arbitrarily large A and for each $\theta < 1/3$ inequality (5) holds.*

Using this theorem and some results from the theory of sieves, one can prove that for each sufficiently large odd N equation (1) has a solution in primes such that p_1+2 has at most H prime factors, where H is a fixed integer. In order to prove this with smaller H, we need estimate (5) for θ as large as possible. At the end of the paper we present an approach that can be used to improve the upper bound of θ for which (5) holds true.

2. NOTATION AND THE OUTLINE OF THE PROOF

We denote real numbers by x, y, r, u, α, θ, prime numbers by p, p_1, p_2, p_3, and integers by m, n, d, t, a, q, k, l, s, h, f. We always have $(k,l) = 1$. As usual, $\mu(n)$, $\varphi(n)$, $\tau(n)$ denote Möbius' function, Euler's function, and the number of positive divisors of n. We denote by (m,n) and $[m,n]$ the greatest common divisor and the least common multiple of m and n respectively. For real x, y, however, (x,y) and $[x,y]$ denote the open interval (closed interval) on the real line with endpoints x and y. The meaning is always clear from the context. Instead of $m \equiv n; (\operatorname{mod} k)$, we shall write, for simplicity, $m \equiv n\ (k)$. We shall also use the notation $e(x) = \exp(2\pi i x)$ and $\|x\| = \min_{n\in\mathbb{Z}}|x-n|$. The letter c denotes some positive real number, not the same in all appearances. This convention allows us to write

$$(\log x)e^{-c\sqrt{\log x}} \ll e^{-c\sqrt{\log x}},$$

for example.

Let N be a sufficiently large odd integer. Let $A > 100$ be a constant and B be a constant, sufficiently large in terms of A, which we shall define later by (32) and (64). Let us denote $Q = (\log N)^B$, $\tau = NQ^{-1}$,

$$E_1 = \bigcup_{q \le Q} \bigcup_{\substack{a=0 \\ (a,q)=1}}^{q-1} \left(\frac{a}{q} - \frac{1}{q\tau}, \frac{a}{q} + \frac{1}{q\tau}\right), \qquad E_2 = \left(-\frac{1}{\tau}, 1 - \frac{1}{\tau}\right) \Big\backslash E_1,$$

$$S_{k,l}(\alpha) = \sum_{\substack{p \le N \\ p \equiv l\ (k)}} (\log p) e(\alpha p), \qquad S(\alpha) = S_{1,0}(\alpha), \qquad M(\alpha) = \sum_{m=1}^{N} e(\alpha m).$$

We have

$$J_{k,l}(N) = \int_{-1/\tau}^{1-1/\tau} S_{k,l}(\alpha) S^2(\alpha) e(-N\alpha) d\alpha = J_{k,l}^{(1)}(N) + J_{k,l}^{(2)}(N),$$

where

$$J_{k,l}^{(i)}(N) = \int_{E_i} S_{k,l}(\alpha) S^2(\alpha) e(-N\alpha) d\alpha, \qquad i = 1, 2.$$

Consequently

$$\mathcal{E} \le \mathcal{E}_1 + \mathcal{E}_2, \tag{6}$$

where

$$\mathcal{E}_1 = \mathcal{E}_1(N, \theta) = \sum_{k \le N^\theta} \max_{(l,k)=1} \left| J_{k,l}^{(1)}(N) - \frac{N^2}{2\varphi(k)} \mathfrak{S}_{k,l}(N) \right|, \tag{7}$$

$$\mathcal{E}_2 = \mathcal{E}_2(N, \theta) = \sum_{k \le N^\theta} \max_{(l,k)=1} \left| J_{k,l}^{(2)}(N) \right|. \tag{8}$$

In Section 3, we use the Hardy–Littlewood circle method and the theorem of Bombieri and Vinogradov (see [1] and [7]) to prove that

$$\mathcal{E}_1 \ll \frac{N^2}{(\log N)^A} \qquad \text{for each} \quad \theta < \frac{1}{2}. \tag{9}$$

In Section 4, we estimate $\mathcal{E}_2$ using the method of Heath-Brown from [3]. Two conjectures are stated in which the real parameter u appears. It is proved that the validity of some of the conjectures for a certain $u > 2$ implies the estimate

$$\mathcal{E}_2 \ll \frac{N^2}{(\log N)^A} \qquad \text{for each} \quad \theta < \frac{1}{u}. \tag{10}$$

Hence, by (6), (9), (10) we get (5) for each $\theta < 1/u$. It is also shown that Lemma 3 of [3] immediately implies the validity of Conjecture 1 for $u = 4$, and therefore we have (5) provided that $\theta < 1/4$.

In Section 5 we prove Conjecture 2 for $u = 3$ which implies the proof of the Theorem. An approach used to prove some of the conjectures for $u > 2$ and u which is close to 2 is also presented.

3. THE ESTIMATE OF $\mathcal{E}_1$

It is clear that

$$J_{k,l}^{(1)}(N) = \sum_{q \le Q} \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} I(a,q), \tag{11}$$

where

$$I(a,q) = \int_{-1/(q\tau)}^{1/(q\tau)} S_{k,l}\left(\frac{a}{q}+\alpha\right) S^2\left(\frac{a}{q}+\alpha\right) e\left(-N\left(\frac{a}{q}+\alpha\right)\right) d\alpha. \tag{12}$$

If

$$q \le Q, \qquad (a,q)=1, \qquad |\alpha| \le \frac{1}{q\tau} \tag{13}$$

then we have

$$S\left(\frac{a}{q}+\alpha\right) = \frac{\mu(q)}{\varphi(q)} M(\alpha) + O\left(Ne^{-c\sqrt{\log N}}\right) \tag{14}$$

(see [6, Lemma 3.1] for the proof).

Let us consider $S_{k,l}(a/q+\alpha)$ for a, q, α satisfying (13). We cannot find an asymptotic formula for this sum for a particularly large k (unless we use some hypotheses which have not been proved so far). We shall find, however, an asymptotic formula with an error term which is small on the average.

