A FEW REMARKS ON LINEAR FORMS INVOLVING CATALAN'S CONSTANT

Wadim Zudilin (Moscow)

E-print `math.NT/0210423`

21 October 2002

To N. M. Korobov on the occasion of his 85th birthday

Abstract. In the joint work [RZ] of T. Rivoal and the author, a hypergeometric construction was proposed for studing arithmetic properties of the values of Dirichlet's beta function $\beta(s)$ at even positive integers. The construction gives some bonuses [RZ], Section 9, for Catalan's constant $G=\beta(2)$, such as a second-order Apéry-like recursion and a permutation group in the sense of G. Rhin and C. Viola [RV]. Here we prove expected integrality properties of solutions to the above recursion as well as suggest a simpler (also second-order and Apéry-like) one for G. We 'enlarge' the permutation group of [RZ], Section 9, by showing that the total 120-permutation group of [RV] for $\zeta(2)$ can be applied in arithmetic study of Catalan's constant. These considerations have computational meanings and do not allow us to prove the (presumed) irrationality of G. Finally, we suggest a conjecture yielding the irrationality property of numbers (e.g., of Catalan's constant) from existence of suitable second-order difference equations (recursions).

Recently, T. Rivoal and the author [RZ] proved several partial results on the irrationality of the numbers

$$\beta(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}, \qquad s=2,4,6,8,\dots .$$

We did not succeed in proving the (expected) irrationality of Catalan's constant $G=\beta(2)$. However, the general analytic construction in [RZ] allows one to derive a certain Apéry-like second-order recursion for Catalan's constant; this was done by semi-human application of Zeilberger's creative telescoping in [Zu2] and completely automatically, thanks to Apéry's 'accélération de la convergence' approach, in [Ze].

I would like to thank S. Fischler, T. Rivoal, and J. Sondow for suggestions that allowed me to improve the text of the article.

Key words and phrases. Catalan's constant, generalized hypergeometric series.
2000 *Mathematics Subject Classification.* Primary 11J72; Secondary 11J70, 30B50, 33C60.

1. Hypergeometric series. Recall the $\mathbb{Q}$-linear forms in 1 and G constructed in [RZ] and [Zu2]:

$$\begin{aligned} r_n &:= u_n G - v_n \qquad (1) \\ &= \frac{n!}{8}\sum_{t=0}^{\infty}(2t+n+1)\frac{\prod_{j=1}^{n}(t-j-1)\cdot\prod_{j=1}^{n}(t+j+n)}{\big(\prod_{j=0}^{n}(t+j+\frac12)\big)^3}(-1)^t \\ &= \frac{(-1)^n n!}{8}\,\frac{\Gamma(3n+2)\,\Gamma(n+\frac12)^3\Gamma(n+1)}{\Gamma(2n+\frac32)^3\Gamma(2n+1)} \\ &\quad\times {}_6F_5\left(\begin{matrix} 3n+1, \frac32 n+\frac32, \ n+\frac12, \ n+\frac12, \ n+\frac12, \ n+1 \\ \frac32 n+\frac12, 2n+\frac32, 2n+\frac32, 2n+\frac32, 2n+1 \end{matrix}\,\middle|\, -1\right), \end{aligned}$$

$$\lim_{n\to\infty}|r_n|^{1/n} = \left|\frac{1-\sqrt5}{2}\right|^5, \qquad \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} v_n^{1/n} = \left(\frac{1+\sqrt5}{2}\right)^5.$$

Note that Theorem 1 in [Zu2] states the following:

$$2^{4n+3}D_n u_n \in \mathbb{Z}, \qquad 2^{4n+3}D_{2n-1}^3 v_n \in \mathbb{Z}, \qquad (2)$$

where D_N denotes the least common multiple of the numbers $1, 2, \dots, N$, although the better inclusions

$$2^{4n}u_n \in \mathbb{Z}, \qquad 2^{4n}D_{2n-1}^2 v_n \in \mathbb{Z}$$

hold for $n = 1, 2, \dots, 1000$ by numerical verification (see [Zu2], Section 4). The aim of this section is to prove (at least asymptotically, as $n \to \infty$, i.e., sufficient for all practical purposes) this experimental observation.

Theorem 1. *For $n = 0, 1, 2, \dots,$ we have*

$$2^{4n+o(n)}u_n \in \mathbb{Z}, \qquad 2^{4n+o(n)}D_{2n-1}^2 v_n \in \mathbb{Z}. \qquad (3)$$

Remark. As follows from the proof below, the $o(n)$-term in the inclusions (3) is of order $\log_2(2n)$.

