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On the Algebraic Structure of Functional Matrices of Special Form

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Abstract. Algebraic properties of functional matrices arising in the construction of graded Padé approximations are established. This construction plays an important role in the theory of transcendental numbers.

Key words: Padé approximation, first-order homogeneous linear differential equations, transcendental numbers.

In 1984, in the brief communication [1], the description of a new analytic construction was given, namely, *graded Padé approximations* were introduced. In the same note substantially new results obtained via the proposed approach were indicated. In fact, the construction suggested by Chudnovsky [1] was a further generalization of the known Siegel–Shidlovskii method in the theory of transcendental numbers. However, the proof of the results in [1] contained a gap, which was filled by the author of the present paper in [2] (where certain details related to the history of the problem can be found), and stronger results were obtained. Note that a variation of the method of graded Padé approximations leads to even stronger results in particular cases [3, 4].

The aim of the present paper is to indicate certain interesting algebraic and combinatorial laws arising in the construction of graded Padé approximations.

§1. Description of the construction

Consider m different systems of first-order homogeneous linear differential equations

$$\begin{gathered} \frac{d}{dz}y_{il}=\sum_{j=1}^{m_i}Q_{lj}^{(i)}y_{ij}\,, \qquad Q_{lj}^{(i)}=Q_{lj}^{(i)}(z)\in\mathbb{C}(z), \\ l,j=1,\dots,m_i, \quad m_i\ge 2, \qquad i=1,\dots,m, \quad m\ge 2. \end{gathered} \tag{1}$$

Denote by $T(z)\in\mathbb{C}[z]$ the least common denominator of the coefficients of systems (1). The collection of these systems is needed only when linear functional forms constructed below proliferate.

In the construction below, due to Chudnovsky, the approximating linear topological forms (*graded Padé approximations*) depending on a positive integral parameter N are described. Let us immediately introduce some notation. We set $\bar a=(\bar a_1,\dots,\bar a_m)$, where $\bar a_i=(a_{i1},\dots,a_{im_i})$, $i=1,\dots,m$, and $\bar\kappa=(\bar\kappa_1,\dots,\bar\kappa_m)$ is a multi-index, where $\bar\kappa_i=(\kappa_{i1},\dots,\kappa_{im_i})$, $i=1,\dots,m$, all components κ_{ij} of any multi-index are nonnegative, and if a sum contains a term with at least one component for which $\kappa_{ij}<0$, then this term is treated as missing (equal to zero). For brevity, in the formulas we write

$$\bar a^{\bar\kappa}=\prod_{\substack{i=1,\dots,m\\ j=1,\dots,m_i}}a_{ij}^{\kappa_{ij}}, \quad |\bar\kappa_i|=\sum_{j=1}^{m_i}\kappa_{ij}, \quad i=1,\dots,m.$$

Let us introduce the sets

$$\Omega_i=\Omega_i(N)=\bigl\{\bar\kappa:|\bar\kappa_l|=N-\delta_{il},l=1,\dots,m\bigr\}, \qquad \Omega=\Omega(N)=\bigcup_{i=1}^m\Omega_i,$$
$$\Theta=\Theta(N)=\bigl\{\bar s:|\bar s_l|=N,l=1,\dots,m\bigr\},$$

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where δ_{il} is the ordinary Kronecker delta. Moreover, we write

$$\omega_i = \operatorname{Card}\Omega_i, \qquad \omega = \operatorname{Card}\Omega = \sum_{i=1}^{m} \omega_i, \qquad \theta = \operatorname{Card}\Theta.$$

