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On Interpolation Formulas

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NOTATION

Suppose that $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers, and the set of complex numbers, respectively, K is a finite algebraic extension of the field $\mathbb{Q}$, $\mathbb{Z}_K$ is the set of integers of the field K, $\omega_1, \omega_2, \dots, \omega_m$ is a certain fundamental basis of the field K, $N(\alpha)$ is the norm of the number $\alpha \in K$ with respect to $\mathbb{Q}$, $\mathbb{P}$ is the set of prime numbers, $p, p_1, \dots, p_k$ are prime numbers, $\mathbb{D}_K$ is the semigroup of integer divisors of the field K. Suppose that for $\mathfrak{a} \in \mathbb{D}_K$ $N(\mathfrak{a})$ denotes the norm of the divisor of $\mathfrak{a}$ with respect to $\mathbb{Q}$; $\mathbb{P}_K$ is the set of prime divisors of the field K; $\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2, \dots$ are prime divisors of the field K; for a finite set $\mathcal{M}$, the notation $|\mathcal{M}|$ denotes the number of elements belonging to $\mathcal{M}$; if $\mathbf{x}$ and $\mathbf{y}$ belong to $\mathbb{R}^m$, then $\langle \mathbf{x}, \mathbf{y} \rangle$ denotes the inner product of $\mathbf{x}$ and $\mathbf{y}$; for the Fourier polynomial

$$f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^m} c_{\mathbf{x}} e^{2\pi i \langle \mathbf{n}, \mathbf{x} \rangle}, \qquad \mathbf{x} \in \mathbb{R}^m,$$

we set

$$\operatorname{Spec} f = \{\mathbf{n} : \mathbf{n} \in \mathbb{Z}^m,\ c_{\mathbf{n}} \neq 0\};$$

for $\mathcal{M} \in \mathbb{Z}^m$, $|\mathcal{M}| < \infty$, we set

$$L_{\mathcal{M}} = \left\{ f(\mathbf{x}) : f(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^m} c_{\mathbf{n}} e^{2\pi i \langle \mathbf{n}, \mathbf{x} \rangle},\ \operatorname{Spec} f \subset \mathcal{M} \right\};$$

if $\mathcal{M} \subset \mathbb{Z}^m$, $|\mathcal{M}| < \infty$, then

$$D_{\mathcal{M}}(\mathbf{x}) = D(\mathbf{x}, \mathcal{M}) = \sum_{\mathbf{n} \in \mathcal{M}} e^{2\pi i \langle \mathbf{n}, \mathbf{x} \rangle};$$

for $\mathfrak{p} \in \mathbb{P}_K$, $\alpha \in \mathbb{Z}_K \setminus \{0\}$, the function $\operatorname{ord}_{\mathfrak{p}} \alpha$ is defined by the relation

$$(\alpha) = \prod_{\mathfrak{p} \in \mathbb{P}_K} \mathfrak{p}^{\operatorname{ord}_{\mathfrak{p}} \alpha},$$

where (α) denotes the principal divisor corresponding to α; for $\alpha \in \mathbb{Z}_K$, $\mathfrak{p} \in \mathbb{P}_K$, the function $\delta(\alpha, \mathfrak{p})$ is defined by the relation

$$\delta(\alpha, \mathfrak{p}) = \delta_{\mathfrak{p}}(\alpha) = \begin{cases} 1 & \text{if } \mathfrak{p} \text{ is a divisor of } \alpha \\ 0 & \text{otherwise;} \end{cases}$$

for $\alpha, \beta \in \mathbb{Z}_K$, $\mathfrak{p} \in \mathbb{P}_K$,

$$\chi(\alpha; \mathfrak{p}, \beta) = \delta_{\mathfrak{p}}(\alpha - \beta);$$

for $\mathbf{n} = (n_1, n_2, \ldots, n_m)$, we set

$$\widetilde{\mathbf{n}} = n_1\omega_1 + n_2\omega_2 + \cdots + n_m\omega_m;$$

for $\alpha \in \mathbb{Z}_K$, we set $\overline{\alpha} = (n_1, \ldots, n_m)$, where n_j are defined by the relation $\alpha = n_1\omega_1 + \cdots + n_m\omega_m$; for $\alpha \in \mathbb{Z}_K$ and $\mathcal{M} \subset \mathbb{Z}^m$ the notation $\alpha \in \mathcal{M} \pmod{\mathfrak{p}}$ means that there exists a vector $\mathbf{n} \in \mathcal{M}$ for which the relation

$$\alpha \equiv \widetilde{\mathbf{n}} \pmod{\mathfrak{p}}$$

holds; in particular, for $h \in \mathbb{Z}$, $\mathcal{M} \subset \mathbb{Z}$, the notation $h \in \mathcal{M} \pmod{p}$ means that there exists $n \in \mathcal{M}$ for which $h \equiv n \pmod{p}$; $\ll$ is Vinogradov's symbol.

1. FORMULATION OF THE RESULTS

In this paper, we present a method for constructing interpolation formulas for Fourier polynomials with a given spectrum. This method is based on the application of division theory in the ring $\mathbb{Z}$ as well as in the rings of integers in finite algebraic extensions of the field $\mathbb{Q}$. The number of interpolation nodes coincides with the total number of harmonics of the Fourier polynomial with an accuracy up to the degree of the logarithm of the maximal norm of the harmonic entering into the spectrum of this polynomial.

This method allows one to compute effectively the interpolation nodes.

The proofs of theorems are based on Lemmas 3.1 and 3.2, which are of combinatorial character and are close in spirit to the large sieve. The choice of L in Theorems 1.1 and 1.3 is explained by the wish to diminish the number of terms in interpolation formulas. If we choose $L = 2$, then the obtained interpolation formulas become less cumbersome but contain more terms.

It should be noted that the idea of "thinning out" the spectrum of the Fourier polynomial with the subsequent construction of interpolation formulas for polynomials with a spaced out spectrum belongs to K. K. Frolov. The author is very grateful to him.

Theorem 1.1. *Let* $\mathcal{M}_0 \subset \mathbb{Z}$, *and let*

$$N = \max_{n \in \mathcal{M}_0} |n|, \qquad Q_0 = \max\bigl(1000, 2|\mathcal{M}_0| \ln(2N + 20)\bigr).$$

Then

(a) *there exists* $k \ll \dfrac{\ln(|\mathcal{M}_0| + 20)}{\ln\ln(|\mathcal{M}_0| + 20)}$,

(b) *for every* $j = 1, 2, \ldots, k$, *there exist numbers* $p_j \in \mathbb{P}$, $p_j \in (Q_0, 2Q_0]$, *and sets* $\mathcal{M}_j \subset \mathcal{M}_0$ *such that*

$$\mathcal{M}_j \subset \mathcal{M}_0 \setminus \bigcup_{l=1}^{j-1} \mathcal{M}_l, \qquad \bigcup_{j=1}^{k} \mathcal{M}_j = \mathcal{M}_0, \tag{1}$$

(c) *for every* $j \in [1, k]$, $j \in \mathbb{N}$, *there exist numbers* $q_j^{(h)}$, $h = 1, 2, \ldots, p_j$, *such that*

$$q_j^{(h)} \in \mathbb{P}, \qquad q_j^{(h)} < \max\bigl(1000, 2\ln^2(|\mathcal{M}_0| + 20)\ln(2N + 3)\bigr);$$

(d) *for every Fourier polynomial $f_0 \in L_{\mathcal{M}_0}$, the relation*

$$f_0(x) = \sum_{l=1}^{k} (-1)^{l-1} \sum_{1\le j_1<\dots<j_l\le k} \frac{1}{p_{j_1}} \sum_{\substack{h=1\\ h\in\mathcal{M}_{j_1} \pmod{p_{j_1}}}}^{p_{j_1}} \frac{1}{q_{j_1}^{(h)}} \sum_{\nu=1}^{p_{j_1}} \exp\Big(-2\pi i \frac{h\nu}{p_{j_1}}\Big)$$
$$\times \sum_{\varkappa=1}^{q_1^{(\varkappa)}} f_0\Big(\frac{\varkappa}{q_{j_1}^{(h)}} + \frac{\nu}{p_{j_1}}\Big) \Big(\mathcal{P}_{j_l}\cdots\mathcal{P}_{j_2} D_{\mathcal{M}_{j_1},h}\Big(* - \frac{\varkappa}{q_{j_1}^{(h)}}\Big)(x)\Big), \tag{2}$$

holds, where the linear operators $\mathcal{P}_j$ are defined by the formula

$$(\mathcal{P}_j\varphi)(x) = \frac{1}{p_j} \sum_{\substack{h=1\\ h\in\mathcal{M}_j \pmod{p_j}}}^{p_j} \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{p_j} \exp\Big(-2\pi i\frac{h\nu}{p_j}\Big) \sum_{\varkappa=1}^{q_j^{(h)}} \varphi\Big(\frac{\nu}{p_j} + \frac{h}{q_j^{(h)}}\Big) D_{\mathcal{M}_j,h}\Big(x - \frac{\varkappa}{q_j^{(h)}}\Big),$$
$$\mathcal{M}_{j,h} = \{n : n\in\mathcal{M}_j,\ n\equiv h \pmod{p_j}\}.$$

Remark. The total number of interpolation nodes in formula (2) does not exceed

$$\ll |\mathcal{M}| \ln^4(|\mathcal{M}_0| + 20)\ln^2(2N+3).$$

Theorem 1.2. *Let the whole notation and all the conditions of Theorem* 1.1 *be preserved. Then we have the relation*

$$f_0(x) = \sum_{l=1}^{k} (-1)^{l-1} \sum_{1\le j_1<\dots<j_l\le k} \frac{1}{p_{j_1}} \sum_{\substack{h=1\\ h\in\mathcal{M}_{j_1} \pmod{p_{j_1}}}}^{p_{j_1}} \frac{1}{q_{j_1}^{(h)}} \sum_{\nu=1}^{p_{j_1}} \exp\Big(-2\pi i \frac{h\nu}{p_{j_1}}\Big)$$
$$\times \sum_{\varkappa=1}^{q_1^{(\varkappa)}} f_0\Big(\frac{\varkappa}{q_{j_1}^{(h)}} + \frac{\nu}{p_{j_1}}\Big) \sum_{n_1\in\mathcal{M}_{j_1,h}} \exp\Big(-2\pi i n_1 \frac{\varkappa}{q_{j_1}^{(h)}}\Big) A(x),$$

where

$$A(x) = \sum_{\substack{h_2=1\\ h_2\in\mathcal{M}_{j_2} \pmod{p_{j_2}}}}^{p_{j_2}} \sum_{n_2\in\mathcal{M}_{j_2,h_2}} \cdots \sum_{\substack{h_l=1\\ h_l\in\mathcal{M}_{j_l} \pmod{p_{j_l}}}}^{p_{j_l}} \sum_{n_l\in\mathcal{M}_{j_l,h_l}} \chi(h_2;p_2,n_1)\chi(n_1;q_{j_2}^{(h_1)},n_2)$$
$$\times \chi(h_3;p_{j_3},n_2)\chi(n_2;q_{j_3}^{(h_3)},n_3)\cdots\chi(h_l;p_l,n_{l-1})\chi(n_{l-1};q_{j_l}^{(h_l)},n_l)\exp(2\pi i n_l x).$$