We have

$$S_{k,l}\left(\frac{a}{q}+\alpha\right) = \sum_{\substack{1 \le m \le q \\ (m,q)=1 \\ m \equiv l\ ((k,q))}} e\left(\frac{am}{q}\right) T(\alpha) + O(q \log N), \tag{15}$$

where

$$T(\alpha) = \sum_{\substack{p \le N \\ p \equiv l\ (k) \\ p \equiv m\ (q)}} (\log p) e(\alpha p).$$

Using the elementary theory of congruences, one can easily prove that if the integers k, l, m, q satisfy $(k,l)=(m,q)=1$ and $m \equiv l\ ((k,q))$, then there exists an integer $f=f(k,l,m,q)$ such that $(f,[k,q])=1$ and such that for any integer n the congruence $n \equiv f\ ([k,q])$ is equivalent to the system $n \equiv l\ (k)$, $n \equiv m\ (q)$. Hence, we have

$$T(\alpha) = \sum_{\substack{p \le N \\ p \equiv f\ ([k,q])}} (\log p) e(\alpha p).$$

Let us denote

$$\Delta(x,h) = \max_{y \le x} \max_{(l,h)=1} \left| \sum_{\substack{p \le y \\ p \equiv l\ (h)}} \log p - \frac{y}{\varphi(h)} \right|. \tag{16}$$

We use Abel's formula and obtain

$$T(\alpha) = -\int_0^N \Bigg(\sum_{\substack{p\le y \\ p\equiv f\ ([k,q])}} \log p \Bigg) \frac{d}{dy} e(\alpha y)\, dy + \Bigg(\sum_{\substack{p\le N \\ p\equiv f\ ([k,q])}} \log p \Bigg) e(\alpha N)$$

$$= -\int_0^N \bigg(\frac{y}{\varphi([k,q])} + O\big(\Delta(N,[k,q])\big) \bigg) \frac{d}{dy} e(\alpha y)\, dy + \bigg(\frac{N}{\varphi([k,q])} + O\big(\Delta(N,[k,q])\big) \bigg) e(\alpha N)$$

$$= \frac{1}{\varphi([k,q])} \bigg(-\int_0^N y\big(\frac{d}{dy} e(\alpha y)\big)\, dy + N e(\alpha N) \bigg) + O\big((1+|\alpha|N)\Delta(N,[k,q])\big).$$

We integrate by parts, and then use (13) and the well-known formula

$$\int_0^N e(\alpha y)\, dy = M(\alpha) + O(1).$$

As a result we get

$$T(\alpha) = \frac{M(\alpha)}{\varphi([k,q])} + O\bigg(\Big(1 + \frac{N}{q\tau}\Big) \Delta(N,[k,q]) \bigg).$$

We substitute the last expression for $T(\alpha)$ into (15) and find out that under condition (13) we have

$$S_{k,l}\Big(\frac{a}{q} + \alpha\Big) = \frac{c_{k,l}(a,q)}{\varphi([k,q])} M(\alpha) + O\big(Q(\log N)\Delta(N,[k,q])\big), \tag{17}$$

where

$$c_{k,l}(a,q) = \sum_{\substack{1\le m\le q \\ (m,q)=1 \\ m\equiv l\ ((k,q))}} e\Big(\frac{am}{q}\Big). \tag{18}$$

Formulas (14), (17) and the trivial estimate

$$\Big| S_{k,l}\Big(\frac{a}{q} + \alpha\Big) \Big| \ll \frac{N \log N}{k}$$

imply that under condition (13) we have

$$S_{k,l}\Big(\frac{a}{q} + \alpha\Big) S^2\Big(\frac{a}{q} + \alpha\Big) e\Big(-N\Big(\frac{a}{q} + \alpha\Big)\Big) = \frac{\mu^2(q)}{\varphi^2(q)} \frac{c_{k,l}(a,q)}{\varphi([k,q])} e\Big(-N\frac{a}{q}\Big) M^3(\alpha) e(-N\alpha)$$
$$+ O\bigg(\frac{N^3}{k} e^{-c\sqrt{\log N}} \bigg) + O\bigg(\frac{QN^2 \log N}{\varphi^2(q)} \Delta(N,[k,q]) \bigg).$$

For the integral $I(a,q)$ defined by (12) we find

$$I(a,q) = \frac{\mu^2(q)}{\varphi^2(q)} \frac{c_{k,l}(a,q)}{\varphi([k,q])} e\Big(-N\frac{a}{q}\Big) \int\limits_{-1/(q\tau)}^{1/(q\tau)} M^3(\alpha) e(-N\alpha)\, d\alpha$$
$$+ O\Big(\frac{N^2}{k} e^{-c\sqrt{\log N}}\Big) + O\Big(\frac{Q^2 N \log N}{q\varphi^2(q)} \Delta(N,[k,q])\Big).$$

Hence,

$$\sum_{\substack{a=0\\(a,q)=1}}^{q-1} I(a,q) = \frac{\mu^2(q)}{\varphi^2(q)} \frac{b_{k,l}(q)}{\varphi([k,q])} \int\limits_{-1/(q\tau)}^{1/(q\tau)} M^3(\alpha) e(-N\alpha)\, d\alpha$$
$$+ O\Big(\frac{N^2}{k} e^{-c\sqrt{\log N}}\Big) + O\Big(\frac{Q^2 N \log N}{q\varphi(q)} \Delta(N,[k,q])\Big), \tag{19}$$

where

$$b_{k,l}(q) = \sum_{\substack{a=0\\(a,q)=1}}^{q-1} c_{k,l}(a,q) e\Big(-N\frac{a}{q}\Big)$$

and where $c_{k,l}(a,q)$ is defined by (18). It is easy to see that $b_{k,l}(q)$ is a multiplicative function of q and that for the prime p we have

$$b_{k,l}(p) = \begin{cases} 1 & \text{for } p \nmid kN \\ 1-p & \text{for } p \nmid k,\ p \mid N \\ -1 & \text{for } p \mid k,\ p \nmid N-l \\ p-1 & \text{for } p \mid k,\ p \mid N-l. \end{cases} \tag{20}$$

Consequently,

$$\mu^2(q)|b_{k,l}(q)| \le \varphi(q). \tag{21}$$

We also have

$$\int\limits_{-1/(q\tau)}^{1/(q\tau)} M^3(\alpha) e(-N\alpha)\, d\alpha = \frac{1}{2} N^2 + O(q^2\tau^2) \tag{22}$$