Proof. We will require Whipple's transform [Ba], Section 4.4, formula (2),

$$\begin{aligned} &{}_6F_5\left(\begin{matrix} a, 1+\frac12 a, & b, & c, & d, & e \\ \frac12 a, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e \end{matrix}\,\middle|\, -1\right) \\ &\qquad = \frac{\Gamma(1+a-d)\,\Gamma(1+a-e)}{\Gamma(1+a)\,\Gamma(1+a-d-e)}\cdot {}_3F_2\left(\begin{matrix} 1+a-b-c, \ d, \ e \\ 1+a-b, \ 1+a-c \end{matrix}\,\middle|\, 1\right), \qquad (4) \end{aligned}$$

provided that $\operatorname{Re}(1+a-d-e) > 0$, and Bailey's transform [Ba], Section 6.4, formula (1),

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} a, & b, & c, & d \\ & k-b, & k-c, & k-d \end{matrix}\,\middle|\, 1\right) \\ &\quad = \frac{\Gamma(k-b)\,\Gamma(k-c)\,\Gamma(k-d)}{\Gamma(b)\,\Gamma(c)\,\Gamma(d)\,\Gamma(k-b-c)\,\Gamma(k-b-d)\,\Gamma(k-c-d)} \\ &\quad\times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(b+t)\,\Gamma(c+t)\,\Gamma(d+t)\,\Gamma(k-a+2t) \times\Gamma(k-b-c-d-t)\,\Gamma(-t)}{\Gamma(k-a+t)\,\Gamma(k+2t)}\,\mathrm{d}t, \qquad (5) \end{aligned}$$

where the path of integration is parallel to the imaginary axis, except that it is curved, if necessary, so that the decreasing sequences of poles of the functions $\Gamma(k-b-c-d-t)$ and $\Gamma(-t)$ lie to the left of the contour, while the increasing sequences of poles of the functions $\Gamma(b+t)$, $\Gamma(c+t)$, $\Gamma(d+t)$, and $\Gamma(k-a+2t)$ lie to the right.

Applying transform (4) with $a=3n+1$, $b=c=d=n+\frac{1}{2}$, $e=n+1$ and then transform (5) with $a=2n+2$, $b=n+\frac{1}{2}$, $c=d=n+1$, $k=3n+\frac{5}{2}$ we obtain

$$\begin{aligned} r_n &= \frac{(-1)^n}{8}\,\frac{(2n+1)!}{n!^2}\,\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(n+1+t)^2\Gamma(n+\frac{1}{2}+2t)\,\Gamma(-t)^2}{\Gamma(3n+\frac{5}{2}+2t)}\,\mathrm{d}t \\ &= \frac{(-1)^n}{8}\,\frac{(2n+1)!}{n!^2}\,\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(n+1+t)^2\Gamma(n+\frac{1}{2}+2t)}{\Gamma(1+t)^2\Gamma(3n+\frac{5}{2}+2t)}\left(\frac{\pi}{\sin\pi t}\right)^2\mathrm{d}t. \end{aligned} \tag{6}$$

Shifting $n+1+t\mapsto t$ and considering the residues at the increasing sequence of poles of the integrand in (6) (cf. [Ne], Lemma 2) we arrive at the formula

$$r_n = \frac{(-1)^{n+1}}{8}\,\frac{(2n+1)!}{n!^2}\sum_{\nu=1}^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Gamma(t)^2\Gamma(-n-\frac{3}{2}+2t)}{\Gamma(-n+t)^2\Gamma(n+\frac{1}{2}+2t)}\right)\bigg|_{t=\nu} = -\sum_{\nu=1}^{\infty}\frac{\mathrm{d}R_n(t)}{\mathrm{d}t}\bigg|_{t=\nu}, \tag{7}$$

where

$$\begin{aligned} R_n(t) &= \frac{(-1)^n}{8}\,\frac{(2n+1)!}{n!^2}\,\frac{\left(\prod_{j=1}^{n}(t-j)\right)^2}{\prod_{j=0}^{2n+1}(2t+j-n-\frac{3}{2})} \\ &= \sum_{l=0}^{n}\left(\frac{A_l}{t+l-\frac{n}{2}-\frac{3}{4}}+\frac{A_l'}{t+l-\frac{n}{2}-\frac{1}{4}}\right). \end{aligned} \tag{8}$$