Theorem 1.3. *Let $\mathcal{M}_0 \subset \mathbb{Z}^m$, K be the field of algebraic numbers, $[K:\mathbb{Q}] = m$, and let*

$$N = \max_{(n_1,\dots,n_m)\in\mathcal{M}_0} \max_{1\le j\le m} |n_j|.$$

Then there exist positive constants c_0 and c_1 which depend only on K and are such that for $Q_0 \ge \max\big(c_1, 4m|\mathcal{M}_0|\ln(c_0N)\big)$ the following conditions hold:

(a) *there exists $k\in\mathbb{N}$, $k \ll \dfrac{\ln(|\mathcal{M}_0|+20)}{\ln\ln(|\mathcal{M}_0|+20)}$,*

(b) *for every* $j = 1,\dots,k$ *there exist prime divisors* $\mathfrak{p}_j$, $N(\mathfrak{p}_j) = p_j \in \mathbb{P}$, $p_j \in (Q_0, 2Q_0]$, *and sets* $\mathcal{M}_j$ *for which* (1) *holds*,

(c) *for every* $j \in [1,k]$, $j \in \mathbb{N}$, *there exist a vector* $\mathbf{a}_j \in \mathbb{Z}^m$, *prime numbers* $q_j^{(h)}$, $h = 1,2,\dots,p_j$,

$$q_j^{(h)} < \max\bigl(c_1, m\ln^2(|\mathcal{M}_0|+20)\ln(N+20)\bigr),$$

and vectors $\mathbf{a}_j^{(h)} \in \mathbb{Z}^m$ *such that for every Fourier polynomial* $f_0 \in L_{\mathcal{M}_0}$ *the relation*

$$f_0(\mathbf{x}) = \sum_{l=1}^{k}(-1)^{l-1}\sum_{1\le j_1<\dots<j_l\le k}\ \sum_{\substack{h=1\\ h\in\mathcal{M}_{j_1}\ (\mathrm{mod}\,\mathfrak{p}_{j_1})}}^{p_{j_1}} \frac{1}{q_{j_1}^{(h)}}\sum_{\nu=1}^{q_{j_1}}\frac{1}{p_{j_1}}\sum_{\nu_1\,\mathrm{mod}\,p_{j_1}}\exp\Bigl(-2\pi i\langle\overline{h},\mathbf{a}_{j_1}\rangle\frac{\nu}{p_{j_1}}\Bigr)$$
$$\times f_0\Bigl(\frac{\nu}{q_j^{(h)}}\mathbf{a}_j^{(h)} + \frac{\nu_1}{p_{j_1}}\mathbf{a}_j\Bigr)\Bigl(\mathcal{P}_{j_l}\cdots\mathcal{P}_{j_2}D_{\mathcal{M}_{j_1,h}}\Bigl(* - \frac{\nu}{q_j^{(h)}}\mathbf{a}_j^{(h)}\Bigr)(x)\Bigr),$$

holds, where the operators $\mathcal{P}_j$ *are defined by the formula*

$$(\mathcal{P}_j\varphi)(x) = \sum_{\substack{h=1\\ h\in\mathcal{M}_j\ (\mathrm{mod}\,\mathfrak{p}_j)}}^{p_j}\frac{1}{q_j^{(h)}}\sum_{\nu=1}^{q_j^{(h)}}\Bigl(\frac{1}{p_j}\sum_{\nu_j\,\mathrm{mod}\,p_j}\exp\Bigl(-2\pi i\langle\overline{h},\mathbf{a}_j\rangle\frac{\nu}{p_j}\Bigr)\varphi\Bigl(\frac{\nu}{q_j^{(h)}}\mathbf{a}_j^{(h)} + \frac{\nu_j}{p_j}\mathbf{a}_j\Bigr)\Bigr)$$
$$\times D_{\mathcal{M}_j,h}\Bigl(\mathbf{x} - \frac{\nu}{q_j^{(h)}}\mathbf{a}_j^{(h)}\Bigr), \qquad \mathcal{M}_{j,h} = \{\mathbf{n} : \mathbf{n}\in\mathcal{M}_j,\ \langle\mathbf{a}_j,\mathbf{n}\rangle \equiv \langle\mathbf{a}_j,\overline{h}\rangle\ (\mathrm{mod}\,p_j)\}.$$

Theorem 1.4. *In the notation of Theorem* 1.3 *and under its conditions the following formula holds true*:

$$f_0(\mathbf{x}) = \sum_{l=1}^{k}(-1)^{l-1}\sum_{1\le j_1<\dots<j_l\le k}\ \sum_{\substack{h=1\\ h\in\mathcal{M}_{j_1}\ (\mathrm{mod}\,\mathfrak{p}_{j_1})}}^{p_{j_1}} \frac{1}{q_{j_1}^{(h)}}\sum_{\nu=1}^{q_{j_1}}\frac{1}{p_{j_1}}\sum_{\nu_1\,\mathrm{mod}\,p_{j_1}}\exp\Bigl(-2\pi i\langle\overline{h},\mathbf{a}_{j_1}\rangle\frac{\nu}{p_{j_1}}\Bigr)$$
$$\times f_0\Bigl(\frac{\nu}{q_{j_1}^{(h)}}\mathbf{a}_j^{(h)} + \frac{\nu_1}{p_{j_1}}\mathbf{a}_j\Bigr)\Bigl(\sum_{\substack{h_2=1\\ h_2\in\mathcal{M}_{j_2}\ (\mathrm{mod}\,\mathfrak{p}_{j_2})}}^{p_{j_2}}\ \sum_{\mathbf{n}_2\in\mathcal{M}_{j_2,h_2}}\cdots\sum_{\substack{h_l=1\\ h_l\in\mathcal{M}_{j_l}\ (\mathrm{mod}\,\mathfrak{p}_{j_l})}}^{p_{j_l}}$$
$$\times\sum_{\mathbf{n}_l\in\mathcal{M}_{j_l,h_l}}\delta_{p_{j_2}}(\langle\mathbf{n}-\overline{h}_2,\mathbf{a}_{j_2}\rangle)\delta_{q_{j_2}^{(h_2)}}(\langle\mathbf{n}-\mathbf{n}_2,\mathbf{a}_{j_2}^{(h_2)}\rangle)\cdots$$
$$\times\delta_{p_{j_l}}(\langle\mathbf{n}_{l-1}-\overline{h}_l,\mathbf{a}_{j_l}\rangle)\delta_{q_{j_l}^{(h_l)}}(\langle\mathbf{n}_{l-1}-\mathbf{n}_l,\mathbf{a}_{j_l}^{(h_l)}\rangle)\exp(2\pi i\langle\mathbf{n}_l,\mathbf{x}\rangle)\Bigr).$$

2. AUXILIARY LEMMAS

Lemma 2.1. *Let*

$$f(x) = \sum_{n\in\mathcal{M}} c_n e^{2\pi i n x}, \qquad \mathcal{M}\subset\mathbb{Z}, \quad |\mathcal{M}|<\infty, \quad x\in\mathbb{R}.$$

Let $q \in \mathbb{P}$ be such that for every $n \in \mathcal{M} \setminus \{0\}$ we have $q \nmid n$. Then

$$\int_0^1 f(x)\,dx = \frac{1}{q}\sum_{\nu=0}^{q-1} f\Big(\frac{\nu}{q}\Big).$$

Proof. We have the chain of equalities

$$\frac{1}{q}\sum_{\nu=0}^{q-1} f\Big(\frac{\nu}{q}\Big) = \frac{1}{q}\sum_{\nu=0}^{q-1}\sum_{n\in\mathcal{M}} c_n e^{2\pi i n\nu/q} = \sum_{n\in\mathcal{M}} c_n \frac{1}{q}\sum_{\nu=0}^{q-1} e^{2\pi i n\nu/q} = c_0 = \int_0^1 f(x)\,dx.$$

This proves the lemma.

Lemma 2.2. *Let*

$$f(x) = \sum_{n\in\mathcal{M}} c_n e^{2\pi i n x}, \qquad |\mathcal{M}| < \infty. \tag{3}$$

Assume that $q \in \mathbb{P}$ and

$$q \nmid M = \prod_{\substack{n_1,n_2\in\mathcal{M}\\ n_1\neq n_2}} |n_1 - n_2|.$$

Then the relation

$$f(x) = \frac{1}{q}\sum_{\nu=0}^{q-1} f\Big(\frac{\nu}{q}\Big) D_{\mathcal{M}}\Big(x - \frac{\nu}{q}\Big)$$

holds.

Proof. By the conditions of the lemma, it follows from Lemma 2.1 that

$$c_n = \int_0^1 f(x) e^{-2\pi i n x}\,dx = \frac{1}{q}\sum_{\nu=0}^{q-1} f\Big(\frac{\nu}{q}\Big) e^{-2\pi i n\nu/q}$$

for each $n \in \mathcal{M}$. Therefore

$$\begin{aligned} f(x) &= \sum_{n\in\mathcal{M}} c_n e^{2\pi i n x} = \sum_{n\in\mathcal{M}} \Big(\frac{1}{q}\sum_{\nu=0}^{q-1} f\Big(\frac{\nu}{q}\Big) e^{-2\pi i n\nu/q}\Big) e^{2\pi i n x} \\ &= \frac{1}{q}\sum_{\nu=0}^{q-1} f\Big(\frac{\nu}{q}\Big) \sum_{n\in\mathcal{M}} e^{2\pi i n(x-\nu/q)} = \frac{1}{q}\sum_{\nu=0}^{q-1} f\Big(\frac{\nu}{q}\Big) D_{\mathcal{M}}\Big(x - \frac{\nu}{q}\Big). \end{aligned}$$

The lemma is proved.

Lemma 2.3. *Let $\mathcal{M} \subset \mathbb{Z}$, $p \in \mathbb{P}$, and $h \in \mathbb{N}$. Let*

$$f_{p,h}(x) = \sum_{\substack{n\in\mathcal{M}\\ n\equiv h \pmod p}} c_n e^{2\pi i n x}$$

for $f(x)$ satisfying (3). *Then the relation*

$$f_{p,h}(x) = \frac{1}{p}\sum_{\nu=1}^{p} e^{-2\pi i h\nu/p} f\Big(x + \frac{\nu}{p}\Big)$$

holds.