Let $f_{i1}(z), \dots, f_{im_i}(z)$, $i = 1, \dots, m$, be an arbitrary solution of the collection of systems (1). The desired linear forms can be expressed as follows:

$$R(z\,;\,\bar a) = \sum_{i=1}^{m} P_i(z\,;\,\bar a)\langle \bar a_i, \bar f_i(z)\rangle = \sum_{i=1}^{m} P_i(z\,;\,\bar a) \sum_{j=1}^{m_i} a_{ij} f_{ij}(z), \tag{2}$$

where the polynomials $P_i(z\,;\,\bar a)$ are homogeneous with respect to each of the components $\bar a_l$, $l = 1, \dots, m$, and have the form

$$P_i(z\,;\,\bar a) = \sum_{\bar\kappa \in \Omega_i} \bar a^{\bar\kappa} P_{\bar\kappa}(z), \qquad i = 1, \dots, m. \tag{3}$$

The functional linear form (2) can be represented as follows:

$$R(z\,;\,\bar a) = \sum_{\bar s \in \Theta} \bar a^{\bar s} R_{\bar s}(z), \qquad \text{where} \quad R_{\bar s}(z) = \sum_{i=1}^{m} \sum_{j=1}^{m_i} P_{\bar s - \bar e_{ij}}(z) f_{ij}(z), \quad \bar s \in \Theta, \tag{4}$$

and $\bar e_{ij}$ is the multi-index with one on the ijth place and with zeros on the other places.

Furthermore, let $M_1, M_2, \dots, M_m$ be arbitrary positive integers and let $\varepsilon > 0$. Without loss of generality, we may assume that $M_1 \ge M_2 \ge \cdots \ge M_m$ (because we can renumerate the systems in collection (1) if necessary). Moreover, we set $M = M_1$.

For the case in which the functions $f_{i1}(z), \dots, f_{im_i}(z)$, $i = 1, \dots, m$, are linearly independent over $\mathbb{C}(z)$ and the coefficients of their Taylor series satisfy certain arithmetic conditions (for instance, are E-functions or G-functions), we can apply the Siegel lemma and construct a nontrivial form (2) such that

$$\begin{gathered} \deg_z P_i(z\,;\,\bar a) < M, \quad \operatorname*{ord}_{z=0} P_i(z\,;\,\bar a) \ge M - M_i, \qquad i = 1, \dots, m, \\ \operatorname*{ord}_{z=0} R(z\,;\,\bar a) \ge K = \left[\frac{\omega_1 M_1 + \omega_2 M_2 + \cdots + \omega_m M_m - \varepsilon M}{\theta} \right], \end{gathered} \tag{5}$$

and whose polynomials $P_{\bar\kappa}(z)$, $\bar\kappa \in \Omega_i$, $i = 1, \dots, m$, have the appropriate arithmetic properties. Now we consider the differential operator

$$D = \frac{\partial}{\partial z} - \sum_{i=1}^{m} \left(\sum_{j=1}^{m_i} \left(\sum_{l=1}^{m_i} Q_{lj}^{(i)}(z) a_{il} \right) \frac{\partial}{\partial a_{ij}} \right),$$

related to the system of homogeneous linear differential equations adjoint to system (1). This operator has the following (easily verified) property:

$$D \sum_{j=1}^{m_i} a_{ij} f_{ij}(z) \equiv 0, \qquad i = 1, \dots, m.$$

Now we write

$$R^{[n]}(z\,;\,\bar a) = \bigl(T(z) \cdot D\bigr)^n R(z\,;\,\bar a), \qquad n = 0, 1, 2, \dots,$$

and denote by $R_{\bar s}^{[n]}(z)$ and $P_{\bar\kappa}^{[n]}(z)$ the forms and the polynomials, respectively, that appear in these functions. Consider the functional determinant

$$\Delta(z) = \det\bigl(P_{\bar\kappa}^{[n]}(z)\bigr)_{n=0,1,\dots,\omega-1\,;\;\bar\kappa\in\Omega}. \tag{6}$$

