AN ELEMENTARY PROOF OF APÉRY'S THEOREM

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ABSTRACT. We present a new 'elementary' proof of the irrationality of $\zeta(3)$ based on some recent 'hypergeometric' ideas of Yu. Nesterenko, T. Rivoal, and K. Ball, and on Zeilberger's algorithm of creative telescoping.

A question of an arithmetic nature of the values of Riemann's zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

at odd integral points $s = 3, 5, 7, \dots$ looks like a challenge for Number Theory. An expected answer '*each odd zeta value is transcendental*' is still far from being proved. We dispose of a particular information on the *irrationality* of odd zeta values, namely:

- $\zeta(3)$ is irrational (R. Apéry [Ap], 1978);
- infinitely many of the numbers $\zeta(3), \zeta(5), \zeta(7), \dots$ are irrational (T. Rivoal [Ri1], [BR], 2000);
- each set $\zeta(s+2), \zeta(s+4), \dots, \zeta(8s-3), \zeta(8s-1)$ with odd $s > 1$ contains at least one irrational number (this author [Zu1], [Zu2], 2001);
- at least one of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational (this author [Zu3], [Zu4], 2001).

All these results have a *classical* well-poised-hypergeometric origin, and we refer the reader roused the curiosity of this terminology to the forthcoming works [Zu4], [Zu5], [RZ] for details. The aim of this note is to prove Apéry's famous result by 'elementary means'.

Apéry's theorem. *The number $\zeta(3)$ is irrational.*

The idea of the following proof is due to T. Rivoal [Ri2], [Ri3], who mixed approaches of Yu. Nesterenko [Ne] and K. Ball, and our contribution here is to make a use of Zeilberger's algorithm of creative telescoping in the most elementary manner.

Key words and phrases. Zeta value, hypergeometric series, Apéry's theorem, Zeilberger's algorithm of creative telescoping.

Our starting point is repetition of [Ne, Section 1]. For each integer $n=0,1,2,\dots$ define the rational function

$$R_n(t):=\biggl(\frac{(t-1)\cdots(t-n)}{t(t+1)\cdots(t+n)}\biggr)^2$$

and denote by D_n the least common multiple of the numbers $1,2,\dots,n$ (and $D_0=1$ for completeness).

Lemma 1 (cf. [Ne, Lemma 1]). *There holds the equality*

$$F_n:=-\sum_{t=1}^\infty R_n'(t)=u_n\zeta(3)-v_n, \tag{1}$$

where $u_n\in\mathbb Z$, $D_n^3v_n\in\mathbb Z$.

Proof. Taking square of the partial-fraction expansion

$$\frac{(t-1)\cdots(t-n)}{t(t+1)\cdots(t+n)}=\sum_{k=0}^n\frac{(-1)^{n-k}\binom{n+k}{n}\binom nk}{t+k}$$

with a help of the relation

$$\frac1{t+k}\cdot\frac1{t+l}=\frac1{l-k}\cdot\biggl(\frac1{t+k}-\frac1{t+l}\biggr)\qquad\text{for}\quad k\ne l,$$

we arrive at the formula

$$R_n(t)=\sum_{k=0}^n\biggl(\frac{A_{2k}^{(n)}}{(t+k)^2}+\frac{A_{1k}^{(n)}}{t+k}\biggr),$$

with $A_{jk}=A_{jk}^{(n)}$ satisfying the inclusions

$$A_{2k}=\binom{n+k}{n}^2\binom nk^2\in\mathbb Z\quad\text{and}\quad D_nA_{1k}\in\mathbb Z,\qquad k=0,1,\dots,n. \tag{2}$$