The coefficients A_l and A_l', $l=0,1,\dots,n$, in the partial-fraction decomposition of the function $R_n(t)$ can be easily determined by the standard procedure:

$$\begin{aligned} A_l &= \frac{(-1)^n}{16}\,\frac{(2n+1)!}{(2l)!\,(2n-2l+1)!}\cdot\left(\frac{(t-1)(t-2)\cdots(t-n)}{n!}\bigg|_{t=-l+\frac{n}{2}+\frac{3}{4}}\right)^2, \\ A_l' &= \frac{(-1)^{n+1}}{16}\,\frac{(2n+1)!}{(2l+1)!\,(2n-2l)!}\cdot\left(\frac{(t-1)(t-2)\cdots(t-n)}{n!}\bigg|_{t=-l+\frac{n}{2}+\frac{1}{4}}\right)^2, \\ &\qquad l=0,1,\dots,n, \end{aligned}$$

hence

$$A_l\cdot 2^{6n+4}\in\mathbb{Z} \quad\text{and}\quad A_l'\cdot 2^{6n+4}\in\mathbb{Z}, \qquad l=0,1,\dots,n, \tag{9}$$

by well-known properties of the integer-valued polynomials $(t-1)(t-2)\cdots(t-n)/n!$.

Using this decomposition we can continue formula (7) as follows:

$$
\begin{aligned}
r_n &= 16\sum_{l=0}^{n} A_l\cdot\sum_{\mu=0}^{\infty}\frac{1}{(4\mu+\epsilon)^2} + 16\sum_{l=0}^{n} A'_l\cdot\sum_{\mu=0}^{\infty}\frac{1}{(4\mu+\epsilon')^2} \\
&\qquad + 16\sum_{l=0}^{m-1} A_l\sum_{\mu=1}^{m-l}\frac{1}{(4\mu-\epsilon)^2} + 16\sum_{l=0}^{m'-1} A'_l\sum_{\mu=1}^{m'-l}\frac{1}{(4\mu-\epsilon')^2} \\
&\qquad - 16\sum_{l=m+1}^{n} A_l\sum_{\mu=0}^{l-m-1}\frac{1}{(4\mu+\epsilon)^2} - 16\sum_{l=m'+1}^{n} A'_l\sum_{\mu=0}^{l-m'-1}\frac{1}{(4\mu+\epsilon')^2},
\end{aligned} \tag{10}
$$

where $m=\lfloor (n+1)/2\rfloor$, $m'=\lfloor n/2\rfloor$; $\epsilon=1$ for n even and $\epsilon=3$ for n odd; $\epsilon'=4-\epsilon$; $\lfloor\,\cdot\,\rfloor$ denotes the integer part of a number. (For instance, if n is even, we have

$$
\begin{aligned}
&\sum_{\nu=1}^{\infty}\sum_{l=0}^{n}\frac{A_l}{(\nu+l-\frac{n}{2}-\frac{3}{4})^2} \\
&\quad = 16\sum_{l=0}^{2m} A_l\sum_{\nu=1}^{\infty}\frac{1}{(4(\nu+l-m-1)+1)^2} = 16\sum_{l=0}^{2m} A_l\sum_{\mu=l-m}^{\infty}\frac{1}{(4\mu+1)^2} \\
&\quad = 16\sum_{l=0}^{m-1} A_l\biggl(\sum_{\mu=l-m}^{-1}+\sum_{\mu=0}^{\infty}\biggr)\frac{1}{(4\mu+1)^2} + 16A_m\sum_{\mu=0}^{\infty}\frac{1}{(4\mu+1)^2} \\
&\qquad + 16\sum_{l=m+1}^{2m} A_l\biggl(\sum_{\mu=0}^{\infty}-\sum_{\mu=0}^{l-m-1}\biggr)\frac{1}{(4\mu+1)^2} \\
&\quad = 16\sum_{l=0}^{2m} A_l\cdot\sum_{\mu=0}^{\infty}\frac{1}{(4\mu+1)^2} \\
&\qquad + 16\sum_{l=0}^{m-1} A_l\sum_{\mu=1}^{m-l}\frac{1}{(4\mu-1)^2} - 16\sum_{l=m+1}^{2m} A_l\sum_{\mu=0}^{l-m-1}\frac{1}{(4\mu+1)^2}
\end{aligned}
$$

and we proceed analogously in the three remaining cases.)