We can readily see that

$$\deg_z P_i^{[n]}(z\,;\,\bar a) < M + tn, \qquad i = 1, \dots, m, \tag{7}$$

$$\operatorname*{ord}_{z=0} P_i^{[n]}(z\,;\,\bar a) \ge M - M_i - n, \qquad i = 1, \dots, m, \tag{8}$$

$$\operatorname*{ord}_{z=0} R^{[n]}(z\,;\,\bar a) \ge K - n, \tag{9}$$

where

$$t = \max\Big\{\deg T, \max_{i,l,j}\{\deg TQ_{lj}^{(i)}\}\Big\}.$$

Applying the Siegel normality condition (however, not to the original systems (1), but to the adjoint systems), Chudnovsky [1] showed that the determinant (6) is nondegenerate. In the same note it was announced that an argument similar to that in the Siegel approach implies the estimate

$$\deg \Delta(z) - \operatorname*{ord}_{z=0} \Delta(z) = o(M), \tag{10}$$

which is an important property used in applications to numerical inequalities. Our objective is just to describe this analogy and to obtain the best possible estimate for the quantity on the left-hand side of (10).

Let us order the elements of the set Ω as follows: an element $\bar\kappa \in \Omega$ is said to be *lexicographically less than* $\bar\kappa' \in \Omega$ (notation: $\bar\kappa \prec \bar\kappa'$) if (for $\bar\kappa \in \Omega_i$ and $\bar\kappa' \in \Omega_{i'}$) we have either $i < i'$ or both $i = i'$ and $\kappa_{11} = \kappa'_{11}$, $\kappa_{12} = \kappa'_{12}$, ..., $\kappa_{1m_1} = \kappa'_{1m_1}$; ...; $\kappa_{l1} = \kappa'_{l1}$, ..., $\kappa_{l,j-1} = \kappa'_{l,j-1}$, and $\kappa_{lj} < \kappa'_{lj}$ for some l, $1 \le l \le m$, and j, $1 \le j \le m_l$. We introduce a similar ordering for the elements of the set Θ and can now express the elements of the row $\big(R_{\bar s}(z)\big)_{\bar s\in\Theta}$ via the elements of the row $\big(P_{\bar\kappa}(z)\big)_{\bar\kappa\in\Omega}$:

$$\begin{aligned}\sum_{\bar s\in\Theta} \bar a^{\bar s} R_{\bar s}(z) = R(z\,;\,\bar a) &= \sum_{i=1}^m P_i(z\,;\,\bar a) \sum_{j=1}^{m_i} a_{ij} f_{ij}(z) = \sum_{i=1}^m \sum_{\bar\kappa\in\Omega_i} P_{\bar\kappa}(z) \sum_{j=1}^{m_i} \bar a^{\bar\kappa+\bar e_{ij}} f_{ij}(z) \\ &= \sum_{i=1}^m \sum_{\bar\kappa\in\Omega_i} P_{\bar\kappa}(z) \sum_{j=1}^{m_i} \sum_{\bar s\in\Theta} \delta_{\bar\kappa+\bar e_{ij},\bar s} \bar a^{\bar s} f_{ij}(z) \sum_{\bar s\in\Theta} \bar a^{\bar s} \sum_{i=1}^m \sum_{\bar\kappa\in\Omega_i} P_{\bar\kappa}(z) \sum_{j=1}^{m_i} \delta_{\bar\kappa+\bar e_{ij},\bar s} f_{ij}(z),\end{aligned}$$

and hence, after equating the terms at $\bar a^{\bar s}$, $\bar s \in \Theta$, we obtain

$$\big(R_{\bar s}(z)\big)_{\bar s\in\Theta} = \big(P_{\bar\kappa}(z)\big)_{\bar\kappa\in\Omega} \cdot \left(\sum_{j=1}^{m_i} \delta_{\bar\kappa+\bar e_{ij},\bar s} f_{ij}(z)\right)_{i=1,\dots,m,\bar\kappa\in\Omega_i\,;\,\bar s\in\Theta}. \tag{11}$$

