THETA CONSTANTS AND DIFFERENTIAL EQUATIONS

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The proof of well-known (since 1996) Nesterenko's Theorem [1] leans on three important points:

(i) functions

$$P(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$
$$Q(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \qquad R(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \tag{1}$$

where $\sigma_k(n) = \sum_{d|n} d^k$, are algebraically independent over $\mathbb{C}(q)$ (corollary from Mahler's Theorem, see [2]);

(ii) functions (1) satisfy the system of differential equations

$$\delta P = \frac{1}{12}(P^2 - Q), \quad \delta Q = \frac{1}{3}(PQ - R), \quad \delta R = \frac{1}{2}(PR - Q^2),$$

where $\delta = q \frac{d}{dq}$ (Ramanujan's Theorem, see [3]);

(iii) functions (1) are holomorphic in the unit circle $|q| < 1$ and have integral coefficients of polynomial growth in their Taylor expansions.

Functions (1) are algebraic over the field $\mathcal{K}$ which is the algebraic extension of $\mathbb{C}(q)$ generated by (*one-dimensional*) *theta constants*

$$\vartheta_2 = \vartheta_2(q) = 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2} = 2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \cdots,$$
$$\vartheta_3 = \vartheta_3(q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots, \tag{2}$$
$$\vartheta_4 = \vartheta_4(q) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} = 1 - 2q + 2q^4 - 2q^9 + \cdots$$

and all their δ-derivatives (see, e.g., [4], Chapter 6). Note that the variable q is connected with a modular parameter τ by the rule $q = e^{\pi i \tau}$, $\operatorname{Im} \tau > 0$ (see [5],

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Chapter 21) and this assignment transforms Taylor expansions to Fourier ones. Jacobi's third order differential equation for functions (2) is too complicated for applications, but there is another "basis" of the field $\mathcal{K}$, namely logarithmic δ-derivatives of theta constants

$$\psi_2=\frac{\delta\vartheta_2}{\vartheta_2},\qquad \psi_3=\frac{\delta\vartheta_3}{\vartheta_3},\qquad \psi_4=\frac{\delta\vartheta_4}{\vartheta_4},\tag{3}$$

and another form of "three important points":

(i) functions (3) are algebraically independent over $\mathbb{C}(q)$;

(ii) functions (3) satisfy the system of differential equations

$$\delta(\psi_2+\psi_3)=4\psi_2\psi_3,\quad \delta(\psi_3+\psi_4)=4\psi_3\psi_4,\quad \delta(\psi_4+\psi_2)=4\psi_4\psi_2\tag{4}$$

(Halpen's Theorem proved before Ramanujan, see [6]);

(iii) functions (3) are holomorphic in the unit circle $|q|<1$ and have integral coefficients (except $\psi_2(0)=1/4$) of polynomial growth in their Taylor expansions.

Moreover, functions (3) are connected with theta constants by formulas

$$\vartheta_2^4=\frac14(\psi_3-\psi_4),\quad \vartheta_3^4=\frac14(\psi_2-\psi_4),\quad \vartheta_4^4=\frac14(\psi_2-\psi_3);\tag{5}$$

(see, in slightly different notation, [7], Chapter 11). Recently Yu. Nesterenko [8] proved an equivalent to his theorem statement in terms of functions (3) and discovered this way easier than previous one because of known information on the structure of solutions of Halphen's system (4).

My own interest to theta constants and their differential properties is initiated by Nesterenko's Theorem. In this talk I state some multi-dimensional analogues of (ii)–(iii) for generalized Halphen's functions.

We fix a *dimension* (or a *genus*) g and define *g-dimensional theta function* (*with characteristic* $\boldsymbol{a}=(\boldsymbol{a}',\boldsymbol{a}'')\in\mathbb{Z}^{2g}$) by the series

$$\vartheta_{\boldsymbol{a}}(\boldsymbol{z},\mathrm{T})=\sum_{\boldsymbol{n}\in\mathbb{Z}^g}\exp\bigl(\pi i\bigl(\boldsymbol{n}+\tfrac12\boldsymbol{a}'\bigr)^t\mathrm{T}\bigl(\boldsymbol{n}+\tfrac12\boldsymbol{a}'\bigr)+2i\bigl(\boldsymbol{n}+\tfrac12\boldsymbol{a}'\bigr)^t\bigl(\boldsymbol{z}+\tfrac{\pi}{2}\boldsymbol{a}''\bigr)\bigr)\tag{6}$$