By (8), $R_n(t)=O(t^{-2})$ as $t\to\infty$, hence $\sum_{l=0}^{n}A_l+\sum_{l=0}^{n}A'_l=0$. Therefore, formula (10) can be written in the desired form $r_n=u_nG-v_n$, where

$$
\begin{aligned}
u_n &= 16(-1)^n\sum_{l=0}^{n}A_l = 16(-1)^{n+1}\sum_{l=0}^{n}A'_l, \\
v_n &= -16\sum_{l=0}^{m-1} A_l\sum_{\mu=1}^{m-l}\frac{1}{(4\mu-\epsilon)^2} - 16\sum_{l=0}^{m'-1} A'_l\sum_{\mu=1}^{m'-l}\frac{1}{(4\mu-\epsilon')^2} \\
&\qquad + 16\sum_{l=m+1}^{n} A_l\sum_{\mu=0}^{l-m-1}\frac{1}{(4\mu+\epsilon)^2} + 16\sum_{l=m'+1}^{n} A'_l\sum_{\mu=0}^{l-m'-1}\frac{1}{(4\mu+\epsilon')^2}.
\end{aligned} \tag{11}
$$

(Again, Zeilberger's creative telescoping applied to the sequences u_n, v_n in (11) yields the Apéry-like recursion from [Zu2] for the old sequences u_n, v_n in (1); this fact implies the coincidence of the two representations (1) and (11) for the numbers u_n and v_n in the sequence $r_n = u_n G - v_n$.) Formulae (11) for u_n, v_n and relations (9) imply $2^{6n}u_n, 2^{6n}D_{2n-1}^2 v_n \in \mathbb{Z}$. Finally, using (2) and the fact that the order of 2 in D_N is $\lfloor \log_2 N \rfloor$ we arrive at the desired inclusions (3), and the theorem is proved.

2. A new Apéry-like recursion for Catalan's constant. The recursion in [RZ], [Zu2] allows one to do fast computation of G with high accuracy. Interpreting the solution to the recursion in [RZ], [Zu2] as in (7) prompted us to modify slightly the parameters of the above construction. Thus we take the sequence

$$\tilde r_n = \tilde u_n G - \tilde v_n = -\sum_{\nu=1}^{\infty} \frac{\mathrm{d}\tilde R_n(t)}{\mathrm{d}t}\bigg|_{t=\nu}, \tag{12}$$

$$\tilde R_n(t) = \frac{(-1)^n}{2}\,\frac{(2n)!}{(n-1)!^2}\,\frac{\prod_{j=1}^{n-1}(t-j)\cdot\prod_{j=1}^{n}(t-j)}{\prod_{j=0}^{2n}(2t+j-n-\frac12)}$$

and apply Zeilberger's algorithm of creative telescoping in order to prove the following result.

Theorem 2. *The numbers $\tilde u_n$ and $\tilde v_n$ satisfy the second-order recursion*

$$\begin{aligned}(2n)^2(2n+1)^2(20n^2-20n+3)\tilde u_{n+1} - (3520n^6-2672n^4+196n^2-9)\tilde u_n\\ - (2n)^2(2n+1)(2n-3)(20n^2+20n+3)\tilde u_{n-1} = 0, \qquad n=1,2,3,\dots,\end{aligned} \tag{13}$$

with the initial data $\tilde u_0 = 0$, $\tilde u_1 = 6$, and $\tilde v_0 = -1$, $\tilde v_1 = 5$. In addition, the limit relations

$$\lim_{n\to\infty} |\tilde u_n G - \tilde v_n|^{1/n} = \left|\frac{1-\sqrt5}{2}\right|^5, \qquad \lim_{n\to\infty}\tilde u_n^{1/n} = \lim_{n\to\infty}\tilde v_n^{1/n} = \left(\frac{1+\sqrt5}{2}\right)^5,$$

hold and

$$2^{4n+o(n)}\tilde u_n \in \mathbb{Z}, \quad 2^{4n+o(n)}D_{2n-1}^2\tilde v_n \in \mathbb{Z} \qquad \text{for } n=0,1,2,\dots. \tag{14}$$

(The proof of the inclusions (14) is a word-by-word repetition of what was done in Section 1.)