Proof. We have the chain of equalities

$$\frac{1}{p}\sum_{\nu=1}^{p} e^{-2\pi i h\nu/p} f\Big(x+\frac{\nu}{p}\Big) = \frac{1}{p}\sum_{\nu=0}^{p-1} e^{-2\pi i h\nu/p} \sum_{n\in\mathcal{M}} c_n e^{2\pi i n(x+\nu/p)}$$
$$= \sum_{n\in\mathcal{M}} c_n e^{2\pi i n x}\Big(\frac{1}{p}\sum_{\nu=0}^{p-1}\exp\Big(2\pi i\frac{n-h}{p}\nu\Big)\Big) = \sum_{\substack{n\in\mathcal{M}\\ n\equiv h \ (\mathrm{mod}\, p)}} c_n e^{2\pi i n x} = f_{p,h}(x).$$

This proves the lemma.

Lemma 2.4. *Let $\alpha_1, \alpha_2, \dots, \alpha_m \in K$, and let $(n_1, \dots, n_m) \in \mathbb{Z}^m$. Then*

$$|N(n_1\alpha_1 + \dots + n_m\alpha_m)| \le C(\alpha_1, \dots, \alpha_m)\Big(\max_{1\le j\le m} |n_j|\Big)^m,$$

where $C(\partial_1, \dots, \alpha_m)$ does not depend on $n_1, \dots, n_m$.

Proof. Since

$$N(n_1\alpha_1 + \dots + n_m\alpha_m) = \prod_{\sigma} \sigma(n_1\alpha_1 + \dots + n_m\alpha_m),$$

where σ runs through all isomorphisms of the field K into the field $\mathbb{C}$, we have

$$N(n_1\alpha_1 + \dots + n_m\alpha_m) = \prod_{\sigma}\big(n_1\sigma(\alpha_1) + \dots + n_m\sigma(\alpha_m)\big).$$

This proves the lemma.

Lemma 2.5. *The norm of an integer divisor $\mathfrak{a}$ of the field K with respect to $\mathbb{Q}$ is equal to the number of residue classes of $\mathbb{Z}_K$* mod $\mathfrak{a}$, *i.e.*,

$$N_{K/\mathbb{Q}}(\mathfrak{a}) = |\mathbb{Z}_K/(\mathfrak{a})|,$$

where $(\mathfrak{a})$ is the ideal of the ring $\mathbb{Z}_K$ consisting of all $\alpha \in \mathbb{Z}_K$ which are divisible by $\mathfrak{a}$.

The proof can be found in [1, p. 242].

Lemma 2.6. *For any algebraic number field K, $[K:\mathbb{Q}] < \infty$, there exists a constant $x_0 = x_0(K)$ such that*

$$\sum_{\substack{\mathfrak{p}\in\mathbb{P}_K\\ N(\mathfrak{p})=p\in\mathbb{P}\\ x<N(\mathfrak{p})\le 2x}} 1 \ge \frac{9}{10}\,\frac{x}{\ln x}$$

for all $x \ge x_0$.

See the proof in [2, p. 345].

Lemma 2.7. *Let $\omega_1, \ldots, \omega_m$ be the fundamental basis of the field K, $\mathfrak{p} \in \mathbb{P}_K$, and let $N(\mathfrak{p}) = p \in \mathbb{P}$. Then there exists a vector $\mathbf{a} \in \mathbb{Z}^m$ such that*

$$\delta_{\mathfrak{p}}(n_1\omega_1 + \cdots + n_m\omega_m) = \frac{1}{p} \sum_{\nu \bmod p} \exp\Big(2\pi i \langle \mathbf{a}, \mathbf{n} \rangle \frac{\nu}{p}\Big)$$

for any $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}^m$.

The proof is given in [3, p. 133].

3. FUNDAMENTAL LEMMAS

Lemma 3.1. *Let $\mathcal{M} \subset \mathbb{Z}$, $|\mathcal{M}| \geq 1$, $\max_{m \in \mathcal{M}} |m| \leq N$, $L \geq 2$, and let*

$$Q \geq \max\big(1000, 2L|\mathcal{M}| \ln(2N)\big). \tag{4}$$

Then there exists $p \in \mathbb{P}$ such that

(a) $p \in (Q, 2Q]$;

(b) *among the numbers $h = 1, 2, ..., p$ there are no less than $|\mathcal{M}|\left(1 - \dfrac{2L+1}{L(L+1)}\right)$ numbers such that*

$$1 \leq \sum_{\substack{n \in \mathcal{M} \\ n \equiv h \ (\mathrm{mod}\, p)}} 1 < L.$$

Proof. We set

$$M = \prod_{\substack{n_1, n_2 \in \mathcal{M} \\ n_1 \neq n_2}} |n_1 - n_2| \qquad \text{for} \quad |\mathcal{M}| > 1 \qquad \text{and} \qquad M = 1 \qquad \text{for} \quad |\mathcal{M}| = 1. \tag{5}$$

We have the estimate

$$M \leq (2N)^{|\mathcal{M}|^2} = \exp\big(|\mathcal{M}|^2 \ln(2N)\big). \tag{6}$$

Let

$$Z(\mathcal{M}; p, h) = \sum_{\substack{n \in \mathcal{M} \\ n \equiv h \ (\mathrm{mod}\, p)}} 1.$$

We shall show that the inequality

$$\mathrm{ord}_p\, M \geq \sum_{h=1}^{p} Z^2(\mathcal{M}; p, h) - |\mathcal{M}| \tag{7}$$

holds for any $p \in \mathbb{P}$.

In fact, by the definition of $Z(\mathcal{M}; p, h)$ there are exactly $Z^2(\mathcal{M}; p, h)$ pairs among $(n_1, n_2) \in \mathcal{M} \times \mathcal{M}$ such that $n_1 \equiv n_2 \equiv h \ (\mathrm{mod}\, p)$. Therefore there exist exactly

$$\sum_{h=1}^{p} \big(Z^2(\mathcal{M}; p, h) - 1\big) = \sum_{h=1}^{p} Z^2(\mathcal{M}; p, h) - |\mathcal{M}|$$

factors in product (5) which are divisible by p. Therefore inequality (7) is proved in the case $|\mathcal{M}| > 1$. Now if $|\mathcal{M}| = 1$, then inequality (7) is obvious. Next, we shall show that there exists a number $p \in \mathbb{P}$ such that

(a) the estimate

$$\sum_{h=1}^{p} Z^2(\mathcal{M}; p, h) \leq \Big(1 + \frac{1}{L}\Big)|\mathcal{M}| \tag{8}$$

hold,

(b) $p \in (Q, 2Q]$.

Assume the contrary. Then the inequality

$$\sum_{h=1}^{p} Z^2(\mathcal{M}; p, h) > \Big(1 + \frac{1}{L}\Big)|\mathcal{M}|$$

holds for all $p \in (Q, 2Q]$. Consequently, by inequality (7), the inequality

$$\operatorname{ord}_p M \geq \sum_{h=1}^{p} Z^2(\mathcal{M}; p, h) - |\mathcal{M}| > \frac{1}{L}|\mathcal{M}|$$

holds for all $p \in (Q, 2Q]$. Therefore the relations

$$M \geq \prod_{p \in (Q,2Q]} p^{\operatorname{ord}_p M} > \prod_{p \in (Q,2Q]} p^{|\mathcal{M}|/L} = \Big(\prod_{Q<p\leq 2Q} p\Big)^{|\mathcal{M}|/L} = \exp\Big(\frac{1}{L}|\mathcal{M}|\Big(\sum_{p\in(Q,2Q]} \ln p\Big)\Big) \tag{9}$$

hold. Since

$$\sum_{Q<p\leq 2Q} \ln p > \frac{1}{2}Q$$

by the Cebyshev theorem on the distribution of prime numbers in the series of positive integers (by virtue of the choice of Q), we infer from relations (9) that

$$M > \exp\Big(\frac{1}{2L}|\mathcal{M}|Q\Big).$$

By estimate (6), this implies the inequality

$$|\mathcal{M}|^2 \ln(2N) > \frac{1}{2L}|\mathcal{M}|Q.$$

Since by the hypothesis of Lemma 3.1, we have (4), we obtain a contradiction. Thus, we have proved the existence of $p \in \mathbb{P}$, $p \in (Q, 2Q]$, such that estimate (8) is valid. We fix this p.

Let the set $H \subset \{1, 2, \dots, p\}$ consists of those and only those $h \in \{1, 2, \dots, p\}$ for which $Z(\mathcal{M}; p, h) \neq 0$, i.e., for these h there exists a number $n \in \mathcal{M}$ such that

$$n \equiv h \pmod{p}.$$

Using the Cauchy inequality, we obtain

$$
\begin{aligned}
|\mathcal{M}| &= \sum_{h\in H} Z(\mathcal{M};p,h) = \sum_{h\in H} 1\times Z(\mathcal{M};p,h)\\
&\le |H|^{1/2}\Big(\sum_{h\in H} Z^2(\mathcal{M};p,h)\Big)^{1/2} \le |H|^{1/2}\Big(\sum_{h=1}^{p} Z^2(\mathcal{M};p,h)\Big)^{1/2}.
\end{aligned}
$$

By estimate (8), this implies

$$
|H| \ge |\mathcal{M}|^2\Big(\sum_{h=1}^{p} Z^2(\mathcal{M};p,h)\Big)^{-1} \ge |\mathcal{M}|^2\Big(|\mathcal{M}|\Big(1+\frac{1}{L}\Big)\Big)^{-1} = |\mathcal{M}|\frac{L}{L+1}. \tag{10}
$$

We set

$$
H_0 = \{h : h\in H,\ Z(\mathcal{M};p,h) < L\}.
$$

In order to prove Lemma 3.1, it suffices to estimate $|H_0|$ from below in an appropriate way. Note that the inequalities

$$
|\mathcal{M}| \ge \sum_{h=1}^{p} Z(\mathcal{M};p,h) \ge \sum_{h\in H\setminus H_0} Z(\mathcal{M};p,h) \ge L|H\setminus H_0|
$$

hold, since, by the definition of H_0, we have $Z(\mathcal{M};p,h)\ge L$ for $h\in H$ and $h\notin H_0$. Therefore

$$
|H\setminus H_0| \le |\mathcal{M}|L^{-1}. \tag{11}
$$

From inequalities (10) and (11) we find that

$$
|H_0| = |H| - |H\setminus H_0| \ge |\mathcal{M}|\frac{L}{L+1} - |\mathcal{M}|L^{-1} = |\mathcal{M}|\Big(\frac{L}{L+1}-\frac{1}{L}\Big) = |\mathcal{M}|\Big(1-\frac{2L+1}{L(L+1)}\Big), \tag{12}
$$

and this proves the lemma.