Here $\delta_{\bar s',\bar s}$, $\bar s', \bar s \in \Theta$, stands for the "generalized" Kronecker delta of the set Θ, i.e.,

$$\delta_{\bar s',\bar s} = \begin{cases} 1 & \text{for } \bar s' = \bar s, \\ 0 & \text{otherwise.} \end{cases}$$

In what follows, the matrix

$$\left(\sum_{j=1}^{m_i} \delta_{\bar\kappa+\bar e_{ij},\bar s} f_{ij}(z)\right)_{i=1,\dots,m,\bar\kappa\in\Omega_i\,;\,\bar s\in\Theta} \tag{12}$$

is called a *transition matrix*. In it, the elements of the sets Ω and Θ are lexicographically ordered, the elements of Ω index the rows, and the elements of Θ index the columns. By formula (11), this matrix is the transition matrix from the row $\big(P_{\bar\kappa}(z)\big)_{\bar\kappa\in\Omega}$ to the row $\big(R_{\bar s}(z)\big)_{\bar s\in\Theta}$.

§2. Algebraic relationships and the rank of the transition matrix

Lemma 1 (see [5, Chap. 2, §7]). *For any* $m, N \in \mathbb{N}$, *the following relation holds*:

$$\operatorname{Card}\bigl\{n_i \in \mathbb{Z},\ n_i \geq 0,\ i = 1, \dots, m : n_1 + n_2 + \dots + n_m = N\bigr\} = \binom{N+m-1}{N}.$$

Hence,

$$\omega_i = \binom{N+m_i-2}{N-1} \prod_{\substack{l=1\\ l\neq i}}^{m} \binom{N+m_l-1}{N}, \qquad i = 1, \dots, m, \qquad \theta = \prod_{l=1}^{m} \binom{N+m_l-1}{N}.$$

Everywhere below we assume that the functions $f_{ij}(z)$, $i = 1, \dots, m$, $j = 1, \dots, m_i$, are algebraically independent.

Denote by $\{Y_{\bar\kappa}\}_{\bar\kappa\in\Omega}$ the rows of the transition matrix (12). Let us indicate a maximal linearly independent system of these rows that contains as many rows with indices from Ω_1 as possible, followed by as many rows with indices from Ω_2 as possible, etc., up to Ω_m. To this end, we consider the set

$$\Omega^* = \bigcup_{i=1}^{m} \Omega_i^*, \qquad \Omega_i^* = \bigl\{\bar\kappa \in \Omega : \kappa_{11} = \kappa_{21} = \dots = \kappa_{i-1,1} = 0\bigr\} \subset \Omega_i, \quad i = 1, \dots, m,$$

and show that if a row $\{Y_{\bar\kappa'}\}$ has the index $\bar\kappa' \notin \Omega^*$, then it can be linearly expressed via rows $\{Y_{\bar\kappa}\}_{\bar\kappa\in\Omega^*}$ with indices less than $\bar\kappa'$. To this end, we need the following identity.

Lemma 2. *Let* $\bar\kappa \in \Omega_i \setminus \Omega_i^*$, *that is, let there be an index* $i' = i'(i, \bar\kappa) < i$ *such that* $\kappa_{i'1} > 0$. *Then*

$$Y_{\bar\kappa} = \frac{1}{f_{i'1}(z)} \left(\sum_{j=1}^{m_i} f_{ij}(z) Y_{\bar\kappa - \bar e_{i'1} + \bar e_{ij}} - \sum_{j=2}^{m_{i'}} f_{i'j}(z) Y_{\bar\kappa - \bar e_{i'1} + \bar e_{i'j}} \right). \tag{13}$$