The polynomials in (13) are polynomials in $2n$ with integer coefficients; the recursion (13) looks a little simpler than in [RZ], [Zu2]. As in [Zu2], Theorems 2 and 3, the constraint (12) also leads to the continued-fraction expansion

$$6G = 5 + \frac{516|}{|q(2)} + \frac{p(3)|}{|q(4)} + \frac{p(5)|}{|q(6)} + \dots + \frac{p(2n-1)|}{|q(2n)} + \cdots,$$

$$p(n) = (5n^2-20n+18)(n-2)(n-1)^2n^2(n+1)^2(n+2)(5n^2+20n+18),$$

$$q(n) = 55n^6 - 167n^4 + 49n^2 - 9,$$

and to the multiple Euler-type integral

$$\tilde u_n G - \tilde v_n = \frac{(-1)^{n-1}n}{2}\int_0^1\!\!\int_0^1 \frac{x^{n-3/2}(1-x)^n y^{n-1}(1-y)^{n-1/2}}{(1-xy)^n}\,\mathrm{d}x\,\mathrm{d}y,$$
$$n=1,2,3,\dots.$$

3. A permutation group related to Catalan's constant. Take the parameters h_0, h_1, h_2, h_3, h_4 satisfying the conditions

$$h_0, h_4 \in \mathbb{Z}, \qquad h_1, h_2, h_3 \in \mathbb{Z} + \tfrac{1}{2}, \tag{15}$$

$$h_j > 0 \quad \text{and} \quad 1 + h_0 - h_j - h_l > 0 \qquad \text{for } j, l = 1, 2, 3, 4. \tag{16}$$

As shown in [RZ], Lemma 2, the quantity

$$\frac{\Gamma(1+h_0)\,\Gamma(h_3)\,\Gamma(h_4)\,\Gamma(1+h_0-h_1-h_3)\,\Gamma(1+h_0-h_2-h_4)\,\Gamma(1+h_0-h_3-h_4)}{\Gamma(1+h_0-h_1)\,\Gamma(1+h_0-h_2)\,\Gamma(1+h_0-h_3)\,\Gamma(1+h_0-h_4)} \times {}_6F_5\left(\begin{matrix} h_0, 1+\frac{1}{2}h_0, & h_1, & h_2, & h_3, & h_4 \\ \frac{1}{2}h_0, & 1+h_0-h_1, & 1+h_0-h_2, & 1+h_0-h_3, & 1+h_0-h_4 \end{matrix} \,\middle|\, -1\right) \tag{17}$$

belongs to the space $\mathbb{Q}G + \mathbb{Q}$. (In [RZ], a different ratio of gamma factors multiplies the ${}_6F_5$-series, but one ratio is a rational multiple of the other.)

By means of the new parameters

$$a_1 = 1 + h_0 - h_1 - h_2, \quad a_2 = h_3, \quad a_3 = h_4,$$
$$b_2 = 1 + h_0 - h_1, \quad b_3 = 1 + h_0 - h_2$$

and thanks to Whipple's transform (4) we can represent the quantity (17) as follows:

$$\frac{\Gamma(a_2)\,\Gamma(a_3)\,\Gamma(b_2-a_2)\,\Gamma(b_3-a_3)}{\Gamma(b_2)\,\Gamma(b_3)} \cdot {}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_2, b_3 \end{matrix} \,\middle|\, 1\right) = \int_0^1\int_0^1 \frac{x^{a_2-1}(1-x)^{b_2-a_2-1}y^{a_3-1}(1-y)^{b_3-a_3-1}}{(1-xy)^{a_0}}\,\mathrm{d}x\,\mathrm{d}y. \tag{18}$$

Finally, take the third 10-element set $\boldsymbol{c}$:

$$c_{00} = (b_2 + b_3) - (a_1 + a_2 + a_3) - 1,$$
$$c_{jl} = \begin{cases} a_j - 1 & \text{if } l = 1, \\ b_l - a_j - 1 & \text{if } l = 2, 3, \end{cases} \qquad j, l = 1, 2, 3 \tag{19}$$

(hence all $c_{jl} > -1$ by (16)), in order to get that the double integral

$$H(\boldsymbol{c}) = \int_0^1\int_0^1 \frac{x^{c_{21}}(1-x)^{c_{22}}y^{c_{31}}(1-y)^{c_{33}}}{(1-xy)^{c_{11}+1}}\,\mathrm{d}x\,\mathrm{d}y \tag{20}$$

lies in $\mathbb{Q}G + \mathbb{Q}$. It will be useful to split the set (19) as $\boldsymbol{c} = (\boldsymbol{c}', \boldsymbol{c}'')$, where

$$\boldsymbol{c}' = (c_{00}, c_{21}, c_{22}, c_{33}, c_{31}) \quad \text{and} \quad \boldsymbol{c}'' = (c_{11}, c_{23}, c_{13}, c_{12}, c_{32})$$

will be interpreted as cyclically ordered sets (i.e., c_{00} follows c_{31} in $\boldsymbol{c}'$ and c_{11} follows c_{32} in $\boldsymbol{c}''$). Obviously, each element in $\boldsymbol{c}''$ can be expressed in terms of elements in $\boldsymbol{c}'$, and vice versa. Using relations (15) and summarizing what we said above we obtain the following result.

