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On Trigonometric Series with Positive Partial Sums

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In this paper, we study trigonometric series (TS) with nonnegative partial sums whose spectrum is $A \cup (-A)$, where A is the set of natural numbers free of sums which is well-known in number theory, i.e., the equation $x + y = z$ has no solutions in A. We state that TS of this type,

$$1 + \sum_{\substack{\nu \in \mathbb{Z} \\ |\nu| \in A}} c_\nu e^{i\nu x}, \qquad c_\nu = \overline{c_{-\nu}}, \tag{1}$$

are Fourier series of functions from L_3. It was stated earlier [1] that these series belong to L_2.

We shall begin with considering polynomial mappings of a unit circle in a complex plane. Let $f(z)$, $z \in \mathbb{C}$, be a polynomial and

$$f(z) = \sum_{k \geq 1} c_k z^k, \qquad \|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})|^p \, dt \right)^p, \quad p > 0;$$

let

$$f(\Gamma) = \{ z \in \mathbb{C} : z = f(e^{it}),\ 0 \leq t < 2\pi \}$$

be the image of the unit circle. Let

$$P(\alpha, R) = \{ z = x + iy \in \mathbb{C} : x \cos \alpha + y \sin \alpha \geq R \}.$$

We denote by $A^* = \{ z : z = x - y,\ x \neq y,\ x, y \in A \}$ the difference set A. In order to carry out the estimation, we shall need the following assertion which is a corollary of the Riesz theorem [2]: for any $1 < p < \infty$, $m \in \mathbb{N}$, we have

$$\left\| \sum_{1 \leq k \leq m} c_k z^k \right\|_p \leq \lambda_p \|f\|_p.$$

For the constant λ_p we have the inequality

$$\lambda_p \leq p, \qquad p \geq 2, \tag{2}$$

whose elementary proof can be found in [3, 4]. Some arguments connected with the orthogonality of the degrees of trigonometric polynomials will be used in this paper.

Since $f(0) = 0$, it follows from the argument principle that the convex hull of the curve $f(\Gamma)$ in the complex plane contains a certain neighborhood of zero. Using the structure of the set A, we shall show that this hull contain a disk of a "sufficiently large" radius.

We have the following

Theorem 1. *Assume that the following conditions hold:*
(1) *all coefficients c_k of the polynomial $f(z)$ with numbers that do not belong to A vanish,*
(2) $A^* \cap A = \varnothing$.
Then the curve $f(\Gamma)$ intersects any half-plane $P(\alpha, R)$, where

$$R = \frac{\|f\|_3^3}{216\|f\|_2^2}. \tag{3}$$

Proof. We set

$$g(t) = f(e^{it})e^{i\alpha}, \qquad G(t) = \operatorname{Re} g(t) = \frac{1}{2}\big(g(t) + \overline{g}(t)\big), \qquad M = \min_t G(t).$$

Since $M + G(t) \geq 0$, we have

$$\begin{aligned}\|G\|_3^3 &= \frac{1}{2\pi}\int_0^{2\pi} |G(t)|^3\,dt = \frac{1}{2\pi}\int_0^{2\pi} |G(t) + M - M| \times |G(t)|^2\,dt \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \big(G(t) + 2M\big)|G(t)|^2\,dt = 2M\|G\|_2^2 + \frac{1}{16\pi}\int_0^{2\pi} \big(g(t) + \overline{g}(t)\big)^3\,dt \\ &= 2M\|G\|_2^2 + \frac{1}{16\pi}\int_0^{2\pi} \big(g^3(t) + \overline{g}^3(t)\big)\,dt + \frac{3}{16\pi}\int_0^{2\pi} |g(t)|^2\big(g(t) + \overline{g}(t)\big)\,dt.\end{aligned}$$