(see the proof of Theorem 3.3 in [6]). Using (19), (21), (22), we get

$$\sum_{\substack{a=0\\(a,q)=1}}^{q-1} I(a,q) = \frac{N^2}{2\varphi(k)} \lambda_{k,l}(q) + O\Big(\frac{q^2\tau^2}{\varphi(q)\varphi([k,q])}\Big) + O\Big(\frac{N^2}{k} e^{-c\sqrt{\log N}}\Big) + O\Big(\frac{Q^2 N \log N}{q\varphi(q)} \Delta(N,[k,q])\Big), \tag{23}$$

where

$$\lambda_{k,l}(q) = \frac{\mu^2(q) b_{k,l}(q) \varphi(k)}{\varphi^2(q)\varphi([k,q])}.$$

It is easy to see that

$$\frac{\varphi(k)}{\varphi([k,q])} = \frac{\varphi((k,q))}{\varphi(q)}, \tag{24}$$

which, as a function of q, is multiplicative. Hence, $\lambda_{k,l}(q)$ is multiplicative with respect to q too. We use Euler's identity (see [2, Theorem 286]) and also (3), (20), (21), (24) and get

$$\begin{aligned}\sum_{q\le Q}\lambda_{k,l}(q) &= \sum_{q=1}^{\infty}\lambda_{k,l}(q) + O\Big(\sum_{q>Q}|\lambda_{k,l}(q)|\Big) = \prod_p\big(1+\lambda_{k,l}(p)\big) + O\Big(\sum_{q>Q}\frac{\varphi((k,q))}{\varphi^2(q)}\Big)\\ &= \mathfrak{S}_{k,l}(N) + O\Big(\sum_{q>Q}\frac{\varphi((k,q))}{\varphi^2(q)}\Big).\end{aligned} \tag{25}$$

From (11), (23)–(25) we find

$$\begin{aligned}J_{k,l}^{(1)}(N) &= \frac{N^2}{2\varphi(k)}\mathfrak{S}_{k,l}(N) + O\Big(\frac{N^2}{\varphi(k)}\sum_{q>Q}\frac{\varphi((k,q))}{\varphi^2(q)}\Big) + O\Big(\sum_{q\le Q}\frac{q^2\tau^2\varphi((k,q))}{\varphi^2(q)\varphi(k)}\Big)\\ &\quad + O\Big(\frac{N^2}{k}e^{-c\sqrt{\log N}}\Big) + O\Big(\sum_{q\le Q}\frac{Q^2N\log N}{q\varphi(q)}\Delta(N,[k,q])\Big).\end{aligned}$$

We substitute the last expression for $J_{k,l}^{(1)}(N)$ into (7) and use the well-known estimate $\varphi(n)\gg n(\log\log n)^{-1}$ to obtain

$$\mathcal{E}_1 \ll \big(N^2\Sigma_1 + \tau^2\Sigma_2 + Q^2N\Sigma_3\big)\log^2 N + N^2e^{-c\sqrt{\log N}}, \tag{26}$$

where

$$\Sigma_1 = \sum_{k\le N^\theta}\sum_{q>Q}\frac{(k,q)\log q}{kq^2}, \qquad \Sigma_2 = \sum_{k\le N^\theta}\sum_{q\le Q}\frac{(k,q)}{k}, \qquad \Sigma_3 = \sum_{k\le N^\theta}\sum_{q\le Q}\frac{1}{q^2}\Delta(N,[k,q]).$$

Let us estimate Σ_1. We have

$$\begin{aligned}\Sigma_1 &= \sum_{d\le N^\theta} d\sum_{k\le N^\theta}\sum_{\substack{q>Q\\(k,q)=d}}\frac{\log q}{kq^2} \ll \sum_{d\le N^\theta} d\sum_{k\le N^\theta/d}\sum_{q>Q/d}\frac{\log(qd)}{kdq^2d^2}\\ &\ll \log N\sum_{d\le Q}\frac{1}{d^2}\sum_{k\le N^\theta/d}\frac{1}{k}\sum_{q>Q/d}\frac{\log q}{q^2} + \log N\sum_{Q<d\le N^\theta}\frac{1}{d^2}\sum_{k\le N^\theta/d}\frac{1}{k}\sum_{q=1}^{\infty}\frac{\log q}{q^2}\\ &\ll \log^2 N\sum_{d\le Q}\frac{1}{d^2}\times\frac{\log Q}{Q/d} + \log^2 N\sum_{Q<d}\frac{1}{d^2} \ll \frac{\log^3 N}{Q}.\end{aligned} \tag{27}$$

For the sum Σ_2 we find

$$\Sigma_2 = \sum_{d\le Q} d\sum_{k\le N^\theta}\frac{1}{k}\sum_{\substack{q\le Q\\(k,q)=d}} 1 \ll \sum_{d\le Q} d\sum_{k\le N^\theta/d}\frac{1}{kd}\frac{Q}{d} \ll Q\log^2 N. \tag{28}$$

We shall now consider Σ_3. Obviously,

$$\Sigma_3 = \sum_{h\le QN^\theta} \omega(h)\Delta(N,h), \tag{29}$$

where

$$\omega(h) = \sum_{k\le N^\theta} \sum_{\substack{q\le Q\\ [k,q]=h}} \frac{1}{q^2}.$$

We have

$$\omega(h) = \sum_{d\le Q} \sum_{k\le N^\theta} \sum_{\substack{q\le Q\\ (k,q)=d\\ kq=hd}} \frac{1}{q^2} \ll \sum_{d\le Q} \sum_{\substack{q\le Q\\ q\equiv 0\ (d)}} \frac{1}{q^2} \ll \sum_{d\le Q} \sum_{q\le Q/d} \frac{1}{q^2 d^2} \ll 1. \tag{30}$$

We apply Bombieri–Vinogradov's theorem (see [1, 7]). Since $\theta < 1/2$ and $Q = (\log N)^B$, we get from (16), (29), (30)

$$\Sigma_3 \ll \frac{N}{Q^3}. \tag{31}$$

By virtue of inequalities (26)–(28), (31) and the definition of Q and τ we find that

$$\mathcal{E}_1 \ll \frac{N^2}{(\log N)^A} \qquad \text{for each} \quad \theta < \frac{1}{2}$$

provided that

$$B \ge A + 5. \tag{32}$$

4. THE ESTIMATE OF $\mathcal{E}_2$

We use the identity

$$S_{k,l}(\alpha) = \sum_{p\le N} (\log p) e(\alpha p) \times \frac{1}{k}\sum_{h=1}^{k} e\left(\frac{h(p-l)}{k}\right) = \frac{1}{k}\sum_{h=1}^{k} e\left(-\frac{hl}{k}\right) S\left(\alpha + \frac{h}{k}\right)$$

and get

$$J_{k,l}^{(2)}(N) = \frac{1}{k}\sum_{h=1}^{k} e\left(-\frac{hl}{k}\right) \int_{E_2} S\left(\alpha + \frac{h}{k}\right) S^2(\alpha) e(-N\alpha)\, d\alpha.$$

Consequently,

$$\mathcal{E}_2 \le \mathcal{E}_3 = \sum_{k\le N^\theta} \frac{1}{k} \sum_{h=1}^{k} \int_{E_2} \left| S\left(\alpha + \frac{h}{k}\right) S^2(\alpha) \right| d\alpha. \tag{33}$$

Instead of $\mathcal{E}_3$ we can operate with a similar sum in which $1 \le h \le k$, $(h,k) = 1$. We have to note that this is not necessary for the proof of our theorem, but it may be useful if we have to improve the upper bound of θ for which the theorem holds true.