Suppose that

$$c_{00}, c_{21}, c_{33} \in \mathbb{Z} + \tfrac{1}{2} \quad \textit{and} \quad c_{22}, c_{31} \in \mathbb{Z} \tag{21}$$

for the elements in $\mathbf{c}'$ (*or, equivalently,* $c_{13}, c_{12}, c_{32} \in \mathbb{Z} + \frac{1}{2}$ *and* $c_{11}, c_{23} \in \mathbb{Z}$ *for the elements in* $\mathbf{c}''$) *and that all elements in* $\boldsymbol{c}$ *are* > -1. *Then* $H(\boldsymbol{c}) \in \mathbb{Q}G + \mathbb{Q}$.

Digressing from the demi-integrality of the parameters $\boldsymbol{c}$, let us note that the hypergeometric ${}_3F_2$-representation (18) and the equivalent ${}_6F_5$-representation (17) lead to the following group structure (cf. [Wh] or [Ba], Sections 3.5–3.6). Each permutation of the parameters a_1, a_2, a_3 in (18) or of the parameters h_1, h_2, h_3, h_4 in (17) gives a hypergeometric series of the same kind (but with a different ratio of gamma factors before it). For instance, the transposition $\mathfrak{h} = (h_1\ h_4)$ rearranges the parameters $\boldsymbol{a}$ and $\boldsymbol{b}$ as follows:

$$\mathfrak{h}\colon \begin{pmatrix} a_1, a_2, a_3 \\ b_2, b_3 \end{pmatrix} \mapsto \begin{pmatrix} b_3 - a_3, & a_2, & b_3 - a_1 \\ & b_2 + b_3 - a_1 - a_3, & b_3 \end{pmatrix}$$

and corresponds to Thomae's transformation [Ba], Section 3.2. Hence the group $\mathfrak{G}$ generated by all such permutations appears naturally. An advantage of the superfluous 10-element set $\boldsymbol{c}$ is the fact that $\mathfrak{G}$ acts on the parameters $\boldsymbol{c}$ quite simply—by permutations. As F. J. W. Whipple has shown [Wh], the group $\mathfrak{G}$ is of order 120. A possible choice of generators of $\mathfrak{G}$ consists of the transpositions $\mathfrak{a}_1 = (a_1\ a_3)$, $\mathfrak{a}_2 = (a_2\ a_3)$, $\mathfrak{b} = (b_2\ b_3)$, and the above-cited $\mathfrak{h} = (h_1\ h_4)$ (see [Zu1], Section 6); the action of these permutations on the set $\boldsymbol{c}$ reads as follows:

$$\begin{aligned} \mathfrak{a}_1 &= (c_{11}\ c_{31})(c_{12}\ c_{32})(c_{13}\ c_{33}), & \mathfrak{a}_2 &= (c_{21}\ c_{31})(c_{22}\ c_{32})(c_{23}\ c_{33}), \\ \mathfrak{b} &= (c_{12}\ c_{13})(c_{22}\ c_{23})(c_{32}\ c_{33}), & \mathfrak{h} &= (c_{00}\ c_{22})(c_{11}\ c_{33})(c_{13}\ c_{31}). \end{aligned} \tag{22}$$

Theorem 3. *Let the quantity* $H(\boldsymbol{c})$ *be defined as the double integral in* (20), *or as the* ${}_3F_2$*-series in* (18), *or as the* ${}_6F_5$*-series in* (17). *Let* $\mathfrak{G} \subset \mathfrak{S}_{10}$ *be the* $\boldsymbol{c}$*-permutation group generated by* (22). *Suppose that all elements in the set* $\boldsymbol{c}$ *are* > -1.
Then

(i) *the quantity*

$$\frac{H(\boldsymbol{c})}{\Pi(\boldsymbol{c})}, \qquad \textit{where} \quad \Pi(\boldsymbol{c}) = \Gamma(c_{00})\,\Gamma(c_{21})\,\Gamma(c_{22})\,\Gamma(c_{33})\,\Gamma(c_{31}), \tag{23}$$

is $\mathfrak{G}$*-stable;*

(ii) *if the set* $\boldsymbol{c}$ *is* $\mathfrak{G}$*-equivalent to a set satisfying condition* (21), *we have* $H(\boldsymbol{c}) \in \mathbb{Q}G + \mathbb{Q}$.

