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Difference Equations and the Irrationality Measure of Numbers

V. V. Zudilin

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1. LINEAR DIFFERENCE EQUATIONS

A relation of the form

$$a_{0\eta}u_{\eta+m} + a_{1\eta}u_{\eta+m-1} + \cdots + a_{m\eta}u_\eta = 0, \qquad \eta = n, n+1, n+2, \ldots \tag{1}$$

with given complex numbers $a_{j\eta}$, $j = 1, \ldots, m$, $\eta = n, n+1, \ldots$, is called a *linear* (*homogeneous*) *equation of order* m. Moreover, we assume that $a_{0\eta} \neq 0$ for any integer $\eta \geq n$. By a *solution* to the difference equation (1) we mean a sequence of numbers $u_n, u_{n+1}, u_{n+2}, \ldots$ which satisfies the recursive relation (1).

The solution $\{u_\eta\}$ to equation (1) is a *trivial* (or *zero*) solution if $u_\eta = 0$ for all integers $\eta \geq n_1$. In this case, replacing n by n_1 in (1), we obtain the usual trivial solution. This is one of the reasons why the change of the subscript η begins with a certain $n \in \mathbb{Z}$ (it is traditionally assumed that $n = 0$). Two solutions $\{u_\eta\}$ and $\{v_\eta\}$ to the difference equation (1) are *linearly independent* if any one of their linear combinations

$$\{\alpha u_\eta + \beta v_\eta\}, \qquad |\alpha| + |\beta| > 0, \quad \alpha, \beta \in \mathbb{C},$$

is nontrivial. Otherwise, these solutions are *linearly dependent*.

If the finite limits

$$a_j = \lim_{\eta\to\infty} \frac{a_{j\eta}}{a_{0\eta}}, \quad j = 1, \ldots, m, \qquad a_m \neq 0,$$

are well defined, then the polynomial $\varphi(\lambda) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m$ is a *characteristic polynomial* of the difference equation (1). In the sequel, we shall consider linear difference equations which possess a characteristic polynomial.

The asymptotics of the solution $\{u_\eta\}$ is closely connected with the roots of the characteristic polynomial. We illustrate this relationship by the assertion which generalizes the classical Poincaré theorem.

Proposition 1 [1, Theorem 1]. *Let the sequence* $u_n, u_{n+1}, \ldots$ *be a nontrivial solution to the linear difference equation* (1) *with the characteristic polynomial* $\varphi(\lambda)$. *Then the limit superior*

$$\limsup_{\eta\to\infty} |u_\eta|^{1/\eta} \tag{2}$$

is equal to the modulus of one of the roots of the polynomial $\varphi(\lambda)$, *and the sequence* $\{u_\eta\}$ *is also a solution to the difference equation*

$$u_{\eta+k} + b_{1\eta}u_{\eta+k-1} + \cdots + b_{k\eta}u_\eta = 0, \qquad \eta = n, n+1, n+2, \ldots,$$

of order $k \le m$ *whose characteristic polynomial divides* $\varphi(\lambda)$ *and has all roots whose moduli are equal to* (2).

Corollary. *Assume that under the conditions of Proposition* 1 *we have*

$$\limsup_{\eta\to\infty} |u_\eta|^{1/\eta} = |\lambda_1|; \tag{3}$$

moreover, let the moduli of the other roots of the characteristic polynomial $\varphi(\lambda)$ *be different from* $|\lambda_1|$. *Then*

$$\lim_{\eta\to\infty} \frac{u_{\eta+1}}{u_\eta} = \lambda_1$$

(*in particular, the limit superior in* (3) *can be replaced by the ordinary one*).

Proof. Indeed, according to Proposition 1, the solution $\{u_\eta\}$ satisfies the linear difference equation

$$u_{\eta+1} - b_{1\eta}u_\eta = 0, \qquad \eta = n, n+1, n+2, \ldots,$$

with the characteristic equation $\lambda - b_1$, which divides $\varphi(\lambda)$. Therefore b_1 is a root of the polynomial $\varphi(\lambda)$, and

$$\lim_{\eta\to\infty} \frac{u_{\eta+1}}{u_\eta} = \lim_{\eta\to\infty} b_{1\eta} = b_1;$$

whence we have

$$\lim_{\eta\to\infty} |u_\eta|^{1/\eta} = |b_1|.$$

Comparing the last equation to (3) and taking into account that λ_1 is a unique root of the polynomial $\varphi(\lambda)$ whose modulus is equal to $|\lambda_1|$, we obtain $b_1 = \lambda_1$. The assertion is proved.

Proposition 2. *Let two sequences* $\{u_\eta\}$ *and* $\{v_\eta\}$ *be nontrivial solutions to the linear difference equation* (1),

$$\lim_{\eta\to\infty} \frac{v_\eta}{u_\eta} = \alpha \ne 0,$$

and let the following relation hold for the roots $\lambda_1, \lambda_2, \ldots, \lambda_m$ *of the characteristic polynomial of this equation*:

$$|\lambda_1| > \delta = \max_{2\le j\le m} \{|\lambda_j|\}. \tag{4}$$

Then the sequence

$$r_\eta = u_\eta\alpha - v_\eta, \qquad \eta = n, n+1, n+2, \ldots,$$

which is also a solution to the difference equation (1), *satisfies the limit relation*

$$\limsup_{\eta\to\infty} |r_\eta|^{1/\eta} \le \delta. \tag{5}$$

Proof. Assume that inequality (5) does not hold. According to Proposition 1, this is possible only in the case where

$$\limsup_{\eta\to\infty} |r_\eta|^{1/\eta} = |\lambda_1|. \tag{6}$$

In particular, $\{r_\eta\}$ is a nontrivial solution to equation (1). Since

$$\lim_{\eta\to\infty} \frac{r_\eta}{u_\eta} = \lim_{\eta\to\infty} \frac{u_\eta\alpha - v_\eta}{u_\eta} = \alpha \lim_{\eta\to\infty} \frac{u_\eta}{u_\eta} - \lim_{\eta\to\infty} \frac{v_\eta}{u_\eta} = 0, \tag{7}$$

we find that

$$\limsup_{\eta\to\infty} |u_\eta|^{1/\eta} \ge \limsup_{\eta\to\infty} |r_\eta|^{1/\eta} = |\lambda_1|.$$

