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Thetanulls and differential equations

V. V. Zudilin

Abstract. The closedness of the system of thetanulls (and the Siegel modular forms) and their first derivatives with respect to differentiation is well-known in the one-dimensional case. It is shown in the present paper that thetanulls and their various logarithmic derivatives satisfy a non-linear system of differential equations; only one and two-dimensional versions of this result were known before. Several distinct examples of such systems are presented, and a theorem on the transcendence degree of the differential closure of the field generated by all thetanulls is established. On the basis of a study of the modular properties of logarithmic derivatives of thetanulls (previously unknown) relations between these functions and thetanulls themselves are obtained in dimensions 2 and 3.

Bibliography: 26 titles.

Introduction

Theta functions is a classical domain of mathematics, marked with beauty. Many results in mathematical analysis, algebraic geometry, differential equations, and other areas owe their existence to its development. There are quite a few monographs (sound examples are [1]–[3]) and papers dedicated to theta functions, of which a small part can be found in our list of literature.

The first systematic study of one-dimensional theta functions was carried out by Jacobi (see [4], [5]), although his notation is distinct from the following notation, which was used in later papers of Frobenius, Krazer, Wirtinger, and other authors:

$$
\begin{aligned}
\vartheta_1(z,q) &= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin(2n+1)z \\
&= 2q^{1/4}\sin z - 2q^{9/4}\sin 3z + 2q^{25/4}\sin 5z - \cdots, \\
\vartheta_2(z,q) &= 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos(2n+1)z \\
&= 2q^{1/4}\cos z + 2q^{9/4}\cos 3z + 2q^{25/4}\cos 5z + \cdots,
\end{aligned}
$$

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$$\vartheta_3(z,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz = 1 + 2q\cos 2z + 2q^4\cos 4z + 2q^9\cos 6z + \cdots,$$

$$\begin{aligned}\vartheta_4(z,q) &= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz \\ &= 1 - 2q\cos 2z + 2q^4\cos 4z - 2q^9\cos 6z + \cdots. \end{aligned} \tag{0.1}$$

The functions (0.1) are entire functions of the z-variable for each $|q| < 1$. The parameter q is related to the modular parameter τ by the formula $q = e^{\pi i\tau}$, $\operatorname{Im}\tau > 0$ (see [6]; § 21.1 and § 21.7). Each function (0.1) satisfies the differential *heat equation*

$$q\frac{\partial\vartheta(z,q)}{\partial q} = -\frac{1}{4}\frac{\partial^2\vartheta(z,q)}{\partial z^2} \tag{0.2}$$

(see [6]; § 21.4). In addition, the ratios of theta functions (0.1) make up a closed system with respect to z-differentiation (see [6]; § 21.6).

In formulae containing theta functions with the same parameter q this parameter is usually dropped, as if it were a constant rather than a variable. Consistent with this ideology is the definition of the *thetanulls* (or *theta constants*)

$$\vartheta_2 = \vartheta_2(0), \qquad \vartheta_3 = \vartheta_3(0), \qquad \vartheta_4 = \vartheta_4(0) \tag{0.3}$$

as the values of even theta functions (0.1) at $z = 0$. Of course, the functions (0.3) are not constant: they depend on the q-variable.

Back in 1848, Jacobi [7] showed that each thetanull in (0.3) satisfies the same third-order differential equation. Jacobi's equation has the following form in terms of δ-differentiation:

$$\begin{aligned}&\big(\vartheta^2\cdot\delta^3\vartheta - 15\vartheta\cdot\delta\vartheta\cdot\delta^2\vartheta + 30(\delta\vartheta)^3\big)^2 + 32\big(\vartheta\cdot\delta^2\vartheta - 3(\delta\vartheta)^2\big)^3 \\ &\qquad = 4\vartheta^{10}\big(\vartheta\cdot\delta^2\vartheta - 3(\delta\vartheta)^2\big)^3, \qquad \delta = q\frac{d}{dq} = \frac{1}{\pi i}\frac{d}{d\tau}\end{aligned} \tag{0.4}$$

(see, for instance, [8]). A hundred years later, Mahler [9] proved a result on the algebraic independence of functions $\vartheta, \delta\vartheta, \delta^2\vartheta$, which showed, in particular, that none of the thetanulls satisfies a second-order algebraic differential equation with coefficients in $\mathbb{C}[\tau]$ or $\mathbb{C}[q]$. We can formulate Mahler's result in [9] as follows.

Mahler's theorem. *The variable τ and the functions $q = e^{\pi i\tau}$, ϑ_j, $\delta\vartheta_j$, and $\delta^2\vartheta_j$ are algebraically independent over the field $\mathbb{C}$ for each $j = 2, 3, 4$.*

Let $\mathcal{K}$ be the differential closure of the field generated over $\mathbb{C}(q)$ by the functions (0.3) with respect to δ-differentiation. By Mahler's theorem the transcendence degree of $\mathcal{K}$ over $\mathbb{C}(q)$ is 3 (see also [10] and [11] on this issue). The bulkiness of (0.4) makes it expedient to discuss the choice of the *generators of the field* $\mathcal{K}$, a system of three elements of $\mathcal{K}$ algebraically independent over $\mathbb{C}(q)$, such that the differential equations for them are possibly simple.