Proof. (i) The $\mathfrak{G}$-stability of the quantity (23) has to be verified for the permutations in the list (22); this is routine using Whipple's transform for verification of the $\mathfrak{h}$-stability.

(ii) In order to deduce the inclusion $H(\boldsymbol{c}) \in \mathbb{Q}G + \mathbb{Q}$ from the above claim (i), it remains to show that $\Pi(\sigma\boldsymbol{c})/\Pi(\boldsymbol{c}) \in \mathbb{Q}$ for a set $\boldsymbol{c}$ satisfying (21) and for all $\sigma \in \mathfrak{G}$ or, equivalently, for $\sigma \in \{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}, \mathfrak{h}\}$ (by $\sigma\boldsymbol{c}$ we mean the action of a permutation $\sigma \in \mathfrak{G}$ on the set $\boldsymbol{c}$). This follows easily from the fact that the gamma factors in

$$\Pi(\boldsymbol{c}), \quad \Pi(\mathfrak{a}_1\boldsymbol{c}), \quad \Pi(\mathfrak{a}_2\boldsymbol{c}), \quad \Pi(\mathfrak{b}\boldsymbol{c}), \quad \Pi(\mathfrak{h}\boldsymbol{c})$$

have exactly three arguments belonging to $\mathbb{Z} + \frac{1}{2}$ and two arguments belonging to $\mathbb{Z}$.

Another (very remarkable) description of the group $\mathfrak{G}$ by means of the double integrals (20) and their birational transformations can be found in the work [RV].

By [RV], when all elements in $\boldsymbol{c}$ are non-negative integers, one has $H(\boldsymbol{c}) \in \mathbb{Q}\zeta(2)+\mathbb{Q}$, where $\zeta(2) = \pi^2/6$. Moreover, in this case, $D_{m_1}D_{m_2}H(\boldsymbol{c}) \in \mathbb{Z}\zeta(2) + \mathbb{Z}$, where $m_1 \geqslant m_2$ are the two successive maxima of the set $\boldsymbol{c}$. This inclusion and the $\mathfrak{G}$-stability of the quantity $H(\boldsymbol{c})/\Pi(\boldsymbol{c})$ make it possible to deduce a nice irrationality measure for $\zeta(2)$ (for details, see [RV]).

Theorems 1–3 allow us to expect a similar inclusion

$$2^{2M+o(M)}D_{m_1}D_{m_2}H(\boldsymbol{c}) \in \mathbb{Z}G + \mathbb{Z} \tag{24}$$

if the set $\boldsymbol{c}$ is $\mathfrak{G}$-equivalent to a set satisfying (21); here M is the sum of two integers in $\boldsymbol{c}' = (c_{00}, c_{21}, c_{22}, c_{33}, c_{31})$ and $m_1 \geqslant m_2$ are the two successive maxima of the set $2\boldsymbol{c}$. Unfortunately, the inclusion (24) is beyond the reach of even the powerful group-structure approach to proving irrationality results developed in [RV] (see also [Zu1]).

4. Difference equations and irrationality. Since

$$\lim_{n\to\infty} D_{2n-1}^{1/n} = e^2$$

by the prime number theorem, Theorem 1 (supplemented with equation (1)) or Theorem 2 do not yield the irrationality of Catalan's constant. What is the connection between irrationality and Apéry-like difference equations? We would like to conclude this note by pointing out the following expectation.

A sequence $\{x_n\} = \{x_n\}_{n=0}^{\infty} \subset \mathbb{Q}$ is said to satisfy the geometric condition[1] if the least common denominator of the numbers $x_0, x_1, \dots, x_n$ grows at most geometrically as $n \to \infty$.

Given a second-order recursion

$$x_{n+1} + a(n)x_n + b(n)x_{n-1} = 0, \qquad \lim_{n\to\infty} a(n) = a_0 \in \mathbb{Q}, \quad \lim_{n\to\infty} b(n) = b_0 \in \mathbb{Q}, \tag{25}$$

[1]We should replace the standard term 'G-condition' by the phrase 'geometric condition' since the capital letter G is reserved for Catalan's constant here.

suppose that the characteristic polynomial $\lambda^2 + a_0\lambda + b_0$ has roots λ_1 and λ_2 satisfying $0 < |\lambda_1| < |\lambda_2|$. Perron's theorem (see, e.g., [Ge], Chapter V, Section 5) then guarantees the existence of two linearly independent solutions $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda_1, \qquad \lim_{n\to\infty} \frac{y_{n+1}}{y_n} = \lambda_2. \tag{26}$$

Conjecture. *In the above notation, suppose that both solutions $\{x_n\}$ and $\{y_n\}$ of the recursion* (25) *are rational and satisfy the geometric condition. Then λ_1 and λ_2 are rational numbers.*

This conjecture is trivially true in the case of constant coefficients $a(n) = a_0$ and $b(n) = b_0$ of the recursion (25); we leave this observation as an exercise to the reader.