Proof. Indeed, we have

$$\begin{aligned}
\sum_{l=1}^{m_{i'}} f_{i'l}(z) Y_{\bar\kappa - \bar e_{i'1} + \bar e_{i'l}} &= \sum_{l=1}^{m_{i'}} f_{i'l}(z) \left(\sum_{j=1}^{m_i} \delta_{\bar\kappa - \bar e_{i'1} + \bar e_{i'l} + \bar e_{ij}, \bar s} f_{ij}(z) \right)_{\bar s\in\Omega} \\
&= \left(\sum_{l=1}^{m_{i'}} \sum_{j=1}^{m_i} \delta_{\bar\kappa - \bar e_{i'1} + \bar e_{i'l} + \bar e_{ij}, \bar s} f_{ij}(z) f_{i'l}(z) \right)_{\bar s\in\Omega} = \left(\sum_{j=1}^{m_i} \sum_{l=1}^{m_{i'}} \delta_{\bar\kappa - \bar e_{i'1} + \bar e_{ij} + \bar e_{i'l}, \bar s} f_{i'l}(z) f_{ij}(z) \right)_{\bar s\in\Omega} \\
&= \sum_{j=1}^{m_i} f_{ij}(z) \left(\sum_{l=1}^{m_{i'}} \delta_{\bar\kappa - \bar e_{i'1} + \bar e_{ij} + \bar e_{i'l}, \bar s} f_{i'l}(z) \right)_{\bar s\in\Omega} = \sum_{j=1}^{m_i} f_{ij}(z) Y_{\bar\kappa - \bar e_{i'1} + \bar e_{ij}},
\end{aligned}$$

and this implies (13). □

Now it remains to note that the right-hand side of formula (13) contains only rows whose indices are less than $\bar\kappa$ with respect to the lexicographic order. Indeed, we have $\bar\kappa - \bar e_{i'1} + \bar e_{ij} \prec \bar\kappa$ because $\bar\kappa - \bar e_{i'1} + \bar e_{ij} \in \Omega_{i'}$, $\bar\kappa \in \Omega_i$, and $i' < i$, $j = 1, \dots, m_i$, and $\bar\kappa - \bar e_{i'1} + \bar e_{i'j} \prec \bar\kappa$ because $\bar e_{i'j} \prec \bar e_{i'1}$, $j = 2, \dots, m_{i'}$.

Thus, if $\bar\kappa' \notin \Omega^*$, then the row $Y_{\bar\kappa'}$ is a linear combination of the rows $Y_{\bar\kappa}$, $\bar\kappa \in \Omega^*$, $\bar\kappa \prec \bar\kappa'$.

Now let us show that the rows $\{Y_{\bar\kappa}\}_{\bar\kappa\in\Omega^*}$ are linearly independent. Assume the contrary: suppose a linear combination of the $Y_{\bar\kappa}$ vanishes, i.e.,

$$\sum_{\bar\kappa\in\Omega^*} \gamma_{\bar\kappa} Y_{\bar\kappa} \equiv 0. \tag{14}$$

Since the functions f_{ij}, $i = 1, \dots, m$, $j = 1, \dots, m_i$, are algebraically independent, it follows that relation (14) remains valid if we set $f_{i1} = 1$ and $f_{ij} = 0$ for $i = 1, \dots, m$ and $j = 2, \dots, m_i$ in this relation. After this substitution, relation (14) becomes

$$Y = \sum_{i=1} \sum_{\bar{\kappa} \in \Omega_i^*} \gamma_{\bar{\kappa}} \left(\delta_{\bar{\kappa} + \bar{e}_{i1}, \bar{s}} \right)_{\bar{s} \in \Omega} \equiv 0. \tag{15}$$