where $\boldsymbol{z}\in\mathbb{C}^g$ and T is a symmetric complex matrix with a positive imaginary part (the space $\mathfrak{H}_g$ of dimension $g(g+1)/2$ of such matrices is an open set in $\mathbb{C}^{g\times g}$ and is called *Siegel upper half-space*). We write vectors as columns and denote the transposition by index t; this means that $\boldsymbol{n}^t\boldsymbol{z}$ is the scalar product of vectors $\boldsymbol{n}$ and $\boldsymbol{z}$. The series (6) converges absolutely and uniformly with respect to $\boldsymbol{z}$ and T on any compact subset of $\mathbb{C}^g\times\mathfrak{H}_g$; therefore it determines the holomorphic function on $\mathbb{C}^g\times\mathfrak{H}_g$ (see [9]).

The quasi-periodicity of (6) with respect to the lattice $\pi\mathbb{Z}^g+\pi\mathrm{T}\mathbb{Z}^g\subset\mathbb{C}^g$ allows us to consider the only functions with *reduced* characteristics $\boldsymbol{a}\in\mathbb{Z}^{2g}/2\mathbb{Z}^{2g}$. Usually we omit the parameter T writing theta functions and define *theta constants* as the values of even theta functions (6) at the point $\boldsymbol{z}=\boldsymbol{0}$. Since

$$\vartheta_{\boldsymbol{a}}(-\boldsymbol{z})=(-1)^{|\boldsymbol{a}|}\vartheta_{\boldsymbol{a}}(\boldsymbol{z}),\qquad |\boldsymbol{a}|=\boldsymbol{a}'^t\boldsymbol{a}'',\quad \boldsymbol{a}=(\boldsymbol{a}',\boldsymbol{a}'')\in\mathbb{Z}^{2g}$$

(see, e.g., [9]), we see that the function $\vartheta_{\boldsymbol{a}}(\boldsymbol{z})$ is even if and only if the number $|\boldsymbol{a}|$ is even. Hence all non-trivial theta constants are assigned to *even* characteristics from the set $\mathfrak{K}_0 = \{\boldsymbol{a} \in \mathbb{Z}^{2g}/2\mathbb{Z}^{2g} : |\boldsymbol{a}| \equiv 0 \pmod 2\}$, $\#\mathfrak{K}_0 = 2^{g-1}(2^g+1)$.

Consider differential operators

$$\delta_{jk} = \frac{1}{\pi i}\frac{\partial}{\partial \tau_{jk}}, \qquad j,k = 1,\dots,g,$$

and all possible "logarithmic" derivatives of theta constants:

$$\psi_{\boldsymbol{a},jk} = \psi_{\boldsymbol{a},jk}(\mathrm{T}) = \frac{\delta_{jk}\vartheta_{\boldsymbol{a}}}{\vartheta_{\boldsymbol{a}}}, \qquad \boldsymbol{a} \in \mathfrak{K}_0, \quad j,k = 1,\dots,g. \tag{7}$$

From the definition given above, we get the almost obvious result on arithmetical properties of functions (7).

Theorem 1. *Functions* (7) *have the following Fourier expansions*:

$$\psi_{\boldsymbol{a},jk}(\mathrm{T}) = \psi^{(0)}_{\boldsymbol{a},jk} + \sum_{\mathrm{N}} \psi^{(\mathrm{N})}_{\boldsymbol{a},jk} \cdot \exp(\pi i \cdot \mathrm{tr}(\mathrm{NT})), \tag{8}$$

where tr *is a matrix trace and the summation in* (8) *is over non-negative symmetric matrices with integral elements*,

$$\psi^{(0)}_{\boldsymbol{a},jk} = \begin{cases} 0 & \textit{if } a'_j a'_k = 0, \\ \dfrac{1}{4} & \textit{if } j = k \textit{ and } a'_j = 1, \\ \dfrac{1}{2}\dfrac{e^{\pi i \tau_{jk}/2} - e^{-\pi i \tau_{jk}/2}}{e^{\pi i \tau_{jk}/2} + e^{-\pi i \tau_{jk}/2}} & \textit{if } j \ne k \textit{ and } a'_j = a'_k = 1, \end{cases}$$

and $\psi^{(\mathrm{N})}_{\boldsymbol{a},jk} \in \mathbb{Z}$.