Furthermore,

$$\sum_{k=0}^nA_{1k}=\sum_{k=0}^n\operatorname{Res}_{t=-k}R_n(t)=-\operatorname{Res}_{t=\infty}R_n(t)=0$$

since $R_n(t)=O(t^{-2})$ as $t\to\infty$, hence the quantity

$$\begin{aligned}F_n&=\sum_{t=1}^\infty\sum_{k=0}^n\biggl(\frac{2A_{2k}}{(t+k)^3}+\frac{A_{1k}}{(t+k)^2}\biggr)=\sum_{k=0}^n\sum_{l=k+1}^\infty\biggl(\frac{2A_{2k}}{l^3}+\frac{A_{1k}}{l^2}\biggr)\\&=2\sum_{k=0}^nA_{2k}\biggl(\sum_{l=1}^\infty-\sum_{l=1}^k\biggr)\frac1{l^3}+\sum_{k=0}^nA_{1k}\biggl(\sum_{l=1}^\infty-\sum_{l=1}^k\biggr)\frac1{l^2}\end{aligned}$$

has the desired form (1), with

$$u_n = 2\sum_{k=0}^{n} A_{2k}, \qquad v_n = 2\sum_{k=0}^{n} A_{2k}\sum_{l=1}^{k}\frac{1}{l^3} + \sum_{k=0}^{n} A_{1k}\sum_{l=1}^{k}\frac{1}{l^2}. \tag{3}$$

Finally, using the inclusions (2) and

$$D_n^j \cdot \sum_{l=1}^{k}\frac{1}{l^j} \in \mathbb{Z} \qquad \text{for} \quad k = 0,1,\dots,n, \quad j = 2,3,$$

we deduce that $u_n \in \mathbb{Z}$ and $D_n^3 v_n \in \mathbb{Z}$ as required.

Since

$$R_0(t) = \frac{1}{t^2}, \qquad R_1(t) = \frac{1}{t^2} + \frac{4}{(t+1)^2} - \frac{4}{t} + \frac{4}{t+1},$$

in accordance with formulae (3) we find that

$$F_0 = 2\zeta(3) \qquad \text{and} \qquad F_1 = 10\zeta(3) - 12. \tag{4}$$

Now, with a help of Zeilberger's algorithm of creative telescoping [PWZ, Chapter 6] we get the rational function $S_n(t) := s_n(t)R_n(t)$, where

$$s_n(t) := 4(2n+1)(-2t^2 + t + (2n+1)^2), \tag{5}$$

satisfying the following property.

Lemma 2. *For each $n = 1, 2, \dots,$ there holds the identity*

$$(n+1)^3 R_{n+1}(t) - (2n+1)(17n^2+17n+5)R_n(t) + n^3 R_{n-1}(t) = S_n(t+1) - S_n(t). \tag{6}$$

'One-line' proof. Divide both sides of (6) by $R_n(t)$ and verify numerically the identity

$$(n+1)^3\left(\frac{t-n-1}{t+n+1}\right)^2 - (2n+1)(17n^2+17n+5) + n^3\left(\frac{t+n}{t-n}\right)^2$$
$$= s_n(t+1)\left(\frac{t^2}{(t-n)(t+n+1)}\right)^2 - s_n(t),$$

where $s_n(t)$ is given in (5).

Lemma 3. *The quantity* (1) *satisfies the difference equation*

$$(n+1)^3 u_{n+1} - (2n+1)(17n^2+17n+5)u_n + n^3 u_n = 0 \tag{7}$$

for $n = 1, 2, \dots$.

Proof. Since $R_n'(t) = O(t^{-3})$ and $S_n'(t) = O(t^{-2})$, differentiating identity (6) and summing the result over $t = 1, 2, \dots$ we arrive at the equality

$$(n+1)^3 F_{n+1} - (2n+1)(17n^2+17n+5)F_n + n^3 F_{n-1} = S_n'(1).$$

It remains to note that, for $n \geqslant 1$, both functions $R_n(t)$ and $S_n(t) = s_n(t)R_n(t)$ have second-order zero at $t = 1$. Thus $S_n'(1) = 0$ for $n = 1, 2, \dots$ and we obtain the desired recurrence (7) for the quantity (1).