On the other hand, λ_1 is a root of the characteristic polynomial, maximal in absolute value, and, according to Proposition 1, the last inequality becomes an equality

$$\limsup_{\eta\to\infty} |u_\eta|^{1/\eta} = |\lambda_1|. \tag{8}$$

In order to continue the proof, we need two auxiliary assertions from [2]. We give these assertions in the notation used in our paper.

Lemma 1. *If $s_\eta \ne 0$, $\eta = n, n+1, \dots,$ and $\{u_\eta\}$ is a solution to the linear difference equation*

$$u_{\eta+m} + a_{1\eta}u_{\eta+m-1} + \dots + a_{m\eta}u_\eta = 0, \qquad \eta = n, n+1, n+2, \dots, \tag{9}$$

then

$$v_\eta = \frac{u_\eta}{s_\eta}, \qquad \eta = n, n+1, n+2, \dots,$$

is a solution to the linear difference equation

$$v_{\eta+m} + a'_{1\eta}v_{\eta+m-1} + \dots + a'_{m\eta}v_\eta = 0, \qquad \eta = n, n+1, n+2, \dots, \tag{10}$$

$$a'_{j\eta} = a_{j\eta}\frac{s_{\eta+m-j}}{s_{\eta+m}}, \qquad \eta = n, n+1, n+2, \dots, \quad j = 1, \dots, m.$$

If, in addition,

$$\lim_{\eta\to\infty} \frac{s_{\eta+1}}{s_\eta} = 1,$$

then the characteristic polynomials of equations (9) *and* (10) *coincide.*

Lemma 2. *If equation* (9) *has a solution $u_\eta^{(1)} = \lambda_1^\eta$, $\eta = n, n+1, \dots,$ then this equation can be written in the form*

$$w_{\eta+m-1} + b_{1\eta}w_{\eta+m-2} + \dots + b_{m-1,\eta}w_\eta = 0, \qquad \eta = n, n+1, n+2, \dots, \tag{11}$$

where

$$w_\eta = u_{\eta+1} - \lambda_1 u_\eta, \qquad \eta = n, n+1, n+2, \dots .$$

Moreover, the characteristic polynomials of equations (11) *and* (9) *are related as*

$$(\lambda - \lambda_1)(\lambda^{m-1} + b_1\lambda^{m-2} + \dots + b_{m-1}) = \lambda^m + a_1\lambda^{m-1} + \dots + a_m.$$

Let us continue the proof of Proposition 2. The limit relation (8) and the corollary to Proposition 1 allow us to assert that

$$\lim_{\eta\to\infty} \frac{u_{\eta+1}}{u_\eta} = \lambda_1.$$

Therefore the sequence

$$s_\eta = \frac{u_\eta}{\lambda_1^\eta}, \qquad \eta = n, n+1, n+2, \dots$$

satisfies the conditions of Lemma 1, and the difference equation

$$v_{\eta+m} + a'_{1\eta} v_{\eta+m-1} + \dots + a'_{m\eta} v_\eta = 0, \qquad \eta = n, n+1, n+2, \dots, \tag{12}$$

where

$$a'_{j\eta} = \frac{a_{j\eta}}{a_{0\eta}} \frac{s_{\eta+m-j}}{s_{\eta+m}}, \qquad \eta = n, n+1, n+2, \dots, \quad j = 1, \dots, m,$$

has a solution

$$v_\eta^{(1)} = \frac{u_\eta}{u_\eta/\lambda_1^\eta} = \lambda_1^\eta, \qquad \eta = n, n+1, n+2, \dots . \tag{13}$$

By Lemma 2, equation (12) can be represented in the form

$$w_{\eta+m-1} + b_{1\eta} w_{\eta+m-2} + \dots + b_{m-1,\eta} w_\eta = 0, \qquad \eta = n, n+1, n+2, \dots, \tag{14}$$

where

$$w_\eta = v_{\eta+1} - \lambda_1 v_\eta, \qquad \eta = n, n+1, n+2, \dots .$$

The roots of the characteristic polynomial of equation (14) are equal to $\lambda_2, \dots, \lambda_m$. We choose a certain basis

$$\{w_\eta^{(1)}\}, \dots, \{w_\eta^{(m-1)}\} \tag{15}$$

in the space of solutions of the linear difference equation (14). According to Proposition 1, we have

$$\limsup_{\eta\to\infty} \bigl|w_\eta^{(j)}\bigr|^{1/\eta} \le \delta, \qquad j = 1, \dots, m-1. \tag{16}$$

For each $j = 2, \dots, m$, we define the solution $\{v_\eta^{(j)}\}$ to equation (12) in such a way that

$$v_{\eta+1}^{(j)} - \lambda_1 v_\eta^{(j)} = w_\eta^{(j-1)}, \qquad \eta = n, n+1, n+2, \dots . \tag{17}$$

To this end, we set

$$v_\eta^{(j)} = -\frac{w_\eta^{(j-1)}}{\lambda_1} - \frac{w_{\eta+1}^{(j-1)}}{\lambda_1^2} - \frac{w_{\eta+2}^{(j-1)}}{\lambda_1^3} - \cdots, \qquad \eta = n, n+1, n+2, \dots, \quad j = 2, \dots, m. \tag{18}$$