Lemma 3.2. *Let $\mathcal{M}\subset\mathbb{Z}^m$, $1\le|\mathcal{M}|<\infty$. Assume that*

$$
N = \max_{\mathbf{n}=(n_1,\dots,n_m)\in\mathcal{M}}\ \max_{1\le j\le m} |n_j|, \qquad L\ge 2.
$$

Then there exist two positive constants c_0 and c_1 which are dependent only on the field K and the fundamental basis $\omega_1,\dots,\omega_m$ of the field K and are such that for any $Q\ge\max\big(c_1, 2m|\mathcal{M}|\ln(c_0N)\big)$ there exists $\mathfrak{p}\in\mathbb{P}_K$ such that

(a) $N(\mathfrak{p}) = p\in(Q,2Q]$;

(b) *among the numbers $h\in\{1,2,...,p\}$ there are no less than $|\mathcal{M}|\Big(1-\dfrac{2L+1}{L(L+1)}\Big)$ numbers for which*

$$
1 \le \sum_{\substack{\mathbf{n}\in\mathcal{M}\\ \tilde{\mathbf{n}}\equiv h \ (\mathrm{mod}\ \mathfrak{p})}} 1 < L.
$$

Proof. We set

$$M = \prod_{\substack{\mathbf{n}_1,\mathbf{n}_2\in\mathcal{M}\\ \mathbf{n}_1\neq\mathbf{n}_2}} (\widetilde{\mathbf{n}}_1 - \widetilde{\mathbf{n}}_2).$$

Let us estimate $|N(M)|$ from above. By Lemma 2.4, we have

$$|N(\widetilde{\mathbf{n}}_1 - \widetilde{\mathbf{n}}_2)| \le C(\omega_1,\dots,\omega_m)(2N)^m \le (c_0 N)^m.$$

Therefore

$$|N(M)| = \prod_{\substack{\mathbf{n}_1,\mathbf{n}_2\in\mathcal{M}\\ \mathbf{n}_1\neq\mathbf{n}_2}} |N(\widetilde{\mathbf{n}}_1 - \widetilde{\mathbf{n}}_2)| \le (c_0 N)^{m|\mathcal{M}|^2}. \tag{13}$$

Let

$$Z(\mathcal{M};\mathfrak{a},\beta) = \sum_{\substack{\mathbf{n}\in\mathcal{M}\\ \widetilde{\mathbf{n}}\equiv\beta \pmod{\mathfrak{a}}}} 1$$

for $\mathfrak{a}\in D_K$ (D_K is an integer divisor of the field K) and $\beta\in\mathbb{Z}_K$. We shall show that the inequality

$$\operatorname{ord}_{\mathfrak{p}} M \ge \sum_{h=1}^{p} Z^2(\mathcal{M};\mathfrak{p},h) - |\mathcal{M}| \ge 0 \tag{14}$$

holds for any $\mathfrak{p}\in\mathbb{P}_K$, $N(\mathfrak{p}) = p \in \mathbb{P}$.

Indeed, if $\mathfrak{p} \mid h \in \mathbb{N}$, then $N(\mathfrak{p}) = p$ is also a divisor of n. Consequently, the numbers $1,2,\dots,p$ are incomparable mod $\mathfrak{p}$. By Lemma 2.5, the numbers $1,2,\dots,p$ are the complete collection of representatives of all residue classes mod $\mathfrak{p}$. Furthermore, if $\widetilde{\mathbf{n}}_1 \equiv h \pmod{\mathfrak{p}}$ and $\widetilde{\mathbf{n}}_2 \equiv h \pmod{\mathfrak{p}}$ for $h\in\{1,2,\dots,p\}$, then $\mathfrak{p} \mid \widetilde{\mathbf{n}}_1 - \widetilde{\mathbf{n}}_2$. Hence, there are at least

$$\sum_{h=1}^{p} Z^2(\mathcal{M};\mathfrak{p},h)$$

pairs among $(\mathbf{n}_1,\mathbf{n}_2)$, $\mathbf{n}_1,\mathbf{n}_2\in\mathcal{M}$, such that $\mathfrak{p} \mid \widetilde{\mathbf{n}}_1 - \widetilde{\mathbf{n}}_2$. Rejecting the pairs $(\mathbf{n}_1,\mathbf{n}_2)$ for which $\mathbf{n}_1 = \mathbf{n}_2$, we obtain the estimate

$$\operatorname{ord}_{\mathfrak{p}} M \ge \sum_{h=1}^{p} Z^2(\mathcal{M};\mathfrak{p},h) - |\mathcal{M}|.$$

Finally, the inequality

$$Z^2(\mathcal{M};\mathfrak{p},h) \ge Z(\mathcal{M};\mathfrak{p},h)$$

holds. Therefore

$$\sum_{h=1}^{p} Z^2(\mathcal{M};\mathfrak{p},h) - |\mathcal{M}| \ge \sum_{h=1}^{p} Z(\mathcal{M};\mathfrak{p},h) - |\mathcal{M}| = 0.$$

This completes the proof of inequality (14).

Let $x_0(K)$ be defined as in the formulation of Lemma 2.6, and let the constant $c_0 = c_0(K)$ be defined just as in formula (13). Let us now prove that if

$$Q \ge \max\bigl(x_0(K), 2mL|\mathcal{M}|\ln(c_0 N)\bigr), \tag{15}$$

then, among the divisors $\mathfrak{p} \in \mathbb{P}$, $N(\mathfrak{p}) = p \in (Q, 2Q]$, there exists a prime divisor $\mathfrak{p}$, $N(\mathfrak{p}) = p$, for which

$$\sum_{h=1}^{p} Z^2(\mathcal{M}; \mathfrak{p}, h) \le \Big(1 + \frac{1}{L}\Big)|\mathcal{M}|. \tag{16}$$

Indeed, assume the contrary. Then

$$\sum_{h=1}^{p} Z^2(\mathcal{M}; \mathfrak{p}, h) > \Big(1 + \frac{1}{L}\Big)|\mathcal{M}|$$

for all $\mathfrak{p}$ such that $N(\mathfrak{p}) = p \in (Q, 2Q]$. In this case, it follows from formula (14) that

$$\operatorname{ord}_{\mathfrak{p}} M \ge \sum_{h=1}^{p} Z^2(\mathcal{M}; \mathfrak{p}, h) - |\mathcal{M}| > L^{-1}|\mathcal{M}|.$$

Therefore the divisor

$$\mathfrak{a} = \prod_{\substack{\mathfrak{p} \in \mathbb{P}_K \\ N(\mathfrak{p}) = p \in (Q, 2Q]}} \mathfrak{p}^{[L^{-1}|\mathcal{M}|]+1}$$

divides the divisor M, where $[L^{-1}|\mathcal{M}|]$ is the integer part of the number $L^{-1}|\mathcal{M}|$. Therefore $N(\mathfrak{a}) \,\big|\, |N(M)|$. Then we infer that

$$\prod_{\substack{\mathfrak{p} \in \mathbb{P}_K \\ N(\mathfrak{p}) = p \in (Q, 2Q]}} |N(\mathfrak{p})|^{L^{-1}|\mathcal{M}|} < |N(M)|,$$

and therefore

$$L^{-1}|\mathcal{M}| \sum_{\substack{\mathfrak{p} \in \mathbb{P}_K \\ N(\mathfrak{p}) = p \in (Q, 2Q]}} \ln |N(\mathfrak{p})| < \ln |N(M)|. \tag{17}$$

By Lemma 2.6, we have

$$\sum_{\substack{\mathfrak{p} \in \mathbb{P}_K \\ N(\mathfrak{p}) = p \in (Q, 2Q]}} \ln N(\mathfrak{p}) > \ln Q \,\frac{9}{10}\,\frac{Q}{\ln Q} = \frac{9}{10} Q$$

for $Q > x_0(K)$.

On the other hand, by inequality (13), we have

$$\ln |N(M)| \le m|\mathcal{M}|^2 \ln(c_0 N).$$

Therefore we infer from inequalities (17) and (15) that

$$\frac{9}{10} L^{-1}|\mathcal{M}| Q < m|\mathcal{M}|^2 \ln(c_0 N),$$

and, hence,

$$\frac{9}{10} L^{-1}|\mathcal{M}| \times 2mL|\mathcal{M}| \ln(c_0 N) < m|\mathcal{M}|^2 \ln(c_0 N),$$

i.e., $9/5 < 1$. This is a contradiction.

The obtained contradiction proves the existence of $\mathfrak{p} \in \mathbb{P}_K$, $N(\mathfrak{p}) = p \in (Q, 2Q]$, for which inequality (16) holds.

We fix this prime divisor $\mathfrak{p}$. Assume that the set $H \subset \{1, 2, \ldots, p\}$ consists of those and only those h for which $Z(\mathcal{M}; \mathfrak{p}, h) > 0$. Let us estimate $|H|$ from below. Using the Cauchy inequality, we obtain

$$|sM| = \sum_{h\in H} Z(\mathcal{M}; \mathfrak{p}, h) = \sum_{h\in H} 1 \times Z(\mathcal{M}; \mathfrak{p}, h) \le \Big(\sum_{h\in H} 1\Big)^{1/2} \Big(\sum_{h=1}^{p} Z^2(\mathcal{M}; \mathfrak{p}, h)\Big)^{1/2}.$$

It follows from this relation that

$$|H| \ge |\mathcal{M}|^2 \Big(\sum_{h=1}^{p} Z^2(\mathcal{M}; \mathfrak{p}, h)\Big)^{-1}. \tag{18}$$

By inequality (16), it follows from inequality (18) that

$$|H| \ge \Big(\frac{L}{L+1}\Big)|\mathcal{M}|. \tag{19}$$

Let now

$$H_0 = \{h : h \in H,\ Z(\mathcal{M}; \mathfrak{p}, h) < L\}.$$

Let us estimate H_0 from below. We have

$$|\mathcal{M}| = \sum_{h=1}^{p} Z(\mathcal{M}; \mathfrak{p}, h) \ge \sum_{h\in H\setminus H_0} Z(\mathcal{M}; \mathfrak{p}, h) \ge L|H \setminus H_0|.$$

Hence, the inequality

$$|H \setminus H_0| \le |\mathcal{M}|L^{-1} \tag{20}$$

holds. Now we obtain (12) from inequalities (19) and (20). The lemma is proved.

4. PROOF OF THEOREM 1.1

We define the sets $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k \subset \mathcal{M}_0$ and the prime numbers $p_1, \ldots, p_k \in (Q_0, 2Q_0]$ as follows. First, in the hypothesis of Lemma 3.1 we set $\mathcal{M} = \mathcal{M}_0$, $L = \ln(|\mathcal{M}_0| + 20)$, $N = \max_{n\in\mathcal{M}_0} |n|$, and $Q = Q_0$. By Lemma 3.1, there exist a prime number $p_1 \in (Q_0, 2Q_0]$ and a set $\mathcal{M}_1 \subset \mathcal{M}_0$ such that

(a) the estimate

$$|\mathcal{M}_0 \setminus \mathcal{M}_1| \le |\mathcal{M}_0| \frac{2L+1}{L(L+1)}$$

holds,

(b) for any $n_1 \in \mathcal{M}_1$, we have

$$1 \le |\{n_2 : n_2 \in \mathcal{M}_0,\ n_2 \equiv n_1 \pmod{p_1}\}| < L.$$

Next, in order to construct the set $\mathcal{M}_2$, under the conditions of Lemma 3.1, we set

$$\mathcal{M} = \mathcal{M}_0 \setminus \mathcal{M}_1, \qquad L = \ln(|\mathcal{M}_0| + 20), \qquad N = \max_{n \in \mathcal{M}_0} |n|, \qquad Q = Q_0.$$

By Lemma 3.1, we find $p_2 \in (Q_0, 2Q_0]$ and $\mathcal{M}_2 \subset \mathcal{M}_0 \setminus \mathcal{M}_1$ such that

(a) $|\mathcal{M}_0(\mathcal{M}_1 \setminus \mathcal{M}_2)| \le |\mathcal{M}_0| \left(\dfrac{2L+1}{L(L+1)} \right)^2$,

(b) the inequalities

$$1 \le |\{n_2 : n_2 \in \mathcal{M}_0 \setminus \mathcal{M}_1,\ n_2 \equiv n_1 \pmod{p_2}\}| < L$$

hold for any $n_1 \in \mathcal{M}_2$.