Consider another rational function

$$\widetilde{R}_n(t) := n!^2(2t+n)\frac{(t-1)\cdots(t-n)\cdot(t+n+1)\cdots(t+2n)}{(t(t+1)\cdots(t+n))^4} \tag{8}$$

and the corresponding hypergeometric series

$$\widetilde{F}_n := \sum_{t=1}^{\infty} \widetilde{R}_n(t), \tag{9}$$

proposed by K. Ball.

Lemma 4 (cf. [BR, the second proof of Lemma 3]). *For each $n = 0, 1, 2, \dots$, there holds the inequality*

$$0 < \widetilde{F}_n < 20(n+1)^4(\sqrt{2}-1)^{4n}. \tag{10}$$

Proof. Since $\widetilde{R}_n(t) = 0$ for $t = 1, 2, \dots, n$ and $\widetilde{R}_n(t) > 0$ for $t > n$ we deduce that $\widetilde{F}_n > 0$.

With a help of elementary inequality

$$\frac{1}{m}\cdot\frac{(m+1)^m}{m^{m-1}} = \left(1+\frac{1}{m}\right)^m < e < \left(1+\frac{1}{m}\right)^{m+1} = \frac{1}{m}\cdot\frac{(m+1)^{m+1}}{m^m}$$

that yields $(m+1)^m/m^{m-1} < em < (m+1)^{m+1}/m^m$ for $m = 1, 2, \dots$, we deduce that

$$e^{-n}\frac{(m+n)^{m+n-1}}{m^{m-1}} < m(m+1)\dots(m+n-1) < e^{-n}\frac{(m+n)^{m+n}}{m^m}.$$

Therefore, for integers $t \geqslant n+1$,

$$\begin{aligned}\widetilde{R}_n(t)\cdot\frac{(t+n)^5}{(2t+n)(t+2n)} &= n!^2\cdot\frac{(t-1)\cdots(t-n)\cdot(t+n)\cdots(t+2n-1)}{(t(t+1)\cdots(t+n-1))^4} \\ &< (n+1)^{2(n+1)}\cdot\frac{t^{5t-4}(t+2n)^{t+2n}}{(t-n)^{t-n}(t+n)^{5(t+n)-4}}\end{aligned}$$

and, as a consequence,

$$\begin{aligned}\widetilde{R}_n(t)\cdot\frac{t^4(t+n)}{(2t+n)(t+2n)(n+1)^2} &< (n+1)^{2n}\cdot\frac{t^{5t}(t+2n)^{t+2n}}{(t-n)^{t-n}(t+n)^{5(t+n)}} \\ &= \left(1+\frac{1}{n}\right)^{2n}\cdot e^{nf(t/n)} < e^2\cdot\left(\sup_{\tau>1} e^{f(\tau)}\right)^n,\end{aligned} \tag{11}$$

where

$$f(\tau) := \log \frac{\tau^{5\tau}(\tau+2)^{\tau+2}}{(\tau-1)^{\tau-1}(\tau+1)^{5(\tau+1)}}.$$

The unique (real) solution τ_0 of the equation

$$f'(\tau) = \log \frac{\tau^5(\tau+2)}{(\tau-1)(\tau+1)^5} = 0$$

in the region $\tau > 1$ is the zero of the polynomial

$$\tau^5(\tau+2) - (\tau-1)(\tau+1)^5 = -\left(\tau+\frac{1}{2}\right)\left(2\left(\tau+\frac{1}{2}\right)^4 - 5\left(\tau+\frac{1}{2}\right)^2 - \frac{7}{8}\right),$$

hence we can determine it explicitly:

$$\tau_0 = -\frac{1}{2} + \sqrt{\frac{5}{4} + \sqrt{2}}.$$

Thus,

$$\begin{aligned}\sup_{\tau>1} f(\tau) = f(\tau_0) &= f(\tau_0) - \tau_0 f'(\tau_0) = 2\log(\tau_0+2) + \log(\tau_0-1) - 5\log(\tau_0+1) \\ &= 4\log(\sqrt{2}-1)\end{aligned}$$

and we can continue the estimate (11) as follows:

$$\widetilde{R}_n(t) \cdot \frac{t^4(t+n)}{(2t+n)(t+2n)} < e^2(n+1)^2(\sqrt{2}-1)^{4n}, \tag{12}$$

