Very well-poised hypergeometric series and multiple integrals

V. V. Zudilin [W. Zudilin]

The purpose of this note is to establish a relationship between two objects: the very well-poised hypergeometric series

$$F_k(\boldsymbol{h}) = F_k(h_0; h_1, \dots, h_k) := \sum_{\mu=0}^{\infty} (h_0 + 2\mu) \frac{\prod_{j=0}^{k} \Gamma(h_j + \mu)}{\prod_{j=0}^{k} \Gamma(1 + h_0 - h_j + \mu)} (-1)^{(k+1)\mu}$$
$$= \frac{\Gamma(1+h_0) \prod_{j=1}^{k} \Gamma(h_j)}{\prod_{j=1}^{k} \Gamma(1+h_0-h_j)} \, {}_{k+2}F_{k+1}\left(\begin{matrix} h_0, 1+\frac{1}{2}h_0, & h_1, & \dots, & h_k \\ \frac{1}{2}h_0, & 1+h_0-h_1, & \dots, & 1+h_0-h_k \end{matrix} \,\middle|\, (-1)^{k+1} \right) \tag{1}$$

and the multiple integrals

$$J_k(\boldsymbol{a}, \boldsymbol{b}) = J_k\left(\begin{matrix} a_0, a_1, \dots, a_k \\ b_1, \dots, b_k \end{matrix} \right)$$
$$:= \int \cdots \int_{[0,1]^k} \frac{\prod_{j=1}^{k} x_j^{a_j - 1} (1 - x_j)^{b_j - a_j - 1}}{(1 - (1 - (\cdots (1 - (1 - x_k) x_{k-1}) \cdots) x_2) x_1)^{a_0}} \, dx_1 \, dx_2 \cdots dx_k. \tag{2}$$

Theorem. *Suppose that $k \geqslant 1$, the parameters $h_0, h_1, \dots, h_{k+2} \in \mathbb{C}$ satisfy the conditions*

$$1 + \Re h_0 > \frac{2}{k+1} \cdot \sum_{j=1}^{k+2} \Re h_j, \qquad \Re(1 + h_0 - h_{j+1}) > \Re h_j > 0 \quad \text{for } j = 2, \dots, k+1,$$

and $h_1, h_{k+2} \neq 0, -1, -2, \dots$. Then the following identity holds:

$$\frac{\prod_{j=1}^{k+1} \Gamma(1 + h_0 - h_j - h_{j+1})}{\Gamma(h_1)\,\Gamma(h_{k+2})} \cdot F_{k+2}(h_0; h_1, \dots, h_{k+2})$$
$$= J_k\left(\begin{matrix} h_1, & h_2, & h_3, & \dots, & h_{k+1} \\ & 1 + h_0 - h_3, & 1 + h_0 - h_4, & \dots, & 1 + h_0 - h_{k+2} \end{matrix} \right). \tag{3}$$

The proof is carried out by induction. If $k = 1$, then the statement of the theorem follows from the limit case of Dougall's theorem ([1], § 4.4, (1)). If $k \geqslant 2$, then we set $\varepsilon_k = 0$ for k even and $\varepsilon_k = 1$ or -1 for k odd and use the relation

$$J_k\left(\begin{matrix} a_0, a_1, \dots, a_{k-1}, a_k \\ b_1, \dots, b_{k-1}, b_k \end{matrix} \right) = \frac{\Gamma(b_k - a_k)}{\Gamma(a_0)} \cdot \frac{1}{2\pi i} \int_{-t_0 - i\infty}^{-t_0 + i\infty} \frac{\Gamma(a_0 + t)\,\Gamma(a_k + t)\,\Gamma(-t)}{\Gamma(b_k + t)} e^{\varepsilon_k \pi i t}$$
$$\times J_{k-1}\left(\begin{matrix} a_0 + t, a_1 + t, \dots, a_{k-1} + t \\ b_1 + t, \dots, b_{k-1} + t \end{matrix} \right) dt, \tag{4}$$

where $t_0 \in \mathbb{R}$, $\Re a_0 > t_0 > 0$, $\Re a_k > t_0 > 0$, $\Re b_k > \Re a_0 + \Re a_k$, provided that the integral on the left-hand side of (4) converges. Representing the hypergeometric series (1) in the form of a Barnes-type contour integral and applying the inductive hypothesis to the integrand on the right-hand side of (4), we obtain the desired identity (3).