We have

$$\mathcal{E}_3 = \sum_{d \le N^\theta} \sum_{k \le N^\theta} \sum_{\substack{h=1 \\ (h,k)=d}}^{k} \int_{E_2} \left| S\left(\alpha + \frac{h}{k}\right) S^2(\alpha) \right| d\alpha$$

$$= \sum_{d \le N\theta} \sum_{k \le N\theta/d} \frac{1}{kd} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \int_{E_2} \left| S\left(\alpha + \frac{h}{k}\right) S^2(\alpha) \right| d\alpha \ll (\log N)\mathcal{E}_4, \tag{34}$$

where

$$\mathcal{E}_4 = \sum_{k \le N^\theta} \frac{1}{k} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \int_{E_2} \left| S\left(\alpha + \frac{h}{k}\right) S^2(\alpha) \right| d\alpha.$$

Clearly,

$$\mathcal{E}_4 \ll (\log N) \times \max_{1/2 \le G \le N^\theta} \left(\frac{1}{G} \mathcal{E}_5 \right), \tag{35}$$

where

$$\mathcal{E}_5 = \mathcal{E}_5(G, N) = \sum_{G < k \le 2G} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \int_{E_2} \left| S\left(\alpha + \frac{h}{k}\right) S^2(\alpha) \right| d\alpha.$$

We set

$$W = \frac{N}{G(\log N)^{A+3}}, \tag{36}$$

$$E_3 = \left\{ \alpha \in E_2 \mid \min\left(\left| S\left(\alpha + \frac{h}{k}\right) \right|, |S(\alpha)| \right) \le W \right\}, \qquad E_4 = E_2 \setminus E_3. \tag{37}$$

Obviously,

$$\mathcal{E}_5 \le \mathcal{E}_6 + \mathcal{E}_7, \tag{38}$$

where

$$\mathcal{E}_6 = \sum_{G < k \le 2G} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \int_{E_3} \left| S\left(\alpha + \frac{h}{k}\right) S^2(\alpha) \right| d\alpha, \tag{39}$$

$$\mathcal{E}_7 = \sum_{G < k \le 2G} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \int_{E_4} \left| S\left(\alpha + \frac{h}{k}\right) S^2(\alpha) \right| d\alpha. \tag{40}$$

If $\alpha \in E_3$, then we have

$$\left| S\left(\alpha + \frac{h}{k}\right) S^2(\alpha) \right| \le W \left(\left| S^2\left(\alpha + \frac{h}{k}\right) \right| + |S^2(\alpha)| \right),$$

and, hence,

$$\int\limits_{E_3} \left| S\left(\alpha + \frac{h}{k}\right) S^2(\alpha) \right| d\alpha \le W \int\limits_0^1 \left(\left| S^2\left(\alpha + \frac{h}{k}\right) \right| + |S^2(\alpha)| \right) d\alpha \ll W N \log N.$$

Using the last estimate, (36), and (39), we get

$$\mathcal{E}_6 \ll \frac{N^2 G}{(\log N)^{A+2}}. \tag{41}$$

Let us consider $\mathcal{E}_7$. We can find a good estimate for the part of $\mathcal{E}_7$ corresponding to the subset of E_4 in which $|S(\alpha + h/k)| \ge NG^{-1}(\log N)^R$ for some constant $R > 0$ (see [3, p. 410]). However, it is impossible to proceed in a similar way with the part of $\mathcal{E}_7$ corresponding to the subset of E_4 in which $|S(\alpha)|$ is large. In fact, as we shall see later, the main contribution to (40) comes from this subset. Fortunately, $|S(\alpha)|$ cannot be too large in E_4 because

$$E_4 \subset E_2, \qquad \sup_{\alpha \in E_2} |S(\alpha)| \ll \frac{N}{(\log N)^{B/2-4}} \tag{42}$$

(see [6, Theorem 3.1]).

We can assume that $G \ge (\log N)^{B/2-A-8}$, otherwise, according to (36), (37), (42), the set E_4 is empty. We divide E_4 into $O(\log^2 N)$ sets of the type

$$E_5(L_1, L_2) = \left\{ \alpha \in E_4 \mid L_1 < \left| S\left(\alpha + \frac{h}{k}\right) \right| \le 2L_1,\ L_2 < |S(\alpha)| \le 2L_2 \right\},$$

where, according to (37) and (42), L_1 and L_2 satisfy

$$W \le L_1 \le N \log N, \qquad W \le L_2 \le \frac{N}{(\log N)^{B/2-5}}. \tag{43}$$

Therefore

$$\mathcal{E}_7 \ll \log^2 N \times \max_{L_1, L_2} \left(L_1 L_2^2 \mathcal{E}_8 \right), \tag{44}$$

where the maximum is over L_1, L_2 satisfying (43),

$$\mathcal{E}_8 = \mathcal{E}_8(G, L_1, L_2) = \sum_{G < k \le 2G} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \mathrm{mes}(E_6) \tag{45}$$

and $\mathrm{mes}(E_6)$ denotes the measure of the set

$$E_6 = E_6(L_1, L_2) = \left\{ \alpha \in [0,1] \mid L_1 \le \left| S\left(\alpha + \frac{h}{k}\right) \right|,\ L_2 \le |S(\alpha)| \right\}.$$