The first example of such a system, the logarithmic δ-derivatives of thetanulls

$$\psi_2 = \frac{\delta\vartheta_2}{\vartheta_2}, \qquad \psi_3 = \frac{\delta\vartheta_3}{\vartheta_3}, \qquad \psi_4 = \frac{\delta\vartheta_4}{\vartheta_4}, \tag{0.5}$$

was presented by Halphen [12] in 1881, who proved the following result.

Halphen's theorem. *The functions* (0.5) *satisfy the system of differential equations*

$$\delta(\psi_2+\psi_3)=4\psi_2\psi_3,\qquad \delta(\psi_3+\psi_4)=4\psi_3\psi_4,\qquad \delta(\psi_4+\psi_2)=4\psi_4\psi_2. \tag{0.6}$$

Writing (0.6) in a less compact form

$$\begin{aligned}\delta\psi_2&=2(\psi_2\psi_3+\psi_2\psi_4-\psi_3\psi_4),\\ \delta\psi_3&=2(\psi_2\psi_3+\psi_3\psi_4-\psi_2\psi_4),\\ \delta\psi_4&=2(\psi_2\psi_4+\psi_3\psi_4-\psi_2\psi_3)\end{aligned} \tag{0.7}$$

one can see the closedness of the system (0.5) with respect to δ-differentiation. The thetanulls (0.3) are connected with the functions (0.5) by the well-known relations

$$\psi_3-\psi_4=\frac{1}{4}\vartheta_2^4,\qquad \psi_2-\psi_4=\frac{1}{4}\vartheta_3^4,\qquad \psi_2-\psi_3=\frac{1}{4}\vartheta_4^4, \tag{0.8}$$

which (in slightly distinct notation) can be found, for instance, in [13]; Chapter 11, formula (93.6).

Another example of a 'simple' system of generators of $\mathcal{K}$ was obtained by Ramanujan [14] (1916).

Ramanujan's theorem. *The functions*

$$\begin{gathered}P(q)=1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n,\\ Q(q)=1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n,\qquad R(q)=1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n,\end{gathered} \tag{0.9}$$

where $\sigma_k(n)=\sum_{d|n}d^k$, *satisfy the system of differential equations*

$$\delta P=\frac{1}{12}(P^2-Q),\qquad \delta Q=\frac{1}{3}(PQ-R),\qquad \delta R=\frac{1}{2}(PR-Q^2). \tag{0.10}$$

The algebraic independence of the functions (0.9) follows from above-mentioned Mahler's result, and their algebraicity over $\mathcal{K}$ is obvious from the formulae

$$P(q^2)=1-24\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}=4\biggl(\frac{\delta\vartheta_2}{\vartheta_2}+\frac{\delta\vartheta_3}{\vartheta_3}+\frac{\delta\vartheta_4}{\vartheta_4}\biggr) \tag{0.11}$$

(see, for instance, [6]; § 21.41),

$$Q(q^2)=\frac{1}{2}(\vartheta_2^8+\vartheta_3^8+\vartheta_4^8),\qquad R(q^2)=\frac{1}{2}(\vartheta_4^4-\vartheta_2^4)(\vartheta_2^4+\vartheta_3^4)(\vartheta_3^4+\vartheta_4^4) \tag{0.12}$$

(see, for instance, [15]; Kapitel III, Hilfssatz 1.1_6, where there is a minor misprint, or [16]; Chapter 6, Exercise 4, without misprints); we drop the parameter q of thetanulls on the right-hand sides of (0.11) and (0.12). The fact that the Ramanujan functions in (0.11) and (0.12) depend on q^2 is not a serious problem: in view of the formulae

$$\vartheta_2^2(q^2) = \frac{1}{2}\bigl(\vartheta_3^2(q) - \vartheta_4^2(q)\bigr), \qquad \vartheta_3^2(q^2) = \frac{1}{2}\bigl(\vartheta_3^2(q) + \vartheta_4^2(q)\bigr), \qquad \vartheta_4^2(q^2) = \vartheta_3(q)\vartheta_4(q)$$

(see [16]; Chapter 1, § 1.8) the differential closure of the field generated over $\mathbb{C}(q)$ by the functions $\vartheta_2(q^2)$, $\vartheta_3(q^2)$, and $\vartheta_4(q^2)$ with respect to δ-differentiation is the same (up to an algebraic extension) as $\mathcal{K}$.

Our interest in thetanulls and their differential properties originated from the following result of Nesterenko [17], [18] in transcendental number theory.

Nesterenko's theorem. *For each $q_0 \in \mathbb{C}$, $|q_0| < 1$, the collection*

$$q_0, P(q_0), Q(q_0), R(q_0) \tag{0.13}$$

contains at least three algebraically independent numbers over the field $\mathbb{Q}$.