In order to show how the irrationality of G follows from the above conjecture, we have only to mention that, if G is rational, the solutions $\{\tilde{u}_n\}$ and $\{\tilde{r}_n\} = \{\tilde{u}_n G - \tilde{v}_n\}$ to the recursion (13) are also rational numbers satisfying the geometric condition and form Perron's basis, while the roots $(11\pm5\sqrt{5})/2 = \big((1\pm\sqrt{5})/2\big)^5$ of the characteristic polynomial are clearly irrational.

The geometric condition cannot be removed from hypothesis of the conjecture[2]. Indeed, taking $\lambda = (11+5\sqrt{5})/2$ and $\lambda_1 = -1/\lambda$, $\lambda_2 = \lambda$, set

$$x_n = \frac{(-1)^n}{\lfloor \lambda^n \rfloor} \in \mathbb{Q}, \quad y_n = \lfloor \lambda^n \rfloor \in \mathbb{Z}, \qquad n = 0, 1, 2, \dots . \tag{27}$$

Then $x_n \sim \lambda_1^n$ and $y_n \sim \lambda_2^n$ as $n \to \infty$, hence relations (26) hold. In addition, the sequences (27) satisfy the recursion (25) with

$$\left.\begin{aligned} b(n) &= -\frac{\lfloor \lambda^{n-1} \rfloor}{\lfloor \lambda^{n+1} \rfloor} \cdot \frac{\lfloor \lambda^n \rfloor^2 + \lfloor \lambda^{n+1} \rfloor^2}{\lfloor \lambda^{n-1} \rfloor^2 + \lfloor \lambda^n \rfloor^2}, \\ a(n) &= \frac{\lfloor \lambda^n \rfloor}{\lfloor \lambda^{n-1} \rfloor} \cdot b(n) + \frac{\lfloor \lambda^n \rfloor}{\lfloor \lambda^{n+1} \rfloor}, \end{aligned}\right. \qquad n = 0, 1, 2, \dots .$$

References

[Ba] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Math. Tracts, vol. 32, Cambridge Univ. Press, Cambridge, 1935; 2nd reprinted edition, Stechert-Hafner, New York–London, 1964.

[Ge] A. O. Gel′fond, *Calculus of finite differences*, International Monographs on Advanced Mathematics and Physics, Hindustan Publishing Corp., Delhi, 1971, translated from the Russian.

[Ne] Yu. V. Nesterenko, *A few remarks on* $\zeta(3)$, Mat. Zametki [Math. Notes] **59** (1996), no. 6, 865–880.

[RV] G. Rhin and C. Viola, *On a permutation group related to* $\zeta(2)$, Acta Arith. **77** (1996), no. 1, 23–56.

[2]It is possible that the conjecture is true if we replace the geometric condition hypothesis by the assumption $a(n), b(n) \in \mathbb{Q}(n)$; however this new conjecture would not cover several known cases (for instance, the recursion corresponding to Nesterenko's continued fraction for $\zeta(3)$ in [Ne], Theorem 2).

[RZ] T. Rivoal and W. Zudilin, *Diophantine properties of numbers related to Catalan's constant*, Prépubl. de l'Institut de Math. de Jussieu, no. 315 (January 2002).

[Wh] F. J. W. Whipple, *A group of generalized hypergeometric series: relations between 120 allied series of the type* $F[a,b,c;d,e]$, Proc. London Math. Soc. (2) **23** (1925), 104–114.

[Ze] D. Zeilberger, *Computerized deconstruction*, Adv. Appl. Math. (2002) (to appear).

[Zu1] W. Zudilin, *Arithmetic of linear forms involving odd zeta values*, E-print `math.NT/0206176` (August 2001).

[Zu2] W. Zudilin, *An Apéry-like difference equation for Catalan's constant*, E-print `math.NT/0201024` (January 2002).

Moscow Lomonosov State University
Department of Mechanics and Mathematics
Vorobiovy Gory, GSP-2, Moscow 119992 RUSSIA
URL: `http://wain.mi.ras.ru/index.html`
E-mail address: `wadim@ips.ras.ru`