Assume that for some $\bar{\kappa}' \in \Omega_{i'}^*$ we have $\gamma_{\bar{\kappa}'} \neq 0$. Consider the component of the row Y with the index $\bar{\kappa}' + \bar{e}_{i'1}$. It is equal to

$$\sum_{i=1}^{m} \sum_{\bar{\kappa} \in \Omega_i^*} \gamma_{\bar{\kappa}} \delta_{\bar{\kappa} + \bar{e}_{i1}, \bar{\kappa}' + \bar{e}_{i'1}} = \gamma_{\bar{\kappa}'} \neq 0,$$

because the relations $\bar{\kappa} + \bar{e}_{i1} = \bar{\kappa}' + \bar{e}_{i'1}$, where $\bar{\kappa} \in \Omega^*$, are possible for $\bar{\kappa} = \bar{\kappa}'$ only. Indeed, if this is not the case, then either $i < i'$ or $i' < i$. In the first case we have $\kappa'_{i1} = \kappa_{i1} + 1 > 0$, and this contradicts the relation $\bar{\kappa}' \in \Omega_{i'}^*$, and in the other case we have $\kappa_{i'1} = \kappa'_{i'1} + 1 > 0$, and this contradicts the relation $\bar{\kappa} \in \Omega_i^*$. Thus, we have a nonzero component of the row Y, and this contradicts relation (15). Thus, the linear independence of the system of rows $\{Y_{\bar{\kappa}}\}_{\bar{\kappa} \in \Omega^*}$ is proved.

Let us summarize the above discussion in the form of the following theorem.

THEOREM 1. *The rows of the matrix* (12) *with indices from the set* Ω^* *form a basis in the system of rows of this matrix.*

COROLLARY. *The rank of the transition matrix* (12) *is equal to*

$$\prod_{l=1}^{m} \binom{N + m_l - 1}{N} - \prod_{l=1}^{m} \binom{N + m_l - 2}{N}.$$

PROOF. It follows from Theorem 1 that the rank of matrix (12) is exactly equal to

$$\omega^* = \operatorname{Card} \Omega^* = \sum_{i=1}^{m} \omega_i^*, \qquad \omega_i^* = \operatorname{Card} \Omega_i^*, \qquad i = 1, \dots, m.$$

By Lemma 1, we have

$$\begin{aligned} \omega_i^* &= \operatorname{Card}\bigl\{\bar{\kappa}_1 : |\bar{\kappa}_1| = N, \kappa_{11} = 0\bigr\} \times \dots \times \operatorname{Card}\bigl\{\bar{\kappa}_{i-1} : |\bar{\kappa}_{i-1}| = N, \kappa_{i-1,1} = 0\bigr\} \\ &\quad \times \operatorname{Card}\bigl\{\bar{\kappa}_i : |\bar{\kappa}_i| = N - 1\bigr\} \times \operatorname{Card}\bigl\{\bar{\kappa}_{i+1} : |\bar{\kappa}_{i+1}| = N\bigr\} \times \dots \times \operatorname{Card}\bigl\{\bar{\kappa}_m : |\bar{\kappa}_m| = N\bigr\} \\ &= \prod_{l=1}^{i-1} \binom{N + m_l - 2}{N} \cdot \binom{N + m_i - 2}{N - 1} \cdot \prod_{l=i+1}^{m} \binom{N + m_l - 1}{N}, \qquad i = 1, \dots, m, \end{aligned}$$

and hence,

$$\begin{aligned} \omega^* &= \sum_{i=1}^{m} \omega_i^* = \sum_{i=1}^{m} \binom{N + m_i - 2}{N - 1} \cdot \prod_{l=1}^{i-1} \binom{N + m_l - 2}{N} \cdot \prod_{l=i+1}^{m} \binom{N + m_l - 1}{N} \\ &= \prod_{l=1}^{m} \binom{N + m_l - 1}{N} \cdot \sum_{i=1}^{m} \frac{\binom{N + m_i - 2}{N - 1}}{\binom{N + m_i - 1}{N}} \prod_{l=1}^{i-1} \frac{\binom{N + m_l - 2}{N}}{\binom{N + m_l - 1}{N}} \\ &= \theta \sum_{i=1}^{m} \left(1 - \frac{m_i - 1}{N + m_i - 1} \right) \prod_{l=1}^{i-1} \frac{m_l - 1}{N + m_l - 1} \end{aligned}$$

$$= \theta\left(1-\prod_{l=1}^{m}\frac{m_l-1}{N+m_l-1}\right)=\prod_{l=1}^{m}\binom{N+m_l-1}{N}-\prod_{l=1}^{m}\binom{N+m_l-2}{N}.$$