Proceeding in this way, we construct sets $\mathcal{M}_1, \dots, \mathcal{M}_k \subset \mathcal{M}_0$ and the prime numbers $p_1, \dots, p_k \in (Q_0, 2Q_0]$ with the following properties:

(a)

$$\mathcal{M}_j \subset \mathcal{M}_0 \setminus \bigcup_{l=1}^{j-1} \mathcal{M}_l, \qquad \mathcal{M}_j \ne \varnothing, \quad j = 1, 2, \dots, k,$$

(b) $\displaystyle\bigcup_{j=1}^{k} \mathcal{M}_j = \mathcal{M}_0$,

(c) $\left| \mathcal{M}_0 \setminus \displaystyle\bigcup_{j=1}^{l} \mathcal{M}_j \right| \le |\mathcal{M}_0| \left(\dfrac{2L+1}{L(L+1)} \right)^l$, $l = 1, 2, \dots, k$,

(d) for all $n_1 \in \mathcal{M}_l$ we have

$$1 \le \left| \left\{ n_2 : n_2 \in \mathcal{M}_0 \setminus \bigcup_{j=1}^{l-1} \mathcal{M}_j,\ n_2 \equiv n_1 \pmod{p_l} \right\} \right| < L, \tag{21}$$

(e) for sufficiently large $|\mathcal{M}_0|$ we have

$$k \ll \frac{\ln |\mathcal{M}_0|}{\ln \ln |\mathcal{M}_0|}. \tag{22}$$

Indeed, since

$$\frac{2L+1}{L(L+1)} < 2L^{-1} < c < 1,$$

this process terminates at a certain $k < \infty$. Moreover, estimate (22) holds for a sufficiently large $|\mathcal{M}_0|$.

Let us now define the operators $\mathcal{P}_j$. Let

$$\mathcal{P}_j \colon L_{\mathcal{M}_0} \setminus \bigcup_{l=1}^{j-1} \mathcal{M}_l \to L_{\mathcal{M}_j}, \qquad j = 1, 2, \dots, k,$$

be the operator which put the Fourier polynomial

$$(\mathcal{P}_j f)(x) = \sum_{n \in \mathcal{M}_j} a_n e^{2\pi i n x}$$

into correspondence with the Fourier polynomial

$$f(x) = \sum_{n \in \mathcal{M}_0 \setminus \bigcup_{l=1}^{j-1} \mathcal{M}_l} a_n e^{2\pi i n x}.$$

Let us analytically express $\mathcal{P}_j$. For brevity of writing, we introduce the notation

$$\widetilde{\mathcal{M}}_j = \mathcal{M}_0 \setminus \bigcup_{l=1}^{j-1} \mathcal{M}_l.$$

Let

$$H^{(j)} = \big\{h : h \in \{1, 2, \dots, p_j\},\ h \in \mathcal{M}_j \ (\operatorname{mod} p_j)\big\}, \tag{23}$$

$$\widetilde{\mathcal{M}}_{j,h} = \{n : n \in \widetilde{\mathcal{M}}_j,\ n \equiv h \ (\operatorname{mod} p_j)\}, \tag{24}$$

$$\mathcal{P}_{j,h} \colon L_{\widetilde{\mathcal{M}}_j} \to L_{\widetilde{\mathcal{M}}_{j,h}},$$

where, for all $f \in L_{\widetilde{\mathcal{M}}_j}$,

$$f(x) = \sum_{n \in \widetilde{\mathcal{M}}_j} a_n e^{2\pi i n x}, \tag{25}$$

we set by definition

$$(\mathcal{P}_{j,h} f)(x) = \sum_{\substack{n \in \widetilde{\mathcal{M}}_j \\ n \equiv h \ (\operatorname{mod} p_j)}} a_n e^{2\pi i n x}. \tag{26}$$

It follows from (23), (24), and (26) that

$$\mathcal{P}_j = \sum_{h \in H^{(j)}} \mathcal{P}_{j,h}. \tag{27}$$

Let $h \in H^{(j)}$, $f_{j,h} \in L_{\widetilde{\mathcal{M}}_{j,h}}$, and let

$$M_{j,h} = \begin{cases} \displaystyle\prod_{\substack{n_1, n_2 \in \widetilde{\mathcal{M}}_{j,h} \\ n_1 \neq n_2}} |n_1 - n_2| & \text{if } |\mathcal{M}_{j,h}| < 1 \\ 1 & \text{if } |\mathcal{M}_{j,h}| = 1. \end{cases}$$

We denote by $q_j^{(h)}$ the minimal prime number which does not divide $M_{j,h}$. Then, by Lemma 2.2, we have the relation

$$f_{j,h}(x) = \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{q_j^{(h)}} f_{j,h}\Big(\frac{\nu}{q_j^{(h)}}\Big) D_{\widetilde{\mathcal{M}}_{j,h}}\Big(x - \frac{\nu}{q_j^{(h)}}\Big). \tag{28}$$

Let us estimate $q_j^{(h)}$ from above. Since $n_1, n_2 \in \widetilde{\mathcal{M}}_{j,h}$, we have $n_1 \equiv n_2 \equiv h \ (\operatorname{mod} p_j)$. Since $h \in H^{(j)}$, we infer from relations (21) and (23) that $1 \le |\mathcal{M}_{j,h}| < L$.

Consequently, the estimate

$$M_{j,h} \le (2N+3)^{L^2} = \exp\bigl(L^2 \ln(2N+3)\bigr)$$

holds.

Let all prime numbers $q \in (1, x]$ divide $M_{j,h}$. Then the inequality

$$\prod_{\substack{q \le x \\ q \in \mathbb{P}}} q \le M_{j,h} \le (2N+3)^{L^2},$$

must hold, and therefore the inequality

$$L^2 \ln(2N+3) \ge \sum_{q \le x} \ln q \tag{29}$$

is valid. Using the Chebyshev theorem on the distribution of prime numbers in the series of positive integers, we obtain from relation (29) the inequality

$$x < \max\bigl(1000, 2L^2 \ln(2N+3)\bigr).$$

This implies that

$$q_j^{(h)} \le \max\bigl(1000, 2L^2 \ln(2N+3)\bigr) = \max\bigl(1000, 2\ln^2(|\mathcal{M}_0| + 20)\ln(2N+3)\bigr). \tag{30}$$

Note that estimate (30) does not depend on j and h only when $h \in H^{(j)}$.

Furthermore, by Lemma 2.3, the relation

$$(\mathcal{P}_{j,h} f)(x) = \frac{1}{p_j} \sum_{\nu=1}^{p_j} \exp\left(-2\pi i \frac{h\nu}{p_j}\right) f\left(\frac{\nu}{p_j} + x\right)$$

holds for $f(x) \in L_{\widetilde{\mathcal{M}}_j}$. Therefore, it follows from formula (28) that

$$\begin{aligned}
f_{j,h}(x) = (\mathcal{P}_{j,h} f)(x) &= \frac{1}{q_j^{(h)}} \sum_{\nu_h=1}^{q_j^{(h)}} (\mathcal{P}_{j,h} f)\left(\frac{\nu_h}{q_j^{(h)}}\right) D_{\widetilde{\mathcal{M}}_{j,h}}\left(x - \frac{\nu_h}{q_j^{(h)}}\right) \\
&= \frac{1}{q_j^{(h)}} \sum_{\nu_h=1}^{q_j^{(h)}} \left(\frac{1}{p_j} \sum_{\nu=1}^{p_j} \exp\left(-2\pi i \frac{h\nu}{p_j}\right) f\left(\frac{\nu}{p_j} + \frac{\nu_h}{q_j^{(h)}}\right)\right) D_{\widetilde{\mathcal{M}}_{j,h}}\left(x - \frac{\nu_h}{q_j^{(h)}}\right) \\
&= \frac{1}{p_j q_j^{(h)}} \sum_{\nu=1}^{p_j} \sum_{\nu_h=1}^{q_j^{(h)}} \exp\left(-2\pi i \frac{h\nu}{p_j}\right) f\left(\frac{\nu}{p_j} + \frac{\nu_h}{q_j^{(h)}}\right) D_{\widetilde{\mathcal{M}}_{j,h}}\left(x - \frac{\nu_h}{q_j^{(h)}}\right).
\end{aligned} \tag{31}$$

(The substitution of $\mathcal{M}_{j,h}$ for $\widetilde{\mathcal{M}}_{j,h}$ in $D_{\widetilde{\mathcal{M}}_{j,h}}(\,\cdot\,)$ is possible since $h \in \mathcal{M}_j \pmod{p_j}$.)

Now it follows from (27) and (31) that

$$(\mathcal{P}_{j,h} f)(x) = \frac{1}{p_j} \sum_{h \in H^{(j)}} \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{p_j} e^{-2\pi i \frac{h\nu}{p_j}} \sum_{\nu_h=1}^{q_j^{(h)}} f\left(\frac{\nu}{p_j} + \frac{\nu_h}{q_j^{(h)}}\right) D_{\widetilde{\mathcal{M}}_{j,h}}\left(x - \frac{\nu_h}{q_j^{(h)}}\right). \tag{32}$$

By construction, the operator $\mathcal{P}_j$ is defined on the space $L_{\widetilde{\mathcal{M}}_j}$. For convenience, we extend its definition to the space $L_{\mathcal{M}_0}$ with the use of formula (32). Let E denote the identity operator, $E\colon L_{\mathcal{M}_0} \to L_{\mathcal{M}_0}$. By the definition of the linear operator $\mathcal{P}_j$,

$$(E - \mathcal{P}_j)f \in L_{\widetilde{\mathcal{M}}_{j+1}}$$

for any Fourier polynomial (25), where $f(x) \in L_{\widetilde{\mathcal{M}}_j}$ (we use the convention $L_{\widetilde{\mathcal{M}}_{k+1}} = \{0\}$). Therefore the relation

$$(E - \mathcal{P}_k) \cdots (E - \mathcal{P}_1)\big|_{L_{\mathcal{M}_0}} \equiv 0. \tag{33}$$

holds. We infer from (33) that

$$E = \sum_{l=1}^{k} (-1)^{l-1} \sum_{1 \le j_1 < \cdots < j_l \le k} \mathcal{P}_{j_l} \mathcal{P}_{j_{l-1}} \cdots \mathcal{P}_{j_1}.$$