Finally, we apply the inequality (12) to deduce the required estimate (10):

$$\begin{aligned}\widetilde{F}_n = \sum_{t=n+1}^{\infty} \widetilde{R}_n(t) &< e^2(n+1)^2(\sqrt{2}-1)^{4n} \sum_{t=n+1}^{\infty} \frac{(2t+n)(t+2n)}{t^4(t+n)} \\ &< e^2(n+1)^2(\sqrt{2}-1)^{4n} \sum_{t=n+1}^{\infty} \left(\frac{2}{t^5} + \frac{5n}{t^4} + \frac{2n^2}{t^3}\right) \\ &\leqslant e^2(n+1)^2\big(2\zeta(5) + 5n\zeta(4) + 2n^2\zeta(3)\big)(\sqrt{2}-1)^{4n} < 20(n+1)^4(\sqrt{2}-1)^{4n}.\end{aligned}$$

This completes the proof.

For the rational function (8) we obtain Zeilberger's certificate

$$\begin{aligned}\widetilde{S}_n(t) := \frac{\widetilde{R}_n(t)}{(2t+n)(t+2n-1)(t+2n)} &\cdot \big(-t^6 - (8n-1)t^5 + (4n^2+27n+5)t^4 \\ &+ 2n(67n^2+71n+15)t^3 + (358n^4+339n^3+76n^2-7n-3)t^2 \\ &+ (384n^5+396n^4+97n^3-29n^2-17n-2)t \\ &+ n(153n^5+183n^4+50n^3-30n^2-22n-4)\big).\end{aligned} \tag{13}$$

Lemma 5. *For each $n=1,2,\dots$, there holds the identity*

$$(n+1)^3\widetilde R_{n+1}(t)-(2n+1)(17n^2+17n+5)\widetilde R_n(t)+n^3\widetilde R_{n-1}(t)=\widetilde S_n(t+1)-\widetilde S_n(t). \quad (14)$$

'One-line' proof. Divide both sides of (14) by $\widetilde R_n(t)$ and verify the reduced identity.

Lemma 6. *The quantity* (9) *satisfies the difference equation* (7) *for $n=1,2,\dots$.*

Proof. Since $\widetilde R_n(t)=O(t^{-5})$ and $\widetilde S_n(t)=O(t^{-2})$ as $t\to\infty$ for $n\geqslant 1$, summation of equalities (14) over $t=1,2,\dots$ yields the relation

$$(n+1)^3\widetilde F_{n+1}-(2n+1)(17n^2+17n+5)\widetilde F_n+n^3\widetilde F_{n-1}=-\widetilde S_n(1).$$

It remains to note that, for $n\geqslant 1$, both functions (8) and (13) have zero at $t=1$. Thus $\widetilde S_n(1)=0$ for $n=1,2,\dots$ and we obtain the desired recurrence (7) for the quantity (9).

Lemma 7. *For each $n=0,1,2,\dots$, the quantities* (1) *and* (9) *coincide.*

Proof. Since both F_n and $\widetilde F_n$ satisfy the same second-order difference equation (7), we have to verify that $F_0=\widetilde F_0$ and $F_1=\widetilde F_1$. Direct calculations show that

$$\widetilde R_0(t)=\frac{2}{t^3}, \qquad \widetilde R_1(t)=-\frac{2}{t^4}+\frac{2}{(t+1)^4}+\frac{5}{t^3}+\frac{5}{(t+1)^3}-\frac{5}{t^2}+\frac{5}{(t+1)^2},$$

hence $\widetilde F_0=2\zeta(3)$ and $\widetilde F_1=10\zeta(3)-12$, and comparison of this result with (4) yields the desired coincidence.