The last series converges by (4) and (16). Solutions (18) are linearly independent according to (17) and the linear independence of basis (15). It is easy to show that

$$\limsup_{\eta\to\infty} \bigl|v_\eta^{(j)}\bigr|^{1/\eta} = \limsup_{\eta\to\infty} \bigl|w_\eta^{(j-1)}\bigr|^{1/\eta} \le \delta, \qquad j = 2, \dots, m. \tag{19}$$

Taking (13) into account, we find that the sequences

$$\{v_\eta^{(1)}\}, \{v_\eta^{(2)}\}, \dots, \{v_\eta^{(m)}\}$$

form a basis in the solution space of equation (12). Therefore the sequences

$$u_\eta^{(j)} = v_\eta^{(j)} s_\eta, \qquad \eta = n, n+1, n+2, \dots, \qquad j = 1, \dots, m, \tag{20}$$

form a basis in the solution space of the initial equation (1). Moreover, $\{u_\eta^{(1)}\} \equiv \{u_\eta\}$, and, according to (19),

$$\limsup_{\eta\to\infty} \bigl|u_\eta^{(j)}\bigr|^{1/\eta} = \limsup_{\eta\to\infty} \bigl|v_\eta^{(j)}\bigr|^{1/\eta} \le \delta, \qquad j = 2, \dots, m. \tag{21}$$

We express the solution $\{r_\eta\}$ to the difference equation (1) in terms of the basis (20), i.e.,

$$r_\eta = \rho_1 u_\eta + \rho_2 u_\eta^{(2)} + \dots + \rho_m u_\eta^{(m)}, \qquad \eta = n, n+1, n+2, \dots . \tag{22}$$

According to (21), the relation

$$\lim_{\eta\to\infty} \frac{u_\eta^{(j)}}{u_\eta} = 0, \qquad j = 2, \dots, m,$$

holds; this implies

$$\lim_{\eta\to\infty} \frac{r_\eta}{u_\eta} = \rho_1.$$

Comparing the last relation to (7), we infer that $\rho_1 = 0$. Therefore resolution (22), together with the limit relations (21), yields

$$\limsup_{\eta\to\infty} |r_\eta|^{1/\eta} \le \delta.$$

However, this contradicts (6). This means that our supposition is not true, and inequality (5) holds.

Proposition 2 is completely proved.

Let us now formulate the general assertion concerning the connection of linear difference equations with the irrationality measure. We will denote by $\mathbb{I}$ either $\mathbb{Q}$ or $\mathbb{Q}(i)$.

Theorem 1. *Let $\{u_\eta\} \subset \mathbb{I}$ and $\{v_\eta\} \subset \mathbb{I}$ be two linearly independent solutions to the linear difference equation* (1),

$$\lim_{\eta\to\infty} \frac{v_\eta}{u_\eta} = \alpha \ne 0, \tag{23}$$

and let the natural numbers d_η, $\eta = n, n+1, \dots$, be such that

$$d_\eta u_\eta, d_\eta v_\eta \in \mathbb{Z}_{\mathbb{I}}, \quad \eta = n, n+1, n+2, \dots, \qquad \lim_{\eta\to\infty} d_\eta^{1/\eta} \le C, \tag{24}$$

where $C \ge 1$ is a constant. If $\lambda_1, \dots, \lambda_m$ are roots of the characteristic polynomial of equation (1), *and, moreover, if*

$$|\lambda_1| > \delta = \max_{2\le j\le m} \{|\lambda_j|\}$$

and $C\delta < 1$, *then the number* α *is irrational*, $\alpha \notin \mathbb{I}$. *Moreover*, *the irrationality measure of* α *does not exceed*

$$\mu = 1 - \frac{\log C + \log|\lambda_1|}{\log C + \log \delta}, \tag{25}$$

i.e., *for any* $\varepsilon > 0$, *there exists* $q_* = q_*(\varepsilon)$ *such that*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{q^{\mu+\varepsilon}}$$

for all $p, q \in \mathbb{Z}_{\mathbb{I}}$, $q \geq q_*$.

Proof. The numbers

$$r_\eta = \widetilde{u}_\eta \alpha - \widetilde{v}_\eta, \qquad \widetilde{u}_\eta = d_\eta u_\eta, \quad \widetilde{v}_\eta = d_\eta v_\eta, \qquad \eta = n, n+1, n+2, \dots, \tag{26}$$

are approximating linear forms of α and 1 with coefficients in $\mathbb{Z}_{\mathbb{I}}$. According to Proposition 2 and estimate (24) of the denominators d_η, $\eta = n, n+1, \dots$, we have

$$\limsup_{\eta\to\infty} |r_\eta|^{1/\eta} \leq C\delta < 1. \tag{27}$$

If α were a rational number, then either $A\alpha - B = 0$ or $|A\alpha - B| \geq 1/D$ for all $A, B \in \mathbb{Z}_{\mathbb{I}}$, where $D \in \mathbb{N}$ is the denominator of the number α; in particular, $r_\eta = 0$ or $|r_\eta| \geq 1/D$ for all integers $\eta \geq n$. The latter fact contradicts the limit relation (27) and the nontriviality of the sequence $\{r_\eta\}$ for a sufficiently large η (the solutions $\{u_\eta\}$ and $\{v_\eta\}$ are linearly independent). Hence, $\alpha \notin \mathbb{Z}_{\mathbb{I}}$; this also implies $r_\eta \neq 0$, $\eta = n, n+1, \dots$.