Theorem 1 is the point (iii) in our program. To claim results on differential equations, I need a certain compact notation for functions (7). Put

$$\psi_{\boldsymbol{a}} = \boldsymbol{z}^t \Psi_{\boldsymbol{a}} \boldsymbol{z}, \quad \Psi_{\boldsymbol{a}} = (\psi_{\boldsymbol{a},jk})_{j,k=1,\dots,g}, \qquad \boldsymbol{a} \in \mathfrak{K}_0, \tag{9}$$

$$\delta = \boldsymbol{z}^t \Delta \boldsymbol{z}, \qquad \Delta = (\delta_{jk})_{j,k=1,\dots,g}. \tag{10}$$

Theorem 2. *Functions* (9) *satisfy the system of differential equations*

$$\vartheta^4_{\boldsymbol{a}} \cdot \delta\psi_{\boldsymbol{a}} = \frac{1}{2^{g-2}} \sum_{\boldsymbol{b} \in \mathfrak{K}_0} (-1)^{|\boldsymbol{a},\boldsymbol{b}|} \vartheta^4_{\boldsymbol{b}} \cdot \psi^2_{\boldsymbol{b}} - 2\vartheta^4_{\boldsymbol{a}} \cdot \psi^2_{\boldsymbol{a}}, \qquad \boldsymbol{a} \in \mathfrak{K}_0, \tag{11}$$

where $|\boldsymbol{a},\boldsymbol{b}| = \boldsymbol{a}'^t\boldsymbol{b}'' - \boldsymbol{b}'^t\boldsymbol{a}'' \equiv |\boldsymbol{b},\boldsymbol{a}| \pmod 2$.

Notice that non-compact form of the system (11) is

$$\vartheta^4_{\boldsymbol{a}} \cdot \delta_{ls}\psi_{\boldsymbol{a},jk} = \frac{1}{2^{g-2}N_{jkls}} \sum_{\boldsymbol{b} \in \mathfrak{K}_0} (-1)^{|\boldsymbol{a},\boldsymbol{b}|} \vartheta^4_{\boldsymbol{b}} \cdot \sum_{j_1,k_1,l_1,s_1}^{*} \psi_{\boldsymbol{b},j_1k_1}\psi_{\boldsymbol{b},l_1s_1} - \vartheta^4_{\boldsymbol{a}} \cdot \psi_{\boldsymbol{a},ls}\psi_{\boldsymbol{a},jk}$$

$$- \frac{1}{N_{jkls}} \vartheta^4_{\boldsymbol{a}} \cdot \sum_{j_1,k_1,l_1,s_1}^{*} \psi_{\boldsymbol{a},j_1k_1}\psi_{\boldsymbol{a},l_1s_1}, \qquad \boldsymbol{a} \in \mathfrak{K}_0, \qquad j,k,l,s = 1,\dots,g,$$

where denote by $\sum^*$ the summation over all different perturbations of the set $\{j,k,l,s\}$ and by N_{jkls} the number of such perturbations.

It follows from Theorem 2 that the algebraic extension of $\mathbb{C}(\tau_{jk}, e^{\pi i \tau_{jk}})_{j,k=1,\dots,m}$ generated by theta constants and all their partial derivatives has a finite degree of algebraic independence over $\mathbb{C}$.

Theorem 3. *For any (even or not) given characteristic $\boldsymbol{c} \in \mathbb{Z}^g$, $\boldsymbol{c} \ne \boldsymbol{0}$, functions* (9) *satisfy the system of differential equations*

$$\vartheta^2_{\boldsymbol{a}+\boldsymbol{c}}\vartheta^2_{\boldsymbol{a}}\delta(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}}) = \frac{1}{2^{g-1}}\sum_{\boldsymbol{b}\in\mathfrak{K}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{\boldsymbol{c}'^t(\boldsymbol{a}''+\boldsymbol{b}'')}\vartheta^2_{\boldsymbol{b}+\boldsymbol{c}}\vartheta^2_{\boldsymbol{b}}(\psi_{\boldsymbol{b}+\boldsymbol{c}}+\psi_{\boldsymbol{b}})^2 + 4\vartheta^2_{\boldsymbol{a}+\boldsymbol{c}}\vartheta^2_{\boldsymbol{a}}\psi_{\boldsymbol{a}+\boldsymbol{c}}\psi_{\boldsymbol{a}} - 2\vartheta^2_{\boldsymbol{a}+\boldsymbol{c}}\vartheta^2_{\boldsymbol{a}}(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})^2, \qquad \boldsymbol{a}\in\mathfrak{K}, \tag{12}$$