The second term vanishes by the multiplicative properties of the trigonometric system. The polynomial $g^3(t)$ has a positive spectrum. Since

$$|g(t)|^2 = |f(e^{it})|^2 = \sum_{k,l \in A} c_k \overline{c}_l e^{i(k-l)t},$$

the nonzero Fourier coefficients of $|g(t)|^2$ are contained in the difference set A^*. By the hypothesis of the theorem, this set does not intersects A. Hence, the third term also vanishes. Thus

$$\|G\|_3^3 \leq 2M\|G\|_2^2. \tag{4}$$

By the orthogonality of g and $\overline{g}$, we have

$$\|G\|_2^2 = \left\|\frac{1}{2}(g + \overline{g})\right\| = \frac{1}{4}\big(\|g\|_2^2 + \|\overline{g}\|_2^2\big) = \frac{1}{2}\|f\|_2^2.$$

By the Riesz theorem and by inequality (2), we have

$$\|G\|_3^3 \geq (2\lambda_3)^{-3}\|g\|_3^3 = (2\lambda_3)^{-3}\|f\|_3^3 \geq \frac{1}{216}\|f\|_3^3.$$

Consequently, we find from (4) that

$$\|f\|_3^3 \leq 216M\|f\|_2^2,$$

and this implies (3). Theorem 1 is proved.

Corollary. *Under the conditions of Theorem* 1 *imposed on the polynomial* $f(z)$, *the following assertion holds*: *for any* α, $f(\Gamma)$ *intersects* $P(\alpha, R)$ *when* $R = \|f\|_3/216$.

Indeed, by the Hölder inequality, we have

$$\|f\|_3 \geq \|f\|_2, \tag{5}$$

and therefore

$$R = \frac{\|f\|_3^3}{216\|f\|_2^2} \geq \frac{\|f\|_3}{216}.$$

If we apply (5) and inequality (2) once again, then, by the Parseval identity we find that $f(\Gamma)$ intersects any half-plane $P(\alpha, R)$ for

$$R = \frac{1}{216}\Bigl(\sum_{k\in A} |c_k|^2\Bigr)^{1/2}.$$

Theorem 2. *Assume that the following conditions hold for the* TS (1):

(1) *there exists a nonnegative subsequence of partial sums*,

(2) $A^* \cap A = \varnothing$.

Then the TS (1) *is a Fourier series of the function from* L_3.

Proof. Let

$$S_{n_k}(t) = 1 + \sum_{\substack{1\leq|\nu|\leq n_k\\ |\nu|\in A}} c_\nu e^{i\nu t} \geq 0.$$

We set $\alpha = 0$, $R = 1/2$, and

$$f(e^{it}) = \sum_{1\leq\nu\leq n_k} c_\nu e^{i\nu t}$$

in (3); then $\|f\|_3 \leq 108$. By the Minkowski inequality,

$$\|S_{n_k}(t)\|_3 = \bigl\|1 + f(e^{it}) + \overline{f(e^{it})}\bigr\|_3 \leq 1 + 2\|f\|_3 \leq 217.$$

For any $n \in \mathbb{N}$, we choose a number n_k such that $n \leq n_k$. By the Riesz theorem, we find the upper bound of the norm of the partial Fourier sum, namely,

$$\|S_n(t)\|_3 = \bigl\|S_n\bigl(S_{n_k}(t)\bigr)\bigr\|_3 \leq 3\|S_{n_k}(t)\|_3 \leq 651.$$

We find that the norms of partial sums of the TS are uniformly bounded. Hence (see [2]), the TS (1) is a Fourier series of the function from L_3. Theorem 2 is proved

In conclusion, we shall consider one special case where $A = \{k^3 : k \in \mathbb{N}\}$. According to Euler's theorem, $A^* \cap A = \varnothing$. We find from Theorems 1 and 2 that

(i) the curve

$$z = \Bigl\{\sum_{k\geq 1} c_k e^{ik^3 t}\Bigr\}_{t\in\mathbb{R}}$$

intersects any straight line of the form

$$x \cos \alpha + y \sin \alpha = \frac{1}{216} \left(\sum_{k \geq 1} |c_k|^2 \right)^{1/2},$$

(ii) if the TS "with respect to the cubes"

$$\sum_{k \in \mathbb{Z}} c_k e^{ik^3 t}$$

has a nonnegative sequence of partial sums, then it is a Fourier series of the function from L_3.

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