Therefore, we have the following relation for $f \in L_{\mathcal{M}_0}$:

$$\begin{aligned} f(x) = (Ef)(x) &= \sum_{l=1}^{k} (-1)^{l-1} \sum_{1 \le j_1 < \cdots < j_l \le k} (\mathcal{P}_{j_l} \mathcal{P}_{j_{l-1}} \cdots \mathcal{P}_{j_1} f)(x) \\ &= \sum_{l=1}^{k} (-1)^{l-1} \sum_{1 \le j_1 < \cdots < j_l \le k} \big((\mathcal{P}_{j_l} \cdots \mathcal{P}_{j_2}) \mathcal{P}_{j_1} f\big)(x). \end{aligned} \tag{34}$$

Expressing $\mathcal{P}_{j_1} f$ on the right-hand side with the aid of formula (32) of (34), we obtain

$$\begin{aligned} f(x) = \sum_{l=1}^{k} (-1)^{l-1} &\sum_{1 \le j_1 < \cdots < j_l \le k} \frac{1}{p_{j_1}} \sum_{h \in H^{(h)}} \frac{1}{q_{j_1}^{(h)}} \sum_{\nu=1}^{p_{j_1}} \exp\left(-2\pi i \frac{h\nu}{p_{j_1}}\right) f\left(\frac{\nu}{p_{j_1}} + \frac{\varkappa}{q_{j_1}^{(h)}}\right) \\ &\times \mathcal{P}_{j_l} \cdots \mathcal{P}_{j_2} D_{\mathcal{M}_{j_1,h}} \left(* - \frac{\varkappa}{q_{j_1}^{(h)}}\right)(x). \end{aligned}$$

The assertion of Theorem 1.1 follows from this formula.

5. PROOF OF THEOREM 1.2

In order to prove Theorem 1.2, we express

$$\left(\mathcal{P}_{j_l} \mathcal{P}_{j_{l-1}} \cdots \mathcal{P}_{j_2} D_{\mathcal{M}_{j_1,h}} \left(* - \frac{\varkappa}{q_{j_1}^{(h)}}\right)\right)(x)$$

in terms of the characteristic functions of the arithmetic progressions, i.e., in terms of $\chi(a; q, b)$. We have

$$\left(\mathcal{P}_{j_l} \cdots \mathcal{P}_{j_2} D_{\mathcal{M}_{j_1,h}} \left(* - \frac{\varkappa}{q_{j_1}^{(h)}}\right)\right)(x) = \sum_{n_1 \in \mathcal{M}_{j_1,h}} \exp\left(-2\pi i n_1 \frac{\varkappa}{q_{j_1}^{(h)}}\right) \big(\mathcal{P}_{j_l} \cdots \mathcal{P}_{j_2} \exp(2\pi i n_1 *)\big)(x).$$

From the definition of $\mathcal{P}_j$ we have

$$\begin{aligned}
&\big(\mathcal{P}_j \exp(2\pi i n_1 *)\big)(x)\\
&= \frac{1}{p_j} \sum_{\substack{h=1\\ h\in\mathcal{M}_j \,(\mathrm{mod}\, p_j)}}^{p_j} \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{p_j} \exp\Big(-2\pi i \frac{h\nu}{p_j}\Big) \sum_{\varkappa=1}^{q_j^{(h)}} \exp\Big(2\pi i n_1 \Big(\frac{\nu}{p_j} + \frac{\varkappa}{q_j^{(h)}}\Big)\Big) D_{\mathcal{M}_j,h}\Big(x - \frac{\varkappa}{q_j^{(h)}}\Big)\\
&= \frac{1}{p_j} \sum_{\substack{h=1\\ h\in\mathcal{M}_j \,(\mathrm{mod}\, p_j)}}^{p_j} \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{p_j} \exp\Big(-2\pi i \frac{h\nu}{p_j}\Big) \sum_{\varkappa=1}^{q_j^{(h)}} \exp\Big(2\pi i n_1 \frac{\nu}{p_j}\Big) \exp\Big(2\pi i n_1 \frac{\varkappa}{q_j^{(h)}}\Big)\\
&\qquad \times \sum_{n_2\in\mathcal{M}_{j,h}} \exp(2\pi i n_2 x) \exp\Big(-2\pi i n_2 \frac{\varkappa}{q_j^{(h)}}\Big)\\
&= \frac{1}{p_j} \sum_{\substack{h=1\\ h\in\mathcal{M}_j \,(\mathrm{mod}\, p_j)}}^{p_j} \sum_{n_2\in\mathcal{M}_{j,h}} \exp(2\pi i n_2 x) \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{p_j} \exp\Big(2\pi i (n_1 - h)\frac{\nu}{p_j}\Big) \sum_{\varkappa=1}^{q_j^{(h)}} \exp\Big(2\pi i (n_1 - n_2)\frac{\varkappa}{q_j^{(h)}}\Big)\\
&= \sum_{\substack{h=1\\ h\in\mathcal{M}_j \,(\mathrm{mod}\, p_j)}}^{p_j} \sum_{n_2\in\mathcal{M}_{j,h}} \chi(h; p_j, n_1)\chi(n_1; q_j^{(h)}, n_2) \exp(2\pi i n_2 x). \qquad (35)
\end{aligned}$$

Applying formula (35), we use iteration and obtain

$$\begin{aligned}
\big(\mathcal{P}_j \dots \mathcal{P}_{j_2} \exp(2\pi i n_1 *)\big)(x) &= \sum_{\substack{h_2=1\\ h_2\in\mathcal{M}_{j_2} \,(\mathrm{mod}\, p_{j_2})}}^{p_{j_2}} \sum_{n_2\in\mathcal{M}_{j_2,h_2}} \cdots \sum_{\substack{h_l=1\\ h_l\in\mathcal{M}_{j_l} \,(\mathrm{mod}\, p_{j_l})}}^{p_{j_l}} \sum_{n_l\in\mathcal{M}_{j_l,h_l}} \chi(h_2; p_{j_2}, n_1)\\
&\times \chi(n_1; q_{j_2}^{(h_2)}, n_2)\chi(h_3; p_{j_3}, n_2)\chi(n_2; q_{j_3}^{(h_3)}, n_3) \cdots \chi(h_l; p_{j_l}, n_{l-1})\chi(n_{l-1}; q_{j_l}^{(h_l)}, n_l) \exp(2\pi i n_l x). \qquad (36)
\end{aligned}$$

Applying (36), we obtain the assertion of Theorem 1.2 from Theorem 1.1.

6. PROOF OF THEOREM 1.3

Using Lemma 3.2, we find prime divisors $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ and sets $\mathcal{M}_1, \dots, \mathcal{M}_k \subset \mathcal{M}_0$ such that

(a) $N(\mathfrak{p}) = p_j \in \mathbb{P}$ for $j = 1, \dots, k$, $p_j \in (Q, 2Q]$,

(b) $\mathcal{M}_0 = \bigcup_{j=1}^{k} \mathcal{M}_j$,

(c) $\mathcal{M}_j \subset \mathcal{M}_0 \setminus (\mathcal{M}_1 \cup \dots \cup \mathcal{M}_{j-1})$;

(d) if $\mathbf{n}_1 \in \mathcal{M}_j$, then

$$1 \le \Big| \Big\{ \mathbf{n}_2 : \widetilde{\mathbf{n}}_2 \equiv \widetilde{\mathbf{n}}_1 \ (\mathrm{mod}\, \mathfrak{p}_j),\ \mathbf{n}_2 \in \mathcal{M}_0 \setminus \bigcup_{l=1}^{j-1} \mathcal{M}_l \Big\} \Big| < L.$$

We shall construct the sets $\mathcal{M}_1, \mathcal{M}_2, \dots$ recursively. To this end, under the conditions of Lemma 3.2, we first set $\mathcal{M} = \mathcal{M}_0$, $Q = Q_0$, $L = \ln(|\mathcal{M}_0| + 20)$ and find $\mathfrak{p}_1$ and $\mathcal{M}_1$ possessing the corresponding properties. Next, setting $\mathcal{M} = \mathcal{M}_0 \setminus \mathcal{M}_1$ and leaving Q and L invariant, we find $\mathfrak{p}_2$ and $\mathcal{M}_2$. We continue this process until we obtain an empty set. By Lemma 3.2, each operation of this kind diminishes $\mathcal{M}$ at least $L/2$ times. Hence, after

$$k \ll \frac{\ln(|\mathcal{M}_0| + 20)}{\ln\ln(|\mathcal{M}_0| + 20)}$$

steps, we obtain an empty set. We set

$$\widetilde{\mathcal{M}}_j = \mathcal{M}_0 \setminus \bigcup_{l=1}^{j-1} \mathcal{M}_l, \qquad \widetilde{\mathcal{M}}_{j,h} = \{\mathbf{n} : \widetilde{\mathbf{n}} \equiv h \pmod{\mathfrak{p}_j},\ \mathbf{n} \in \widetilde{\mathcal{M}}_j\}.$$

Let the linear operator $\mathcal{P}_j$ put the polynomial

$$(\mathcal{P}_j f)(\mathbf{x}) = \sum_{\mathbf{n} \in \mathcal{M}_j} a_{\mathbf{n}} \exp(2\pi i \langle \mathbf{n}, \mathbf{x} \rangle)$$

into correspondence with the Fourier polynomial

$$f(\mathbf{x}) = \sum_{\mathbf{n} \in \widetilde{\mathcal{M}}_j} a_{\mathbf{n}} \exp(2\pi i \langle \mathbf{n}, \mathbf{x} \rangle). \tag{37}$$

Since $\operatorname{Im}(E - \mathcal{P}_j) \subset L_{\widetilde{\mathcal{M}}_{j+1}}$, we have

$$(E - \mathcal{P}_k)(E - \mathcal{P}_{k-1}) \cdots (E - \mathcal{P}_1)\big|_{L_{\mathcal{M}_0}} = 0. \tag{38}$$

Furthermore, let

$$(\mathcal{P}_{j,h})(\mathbf{x}) = \sum_{\mathbf{n} \in \widetilde{\mathcal{M}}_{j,h}} a_{\mathbf{n}} \exp(2\pi i \langle \mathbf{n}, \mathbf{x} \rangle)$$

for polynomial (37). It follows from the definitions of $\mathcal{P}_j$ and $\mathcal{P}_{j,h}$ that

$$\mathcal{P}_j = \sum_{\substack{h=1 \\ h \in \mathcal{M}_j \pmod{\mathfrak{p}_j}}}^{p} \mathcal{P}_{j,h}. \tag{39}$$

Let us express $\mathcal{P}_{j,h}$ formally. Suppose that $f \in L_{\widetilde{\mathcal{M}}_j}$, and let

$$f(\mathbf{x}) = \sum_{\mathbf{n} \in \widetilde{\mathcal{M}}_j} b_{\mathbf{n}} \exp(2\pi i \langle \mathbf{n}, \mathbf{x} \rangle).$$