where $\mathfrak{K} = \mathfrak{K}_{\boldsymbol{c}} = \{\boldsymbol{a}\in\mathbb{Z}^{2g}/2\mathbb{Z}^{2g} : \boldsymbol{a}, \boldsymbol{a}+\boldsymbol{c}\in\mathfrak{K}_0\}$. Moreover, theta constants and (9) *have relations*

$$\vartheta^2_{\boldsymbol{a}+\boldsymbol{c}}\vartheta^2_{\boldsymbol{a}} = \frac{1}{2^{g}}\sum_{\boldsymbol{b}\in\mathfrak{K}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{\boldsymbol{c}'^t(\boldsymbol{a}''+\boldsymbol{b}'')}\vartheta^2_{\boldsymbol{b}+\boldsymbol{c}}\vartheta^2_{\boldsymbol{b}}, \qquad \boldsymbol{a}\in\mathfrak{K}, \tag{13}$$

$$\vartheta^2_{\boldsymbol{a}+\boldsymbol{c}}\vartheta^2_{\boldsymbol{a}}(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}}) = \frac{1}{2^{g}}\sum_{\boldsymbol{b}\in\mathfrak{K}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{\boldsymbol{c}'^t(\boldsymbol{a}''+\boldsymbol{b}'')}\vartheta^2_{\boldsymbol{b}+\boldsymbol{c}}\vartheta^2_{\boldsymbol{b}}(\psi_{\boldsymbol{b}+\boldsymbol{c}}+\psi_{\boldsymbol{b}}), \qquad \boldsymbol{a}\in\mathfrak{K}. \tag{14}$$

In spite of a symmetry of (11) and (12), these systems have a weak likeness with Halphen's one (4). First there are a lot of relations between participants of (11) or (12); secondly the sets of irregular points of these systems contain the variety of zeros of theta constants.

However, as was shown in the papers [10] and [11], in the case $g=2$ it is possible to state a system similar to (4).

Theorem 4 (Ohyama's Theorem). *Suppose even characteristics $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4 \in \mathbb{Z}^4/2\mathbb{Z}^4$ are "vertices of non-trivial parallelogram" (in other words, $\boldsymbol{a}_1-\boldsymbol{a}_2 \equiv \boldsymbol{a}_3-\boldsymbol{a}_4 \pmod{2\mathbb{Z}^4}$). Then corresponding functions $\psi_{\boldsymbol{a}_1}, \psi_{\boldsymbol{a}_2}, \psi_{\boldsymbol{a}_3}, \psi_{\boldsymbol{a}_4}$ satisfy the system of differential equations*

$$\delta(\psi_{\boldsymbol{a}_1}+\psi_{\boldsymbol{a}_2}+\psi_{\boldsymbol{a}_3}+\psi_{\boldsymbol{a}_4}) = (\psi_{\boldsymbol{a}_1}+\psi_{\boldsymbol{a}_2}+\psi_{\boldsymbol{a}_3}+\psi_{\boldsymbol{a}_4})^2 - 2(\psi^2_{\boldsymbol{a}_1}+\psi^2_{\boldsymbol{a}_2}+\psi^2_{\boldsymbol{a}_3}+\psi^2_{\boldsymbol{a}_4}). \tag{15}$$

Unfortunately, in the case $g>2$, a system of Halphen's type for (9) is not still obtained since nowadays. In the case $g=3$, I can reduce from Theorem 3 that δ-derivatives of functions (9) can be expressed as rational functions in terms of themselves with degree of denominators at most 9.

I skip the proof of Theorem 2 and give hints for proving Theorem 3 and reducing Theorem 4 from it.