Each of the linearly independent solutions $\{u_\eta\}$ and $\{v_\eta\}$ is nontrivial. According to Proposition 1, we have

$$\limsup_{\eta\to\infty} |u_\eta|^{1/\eta} = |\lambda_j|$$

for a certain j, $1 \leq j \leq m$. If $j \neq 1$, then the limit relation

$$\limsup_{\eta\to\infty} |\widetilde{u}_\eta|^{1/\eta} \leq C\delta < 1$$

holds for the numbers $\widetilde{u}_\eta \in \mathbb{Z}_{\mathbb{I}}$, $\eta = n, n+1, \dots$; but this is impossible since $|\widetilde{u}_\eta| \geq 1$ for an infinite set of subscripts η. Therefore $j = 1$, and by the corollary of Proposition 1 we have

$$\lim_{\eta\to\infty} |u_\eta|^{1/\eta} = |\lambda_1|.$$

Thus,

$$\lim_{\eta\to\infty} |\widetilde{u}_\eta|^{1/\eta} = C|\lambda_1|. \tag{28}$$

If we now apply the result by G. V. Chudnovsky (see [3, Lemma 3.5]) to relations (27) and (28) for the approximating linear forms (26), then we obtain estimate (25) of the irrationality measure of α.

The theorem is proved.

As an example to Theorem 1, we consider the Apéry difference equation

$$(\eta+1)^3 u_{\eta+1} - (34\eta^3 + 51\eta^2 + 27\eta + 5)u_\eta + \eta^3 u_{\eta-1} = 0, \qquad \eta = 1, 2, \ldots, \tag{29}$$

and two linearly independent solutions $\{u_\eta\}$ and $\{v_\eta\}$ to this equation which are given by the initial conditions

$$u_0 = 1, \quad u_1 = 5, \qquad v_0 = 0, \quad v_1 = 6.$$

One can show (see, e.g., [4]) that $u_\eta, D_\eta^3 v_\eta \in \mathbb{Z}$, where D_η is the least common multiple of the numbers $1, 2, \ldots, \eta$ and

$$\lim_{\eta\to\infty} \frac{v_\eta}{u_\eta} = \zeta(3) = \sum_{k=1}^{\infty} \frac{1}{k^3}.$$

The characteristic polynomial $\lambda^2 - 34\lambda + 1 = 0$ of the difference equation (29) has the roots $\lambda_1 = (\sqrt{2}+1)^4$ and $\lambda_2 = (\sqrt{2}-1)^4$. Moreover,

$$\lim_{\eta\to\infty} D_\eta^{1/\eta} = e.$$

Since $(\sqrt{2}-1)^4 e^3 < 1$, Theorem 1 yields the irrationality measure

$$1 - \frac{3 + 4\log(\sqrt{2}+1)}{3 + 4\log(\sqrt{2}-1)} = 13.4178202\ldots$$

of the number $\zeta(3)$.

2. THE IRRATIONALITY MEASURE

As an application of Theorem 1, we give an example of a linear difference equation whose characteristic polynomial is

$$(\lambda-\lambda_1)\cdots(\lambda-\lambda_m) = \lambda^m + a_1\lambda^{m-1} + \cdots + a_{m-1}\lambda + a_m, \qquad a_j \in \mathbb{Q}, \quad j = 1, \ldots, m, \quad a_m \neq 0. \tag{30}$$

The root λ_1 which is the largest in absolute value is assumed to be real and positive, i.e.,

$$\lambda_1 = |\lambda_1| > \delta = \max_{2\le j\le m} \{|\lambda_j|\}. \tag{31}$$

We set

$$\tau(z) = (1-\lambda_1 z)\cdots(1-\lambda_m z) = 1 + a_1 z + \cdots + a_m z^m, \qquad a_j \in \mathbb{Q}, \quad j = 1, \ldots, m, \quad a_m \neq 0. \tag{32}$$

The generating function

$$U(z) = \tau(z)^{-s} = \sum_{\nu=0}^{\infty} u_\nu z^\nu, \qquad s \in (0,1) \cap \mathbb{Q}, \tag{33}$$

satisfies the linear differential equation $\tau y' + s\tau' y = 0$. (We take a branch of function (33) which assumes positive values on the cut $[0, 1/\lambda_1)$.) Thus,

$$\left(1 + \sum_{\mu=1}^{m} a_\mu z^\mu\right) \times \sum_{\nu=0}^{\infty} (\nu+1) u_{\nu+1} z^\nu + s \sum_{\mu=1}^{m} \mu a_\mu z^{\mu+1} \times \sum_{\nu=0}^{\infty} u_\nu z^\nu = 0$$

or

$$\sum_{\eta=0}^{\infty}\Bigl((\eta+1)u_{\eta+1}+\sum_{\mu=1}^{m}a_\mu\bigl(\eta+1+(s-1)\mu\bigr)u_{\eta+1-\mu}\Bigr)z^\eta=0,$$

where $u_0=1$ and $u_{-m+1}=\cdots=u_{-2}=u_{-1}=0$.

Thus, the sequence of numbers $\{u_\eta\}$ satisfies the difference equation

$$(\eta+1)u_{\eta+1}+\sum_{\mu=1}^{m}a_\mu\bigl(\eta+1+(s-1)\mu\bigr)u_{\eta+1-\mu}=0,\qquad \eta=0,1,2,\dots,\tag{34}$$

whose characteristic polynomial is (30).