We proceed as in [3, p. 411, 412] and find that

$$\mathrm{mes}(E_6) = \sum_{s=0}^{N-1} \mathrm{mes}(E_7) = \sum_{s \in \mathfrak{N}_2} \mathrm{mes}(E_7), \tag{46}$$

where $E_7 = E_6 \cap [s/N, (s+1)/N]$ and where $\mathfrak{N}_2$ is the set of all integers $s \in \{0, 1, \dots, N-1\}$ such that $|S(\alpha_s)| \geq L_2$ for some $\alpha_s \in [s/N, (s+1)/N]$. Let $\mathfrak{N}'$, $\mathfrak{N}''$ be the subsets of even, respectively odd, integers from $\mathfrak{N}_2$. Suppose that $\mathfrak{N}' = \{s_1, s_2, \dots, s_h\}$ and let α_i, $1 \leq i \leq h$, be numbers such that $\alpha_i \in [s_i/N, (s_i+1)/N]$ and $|S(\alpha_i)| \geq L_2$. It is clear that if $i \neq j$, then $|\alpha_i - \alpha_j| \geq 1/N$. Applying Theorem 2.3 of [4], we get

$$L_2^2 \times h \leq \sum_{i=1}^{h} |S(\alpha_i)|^2 \ll N^2 \log N.$$

We estimate the numbers of elements of $\mathfrak{N}''$ in the same way and find that

$$\#\mathfrak{N}_2 \ll \frac{N^2 \log N}{L_2^2}, \tag{47}$$

where $\#\mathfrak{N}_2$ denotes the number of elements of $\mathfrak{N}_2$.

We have

$$\begin{aligned} \operatorname{mes}(E_7) &\leq \operatorname{mes}\left\{\alpha \in \left[\frac{s}{N}, \frac{s+1}{N}\right] \;\Big|\; L_1 \leq \left|S\left(\alpha + \frac{h}{k}\right)\right|\right\} \\ &= \operatorname{mes}\left\{\alpha \in \left[\frac{s}{N} + \frac{h}{k}, \frac{s+1}{N} + \frac{h}{k}\right] \;\Big|\; L_1 \leq |S(\alpha)|\right\} = \sum_{t=0}^{2N-1} \operatorname{mes}(E_8) = \sum_{t \in \mathfrak{N}_1} \operatorname{mes}(E_8), \end{aligned} \tag{48}$$

where

$$E_8 = \left[\frac{t}{N}, \frac{t+1}{N}\right] \cap \left[\frac{s}{N} + \frac{h}{k}, \frac{s+1}{N} + \frac{h}{k}\right] \cap \{\alpha \mid L_1 \leq |S(\alpha)|\}$$

and where $\mathfrak{N}_1$ is the set of all integers $t \in \{0, 1, \dots, 2N-1\}$ such that $|S(\alpha_t)| \geq L_1$ for some $\alpha_t \in [t/N, (t+1)/N]$. For the number of elements of $\mathfrak{N}_1$ we find, as above, that

$$\#\mathfrak{N}_1 \ll \frac{N^2 \log N}{L_1^2}. \tag{49}$$

Clearly, the measure of the set E_8 is at most $1/N$ and E_8 is empty unless $|s - t + Nh/k| \leq 1$. Therefore we get, by (46) and (48),

$$\operatorname{mes}(E_6) \ll \frac{1}{N} \sum_{\substack{t \in \mathfrak{N}_1 \\ |s-t+Nh/k| \leq 1}} \sum_{s \in \mathfrak{N}_2} 1 \ll \frac{1}{N} \sum_{t \in \mathfrak{N}_1} \sum_{s \in \mathfrak{N}_2} e^{-\pi(s-t+Nh/k)^2}.$$

We use the well-known identity

$$e^{-\pi x^2} = \int_{-\infty}^{\infty} e^{-\pi\alpha^2} e(\alpha x)\, d\alpha \tag{50}$$

and find that

$$\operatorname{mes}(E_6) \ll \frac{1}{N} \sum_{t \in \mathfrak{N}_1} \sum_{s \in \mathfrak{N}_2} \int_{-\infty}^{\infty} e^{-\pi\alpha^2} e\left(\alpha\left(s - t + N\frac{h}{k}\right)\right) d\alpha.$$

We substitute the last estimate into (45) and change the order of summation and integration. As a result we get

$$\mathcal{E}_8 \ll \frac{1}{N} \int_{-\infty}^{\infty} e^{-\pi\alpha^2} |\mathcal{F}_1(\alpha)\mathcal{F}_2(\alpha)\mathcal{K}^*(\alpha)|\, d\alpha, \tag{51}$$

where

$$\mathcal{F}_1(\alpha) = \sum_{t\in\mathfrak{N}_1} e(\alpha t), \qquad \mathcal{F}_2(\alpha) = \sum_{t\in\mathfrak{N}_2} e(\alpha t), \qquad \mathcal{K}^*(\alpha) = \sum_{G<k\le 2G} \sum_{\substack{h=1\\(h,k)=1}}^{k} e\Big(\alpha N \frac{h}{k}\Big). \tag{52}$$

Suppose that $r > 2$, $u > 2$ are real numbers such that

$$\frac{1}{2} + \frac{1}{r} + \frac{1}{u} = 1. \tag{53}$$

To estimate the right-hand side of (51), we apply Hölder's inequality and get

$$\mathcal{E}_8 \ll \frac{1}{N} (I_1)^{1/2} (I_2)^{1/r} (\xi_u)^{1/u}, \tag{54}$$

where

$$I_1 = \int_{-\infty}^{\infty} e^{-\pi\alpha^2} |\mathcal{F}_1(\alpha)|^2\, d\alpha, \qquad I_2 = \int_{-\infty}^{\infty} e^{-\pi\alpha^2} |\mathcal{F}_2(\alpha)|^r\, d\alpha, \qquad \xi_u = \int_{-\infty}^{\infty} e^{-\pi\alpha^2} |\mathcal{K}^*(\alpha)|^u\, d\alpha. \tag{55}$$

It is clear that if we substitute

$$\mathcal{K}(\alpha) = \sum_{G<k\le 2G} \sum_{h=1}^{k} e\Big(\alpha N \frac{h}{k}\Big) \tag{56}$$

for $\mathcal{K}^*(\alpha)$ in (51), then the inequality remains valid, and therefore we also have

$$\mathcal{E}_8 \ll \frac{1}{N} (I_1)^{1/2} (I_2)^{1/r} (\eta_u)^{1/u}, \tag{57}$$

where

$$\eta_u = \int_{-\infty}^{\infty} e^{-\pi\alpha^2} |\mathcal{K}(\alpha)|^u\, d\alpha. \tag{58}$$