By $O(\boldsymbol{z}^n)$, $n=0,1,2,\dots$, denote the class of holomorphic near the point $\boldsymbol{z}=\boldsymbol{0}$ functions $f(\boldsymbol{z})$ that have Taylor expansions starting from degree n, that is

$$\frac{\partial^{m_1+\dots+m_g}f}{\partial z_1^{m_1}\cdots\partial z_g^{m_g}}(\boldsymbol{0}) = 0, \qquad m_1+\dots+m_g<n.$$

Lemma 1. *In the neighborhood of the point $\boldsymbol{z}=\boldsymbol{0}$ following expansions are valid:*

$$\vartheta_{\boldsymbol{a}}(\boldsymbol{z}) = \vartheta_{\boldsymbol{a}}\cdot\Bigl(1-2\psi_{\boldsymbol{a}}+\frac{2}{3}\psi^2_{\boldsymbol{a}}+\frac{2}{3}\delta\psi_{\boldsymbol{a}}\Bigr)+O(\boldsymbol{z}^6), \qquad \boldsymbol{a}\in\mathfrak{K}_0,$$

where functions $\psi_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathfrak{K}_0$, *and differential operator* δ *are defined by* (9) *and* (10).

Lemma 1 follows from the multi-dimensional analogue of the heat equation:

$$\delta_{jk}\vartheta_{\boldsymbol{a}}(\boldsymbol{z}, \mathrm{T}) = -\frac{1}{4}\frac{\partial^2}{\partial z_j\,\partial z_k}\vartheta_{\boldsymbol{a}}(\boldsymbol{z}, \mathrm{T}), \qquad j, k = 1, \dots, g, \quad \boldsymbol{a} \in \mathbb{Z}^{2g}$$

(see, e.g., [12], Teil 1, Kapitel 1).

Fixing $\boldsymbol{c} \in \mathbb{Z}^{2g}$, we introduce the notation

$$\theta_{\boldsymbol{a}}(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3, \boldsymbol{z}_4) = \vartheta_{\boldsymbol{a}+\boldsymbol{c}}(\boldsymbol{z}_1)\vartheta_{\boldsymbol{a}+\boldsymbol{c}}(\boldsymbol{z}_2)\vartheta_{\boldsymbol{a}}(\boldsymbol{z}_3)\vartheta_{\boldsymbol{a}}(\boldsymbol{z}_4),$$

useful from the point of view of Riemann's addition formulas (see [12], Teil 2, Kapitel VII). Two specializations of these formulas give us following assertion.

Lemma 2. *For each* $\boldsymbol{a} \in \mathfrak{K}_{\boldsymbol{c}}$ *we have*

$$(-1)^{\boldsymbol{c}'^t\boldsymbol{a}''}\theta_{\boldsymbol{a}}(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) = \frac{1}{2^g}\sum_{\boldsymbol{b}\in\mathfrak{K}_{\boldsymbol{c}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{\boldsymbol{c}'^t\boldsymbol{b}''}\theta_{\boldsymbol{b}}(2\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}), \tag{16}$$

$$(-1)^{\boldsymbol{c}'^t\boldsymbol{a}''}\theta_{\boldsymbol{a}}(\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{z}, \boldsymbol{0}) = \frac{1}{2^g}\sum_{\boldsymbol{b}\in\mathfrak{K}_{\boldsymbol{c}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{\boldsymbol{c}'^t\boldsymbol{b}''}\theta_{\boldsymbol{b}}(\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{z}, \boldsymbol{0}). \tag{17}$$

Actually the summation in right-hand sides of identities (16) and (17) can be twice decreased, e.g., we can group summands $\theta_{\boldsymbol{b}}(2\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$, $\theta_{\boldsymbol{b}+\boldsymbol{c}}(2\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0})$ in (16), and $\theta_{\boldsymbol{b}}(\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{z}, \boldsymbol{0})$, $\theta_{\boldsymbol{b}+\boldsymbol{c}}(\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{z}, \boldsymbol{0})$ in (17) into pairs. Such a division into pairs is well defined because the map $\boldsymbol{b} \mapsto \boldsymbol{b} + \boldsymbol{c}$ is a (non-trivial) bijection of the set $\mathfrak{K}$. Build a factor-set $\mathfrak{K}^* = \mathfrak{K}^*_{\boldsymbol{c}}$ choosing one element from each pair. Then we rewrite identities (16) and (17) in a form

$$\theta_{\boldsymbol{a}}(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}) = \frac{1}{2^{g-1}}\sum_{\boldsymbol{b}\in\mathfrak{K}^*_{\boldsymbol{c}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{\boldsymbol{c}'^t(\boldsymbol{a}''+\boldsymbol{b}'')}\frac{\theta_{\boldsymbol{b}}(2\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}) + \theta_{\boldsymbol{b}}(\boldsymbol{0}, \boldsymbol{0}, 2\boldsymbol{z}, \boldsymbol{0})}{2}, \tag{18}$$

$$\theta_{\boldsymbol{a}}(\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{z}, \boldsymbol{0}) = \frac{1}{2^{g-1}}\sum_{\boldsymbol{b}\in\mathfrak{K}^*_{\boldsymbol{c}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{\boldsymbol{c}'^t(\boldsymbol{a}''+\boldsymbol{b}'')}\theta_{\boldsymbol{b}}(\boldsymbol{z}, \boldsymbol{0}, \boldsymbol{z}, \boldsymbol{0}) \tag{19}$$

for each $\boldsymbol{a} \in \mathfrak{K}^*$.

