THETA CONSTANTS AND DIFFERENTIAL EQUATIONS

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For any genus g consider the *theta constants* (*thetanulls*, or *Thetanullwerte*)

$$\vartheta_{\boldsymbol{a}}(\boldsymbol{\tau})=\vartheta_{(\boldsymbol{a}',\boldsymbol{a}'')}(\boldsymbol{\tau})=\sum_{\boldsymbol{n}\in\mathbb{Z}^g}\exp\bigl(\pi i^t\bigl(\boldsymbol{n}+\tfrac12\boldsymbol{a}'\bigr)\boldsymbol{\tau}\bigl(\boldsymbol{n}+\tfrac12\boldsymbol{a}'\bigr)+\pi i^t\bigl(\boldsymbol{n}+\tfrac12\boldsymbol{a}'\bigr)\boldsymbol{a}''\bigr)$$

attached to the even 2-characteristics $\boldsymbol{a}=(\boldsymbol{a}',\boldsymbol{a}'')$ from $\mathfrak{K}_+\subset(\mathbb{Z}/2\mathbb{Z})^{2g}=\{\boldsymbol{a}:{}^t\boldsymbol{a}'\boldsymbol{a}''\equiv 0\bmod 2\}$, where $\boldsymbol{\tau}=(\tau_{jl})_{j,l=1,\dots,g}$ is a symmetric $g\times g$-matrix from Siegel half space $\mathfrak{H}_g=\{\boldsymbol{\tau}:\operatorname{Im}\boldsymbol{\tau}>0\}$ of degree g. We generalize well-known results of M. Halphen, R. A. Rankin, K. Mahler for theta constants of genus 1 and their derivatives to the case of arbitrary genus.

Consider partial derivations $\boldsymbol{\delta}$,

$$\delta_{jj}=\frac{1}{\pi i}\frac{\partial}{\partial\tau_{jj}},\quad j=1,\dots,g,\qquad \delta_{jl}=\frac{1}{2\pi i}\frac{\partial}{\partial\tau_{jl}}=\delta_{lj},\quad j,l=1,\dots,g,\ j\ne l,$$

and logarithmic derivatives of the theta constants

$$\psi_{\boldsymbol{a},jl}=\psi_{\boldsymbol{a},jl}(\boldsymbol{\tau}):=\frac{\delta_{jl}\vartheta_{\boldsymbol{a}}}{\vartheta_{\boldsymbol{a}}}=\psi_{\boldsymbol{a},lj},\qquad \boldsymbol{a}\in\mathfrak{K}_+,\quad j,l=1,\dots,g.$$

To the ψ-functions, we assign the quadratic forms

$$\psi_{\boldsymbol{a}}=\psi_{\boldsymbol{a}}[\boldsymbol{z}]:=\sum_{j,l=1}^{g}\psi_{\boldsymbol{a},jl}\cdot z_jz_l;$$

then we define quartic forms

$$\boldsymbol{\delta}\psi_{\boldsymbol{a}}=\boldsymbol{\delta}\psi_{\boldsymbol{a}}[\boldsymbol{z}]:=\sum_{j,l,m,p=1}^{g}\delta_{jl}\psi_{\boldsymbol{a},mp}\cdot z_jz_lz_mz_p.$$

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Theorem 1 [Z]. *The logarithmic derivatives of theta constants satisfy the system of partial differential equations*

$$\vartheta_{\boldsymbol{a}}^4 \cdot \boldsymbol{\delta}\psi_{\boldsymbol{a}} = \frac{1}{2^{g-2}} \sum_{\boldsymbol{b}\in\mathfrak{K}_+} (-1)^{<\boldsymbol{a},\boldsymbol{b}>} \vartheta_{\boldsymbol{b}}^4 \cdot \psi_{\boldsymbol{b}}^2 - 2\vartheta_{\boldsymbol{a}}^4 \cdot \psi_{\boldsymbol{a}}^2, \qquad \boldsymbol{a} \in \mathfrak{K}_+,$$

where $<\boldsymbol{a},\boldsymbol{b}> = {}^t\boldsymbol{a}'\boldsymbol{b}'' - {}^t\boldsymbol{b}'\boldsymbol{a}'' \bmod 2$.

Consider the rings

$$Q_g = \mathbb{Q}[\vartheta_{\boldsymbol{a}}, \psi_{\boldsymbol{a},jl}]_{\boldsymbol{a}\in\mathfrak{K}_+;j,l=1,\dots,g} \quad \text{and} \quad Q'_g = \mathbb{Q}[\psi_{\boldsymbol{a},jl}]_{\boldsymbol{a}\in\mathfrak{K}_+;j,l=1,\dots,g}.$$

Theorem 1 then implies that the *fraction field* of Q_g is stable under δ_{jl}-derivations.

Theorem 2 [BZ]. *The ring Q_g has transcendence degree $2g^2+g$ over $\mathbb{Q}$.*

For small genus $g = 1, 2, 3$ the results above can be sharpened as follows (see [BZ], [Z]):

(i) theta constants in genus ≤ 3 are algebraic over the fraction field of Q'_g;

(ii) the rings Q'_1 and Q'_2 (and the fraction field of Q'_3) are $\boldsymbol{\delta}$-stable.

In particular, the rings Q'_1, Q'_2, and Q'_3 have transcendance degree 3, 10, and 21 over $\mathbb{Q}$ respectively.

The proof of Theorem 1 lies on Riemann's relations for abelian theta functions and the heat equation, while the proof of Theorem 2 combines results from differential Galois theory with a modular interpretation of the period matrix of abelian varieties. The constant $2g^2+g$ is $\dim Sp_{2g}(\mathbb{C})$, and the sympectic group $Sp_{2g}(\mathbb{C})$ occurs as a differential Galois group of a Picard–Vessiot extension for a system of *linear* partial differential equations.

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References

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