Let now $\varkappa(z)\in\mathbb{I}[z]$ be a nonzero polynomial, and let the function $V(z)$ satisfy the differential equation

$$\tau y'+s\tau' y=\varkappa\tag{35}$$

with the initial condition $V(0)=0$. This condition and the relation

$$\frac{d}{dz}\bigl(\tau(z)^sV(z)\bigr)=\tau(z)^{s-1}\bigl(\tau(z)V'(z)+s\tau'(z)V(z)\bigr)=\tau(z)^{s-1}\varkappa(z)$$

yield

$$V(z)=\tau(z)^{-s}\int_0^z\tau(x)^{s-1}\varkappa(x)\,dx=\sum_{\eta=0}^{\infty}v_\eta z^\eta.\tag{36}$$

In terms of the generating function, the differential equation (35) can be rewritten in the form

$$\sum_{\eta=0}^{\infty}\Bigl((\eta+1)v_{\eta+1}+\sum_{\mu=1}^{m}a_\mu\bigl(\eta+1+(s-1)\mu\bigr)v_{\eta+1-\mu}\Bigr)z^\eta=\varkappa(z),$$

where we set $v_{-m+1}=\cdots=v_{-1}=v_0=0$. Thus, beginning with $\eta=\deg\varkappa$, the sequence $\{v_\eta\}$ is a solution to the difference equation (34).

If the solutions $\{u_\eta\}$ and $\{v_\eta\}$ to equation (34) were linearly dependent, then $\alpha U(z)-V(z)=\pi(z)$ would hold for certain $\alpha\in\mathbb{C}$ and $\pi(z)\in\mathbb{C}[z]$. After the multiplication of both sides of this equation by $\tau(z)^s$ and after the differentiation, we find that $\tau(z)^{s-1}\varkappa(z)=\bigl(\tau(z)^s\pi(z)\bigr)'$, i.e., $\pi(z)$ is not a solution to the differential equation (35). Thus, the requirement that equation (35) should have no polynomial solution for a chosen $\varkappa(z)$ entails the linear independence of the solutions $\{u_\eta\}$ and $\{v_\eta\}$.

Lemma 3. *Let the polynomial $\tau(z)$,* (32), *be chosen in accordance with conditions* (31), *$\varkappa(z)$ be an arbitrary polynomial, and let the functions $U(z)$ and $V(z)$ be defined by relations* (33) *and* (36), *respectively. Then*

$$\lim_{\eta\to\infty}\frac{v_\eta}{u_\eta}=\int_0^{1/\lambda_1}\tau(x)^{s-1}\varkappa(x)\,dx.$$

Proof. By virtue of (31), the point $z=1/\lambda_1$ is the only singularity of the functions $U(z)$ and $V(z)$ in the disc $|z|\le r$, where $1/\lambda_1<r<1/\delta$. Hence, by the Cauchy formula, we have

$$u_\eta=\frac{1}{2\pi i}\int\limits_{\mathcal K}\frac{U(z)}{z^{\eta+1}}\,dz,\quad v_\eta=\frac{1}{2\pi i}\int\limits_{\mathcal K}\frac{V(z)}{z^{\eta+1}}\,dz,\qquad \eta=0,1,2,\dots,\tag{37}$$

where the contour of integration $\mathcal K$ is the boundary of the disc $|z|\le r$ with the radial cut $(1/\lambda_1,r)$ (by the condition $s<1$, we can also integrate over the path passing through the point $z=1/\lambda_1$).

The functions $(1-\lambda_1 z)^sU(z)$ and $(1-\lambda_1 z)^sV(z)$ are analytic in the disc $|z|\le r$, and therefore, in the smaller disc $|\lambda_1 z-1|\le\lambda_1 r-1$. Therefore these functions can be represented in the form

$$U(z)=(1-\lambda_1 z)^{-s}\widetilde U(\lambda_1 z-1),\qquad V(z)=(1-\lambda_1 z)^{-s}\widetilde V(\lambda_1 z-1),$$

where the functions $\widetilde U(t)$ and $\widetilde V(t)$ are analytic in the disc $|t|\le\lambda_1 r-1$. Moreover, we have

$$\begin{aligned}\widetilde U(0)&=\big((1-\lambda_1 z)^sU(z)\big)\big|_{z=1/\lambda_1}=\prod_{j=2}^m\Big(1-\frac{\lambda_j}{\lambda_1}\Big)^{-s},\\ \widetilde V(0)&=\big((1-\lambda_1 z)^sV(z)\big)\big|_{z=1/\lambda_1}=\prod_{j=2}^m\Big(1-\frac{\lambda_j}{\lambda_1}\Big)^{-s}\int\limits_0^{1/\lambda_1}\tau(x)^{s-1}\varkappa(x)\,dx\end{aligned}\tag{38}$$

for these functions.

We pass now to formulas (37) for the function $U(z)$. We have

$$U(z)=e^{\pi is}(\lambda_1 z-1)^{-s}\widetilde U(\lambda_1 z-1)$$

on the upper bank of the cut, and

$$U(z)=e^{-\pi is}(\lambda_1 z-1)^{-s}\widetilde U(\lambda_1 z-1)$$

on the lower one. The integral (37) is equal to $O(r^{-\eta})$ on the circle $|z|=r$; therefore,

$$\begin{aligned}u_\eta&=\frac{e^{\pi is}-e^{-\pi is}}{2\pi i}\int\limits_{1/\lambda_1}^{r}(\lambda_1 z-1)^{-s}\widetilde U(\lambda_1 z-1)\frac{dz}{z^{\eta+1}}+O(r^{-\eta})\\ &=\frac{\sin\pi s}{\pi}\int\limits_{1/\lambda_1}^{r}(\lambda_1 z-1)^{-s}\widetilde U(\lambda_1 z-1)\frac{dz}{z^{\eta+1}}+O(r^{-\eta}).\end{aligned}$$