Proof of Apéry's theorem. Suppose, on the contrary, that $\zeta(3)=p/q$, where p and q are positive integers. Then, using a trivial bound $D_n<3^n$, we deduce that, for each $n=0,1,2,\dots$, the integer $qD_n^3F_n=D_n^3u_np-D_n^3v_nq$ satisfies the estimate

$$0<qD_n^3F_n<20q(n+1)^4 3^{3n}(\sqrt2-1)^{4n} \quad (15)$$

that is not possible since $3^3(\sqrt2-1)^4=0.7948\ldots<1$ and the right-hand side of (15) is less than 1 for a sufficiently large integer n. This contradiction completes the proof of the theorem.

Inspite of its elementary arguments, our proof of Apéry's theorem does not look simpler than the original (also elementary) Apéry's proof well-explained in A. van der Poorten's informal report [Po], or (almost elementary) Beukers's proof [Be] by means of Legendre polynomials and multiple integrals. We want to mention that our way to deduce the recursion (7) for the sequence F_n as well as for the coefficients u_n,v_n[‡]

[‡]Hint: multiply both sides of (6) by $(t+k)^2$, substitute $t=-k$ and sum over *all* integers k to show that the sequence u_n satisfies the difference equation (7); then $v_n=u_n\zeta(3)-F_n$ also satisfies it.

slightly differs from those considered in [Po, Section 8] and [Ze, Section 13] although it is based on the same algorithm of creative telescoping. This algorithm and the above scheme allow us [Zu5], [Zu6] to obtain Apéry-like difference equations for $\zeta(4)$ and Calalan's constant.

The fact that $\widetilde{F}_n = \widetilde{u}_n\zeta(3) - \widetilde{v}_n$ with $D_n\widetilde{u}_n, D_n^4\widetilde{v}_n \in \mathbb{Z}$ was first discovered by K. Ball; the proof follows lines of the proof of Lemma 1 and vanishing the coefficients for $\zeta(4)$ and $\zeta(2)$ is due to a well-poised origin of the series (9). An open question of T. Rivoal here is to get the better inclusions $\widetilde{u}_n, D_n^3\widetilde{v}_n \in \mathbb{Z}$ by elementary means without going back to Apéry's series (1). A solution of this question accompanied with Ball's Lemma 4 can bring the 'most elementary' proof of Apéry's theorem.

Lemma 7 can be proved by specialization of Bailey's identity [Ba, Section 6.3, formula (2)]

$$\begin{aligned}
&{}_7F_6\left(\begin{matrix} a, 1+\frac12 a, & b, & c, & d, & e, & f \\ \frac12 a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f \end{matrix}\,\middle|\, 1\right) \\
&\quad= \frac{\Gamma(1+a-b)\,\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(1+a)\,\Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(1+a-b-c)\,\Gamma(1+a-b-d) \times\Gamma(1+a-c-d)\,\Gamma(1+a-e-f)} \\
&\quad\times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(b+t)\,\Gamma(c+t)\,\Gamma(d+t)\,\Gamma(1+a-e-f+t) \times\Gamma(1+a-b-c-d-t)\,\Gamma(-t)}{\Gamma(1+a-e+t)\,\Gamma(1+a-f+t)}\,\mathrm{d}t,
\end{aligned} \tag{16}$$

provided that the very-well-poised hypergeometric series on the left-hand side converges. Namely, taking $a = 3n+2$ and $b = c = d = e = f = n+1$ in (16) we obtain Ball's sequence (9) on the left and Apéry's sequence (1) on the right (for the last fact see [Ne, Lemma 2]). Identity (16) can be put forward for an explanation how the permutation group from [RV] for linear forms in 1 and $\zeta(3)$ appears (see [Zu5, Sections 4 and 5] for details).

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