The last number gives the rank of matrix (12). $\square$

In conclusion of this section, we also indicate a basis of the system of columns of matrix (12). Consider the set of columns with indices from the set

$$\Theta^* = \{\bar{s}\in\Theta : s_{11}+\cdots+s_{m1}>0\}.$$

THEOREM 2. *The columns of the matrix* (12) *with indices from the set* Θ^* *form a basis in the set of columns of this matrix.*

PROOF. The complement $\Theta\setminus\Theta^* = \big\{\bar{s}\in\Theta : s_{11}=s_{21}=\cdots=s_{m1}=0\big\}$ has the cardinality

$$\operatorname{Card}\Theta\setminus\Theta^* = \prod_{i=1}^{m}\operatorname{Card}\{\bar{s}_i : |\bar{s}_i|=N,\, s_{i1}=0\} = \prod_{i=1}^{m}\binom{N+m_i-2}{N}.$$

Therefore,

$$\theta^* = \operatorname{Card}\Theta^* = \theta - \prod_{i=1}^{m}\binom{N+m_i-2}{N} = \omega^*,$$

and this is exactly the rank of the transition matrix. Denote by $\{X_{\bar{s}}\}_{\bar{s}\in\Theta}$ the columns of matrix (12). We must show that all columns with indices from $\Theta\setminus\Theta^*$ can be expressed via the system $\{X_{\bar{s}}\}_{\bar{s}\in\Theta^*}$, and this means that the columns from the latter set form a basis. We need the relation of "being higher" on the set of multi-indices (in general, not only taken from the set Θ, but of dimension $(m_1, m_2, \dots, m_m)$ and with nonnegative components). We write $\bar{r}\le\bar{s}$, i.e., $\bar{r}$ is not higher than $\bar{s}$, if $r_{ij}\le s_{ij}$ for all $i=1,\dots,m$, $j=1,\dots,m_i$, and $\bar{r}<\bar{s}$, i.e., $\bar{r}$ is lower than $\bar{s}$, if $\bar{r}\le\bar{s}$ and $\bar{r}\ne\bar{s}$. We only note that the relation of being higher is not an order. For any $\bar{s}\in\Theta\setminus\Theta^*$, the following identity holds, which is presented here without proof:

$$\sum_{\bar{r}\le\bar{s}}\left(\prod_{i=1}^{m}\left((-1)^{|\bar{r}_i|}\frac{(|\bar{r}_i|)!}{r_{i2}!\cdots r_{im_i}!}f_{i1}^{-|\bar{r}_i|}(z)f_{i2}^{r_{i2}}(z)\cdots f_{im_i}^{r_{im_i}}(z)\right)\right)X_{\bar{s}-\bar{r}+|\bar{r}_1|\bar{e}_{11}+|\bar{r}_2|\bar{e}_{21}+\cdots+|\bar{r}_m|\bar{e}_{m1}}\equiv 0. \tag{16}$$

In this relation we can carry over the term from the sum for the zero value of $\bar{r}$ to the right-hand side and obtain the required expression for the column $X_{\bar{s}}$, $\bar{s}\in\Theta\setminus\Theta^*$, via the columns $\{X_{\bar{s}}\}_{\bar{s}\in\Theta^*}$. $\square$

§3. Estimate for the difference between the degree and the order of a zero of a functional determinant

We first note that by applying the above-mentioned Siegel lemma, we can make the order of zero at the point $z=0$ of the θ^* linear functional forms (4) $R_{\bar{s}}(z)$, $\bar{s}\in\Theta^*$, greater than or equal to

$$K^* = \left[\frac{\omega_1M_1+\omega_2M_2+\cdots+\omega_mM_m-\varepsilon M}{\theta^*}\right].$$