Then, by the definition of $\mathcal{P}_{j,h}$, we have

$$(\mathcal{P}_{j,h} f)(x) = \sum_{\substack{\mathbf{n} \in \mathcal{M}_j \\ \widetilde{\mathbf{n}} \equiv h \pmod{\mathfrak{p}_j}}} b_{\mathbf{n}} \exp(2\pi i \langle \mathbf{n}, \mathbf{x} \rangle) = \sum_{\mathbf{n} \in \widetilde{\mathcal{M}}_j} \delta_{\mathfrak{p}_j}(\widetilde{\mathbf{n}} - h) b_{\mathbf{n}} \exp(2\pi i \langle \mathbf{n}, \mathbf{x} \rangle). \tag{40}$$

We use here Lemma 2.7. By Lemma 2.7, there exists a vector $a_j \in \mathbb{Z}^m$ such that for $\alpha \in \mathbb{Z}_K$ we have

$$\delta_{\mathfrak{p}}(\alpha) = \frac{1}{p_j} \sum_{\nu_j \bmod p_j} \exp\biggl(2\pi i \langle \overline{\alpha}, \mathbf{a}_j \rangle \frac{\nu_j}{p_j}\biggr), \tag{41}$$

where $p_j = N(\mathfrak{p}_j)$, and the vector $\overline{\alpha}$ can be found from the decomposition $\alpha = \alpha_1\omega_1 + \cdots + \alpha_m\omega_m$, i.e., $\overline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$. We find from relations (40) and (41) that

$$\begin{aligned}(\mathcal{P}_{j,h}f)(\mathbf{x}) &= \sum_{\mathbf{n} \in \widetilde{\mathcal{M}}_j} \biggl(\frac{1}{p_j} \sum_{\nu_j \bmod p_j} \exp\biggl(2\pi i \langle \mathbf{n} - \overline{h}, \mathbf{a}_j \rangle \frac{\nu_j}{p_j}\biggr)\biggr) b_{\mathbf{n}} \exp(2\pi i \langle \mathbf{n}, \mathbf{x} \rangle) \\ &= \frac{1}{p_j} \sum_{\nu_j \bmod p_j} \exp\biggl(-2\pi i \langle \overline{h}, \mathbf{a}_j \rangle \frac{\nu_j}{p_j}\biggr) \sum_{\mathbf{n} \in \widetilde{\mathcal{M}}_j} b_{\mathbf{n}} \exp\biggl(2\pi i \biggl\langle \mathbf{n}, \mathbf{x} + \frac{\nu_j \mathbf{a}_j}{p_j} \biggr\rangle\biggr). \end{aligned} \tag{42}$$

Let $h \in \mathcal{M}_j \pmod{\mathfrak{p}_j}$, and let

$$M_{j,h} = \begin{cases} \displaystyle\prod_{\mathbf{n}_1, \mathbf{n}_2 \in \widetilde{\mathcal{M}}_{j,h}} (\widetilde{\mathbf{n}}_1 - \widetilde{\mathbf{n}}_2) & \text{if } |\mathcal{M}_{j,h}| > 1 \\ 1 & \text{if } |sM_{j,h}| = 1. \end{cases}$$

By the definition of the set $\mathcal{M}_j$, we have $1 \le |M_{j,h}| < L$ for $h \in \mathcal{M}_j \pmod{\mathfrak{p}_j}$. Therefore it follows from Lemma 2.4 that

$$|N(M_{j,h})| = \prod_{\mathbf{n}_1, \mathbf{n}_2 \in \widetilde{\mathcal{M}}_{j,h}} |N(\widetilde{\mathbf{n}}_1 - \widetilde{\mathbf{n}}_2)| \le (CN^m)^{L^2} = \exp\bigl(L^2(m \ln N + C)\bigr). \tag{43}$$

Assume that all $\mathfrak{p} \in \mathbb{P}_K$ such that $N(\mathfrak{p}) = p \in \mathbb{P}$, $p \le 2x$, divide $M_{j,h}$. Then, by inequality (43), the inequalities

$$\exp\Biggl(\ln x \sum_{\substack{\mathfrak{p} \in \mathbb{P} \\ N(\mathfrak{p}) = p \in \mathbb{P} \\ x < N(\mathfrak{p}) \le 2x}} 1\Biggr) \le \exp\Biggl(\sum_{\substack{\mathfrak{p} \in \mathbb{P}_K \\ N(\mathfrak{p}) = p \in \mathbb{P} \\ x < N(\mathfrak{p}) \le 2x}} \ln p\Biggr) \le \exp\bigl(L^2(m\, \ln N + C)\bigr) \tag{44}$$

must be valid. By Lemma 2.6, it follows from inequality (44) that

$$\frac{9}{10}x \le L^2(m\, \ln N + C).$$

We infer from this inequality that for a sufficiently large x_0, there exists a divisor $\mathfrak{q}_j^{(h)} \in \mathbb{P}_K$, $N(\mathfrak{q}_j^{(h)}) = q_j^{(h)} \in \mathbb{P}$, for which

$$\mathfrak{q}_j^{(h)} \nmid M_{j,h}$$

and

$$N(\mathfrak{q}_j^{(h)}) = q_j^{(h)} < \max\bigl(x_0, 4L^2(m\, \ln N + C)\bigr).$$

Let

$$\varphi(\mathbf{x}) = \sum_{\mathbf{n} \in \mathcal{M}_{j,h}} \alpha_{\mathbf{n}} \exp(2\pi i \langle \mathbf{n}, \mathbf{x} \rangle) \in L_{\mathcal{M}_{j,h}}.$$

Let us construct an interpolation formula for $\varphi(\mathbf{x})$. We have

$$\varphi(\mathbf{x}) = \sum_{\mathbf{n}_1 \in \mathcal{M}_{j,h}} \alpha_{\mathbf{n}_1} \sum_{\mathbf{n} \in \mathcal{M}_{j,h}} \delta(\mathbf{n} - \mathbf{n}_1) \exp(2\pi i \langle \mathbf{n}, \mathbf{x} \rangle). \tag{45}$$

Since $\mathfrak{q}_j^{(h)}$ does not divide $\mathbf{n} - \mathbf{n}_1$ for $\mathbf{n} \neq \mathbf{n}_1$, it follows from (45) that

$$\varphi(\mathbf{x}) = \sum_{\mathbf{n}_1 \in \mathcal{M}_{j,h}} \alpha_{\mathbf{n}_1} \sum_{\mathbf{n} \in \mathcal{M}_{j,h}} \delta_{\mathfrak{q}_j^{(h)}}(\widetilde{\mathbf{n}} - \widetilde{\mathbf{n}}_1) \exp(2\pi i \langle \mathbf{n}, \mathbf{x} \rangle). \tag{46}$$

By Lemma 2.7, there exists a vector $\mathbf{a}_j^{(h)} \in \mathbb{Z}^m$ such that

$$\delta_{\mathfrak{q}_j^{(h)}}(\widetilde{\mathbf{n}} - \widetilde{\mathbf{n}}_1) = \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{q_j^{(h)}} \exp\left(2\pi i \langle \mathbf{a}_j^{(h)}, \mathbf{n}_1 - \mathbf{n} \rangle \frac{\nu}{q_j^{(h)}}\right). \tag{47}$$

We infer from formulas (46) and (47) that

$$\begin{aligned}
\varphi(\mathbf{x}) &= \sum_{\mathbf{n}_1 \in \mathcal{M}_{j,h}} \alpha_{\mathbf{n}_1} \sum_{\mathbf{n} \in \mathcal{M}_{j,h}} \left(\frac{1}{q_j^{(h)}} \sum_{\nu=1}^{q_j^{(h)}} \exp\left(2\pi i \langle \mathbf{a}_j^{(h)}, \mathbf{n}_1 - \mathbf{n} \rangle \frac{\nu}{q_j^{(h)}}\right) \exp(2\pi i \langle \mathbf{n}, \mathbf{x} \rangle)\right) \\
&= \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{q_j^{(h)}} \left(\sum_{\mathbf{n}_1 \in \mathcal{M}_{j,h}} \alpha_{\mathbf{n}_1} \exp\left(2\pi i \langle \mathbf{a}_j^{(h)}, \mathbf{n}_1 \rangle \frac{\nu}{q_j^{(h)}}\right)\right) \left(\sum_{\mathbf{n} \in \mathcal{M}_{j,h}} \exp\left(2\pi i \left\langle \mathbf{n}, \mathbf{x} - \mathbf{a}_j^{(h)} \frac{\nu}{q_j^{(h)}} \right\rangle\right)\right) \\
&= \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{q_j^{(h)}} \varphi\left(\frac{\nu}{q_j^{(h)}} \mathbf{a}_j^{(h)}\right) D_{\mathcal{M}_{j,h}}\left(\mathbf{x} - \frac{\nu}{q_j^{(h)}} \mathbf{a}_j^{(h)}\right).
\end{aligned} \tag{48}$$

Since $(\mathcal{P}_{j,h} f)(\mathbf{x}) \in L_{\mathcal{M}_{j,h}}$, it follows from formulas (42) and (48) that

$$\begin{aligned}
(\mathcal{P}_{j,h} f)(\mathbf{x}) &= \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{q_j^{(h)}} (\mathcal{P}_{j,h} f)\left(\frac{\nu}{q_j^{(h)}} \mathbf{a}_j^{(h)}\right) D_{\mathcal{M}_{j,h}}\left(\mathbf{x} - \frac{\nu}{q_j^{(h)}} \mathbf{a}_j^{(h)}\right) \\
&= \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{q_j^{(h)}} \left(\frac{1}{p_j} \sum_{\nu_j \bmod p_j} \exp\left(-2\pi i \langle \overline{h}, \mathbf{a}_j \rangle \frac{\nu_j}{p_j}\right)\right. \\
&\quad \times \left.\sum_{\mathbf{n} \in \widetilde{\mathcal{M}}_j} b_{\mathbf{n}} \exp\left(2\pi i \left\langle \mathbf{n}, \frac{\nu}{q_j^{(h)}} \mathbf{a}_j^{(h)} + \frac{\nu_j}{p_j} \mathbf{a}_j \right\rangle\right)\right) D_{\mathcal{M}_{j,h}}\left(\mathbf{x} - \frac{\nu}{q_j^{(h)}} \mathbf{a}_j^{(h)}\right).
\end{aligned} \tag{49}$$

It follows from relations (39) and (49) that

$$\begin{aligned}
(\mathcal{P}_j f)(\mathbf{x}) &= \sum_{\substack{h=1 \\ h \in \mathcal{M}_j \ (\mathrm{mod}\ \mathfrak{p}_j)}}^{p_j} \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{q_j^{(h)}} \left(\frac{1}{p_j} \sum_{\nu_j \bmod p_j} \exp\left(-2\pi i \langle \overline{h}, \mathbf{a}_j \rangle \frac{\nu_j}{p_j}\right)\right. \\
&\quad \times \left.\sum_{\mathbf{n} \in \widetilde{\mathcal{M}}_j} b_{\mathbf{n}} \exp\left(2\pi i \left\langle \mathbf{n}, \frac{\nu}{q_j^{(h)}} \mathbf{a}_j^{(h)} + \frac{\nu_j}{p_j} \mathbf{a}_j \right\rangle\right)\right) D_{\mathcal{M}_{j,h}}\left(\mathbf{x} - \frac{\nu}{q_j^{(h)}} \mathbf{a}_j^{(h)}\right).
\end{aligned}$$