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We note that the series on the right-hand side of (3) admits a 'less economical' representation in the form of an Euler-type multiple integral over the cube $[0,1]^{k+2}$ (see [2], Lemma 1). The above theorem and recent results of Zlobin ([3], [4]) also yield a representation of the very well-poised hypergeometric series (1) in the form of an integral proposed in Sorokin's papers [5], [6].

In spite of the analytic nature of the theorem, the identity (3) is motivated by arithmetic results for the values of the Riemann zeta function (*zeta values*) at positive integers ([5]–[13]). As is known [13], in the case of integral parameters $\boldsymbol{h}$ a very well-poised hypergeometric series of the form (1) is a $\mathbb{Q}$-linear form in even or odd zeta values, depending on the parity of $k \geqslant 4$. Therefore, if the parameters $\boldsymbol{a}$ and $\boldsymbol{b}$ are positive and integral and satisfy the additional condition

$$b_1 + a_2 = b_2 + a_3 = \cdots = b_{k-1} + a_k, \tag{5}$$

then the integral (2) is a $\mathbb{Q}$-linear form in zeta values whose arguments are of the same parity. The specialization $a_j = n+1$ and $b_j = 2n+2$ leads to the coincidence (conjectured by the author in [13], § 9) of multiple integrals and very well-poised hypergeometric series; denoting the corresponding integrals (2) by $J_{k,n}$ and using the arithmetic results in [12], Lemmas 4.2–4.4, we conclude that

$$D_n^{k+1}\Phi_n^{-1} \cdot J_{k,n} \in \mathbb{Z}\zeta(k) + \mathbb{Z}\zeta(k-2) + \cdots + \mathbb{Z}\zeta(3) + \mathbb{Z} \quad \text{for } k \text{ odd}, \tag{6}$$

where D_n is the least common multiple of the numbers $1, 2, \dots, n$ and Φ_n is the product of the primes $p < n$ such that $2/3 \leqslant \{n/p\} < 1$ ($\{\cdot\}$ stands for the fractional part of a number). The relations (6) (with the multiple D_n^k instead of $D_n^{k+1}\Phi_n^{-1}$) were conjectured by Vasil'ev [14] (see also [11], comment to Theorem 2) and proved by him for $k=5$ (the case $k=3$ was treated in [7]). Thus, we give a particular answer to Vasil'ev's conjecture. The choice $a_j = rn+1$ and $b_j = (r+1)n+2$ in (2) (or, equivalently, $h_0 = (2r+1)n+2$ and $h_j = rn+1$ for $j = 1, \dots, k+2$ in (1)) with an integer $r \geqslant 1$ depending on a given odd integer k leads to the linear forms (in odd zeta values) similar to those considered by Rivoal [10] in the proof of his remarkable result that the sequence $\zeta(3)$, $\zeta(5)$, $\zeta(7)$, ... contains infinitely many irrationals.

Moreover, it should be noted that if the assumption (5) holds, then the quantity

$$\frac{F_{k+2}(h_0; h_1, \dots, h_{k+2})}{\prod_{j=1}^{k+2}\Gamma(h_j)} = \frac{J_k(\boldsymbol{a}, \boldsymbol{b})}{\prod_{j=1}^{k}\Gamma(a_j)\cdot\Gamma(b_1 + a_2 - a_0 - a_1)\cdot\prod_{j=1}^{k}\Gamma(b_j - a_j)}$$

is obviously invariant under the action of the ($\boldsymbol{h}$-trivial) group $\mathfrak{G}$ (of order $(k+2)!$) consisting of all permutations of the parameters $h_1, \dots, h_{k+2}$. This result also has number-theoretic applications. For $k=2$ and $k=3$ the change of variables $(x_{k-1}, x_k) \mapsto (1-x_k, 1-x_{k-1})$ in (2) gives an additional transformation $\mathfrak{c}$ of both the integral (2) and the series (1); for $k \geqslant 4$ this transformation is not available, since the condition (5) is violated. The groups $\langle \mathfrak{G}, \mathfrak{c} \rangle$ of orders 120 and 1920 for $k=2$ and $k=3$, respectively, are known ([8], [9]); Rhin and Viola use these groups to obtain nice estimates for the irrationality measures of $\zeta(2)$ and $\zeta(3)$. For $k \geqslant 4$ the group $\mathfrak{G}$ admits a natural interpretation as a permutation group of the parameters $e_{0l} = h_l - 1$, $1 \leqslant l \leqslant k+2$, and $e_{jl} = h_0 - h_j - h_l$, $1 \leqslant j < l \leqslant k+2$ (for details, see [13], § 9).

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Moscow State University
E-mail: wadim@ips.ras.ru

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