Let us consider I_2. We use (50) and the trivial estimate $|\mathcal{F}_2(\alpha)| \ll \#\mathfrak{N}_2$ to get

$$\begin{aligned} I_2 &\ll (\#\mathfrak{N}_2)^{r-2} \int_{-\infty}^{\infty} e^{-\pi\alpha^2} |\mathcal{F}_2(\alpha)|^2\, d\alpha = (\#\mathfrak{N}_2)^{r-2} \int_{-\infty}^{\infty} e^{-\pi\alpha^2} \sum_{t_1,t_2\in\mathfrak{N}_2} e\big(\alpha(t_1-t_2)\big)\, d\alpha \\ &= (\#\mathfrak{N}_2)^{r-2} \sum_{t_1,t_2\in\mathfrak{N}_2} e^{-\pi(t_1-t_2)^2} \ll (\#\mathfrak{N}_2)^{r-1}. \end{aligned} \tag{59}$$

Similarly,

$$I_1 \ll \#\mathfrak{N}_1. \tag{60}$$

From (47), (49), (54), (59), (60) we obtain

$$\mathcal{E}_8 \ll \frac{N^{2-2/r}\log^2 N}{L_1 L_2^{2-2/r}}(\xi_u)^{1/u}.$$

We substitute the last estimate for $\mathcal{E}_8$ in (44) and use (43) to obtain

$$\mathcal{E}_7 \ll N^2(\xi_u)^{1/u}(\log N)^{4-(B-10)/r}. \tag{61}$$

If we consider estimate (57) instead of (54), we also find

$$\mathcal{E}_7 \ll N^2(\eta_u)^{1/u}(\log N)^{4-(B-10)/r}. \tag{62}$$

Let us state now the conjectures that we mentioned at the end of Section 2.

Conjecture 1. *Suppose that $u > 0$ is a real number and ξ_u is defined by* (55). *There exist $c_1(u), c_2(u) \geq 0$ such that if*

$$G \leq \frac{N^{1/u}}{(\log N)^{c_1(u)}},$$

then

$$\xi_u \leq G^u (\log N)^{c_2(u)}.$$

Conjecture 2. *Suppose that $u > 0$ is a real number and η_u is defined by* (58). *There exist $c_1(u), c_2(u) \geq 0$ such that if*

$$G \leq \frac{N^{1/u}}{(\log N)^{c_1(u)}},$$

then

$$\eta_u \leq G^u (\log N)^{c_2(u)}.$$

Let us assume that one of the conjectures stated above is true for a particular $u > 2$. We define the real number r by (53) and substitute the estimate for ξ_u (or η_u) into (61) (or (62)). We obtain

$$\mathcal{E}_7 \ll N^2 G (\log N)^{4+c_2(u)-(B-10)/r} \tag{63}$$

provided that

$$G \leq \frac{N^{1/u}}{(\log N)^{c_1(u)}}.$$

Suppose that $\theta < 1/u$. We apply (33)–(35), (38), (41), and (63) and find that

$$\mathcal{E}_2 \ll N^2\big((\log N)^{-A} + (\log N)^{6+c_2(u)-(B-10)/r}\big).$$

We choose

$$B \geq r\big(10A + c_2(u)\big) \tag{64}$$

and get

$$\mathcal{E}_2 \ll \frac{N^2}{(\log N)^A} \qquad \text{for each} \quad \theta < \frac{1}{u}.$$

At the end of this section we prove Conjecture 1 for $u = 4$. Suppose that $G \le N^{1/4}$. By virtue of (52) and (55), we obtain

$$\xi_4 = \int_{-\infty}^{\infty} e^{-\pi\alpha^2} \mathcal{K}^*(\alpha)^2 \mathcal{K}^*(-\alpha)^2 \, d\alpha = \sum_{k_i, h_i \ (65)} \int_{-\infty}^{\infty} e^{-\pi\alpha^2} e\biggl(\alpha N \Bigl(\frac{h_1}{k_1} + \frac{h_2}{k_2} - \frac{h_3}{k_3} - \frac{h_4}{k_4}\Bigr)\biggr) \, d\alpha.$$

The summation is over $k_1, \dots, k_4$, $h_1, \dots, h_4$ such that

$$G < k_i \le 2G, \quad 1 \le h_i \le k_i, \quad (h_i, k_i) = 1, \qquad 1 \le i \le 4. \tag{65}$$

We denote by Γ the number of solutions of the inequality

$$\left| \frac{h_1}{k_1} + \frac{h_2}{k_2} - \frac{h_3}{k_3} - \frac{h_4}{k_4} \right| \le \frac{\log N}{N}$$

in the integers k_i, h_i satisfying (65). We apply (50) and get

$$\xi_4 = \sum_{k_i, h_i \ (65)} e^{-\pi N^2 (h_1/k_1 + h_2/k_2 - h_3/k_3 - h_4/k_4)^2} \ll \Gamma.$$

Now we use Lemma 3 of [3] and obtain

$$\xi_4 \ll G^4 (\log N)^{17}.$$

Thus inequality (5) holds for each $\theta < 1/4$.

5. PROOF OF CONJECTURE 2 FOR $u = 3$

We can assume that $G \ge 10$. The sum $\mathcal{K}(\alpha)$ defined by (56) satisfies

$$|\mathcal{K}(\alpha)| \le \sum_{G < k \le 2G} \left| \sum_{h=1}^{k} e\Bigl(\alpha N \frac{h}{k}\Bigr) \right| \ll \sum_{G < k \le 2G} \min\Bigl(G, \frac{1}{\|\alpha N/k\|}\Bigr). \tag{66}$$

It is well known that

$$\min\Bigl(G, \frac{1}{\|x\|}\Bigr) = \sum_{m=-\infty}^{\infty} c(m) e(mx), \tag{67}$$

where the coefficients $c(m)$ satisfy

$$|c(m)| \ll \begin{cases} \log G & \text{for all } m \\ G^2/m^2 & \text{for } m \ne 0. \end{cases} \tag{68}$$

We use (66) and (67) and find that

$$|\mathcal{K}(\alpha)| \ll \sum_{G < k \le 2G} \sum_{m=-\infty}^{\infty} c(m) e\Bigl(\alpha N \frac{m}{k}\Bigr) \tag{69}$$

uniformly with respect to α. On the other hand,

$$|\mathcal{K}(\alpha)|^2 = \mathcal{K}(\alpha)\mathcal{K}(-\alpha) = \sum_{G<k_1,k_2\le 2G} \sum_{h_1=1}^{k_1} \sum_{h_2=1}^{k_2} e\left(\alpha N\left(\frac{h_1}{k_1} - \frac{h_2}{k_2}\right)\right). \tag{70}$$