Lemma 3. *The symmetric matrix*

$$\mathrm{M} = \mathrm{M}_{\boldsymbol{c}} = \frac{1}{2^{g-1}}\big((-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{\boldsymbol{c}'^t(\boldsymbol{a}''+\boldsymbol{b}'')}\big)_{\boldsymbol{a},\boldsymbol{b}\in\mathfrak{K}^*}$$

of size $2^{g-2}(2^{g-1}+1)$ *corresponding to identities* (18) *and* (19), *satisfies the equation*

$$\mathrm{M}^2 = \frac{1}{2}(\mathrm{M} + E),$$

where E is an identity matrix of the same size.

Proof of Theorem 3. Using Lemma 1, we have

$$\begin{aligned}
\vartheta_1^2(\boldsymbol z)\vartheta_2^2(\boldsymbol z) &= \vartheta_1^2\vartheta_2^2\cdot\Bigl(1-2\psi_1+\frac{2}{3}\psi_1^2+\frac{2}{3}\delta\psi_1\Bigr)^2\Bigl(1-2\psi_2+\frac{2}{3}\psi_2^2+\frac{2}{3}\delta\psi_2\Bigr)^2+O(\boldsymbol z^6)\\
&= \vartheta_1^2\vartheta_2^2\cdot\Bigl(1-4(\psi_1+\psi_2)+\frac{16}{3}\psi_1\psi_2+\frac{16}{3}(\psi_1+\psi_2)^2+\frac{4}{3}\delta(\psi_1+\psi_2)\Bigr)+O(\boldsymbol z^6),\\
&\frac{\vartheta_1(2\boldsymbol z)\vartheta_2+\vartheta_2(2\boldsymbol z)\vartheta_1}{2}\\
&= \vartheta_1\vartheta_2\cdot\Bigl(1-4(\psi_1+\psi_2)-\frac{32}{3}\psi_1\psi_2+\frac{16}{3}(\psi_1+\psi_2)^2+\frac{16}{3}\delta(\psi_1+\psi_2)\Bigr)+O(\boldsymbol z^6),\\
&\vartheta_1(\boldsymbol z)\vartheta_2(\boldsymbol z)\\
&= \vartheta_1\vartheta_2\cdot\Bigl(1-2(\psi_1+\psi_2)+\frac{8}{3}\psi_1\psi_2+\frac{2}{3}(\psi_1+\psi_2)^2+\frac{2}{3}\delta(\psi_1+\psi_2)\Bigr)+O(\boldsymbol z^6),
\end{aligned}$$

where I omit the numeration of theta constants and their logarithmic derivatives for simplicity. Substituting last expansions in (18) and (19) and comparing the homogeneous components of degree 0, 2, and 4, we obtain (13), (14), and

$$4\boldsymbol W+4\boldsymbol X+\boldsymbol Y=4\mathrm M(-2\boldsymbol W+\boldsymbol X+\boldsymbol Y), \tag{20}$$

$$4\boldsymbol W+\boldsymbol X+\boldsymbol Y=\mathrm M(4\boldsymbol W+\boldsymbol X+\boldsymbol Y), \tag{21}$$

where

$$\boldsymbol W=(\vartheta_{\boldsymbol a+\boldsymbol c}^2\vartheta_{\boldsymbol a}^2\psi_{\boldsymbol a+\boldsymbol c}\psi_{\boldsymbol a})_{\boldsymbol a\in\mathfrak K^*},\qquad \boldsymbol X=\bigl(\vartheta_{\boldsymbol a+\boldsymbol c}^2\vartheta_{\boldsymbol a}^2(\psi_{\boldsymbol a+\boldsymbol c}+\psi_{\boldsymbol a})^2\bigr)_{\boldsymbol a\in\mathfrak K^*},$$
$$\boldsymbol Y=\bigl(\vartheta_{\boldsymbol a+\boldsymbol c}^2\vartheta_{\boldsymbol a}^2\delta(\psi_{\boldsymbol a+\boldsymbol c}+\psi_{\boldsymbol a})\bigr)_{\boldsymbol a\in\mathfrak K^*}$$