After the change of the variable $\lambda_1 z-1=t$, the integral in the last expression reduces to the form

$$\begin{aligned}\int\limits_{1/\lambda_1}^{r}(\lambda_1 z-1)^{-s}\widetilde U(\lambda_1 z-1)\frac{dz}{z^{\eta+1}}&=\int\limits_0^{\lambda_1 r-1}t^{-s}\widetilde U(t)\frac{\lambda_1^\eta\,dt}{(1+t)^{\eta+1}}=\lambda_1^\eta\int\limits_0^{\lambda_1 r-1}t^{-s}\widetilde U(t)e^{-(\eta+1)\log(1+t)}\,dt\\ &=\lambda_1^\eta\int\limits_0^{\lambda_1 r-1}t^{-s}\widetilde U(t)e^{-(\eta+1)t(1+O(t))}\,dt.\end{aligned}$$

Theorem 1.2.1 in [5] yields the asymptotic estimate

$$\int_0^{\lambda_1 r-1} t^{-s}\widetilde{U}(t)e^{-(\eta+1)t(1+O(t))}\,dt \sim \widetilde{U}(0)\Gamma(1-s)\eta^{s-1} \qquad \text{as} \quad \eta\to\infty.$$

Hence,

$$u_\eta \sim \frac{\sin\pi s}{\pi}\lambda_1^\eta \widetilde{U}(0)\Gamma(1-s)\eta^{s-1} \qquad \text{as} \quad \eta\to\infty.$$

By analogy, we find that

$$v_\eta \sim \frac{\sin\pi s}{\pi}\lambda_1^\eta \widetilde{V}(0)\Gamma(1-s)\eta^{s-1} \qquad \text{as} \quad \eta\to\infty.$$

Therefore

$$\lim_{\eta\to\infty}\frac{v_\eta}{u_\eta}=\frac{\widetilde{V}(0)}{\widetilde{U}(0)},$$

and the substitution of values (38) yields the required result.

The lemma is proved.

Remark. The arguments used above can be easily extended to the case of negative λ_1. Therefore condition (31) can be replaced by

$$\lambda_1\in\mathbb{R}, \qquad |\lambda_1|>\delta=\max_{2\le j\le m}\{|\lambda_j|\}. \tag{39}$$

Now we estimate the denominators of the sequences $\{u_\eta\}$ and $\{v_\eta\}$.

Lemma 4. *Let us define natural numbers A, B, and C as follows: B is the denominator of the rational number $s\in(0,1)$; A is the least common denominator of the numbers a_j/B, $j=1,\dots,m$; C is the least common denominator of the coefficients of the polynomial $\varkappa(z)\in\mathbb{I}[z]$. Then, for the sequence*

$$d_\eta = A^\eta\times C\times D_\eta\times\prod_{p|B} p^{\left[\frac{\eta}{p-1}\right]}, \qquad \eta=0,1,2,\dots, \tag{40}$$

and for the functions $U(z)$ and $V(z)$ which are chosen in accordance with (33) *and* (36) *we have*

$$d_\eta u_\eta,\, d_\eta v_\eta\in\mathbb{Z}_{\mathbb{I}}, \qquad \eta=0,1,2,\dots$$

(as before, D_η denotes the least common multiple of the numbers $1,2,\dots,\eta$.)

Proof. By the Taylor formula, we have

$$U(z)=(1+a_1z+\dots+a_mz^m)^{-s}=1+\sum_{\nu=1}^\infty(-1)^\nu\frac{s(s+1)\cdots(s+\nu-1)}{\nu!}(a_1z+\dots+a_mz^m)^\nu.$$

If $s=B_0/B$ is an irreducible fraction and the prime number p does not divide B, then this number enters into each rational number

$$\frac{s(s+1)\cdots(s+\nu-1)}{\nu!}=\frac{1}{B^\nu}\,\frac{B_0(B_0+B)\cdots(B_0+(\nu-1)B)}{\nu!}, \qquad \nu=0,1,2,\dots, \tag{41}$$

in a nonnegative power (see, e.g., [6, Ch. I, Appendix]). Hence, the denominators of numbers (41) contain only prime numbers which divide B. Any prime number p of this kind enters into the first factor on the right-hand side of (41) in the power of ν and in the power not exceeding

$$\left[\frac{\nu}{p}\right]+\left[\frac{\nu}{p^2}\right]+\left[\frac{\nu}{p^3}\right]+\dots\le\nu\left(\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\dots\right)=\frac{\nu}{p-1} \tag{42}$$

the second factor, $\nu=0,1,2,\dots$. Thus, when the νth term of the sum

$$U(z)=1+\sum_{\nu=1}^{\infty}(-1)^{\nu}\frac{B_0(B_0+B)\cdots(B_0+(\nu-1)B)}{\nu!}\left(\frac{a_1}{B}z+\dots+\frac{a_m}{B}z^m\right)^{\nu}$$

is multiplied by the number

$$f_\nu=A^\nu\times\prod_{p\mid B}p^{\left[\frac{\eta}{p-1}\right]},$$

this term becomes a polynomial with integer coefficients. Therefore the coefficient u_η in decomposition (33) becomes an integral after the multiplication by f_η, $\eta=0,1,2,\dots$. Similar arguments allow us to assert that the coefficient w_η in the decomposition

$$\tau(z)^{s-1}\varkappa(z)=\sum_{\eta=0}^{\infty}w_\eta z^\eta$$

becomes an integer in $\mathbb{Z}_{\mathbb{I}}$ after multiplication by Cf_η, $\eta=0,1,2,\dots$, since $C\varkappa(z)\in\mathbb{Z}_{\mathbb{I}}[z]$. According to (36), we have

$$V(z)=\tau(z)^{-s}\int_0^z\tau(x)^{s-1}\varkappa(x)\,dx=\sum_{\eta=0}^{\infty}u_\eta z^\eta\times\sum_{\eta=1}^{\infty}\frac{w_\eta}{\eta}z^\eta,$$

and therefore the multiplication of v_η by d_η yields a number from $\mathbb{Z}_{\mathbb{I}}$, $\eta=0,1,2,\dots$. The lemma is proved.