Formulas (69) and (70) imply that uniformly with respect to α we have

$$|\mathcal{K}(\alpha)|^3 \ll \sum_{G<k,k_1,k_2\le 2G} \sum_{h_1=1}^{k_1} \sum_{h_2=1}^{k_2} \sum_{m=-\infty}^{\infty} c(m) e\left(\alpha N\left(\frac{h_1}{k_1} - \frac{h_2}{k_2} + \frac{m}{k}\right)\right).$$

We substitute the last estimate for $|\mathcal{K}(\alpha)|^3$ into the expression of η_3 (defined by (58)), change the order of summation and integration, and apply (50) to obtain

$$\eta_3 \ll \sum_{G<k,k_1,k_2\le 2G} \sum_{h_1=1}^{k_1} \sum_{h_2=1}^{k_2} \sum_{m=-\infty}^{\infty} |c(m)| e^{-\pi N^2 (h_1/k_1 - h_2/k_2 + m/k)^2}. \tag{71}$$

We denote by η_3' the contribution of the summands in (71) satisfying

$$\left|\frac{h_1}{k_1} - \frac{h_2}{k_2} + \frac{m}{k}\right| \le \frac{\log N}{N}. \tag{72}$$

If condition (72) does not hold, then the corresponding summand on the right-hand side of (71) is very small. The numbers $|c(m)|$ are not large either (see (68)). Thus we have

$$\eta_3 \ll 1 + \eta_3'. \tag{73}$$

Suppose that

$$G \le \frac{N^{1/3}}{\log N}. \tag{74}$$

It is easy to see that if $G < k, k_1, k_2 \le 2G$, then conditions (72), (74) imply

$$\frac{h_1}{k_1} - \frac{h_2}{k_2} + \frac{m}{k} = 0. \tag{75}$$

In turn, (75) and the conditions imposed on k, k_1, k_2, h_1, h_2 in (71) imply that $|m| \le k$. We also use (68) and get

$$\eta_3' \ll \log G \times \sum_{\substack{G<k,k_1,k_2\le 2G \\ h_1/k_1 - h_2/k_2 + m/k = 0}} \sum_{h_1=1}^{k_1} \sum_{h_2=1}^{k_2} \sum_{|m|\le k} 1 \ll \log G \times (G\eta_3'' + \eta_3'''), \tag{76}$$

where

$$\eta_3'' = \sum_{\substack{G<k_1,k_2\le 2G \\ h_1/k_1 = h_2/k_2}} \sum_{h_1=1}^{k_1} \sum_{h_2=1}^{k_2} 1, \qquad \eta_3''' = \sum_{\substack{G<k,k_1,k_2\le 2G \\ h_1/k_1 + h_2/k_2 = h/k}} \sum_{h_1=1}^{k_1} \sum_{h_2=1}^{k_2} \sum_{h=1}^{k} 1.$$

(For convenience, we change the notation in the expression for η_3'''.) It is clear that

$$\eta_3'' \ll \sum_{n\le 4G^2}\Bigg(\sum_{\substack{G<k\le 2G\\ h\le 2G\\ hk=n}} 1\Bigg)^2 \ll \sum_{n\le 4G^2}\tau^2(n)\ll G^2\log^3 G. \tag{77}$$

It is more difficult to estimate η_3'''. We have

$$\eta_3''' = \sum_{q\le 2G}\sum_{\substack{a=1\\ (a,q)=1}}^{q}\mathcal{L}_1(a,q)\mathcal{L}_2(a,q), \tag{78}$$

where

$$\mathcal{L}_1(a,q)=\sum_{\substack{G<k\le 2G\\ h/k=a/q}}\sum_{h=1}^{k}1,\qquad \mathcal{L}_2(a,q)=\sum_{\substack{G<k_1,k_2\le 2G\\ h_1/k_1+h_2/k_2=a/q}}\sum_{h_1=1}^{k_1}\sum_{h_2=1}^{k_2}1.$$

Since $(a,q)=1$, we have

$$\mathcal{L}_1(a,q)=\sum_{\substack{G<k\le 2G\\ k\equiv 0\ (q),\ h\equiv 0\ (a),\ hq=ak}}\sum_{h=1}^{k}1=\sum_{G/q<k\le 2G/q}\sum_{\substack{1\le h\le kq/a\\ h=k}}1\ll\frac{G}{q}. \tag{79}$$

To estimate $\mathcal{L}_2(a,q)$, we proceed as in the proof of Lemma 3 from [3]. We have

$$\mathcal{L}_2(a,q)=\sum_{d_1,d_2\le 2G}\sum_{\substack{G<k_1,k_2\le 2G\\ (h_1,k_1)=d_1,\ (h_2,k_2)=d_2\\ h_1/k_1+h_2/k_2=a/q}}\sum_{h_1=1}^{k_1}\sum_{h_2=1}^{k_2}1=\sum_{d_1,d_2\le 2G}\sum_{G/d_1<k_1\le 2G/d_1}\sum_{G/d_2<k_2\le 2G/d_2}\mathcal{L}_3, \tag{80}$$

where $\mathcal{L}_3=\mathcal{L}_3(a,q,k_1,k_2)$ is the number of solutions of the equation

$$\frac{h_1}{k_1}+\frac{h_2}{k_2}=\frac{a}{q} \tag{81}$$

in the integers h_1, h_2 such that

$$1\le h_1\le k_1,\qquad 1\le h_2\le k_2,\qquad (h_1,k_1)=(h_2,k_2)=1. \tag{82}$$

Equation (81) is equivalent to

$$q\bigg(h_1\frac{k_2}{(k_1,k_2)}+h_2\frac{k_1}{(k_1,k_2)}\bigg)=a\frac{k_1}{(k_1,k_2)}\,\frac{k_2}{(k_1,k_2)}(k_1,k_2). \tag{83}$$