are corresponding vector-columns. Since

$$\delta\bigl(\vartheta_{\boldsymbol a+\boldsymbol c}^2\vartheta_{\boldsymbol a}^2(\psi_{\boldsymbol a+\boldsymbol c}+\psi_{\boldsymbol a})\bigr)=2\vartheta_{\boldsymbol a+\boldsymbol c}^2\vartheta_{\boldsymbol a}^2(\psi_{\boldsymbol a+\boldsymbol c}+\psi_{\boldsymbol a})^2+\vartheta_{\boldsymbol a+\boldsymbol c}^2\vartheta_{\boldsymbol a}^2\delta(\psi_{\boldsymbol a+\boldsymbol c}+\psi_{\boldsymbol a}),\quad \boldsymbol a\in\mathfrak K^*,$$

δ-differentiation of (14) gives us

$$2\boldsymbol X+\boldsymbol Y=\mathrm M(2\boldsymbol X+\boldsymbol Y). \tag{22}$$

Using (20)–(22) and Lemma 3, we obtain

$$\boldsymbol Y=4\boldsymbol W+2(\mathrm M-E)\boldsymbol X, \tag{23}$$

that is, exactly, the matrix form of the system (12).

This completes the proof of Theorem 3.

Proof of Theorem 4. Now I show how to reduce Theorem 4 from Theorem 3 if $g=2$. Choose any characteristic $\boldsymbol c\in\mathbb Z^4/2\mathbb Z^4$, $\boldsymbol c\ne\boldsymbol 0$. The corresponding set $\mathfrak K^*=\mathfrak K_{\boldsymbol c}^*$ contains three elements, say $\boldsymbol a_1,\boldsymbol a_2,\boldsymbol a_3$. It can be assumed that the order of these elements is chosen such that the matrix $\mathrm M=\mathrm M_{\boldsymbol c}$ has a form

$$\mathrm M=\frac{1}{2}\begin{pmatrix}1&1&1\\1&1&-1\\1&-1&1\end{pmatrix}.$$

Putting

$$\theta_j=\vartheta^2_{\boldsymbol{a}_j+\boldsymbol{c}}\vartheta^2_{\boldsymbol{a}_j},\quad S_j=\psi_{\boldsymbol{a}_j+\boldsymbol{c}}+\psi_{\boldsymbol{a}_j},\quad P_j=\psi_{\boldsymbol{a}_j+\boldsymbol{c}}\psi_{\boldsymbol{a}_j},\qquad j=1,2,3,$$

we rewrite the system of differential equations (12) (equivalently, (23)) as

$$\begin{aligned}\theta_1\delta S_1&=4\theta_1P_1-(\theta_1S_1^2-\theta_2S_2^2-\theta_3S_3^2),\\ \theta_2\delta S_2&=4\theta_2P_2+(\theta_1S_1^2-\theta_2S_2^2-\theta_3S_3^2),\\ \theta_3\delta S_3&=4\theta_3P_3+(\theta_1S_1^2-\theta_2S_2^2-\theta_3S_3^2).\end{aligned}\tag{24}$$

In our case, relations (13), (14) look like

$$\theta_1-\theta_2-\theta_3=0,\qquad \theta_1S_1-\theta_2S_2-\theta_3S_3=0,$$

hence,

$$\frac{S_2-S_3}{\theta_1}=\frac{S_1-S_3}{\theta_2}=\frac{S_2-S_1}{\theta_3}.$$

This implies that

$$\begin{aligned}\theta_1S_1^2-\theta_2S_2^2-\theta_3S_3^2=&\ \ \theta_1(S_1-S_2)(S_1-S_3)\\ =&-\theta_2(S_2-S_1)(S_2-S_3)\\ =&-\theta_3(S_3-S_1)(S_3-S_2).\end{aligned}\tag{25}$$

Substituting (25) in (24) and cancellating theta constants, we obtain the system

$$\begin{aligned}\delta S_1&=4P_1-(S_1-S_2)(S_1-S_3),\\ \delta S_2&=4P_2-(S_2-S_1)(S_2-S_3),\\ \delta S_3&=4P_3-(S_3-S_1)(S_3-S_2).\end{aligned}\tag{26}$$