Applying relations (11) and (16), we obtain an expression for the linear forms $R_{\bar{s}}(z)$, $\bar{s}\in\Theta\setminus\Theta^*$, in terms of the forms already constructed:

$$\sum_{\bar{r}\le\bar{s}}\left(\prod_{i=1}^{m}\left((-1)^{|\bar{r}_i|}\frac{(|\bar{r}_i|)!}{r_{i2}!\cdots r_{im_i}!}f_{i1}^{-|\bar{r}_i|}(z)f_{i2}^{r_{i2}}(z)\cdots f_{im_i}^{r_{im_i}}(z)\right)\right)R_{\bar{s}-\bar{r}+|\bar{r}_1|\bar{e}_{11}+|\bar{r}_2|\bar{e}_{21}+\cdots+|\bar{r}_m|\bar{e}_{m1}}(z)\equiv 0.$$

By this relation we have

$$\operatorname*{ord}_{z=0} R_{\bar{s}}(z) \geq K = K^* - N \max_{1\leq i\leq m} \Big\{ \operatorname*{ord}_{z=0} f_{i1}(z) \Big\}, \qquad \bar{s} \in \Theta \setminus \Theta^*.$$

The new value of K significantly exceeds the older one in condition (5) for sufficiently large N and M. Below it is the new value that we use.

To obtain the best possible estimate of the order of the zero at the point $z = 0$ of the determinant (6), we must replace as many columns of this determinant (with indices from a set $\Omega' \subset \Omega$) as possible, by means of a nondegenerate linear transformation, by as many columns of the linear forms $R_{\bar{s}}^{[n]}(z)$, $n = 0, 1, \dots, \omega - 1$, $\bar{s} \in \Theta$, as possible. Here it is first required to replace the columns of the polynomials whose order of the zero at the point $z = 0$ is smaller. As an example of such a set Ω', we can take the set Ω^* because of what was said in the preceding section. Under the corresponding linear transformation, the columns with indices from Ω_1^*, ..., Ω_m^* are replaced by linear functional forms for which estimate (9) holds. Under this transformation, the order of the zero of the determinant increases (as compared to the order of the zero of $\Delta(z)$) by $O(\omega^*)$, and relations (8) and (9) imply that it is at most

$$\theta^* K + \sum_{i=1}^{m} (\omega_i - \omega_i^*)(M - M_i) - O(\omega^2) \geq \omega M - \varepsilon M - \sum_{i=2}^{m} \omega_i^* (M - M_i) - O(\omega^2),$$

whence

$$\operatorname*{ord}_{z=0} \Delta(z) \geq \omega M - \varepsilon M - \sum_{i=2}^{m} \omega_i^* (M - M_i) - O(\omega^2).$$

Moreover, by (7) we have the trivial estimate $\deg \Delta(z) \leq \omega M + O(\omega^2)$. Hence,

$$\deg \Delta(z) - \operatorname*{ord}_{z=0} \Delta(z) \leq \varepsilon M + \sum_{i=2}^{m} \omega_i^* (M - M_i) + O(\omega^2).$$

The quantity on the right-hand side of the last estimate is not $o(M)$ because of the second summand. In fact, in the Siegel approach, relation (10) gives an upper estimate of the order of the zero of the determinant $\Delta(z)$ at a rational point α that differs from zero and from the singularities of system (1). It is precisely this fact that is used in numerical applications. In [6], Galochkin proposed a new approach in principle to estimating the value $\operatorname{ord}_{z=\alpha} \Delta(z)$ that makes use of arithmetical (and not algebraic, as above) properties of the constructed polynomials (3). Here relation (10) becomes unnecessary. The realization of the graded Padé approximations with the help of Galochkin's result [6] is described in detail in [2].

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