The integers

$$\frac{k_1}{(k_1,k_2)},\qquad \frac{k_2}{(k_1,k_2)},\qquad h_1\frac{k_2}{(k_1,k_2)}+h_2\frac{k_1}{(k_1,k_2)}$$

are, obviously, pairwise coprime. Hence, we find from (83) that

$$\frac{k_1 k_2}{(k_1,k_2)^2} \,\Big|\, q, \tag{84}$$

i.e., $q(k_1,k_2)^2/(k_1k_2)$ is an integer dividing q. Equation (83) implies

$$\frac{q(k_1,k_2)^2}{k_1k_2}\left(h_1\frac{k_2}{(k_1,k_2)}+h_2\frac{k_1}{(k_1,k_2)}\right)=a(k_1,k_2). \tag{85}$$

Since $(a,q)=1$, we find that

$$\frac{q(k_1,k_2)^2}{k_1k_2} \,\Big|\, (k_1,k_2), \qquad \text{i.e.,} \qquad q \,\Big|\, \frac{k_1k_2}{(k_1,k_2)}. \tag{86}$$

Using (85), we arrive at

$$h_1\frac{k_2}{(k_1,k_2)}+h_2\frac{k_1}{(k_1,k_2)}=a\frac{k_1k_2}{q(k_1,k_2)}. \tag{87}$$

If $h_1=H_1$, $h_2=H_2$ is a special solution of (87), then all solutions are given by

$$h_1=H_1+\frac{k_1}{(k_1,k_2)}t, \quad h_2=H_2-\frac{k_2}{(k_1,k_2)}t, \qquad t\in\mathbb{Z}.$$

Using (82), we infer that the integer t takes $\ll (k_1,k_2)$ values. Hence, we have $\mathcal{L}_3 \ll (k_1,k_2)$. We have found that $\mathcal{L}_3=0$ unless conditions (84), (86) hold. Therefore we get from (80)

$$\begin{aligned}\mathcal{L}_2(a,q) &\ll \sum_{d_1,d_2\le 2G}\ \sum_{\substack{k_1\le 2G/d_1,\ k_2\le 2G/d_2\\ k_1k_2/(k_1,k_2)\equiv 0\ (q),\ k_1k_2/(k_1,k_2)^2\mid q}} (k_1,k_2)\\ &= \sum_{d_1,d_2\le 2G}\ \sum_{d\le \min(2G/d_1,2G/d_2)} d \sum_{\substack{k_1\le 2G/d_1,\ k_2\le 2G/d_2\\ (k_1,k_2)=d,\ k_1k_2/d\equiv 0\ (q),\ k_1k_2/d^2\mid q}} 1\\ &\ll \sum_{d_1,d_2\le 2G}\ \sum_{d\le \min(2G/d_1,2G/d_2)} d \sum_{\substack{k_1\le 2G/(d_1d),\ k_2\le 2G/(d_2d)\\ k_1k_2d\equiv 0\ (q),\ k_1k_2\mid q}} 1. \end{aligned}\tag{88}$$

Formulas (78), (79), (88) give us

$$\begin{aligned}\eta_3''' &\ll \sum_{q\le 2G}\ \sum_{\substack{a=1\\(a,q)=1}}^{q} \frac{G}{q} \sum_{d_1,d_2\le 2G}\ \sum_{d\le \min(2G/d_1,2G/d_2)} d \sum_{\substack{k_1\le 2G/(d_1d),\ k_2\le 2G/(d_2d)\\ k_1k_2d\equiv 0\ (q),\ k_1k_2\mid q}} 1\\ &\ll G \sum_{d_1,d_2\le 2G}\ \sum_{d\le \min(2G/d_1,2G/d_2)} d \sum_{k_1\le 2G/(d_1d),\ k_2\le 2G/(d_2d)}\ \sum_{\substack{q\le 2G\\ q\equiv 0\ (k_1k_2)\\ q\mid k_1k_2d}} 1.\end{aligned}$$

The sum over q is $\ll \tau(q)$, and, hence

$$\eta_3''' \ll G \sum_{d_1,d_2\le 2G} \sum_{d\le\min(2G/d_1,2G/d_2)} d\tau(d) \sum_{k_1\le 2G/(d_1d),\ k_2\le 2G/(d_2d)} 1$$

$$\ll G^3 \sum_{d,d_1,d_2\le 2G} \frac{\tau(d)}{dd_1d_2} \ll G^3\log^4 G. \tag{89}$$

Estimates (73), (76), (77), (89) imply $\eta_3 \le G^3\log^6 N$ provided that

$$G \le \frac{N^{1/3}}{\log N}.$$

Conjecture 2 for $u = 3$ is proved, and therefore the Theorem is proved too.

Consider an approach that is used to prove Conjecture 1 or Conjecture 2 for $u > 2$ and u which is close to 2.

Let $u = 2 + 2/n$, where n is a large integer. We consider, for example, the quantities $\mathcal{K}(\alpha)$, η_u defined by (56) and (58). Assume that uniformly with respect to α we have

$$|\mathcal{K}(\alpha)| \ll |\mathcal{H}(\alpha)|^n + R_0, \tag{90}$$

where

$$\mathcal{H}(\alpha) = \sum_{s=1}^{\infty}\sum_{l=1}^{s} c(s,l)e\Big(\alpha N\frac{l}{s}\Big)$$

and where R_0 and $|c(s,l)|$ are small in a certain sense. Then, by (90), we get

$$\eta_{2+2/n} = \int_{-\infty}^{\infty} e^{-\pi\alpha^2}|\mathcal{K}(\alpha)|^{2+2/n}\,d\alpha \ll \eta^* + R_1,$$

where R_1 is "small" and

$$\eta^* = \int_{-\infty}^{\infty} e^{-\pi\alpha^2}|\mathcal{K}(\alpha)|^2|\mathcal{H}(\alpha)|^2\,d\alpha.$$

According to (50) we have

$$\eta^* \ll \sum_{\substack{G<k_1,\ k_2\le 2G\\ s_1,s_2\ge 1}} \sum_{h_1=1}^{k_1}\sum_{h_2=1}^{k_2}\sum_{l_1=1}^{s_1}\sum_{l_2=1}^{s_2} |c(s_1,l_1)c(s_2,l_2)|e^{-\pi N^2(h_1/k_1-h_2/k_2+l_1/s_1-l_2/s_2)^2}.$$

We would be able to estimate the last expression if we had information about the size of $|c(s,l)|$ and if we could calculate the number of solutions of the inequality

$$\left|\frac{h_1}{k_1}-\frac{h_2}{k_2}+\frac{l_1}{s_1}-\frac{l_2}{s_2}\right| \le \frac{\log N}{N}$$

in integers satisfying some conditions.

Other versions of the same approach are also possible. However, the author cannot realize this idea at present.

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