Thus, in our notation, the system (15) follows from (26) by adding any two of equations:

$$\begin{aligned}\delta(S_1+S_2)&=4P_1+4P_2-(S_1-S_2)^2,\\ \delta(S_1+S_3)&=4P_1+4P_3-(S_1-S_3)^2,\\ \delta(S_2+S_3)&=4P_2+4P_3-(S_2-S_3)^2,\end{aligned}$$

since any two among three pairs $\{\boldsymbol{a}_1,\boldsymbol{a}_1+\boldsymbol{c}\}$, $\{\boldsymbol{a}_2,\boldsymbol{a}_2+\boldsymbol{c}\}$, and $\{\boldsymbol{a}_3,\boldsymbol{a}_3+\boldsymbol{c}\}$ form "non-trivial parallelogram".

This completes the proof of Theorem 4.

The point (i) of our program is still an open question even in the case $g=2$. I do not want to conjecture anything on this subject. To discuss this problem we need a modular point of view on functions (7).

In the case $g>1$ consider the *symplectic group*

$$Sp_{2g}(\mathbb{R})=\left\{\gamma=\begin{pmatrix}A&B\\C&D\end{pmatrix}:\gamma^t\begin{pmatrix}0&E\\-E&0\end{pmatrix}\gamma=\begin{pmatrix}0&E\\-E&0\end{pmatrix}\right\},\tag{27}$$

where A,B,C,D are square matrices of size g and E is the identity matrix. One has the same definition for the *Siegel modular group* $Sp_{2g}(\mathbb{Z})$, the only difference from (27) is in integer-valued matrices A,B,C,D. Both of groups, $Sp_{2g}(\mathbb{R})$ or $Sp_{2g}(\mathbb{Z})$, act on Siegel's upper half-space $\mathfrak{H}_g\subset\mathbb{C}^{g(g+1)/2}$ by the rule

$$\mathrm{T}\mapsto\gamma\mathrm{T}=(A\mathrm{T}+B)(C\mathrm{T}+D)^{-1},\qquad \gamma=\begin{pmatrix}A&B\\C&D\end{pmatrix}.$$

Following statement shows a connection between differentiating and above acting.

Lemma 4. *Suppose a function $F(\mathrm{T})\colon \mathfrak{H}_g \to \mathbb{C}$ is holomorphic, the matrix differential operator Δ is defined by* (10) *(so, ΔF is a symmetric matrix of size g which elements are corresponding partial derivatives of F). Then*

$$(\Delta F)(\gamma\mathrm{T}) = (C\mathrm{T}+D)\cdot\Delta F(\gamma\mathrm{T})\cdot(C\mathrm{T}+D)^t \quad \textit{for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}).$$

Using the functional equation for theta constants

$$\vartheta_{\boldsymbol{a}}(\gamma\mathrm{T}) = \xi\cdot\det{}^{1/2}(C\mathrm{T}+D)\cdot\vartheta_{\boldsymbol{a}}(\mathrm{T}), \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{1,2}, \quad \xi^8 = 1$$

(see [9], Chapter II) and Lemma 4, we have the following result on modular properties of functions (7).

Theorem 5. *For each characteristic $\boldsymbol{a} \in \mathfrak{K}_0$ matrix-valued function $\Psi_{\boldsymbol{a}}$ consisting of functions* (7) *satisfies the functional equation*

$$\Psi_{\boldsymbol{a}}(\gamma\mathrm{T}) = \frac{1}{2}(C\mathrm{T}+D)\cdot C^t + (C\mathrm{T}+D)\cdot\Psi_{\boldsymbol{a}}(\mathrm{T})\cdot(C\mathrm{T}+D)^t, \qquad \gamma\in\Gamma_{1,2}.$$

In particular, for $\boldsymbol{a},\boldsymbol{b} \in \mathfrak{K}_0$ the function $\det(\Psi_{\boldsymbol{a}} - \Psi_{\boldsymbol{b}})(\mathrm{T})$ is a modular form of weight 2 *with respect to $\Gamma_{1,2}$.*

Theorem 5 can help to get the multi-dimensional analogues of formulas (5), at least in cases $g = 2$ and $g = 3$. This problem seems to be more computational than mathematical.

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