In the case where the denominator of the number s is a power of 2, the assertion of Lemma 4 can be strengthened.

Lemma 5. *Let $B=2^k$, $k\in\mathbb{N}$, A be the least common denominator of the numbers $a_j/(2B)$, $j=1,\dots,m$, and let C be the least common denominator of the coefficients of the polynomial $\varkappa(z)\in\mathbb{I}[z]$. Then, for the sequence*

$$d_\eta=A^\eta\times C\times D_\eta,\qquad \eta=0,1,2,\dots, \tag{43}$$

and for the functions $U(z)$ and $V(z)$ which are chosen in accordance with (33) *and* (36) *we have*

$$d_\eta u_\eta,d_\eta v_\eta\in\mathbb{Z}_{\mathbb{I}},\qquad \eta=0,1,2,\dots .$$

Proof. According to (42), the prime number 2 enters into $\nu!$ in the power not higher than ν, $\nu=0,1,\dots$. Therefore the multiplication of the νth term of the decomposition

$$U(z)=1+\sum_{\nu=1}^{\infty}(-1)^{\nu}\frac{2^\nu B_0(B_0+B)\cdots(B_0+(\nu-1)B)}{\nu!}\left(\frac{a_1}{2B}z+\dots+\frac{a_m}{2B}z^m\right)^{\nu}$$

by $f_\nu = A^\nu$ yields a polynomial with integral coefficients. The remaining part of the proof repeats the proof of the preceding lemma with the use of the new value of f_ν.

Let us summarize the results of this section in the following assertion.

Theorem 2. *Let the polynomial $\tau(z) \in \mathbb{Q}[z]$ defined by relation (32) satisfy condition (39), $s \in (0,1)$ be a rational number with denominator B, and let the polynomial $\varkappa(z) \in \mathbb{I}[z]$ be chosen such that any solution to the differential equation (35) is not a polynomial. Let A be the least common denominator of the numbers $a_j/(2B)$ in the case where $B = 2^k$, $k \in \mathbb{N}$, and of the numbers a_j/B for other B, $j = 1, \dots, m$, and let*

$$C = \begin{cases} Ae & \text{if } B = 2^k,\ k \in \mathbb{N} \\ A \exp\left\{1 + \sum\limits_{p|B} \dfrac{\ln p}{p-1}\right\} & \text{in other cases.} \end{cases} \tag{44}$$

If $C\delta < 1$, then the irrationality measure of

$$\alpha = \int_0^{1/\lambda_1} \tau(z)^{s-1} \varkappa(z)\, dz$$

does not exceed

$$\mu = 1 - \frac{\log C + \log|\lambda_1|}{\log C + \log \delta}.$$

Proof. The sequences $\{u_\eta\}$ and $\{v_\eta\}$, which are defined by the generating functions (33) and (36), satisfy the linear difference equation (34) and are linearly independent. In addition, by Lemma 3, we have

$$\lim_{\eta\to\infty} \frac{v_\eta}{u_\eta} = \alpha$$

and $\{d_\eta u_\eta\}, \{d_\eta v_\eta\} \in \mathbb{Z}_{\mathbb{I}}$ according to Lemmas 4 and 5; the sequence $\{d_\eta\}$ is chosen in accordance with (40) or (43). Since

$$\lim_{\eta\to\infty} D_\eta^{1/\eta} = e, \qquad \lim_{\eta\to\infty} \left(\prod_{p|B} p^{\left[\frac{\eta}{p-1}\right]} \right)^{1/\eta} = \prod_{p|B} p^{\frac{1}{p-1}},$$

we obtain

$$\lim_{\eta\to\infty} d_\eta^{1/\eta} = C.$$

The application of Theorem 1 completes the proof of the theorem.

3. THE CHOICE OF THE CHARACTERISTIC POLYNOMIAL

The aim of this section is to present the characteristic polynomial $\varphi(\lambda) \in \mathbb{Q}[\lambda]$ all of whose roots, except for one, lie in the closed disc $|\lambda| \le \delta$ with a sufficiently small δ.

Lemma 6. *Let $m \in \mathbb{N}$, $m \geq 3$, and let the real numbers a and b satisfy the conditions*

$$|a| \geq 1, \qquad |b| \geq 5, \qquad \left|\frac{a}{b}\right| < \frac{1}{2}\sin^{m-1}\frac{\pi}{m-1}. \tag{45}$$

Then the root λ_1 of the polynomial $\varphi(\lambda) = \lambda^m - b\lambda^{m-1} + a$ which is maximal in absolute value is real and $\operatorname{sign}\lambda_1 = \operatorname{sign} b$; *the other roots $\lambda_2, \dots, \lambda_m$ lie in the disc $|\lambda| < r + 2^{1/(m-1)}r^2$, where $r = |a/b|^{1/(m-1)}$.*

Proof. According to (45), the inequality $|b|/2 > |a| \geq 1$ holds. Hence, $(|b|/2)^m > |a|$, and the number

$$\varphi\Big(\frac{b}{2}\Big) = -\Big(\frac{b}{2}\Big)^m + a$$

has the same sign as b^m. Therefore, at least one root of the polynomial $\varphi(\lambda)$ lies in the interval

$$\Big(-\infty, \frac{b}{2}\Big) \quad \text{for} \quad b < 0, \qquad \Big(\frac{b}{2}, +\infty\Big) \quad \text{for} \quad b > 0. \tag{46}$$