We extend the definition of the operator $\mathcal{P}_j$ to the space $L_{\mathcal{M}_0}$ with the use of the formula

$$(\mathcal{P}_j f)(\mathbf{x}) = \sum_{\substack{h=1\\ h\in\mathcal{M}_j \pmod{\mathfrak{p}_j}}}^{p_j} \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{q_j^{(h)}} \biggl(\frac{1}{p_j} \sum_{\nu_j \bmod p_j} \exp\Bigl(-2\pi i\langle \overline{h}, \mathbf{a}_j\rangle \frac{\nu_j}{p_j}\Bigr)$$
$$\times \sum_{\mathbf{n}\in\mathcal{M}_0} b_{\mathbf{n}} \exp\Bigl(2\pi i\Bigl\langle \mathbf{n}, \frac{\nu}{q_j^{(h)}}\mathbf{a}_j^{(h)} + \frac{\nu_j}{p_j}\mathbf{a}_j\Bigr\rangle\Bigr)\biggr) D_{\mathcal{M}_{j,h}}\Bigl(\mathbf{x} - \frac{\nu}{q_j^{(h)}}\mathbf{a}_j^{(h)}\Bigr). \qquad (50)$$

Note that formula (38) is valid. Indeed, the operator $\mathcal{P}_j$ is applied to the function $(E - \mathcal{P}_{j-1}) \times \cdots (E - \mathcal{P}_1) f$ which belongs to the space $L_{\widetilde{\mathcal{M}}_j}$. On the other hand, the extension of the domain of definition of the operators $\mathcal{P}_j$ gives us a possibility to open the brackets in relation (38).

Opening the brackets in relation (38), we obtain

$$E = \sum_{l=1}^{k} (-1)^{l-1} \sum_{1\le j_1<\dots<j_l\le k} \mathcal{P}_{j_l}\mathcal{P}_{j_{l-1}}\cdots\mathcal{P}_{j_1}.$$

Hence, the identity

$$f(\mathbf{x}) = (Ef)(\mathbf{x}) = \sum_{l=1}^{k} (-1)^{l-1} \sum_{1\le j_1<\dots<j_l\le k} (\mathcal{P}_{j_l}\cdots\mathcal{P}_{j_1} f)(\mathbf{x})$$

holds for every $f \in L_{\mathcal{M}_0}$. Applying relation (50), we obtain

$$f(\mathbf{x}) = \sum_{l=1}^{k} (-1)^{l-1} \sum_{1\le j_1<\dots<j_l\le k} \sum_{\substack{h=1\\ h\in\mathcal{M}_{j_1} \pmod{\mathfrak{p}_{j_1}}}}^{p_{j_1}} \frac{1}{q_{j_1}^{(h)}} \sum_{\nu=1}^{q_{j_1}^{(h)}} \frac{1}{p_{j_1}} \sum_{\nu_1 \bmod p_{j_1}} \exp\Bigl(-2\pi i\langle \overline{h}, \mathbf{a}_{j_1}\rangle \frac{\nu_1}{p_{j_1}}\Bigr)$$
$$\times f\Bigl(\frac{\nu}{q_{j_1}^{(h)}}\mathbf{a}_{j_1}^{(h)} + \frac{\nu_1}{p_{j_1}}\mathbf{a}_{j_1}\Bigr) \Bigl(\mathcal{P}_{j_l}\mathcal{P}_{j_{l-1}}\cdots\mathcal{P}_{j_2} D_{\mathcal{M}_{j_1,h}}\Bigl(* - \frac{\nu}{q_{j_1}^{(h)}}\mathbf{a}_{j_1}^{(h)}\Bigr)\Bigr)(\mathbf{x}).$$

We have thus proved Theorem 1.3.

7. PROOF OF THEOREM 1.4

To prove Theorem 1.4, it suffices to express

$$\Bigl(\mathcal{P}_{j_l}\mathcal{P}_{j_{l-1}}\cdots\mathcal{P}_{j_2} D_{\mathcal{M}_{j_1,h}}\Bigl(* - \frac{\nu}{q_{j_1}^{(h)}}\mathbf{a}_{j_1}^{(h)}\Bigr)\Bigr)(\mathbf{x})$$

in terms of the characteristic functions of arithmetic progressions.

By the definition of $D_{\mathcal{M}}$, we have

$$\Bigl(\mathcal{P}_{j_l}\cdots\mathcal{P}_{j_2} D_{\mathcal{M}_{j_1,h}}\Bigl(* - \frac{\nu}{q_{j_1}^{(h)}}\mathbf{a}_{j_1}^{(h)}\Bigr)\Bigr)(\mathbf{x})$$
$$= \sum_{\mathbf{n}_1\in\mathcal{M}_{j_1,h}} \exp\Bigl(-2\pi i\Bigl\langle \mathbf{n}_1, \frac{\nu}{q_{j_1}^{(h)}}\mathbf{a}_{j_1}^{(h)}\Bigr\rangle\Bigr) \bigl(\mathcal{P}_{j_l}\cdots\mathcal{P}_{j_2} \exp(2\pi i\langle \mathbf{n}_1, *\rangle)\bigr)(\mathbf{x}).$$

By the definition of the operator $\mathcal{P}_j$ (formula (50)), we have

$$\begin{aligned}
\big(\mathcal{P}_j \exp(2\pi i\langle \mathbf{n}, *\rangle)\big)(\mathbf{x}) &= \sum_{\substack{h=1\\ h\in\mathcal{M}_j \,(\mathrm{mod}\,\mathfrak{p}_j)}}^{p_j} \frac{1}{q_j^{(h)}} \sum_{\nu=1}^{q_j^{(h)}} \biggl(\frac{1}{p_j} \sum_{\nu_j \,\mathrm{mod}\, p_j} \exp\biggl(-2\pi i\langle \overline{h}, \mathbf{a}_j\rangle \frac{\nu_j}{p_j}\biggr)\\
&\quad \times \exp\biggl(2\pi i \biggl\langle \mathbf{n}, \frac{\nu}{q_j^{(h)}} \mathbf{a}_j^{(h)} + \frac{\nu_j}{p_j}\mathbf{a}_j\biggr\rangle\biggr) D_{\mathcal{M}_{j,h}}\biggl(\mathbf{x} - \frac{\nu}{q_j^{(h)}}\mathbf{a}_j^{(h)}\biggr)\biggr)\\
&= \frac{1}{p_j} \sum_{\substack{h=1\\ h\in\mathcal{M}_j \,(\mathrm{mod}\,\mathfrak{p}_j)}}^{p_j} \sum_{\mathbf{n}_1 \in \mathcal{M}_{j,h}} \exp\big(2\pi i\langle \mathbf{n}_1, \mathbf{x}\rangle\big) \frac{1}{q_j^{(h)}}\\
&\quad \times \sum_{\nu_j \,\mathrm{mod}\, p_j} \exp\biggl(2\pi i\langle \mathbf{n} - \overline{h}, \mathbf{a}_j\rangle \frac{\nu_j}{p_j}\biggr) \sum_{\nu \,\mathrm{mod}\, q_j^{(h)}} \exp\biggl(2\pi i \langle \mathbf{n} - \mathbf{n}_1, \mathbf{a}_j^{(h)}\rangle \frac{\nu}{q_j^{(h)}}\biggr)\\
&= \sum_{\substack{h=1\\ h\in\mathcal{M}_j \,(\mathrm{mod}\,\mathfrak{p}_j)}}^{p_j} \sum_{\mathbf{n}_1 \in \mathcal{M}_{j,h}} \delta_{p_j}\big(\langle \mathbf{n} - \overline{h}, \mathbf{a}_j\rangle\big) \delta_{q_j^{(h)}}\big(\langle \mathbf{n} - \mathbf{n}_1, \mathbf{a}_j^{(h)}\rangle\big) \exp\big(2\pi i\langle \mathbf{n}_1, \mathbf{x}\rangle\big).
\end{aligned} \tag{51}$$

Applying successively formula (51), we obtain

$$\begin{aligned}
\big(\mathcal{P}_{j_l} \cdots \mathcal{P}_{j_2} \exp(2\pi i\langle \mathbf{n}, *\rangle)\big)(\mathbf{x}) &= \sum_{\substack{h_2=1\\ h_2\in\mathcal{M}_{j_2} \,(\mathrm{mod}\,\mathfrak{p}_{j_2})}}^{p_{j_2}} \sum_{\mathbf{n}_2 \in \mathcal{M}_{j_2,h_2}} \cdots \sum_{\substack{h_l=1\\ h_l\in\mathcal{M}_{j_l} \,(\mathrm{mod}\,\mathfrak{p}_{j_l})}}^{p_{j_l}} \sum_{\mathbf{n}_l \in \mathcal{M}_{j_l,h_l}}\\
&\quad \times \delta_{p_{j_2}}\big(\langle \mathbf{n} - \overline{h}_2, \mathbf{a}_{j_2}\rangle\big) \delta_{q_{j_2}^{(h_2)}}\big(\langle \mathbf{n} - \mathbf{n}_2, \mathbf{a}_{j_2}^{(h_2)}\rangle\big)\\
&\quad \times \delta_{p_{j_3}}\big(\langle \mathbf{n}_2 - \overline{h}_3, \mathbf{a}_{j_3}\rangle\big) \delta_{q_{j_3}^{(h_3)}}\big(\langle \mathbf{n}_2 - \mathbf{n}_3, \mathbf{a}_{j_3}^{(h_3)}\rangle\big) \cdots\\
&\quad \times \delta_{p_{j_l}}\big(\langle \mathbf{n}_{l-1} - \overline{h}_l, \mathbf{a}_{j_l}\rangle\big) \delta_{q_{j_l}^{(h_l)}}\big(\langle \mathbf{n}_{l-1} - \mathbf{n}_l, \mathbf{a}_{j_l}^{(h_l)}\rangle\big) \exp\big(2\pi i\langle \mathbf{n}_l, \mathbf{x}\rangle\big).
\end{aligned} \tag{52}$$

The assertion of Theorem 1.4 follows from Theorem 1.3 and formula (52).

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REFERENCES

1. Borevich, Z. I. and Shafarevich, I. R., *Teoriya chisel* (Number Theory), Moscow: Nauka, 1985.
2. *Algebraic Number Theory* (Cassels, J. and Frölich, A., Eds.), London, 1967.
3. Voronin, S. M., On Quadrature Formulas, *Izv. Russian Akad. Nauk, Ser. Mat.*, 1994, vol. 58, no. 5, pp. 189–194.

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