Let us denote by λ_1 one of these roots (below we shall show that this root is unique). Since $m \geq 3$,

$$0 < r = \left|\frac{a}{b}\right|^{1/(m-1)} < \frac{1}{2^{1/(m-1)}}\sin\frac{\pi}{m-1} \leq \frac{1}{\sqrt{2}}$$

and

$$\frac{1}{\sqrt{2}}\frac{1}{|a|} + \sqrt{2}\frac{1}{|b|} \leq \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{5} = \frac{7\sqrt{2}}{10} < 1$$

for $|a| \geq 1$ and $|b| \geq 5$, we obtain

$$|\lambda_1| > \frac{|b|}{2} > \sqrt{2}\bigg(\frac{|b|}{2|a|} + 1\bigg) > r\bigg(\frac{|b|}{2|a|} + 1\bigg). \tag{47}$$

Consider the auxiliary polynomial

$$\psi(\lambda) = \Big(\lambda^{m-1} - \frac{a}{b}\Big)(\lambda - b) = \lambda^m - b\lambda^{m-1} - \frac{a}{b}\lambda + a.$$

One of its roots is equal to $\lambda_1' = b$ and the other roots $\lambda_2', \dots, \lambda_m'$ lie at the vertices of the regular $(m-1)$-gon on the circle $|\lambda| = r$.

We fix some roots λ_j', $j = 2, \dots, m$, of the polynomial $\psi(\lambda)$ and denote

$$\rho = \min_{2 \leq l \leq m}\big\{|\lambda_j' - \lambda_l|\big\}.$$

Since

$$\varphi(\lambda) - \psi(\lambda) = \frac{a}{b}\lambda,$$

we find for $\lambda = \lambda_j'$ that

$$(\lambda_1 - \lambda_j')(\lambda_2 - \lambda_j')\cdots(\lambda_m - \lambda_j') = \varphi(\lambda_j') = \frac{a}{b}\lambda_j';$$

this implies

$$r^m = \left|\frac{a}{b}\right| \times |\lambda'_j| = |\varphi(\lambda'_j)| \geq |\lambda_1 - \lambda'_j|\rho^{m-1} \geq (|\lambda_1| - r)\rho^{m-1} > r\frac{|b|}{2|a|}\rho^{m-1}$$

according to (47). The final result is

$$\rho < r\left(\frac{2|a|}{|b|}\right)^{1/(m-1)} = 2^{1/(m-1)}r^2 = \varepsilon.$$

We have found out that in the ε-neighborhood of any root λ'_j, $j = 2, \ldots, m$, of the polynomial $\psi(\lambda)$, there is a certain root λ_j of the polynomial $\varphi(\lambda)$. These ε-neighborhoods are mutually disjoint and do not intersect interval (46), since

$$|\lambda'_j - \lambda'_l| \geq r\left|e^{2\pi i/(m-1)} - 1\right| = 2r\sin\frac{\pi}{m-1} > 2r\left(2\frac{|a|}{|b|}\right)^{1/(m-1)} = 2\varepsilon$$

according to (45). It remains to note that

$$|\lambda_j| \leq |\lambda'_j| + \rho < r + \varepsilon, \qquad j = 2, \ldots, m,$$

and the root λ_1 is a root of the polynomial $\varphi(\lambda)$ maximal in the absolute value. The lemma is proved.

Remark. Actually, the boundary on the roots $\lambda_2, \ldots, \lambda_m$ of the polynomial

$$\varphi(\lambda) = \lambda^m - b\lambda^{m-1} + a$$

can be considerably refined with a specific choice of the numbers m, a, and b. We do not pose this problem here since Lemma 6 makes it possible to apply Theorem 2.

Let now $\tau(z) = az^m - bz + 1$, $m \geq 3$, and let $\varkappa(z)$ be a nonzero polynomial. Assume that the polynomial $\pi(z)$ of degree $k \geq 0$ is a solution to equation (35). Then $\pi(z) = cz^k + \cdots$, $c \neq 0$, and

$$\varkappa(z) = \tau(z)\pi'(z) + s\tau'(z)\pi(z) = ac(k + sm)z^{m+k-1} + \cdots;$$

this implies that $\deg\varkappa = m + k - 1 \geq m - 1$. Hence, if we assume additionally that $\deg\varkappa < m - 1$, then any solution to equation (35) is not a polynomial.

The following assertion is a corollary of Theorem 2 and Lemma 6.

Theorem 3. *Let $m \in \mathbb{N}$, $m \geq 3$, and let the rational numbers a and b satisfy conditions* (45). *Assume that B is the denominator of the rational number $s \in (0, 1)$, A is the least common denominator of the numbers $a/(2B)$ and $b/(2B)$ in the case $B = 2^k$, $k \in \mathbb{N}$, and of the numbers a/B and b/B for other B, the constant C is chosen in accordance with* (44), *and*

$$\delta = \left|\frac{a}{b}\right|^{1/(m-1)} + 2^{1/(m-1)}\left|\frac{a}{b}\right|^{2/(m-1)}.$$

If $C\delta < 1$ and the degree of the polynomial $\varkappa(z) \in \mathbb{I}[z]$ is lower than $m-1$, then the irrationality measure of the number

$$\int\limits_0^{z_1} \frac{\varkappa(z)}{(az^m - bz + 1)^s}\, dz,$$

where z_1 is a root of the polynomial $az^m - bz + 1$ which is minimal in the absolute value, does not exceed the value

$$\mu = 1 - \frac{\log C - \log|z_1|}{\log C + \log\delta}.$$

In [7], a special case ($m = 3$, $s = 1/2$, $a = 4$, b is a multiple of 4) of Theorem 3 was considered. The exact localization of the roots of the characteristic polynomial $\varphi(\lambda)$ in this case allows one to obtain the irrationality measure of values of certain elliptic integral with a high degree of accuracy.

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