Of course, one can replace the collection (0.13) by $q_0, f_1(q_0), f_2(q_0), f_3(q_0)$ for each choice of a system of generators f_1, f_2, f_3 in the field $\mathcal{K}$ (although the above result was established specifically for the Ramanujan functions!). One consequence of Nesterenko's theorem is the algebraic independence over $\mathbb{Q}$ of the quantities π, e^{π}, and $\Gamma(1/4)$.

The proof of this result is technically not simple; it is based upon 'nice' algebraic properties of the non-linear system of differential equations (0.10) and 'nice' arithmetic properties of the expansions (0.9). We shall not elaborate on this, but point out instead that the system (0.7) and the expansions of the functions (0.5) in powers of q have similar 'nice' properties (see [19]).

Whether there are other examples of such 'nice' systems of functions and non-linear differential equations connecting them, is an open question. In [20] the author shows a way to the derivation of such systems of differential equations (similar to Halphen's system) using second-order Fuchsian linear differential equations, but their solutions, by contrast to the functions (0.5), do not seem to have 'nice' arithmetic properties.

In the present paper we obtain closed systems of partial differential equations for multidimensional thetanulls and their logarithmic derivatives and study the algebraic and modular properties of these systems.

We now fix a dimension (genus) g and define a g-dimensional theta function by means of a series:

$$\vartheta(\boldsymbol{z}, \mathrm{T}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^g} \exp(\pi i {}^t\boldsymbol{n}\mathrm{T}\boldsymbol{n} + 2i {}^t\boldsymbol{n}\boldsymbol{z}), \tag{0.14}$$

$$\boldsymbol{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_g \end{pmatrix} \in \mathbb{C}^g, \qquad \mathrm{T} = \begin{pmatrix} \tau_{11} & \cdots & \tau_{1g} \\ \vdots & \ddots & \vdots \\ \tau_{g1} & \cdots & \tau_{gg} \end{pmatrix},$$

where T is a symmetric complex matrix with positive definite imaginary part (the space $\mathfrak{H}_g$ of such matrices of order $g(g+1)/2$ is an open subset of $Sym_g(\mathbb{C})$ and is called the *Siegel upper half-space*). We write vectors as columns and use the superscript t to denote transposition: thus, ${}^t\boldsymbol{n}\boldsymbol{z}$ is the scalar product of the vectors $\boldsymbol{n}$ and $\boldsymbol{z}$.

Note that the series (0.14) converges absolutely and uniformly in $\boldsymbol{z}$ and T on each compact subset of $\mathbb{C}^g \times \mathfrak{H}_g$, that is, it defines a holomorphic function on this set (see [3]; Chapter II, Proposition 1.1).

With each matrix $\mathrm{T} \in \mathfrak{H}_g$ one can associate the lattice $\mathfrak{L}_{\mathrm{T}} = \pi\mathrm{T}\mathbb{Z}^g + \pi\mathbb{Z}^g \subset \mathbb{C}^g$. The theta function has the property of quasi-periodicity with respect to $\mathfrak{L}_{\mathrm{T}}$, namely ([3]; Chapter II, § 1),

$$\vartheta(\boldsymbol{z} + \pi\mathrm{T}\boldsymbol{m} + \pi\boldsymbol{n}) = \exp(-\pi i\,{}^t\boldsymbol{m}\mathrm{T}\boldsymbol{m} - 2i\,{}^t\boldsymbol{m}\boldsymbol{z})\vartheta(\boldsymbol{z}), \qquad \boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^g.$$

Along with the function (0.14) we consider its shifts by half-periods of the lattice $\mathfrak{L}_{\mathrm{T}}$, *theta functions with characteristics*

$$\begin{aligned}\vartheta_{\boldsymbol{a}}(\boldsymbol{z}, \mathrm{T}) &= \sum_{\boldsymbol{n}\in\mathbb{Z}^g} \exp\bigl(\pi i\,{}^t\bigl(\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{a}'\bigr)\mathrm{T}\bigl(\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{a}'\bigr) + 2i\,{}^t\bigl(\boldsymbol{n} + \tfrac{1}{2}\boldsymbol{a}'\bigr)\bigl(\boldsymbol{z} + \tfrac{\pi}{2}\boldsymbol{a}''\bigr)\bigr)\\ &= \exp\bigl(\tfrac{\pi i}{4}\,{}^t\boldsymbol{a}'\mathrm{T}\boldsymbol{a}' + i\,{}^t\boldsymbol{a}'\bigl(\boldsymbol{z} + \tfrac{\pi}{2}\boldsymbol{a}''\bigr)\bigr)\vartheta\bigl(\boldsymbol{z} + \tfrac{\pi}{2}\mathrm{T}\boldsymbol{a}' + \tfrac{\pi}{2}\boldsymbol{a}'', \mathrm{T}\bigr),\\ &\qquad \boldsymbol{a} = (\boldsymbol{a}', \boldsymbol{a}'') \in \mathbb{Z}^{2g},\end{aligned} \tag{0.15}$$

which we shall also call theta functions. The property of the quasi-periodicity of the functions (0.15) with respect to the lattice $\mathfrak{L}_{\mathrm{T}}$ (see [3]; Chapter II, § 1) allows one to consider only functions with characteristics $\boldsymbol{a} \in \mathbb{Z}^{2g}/2\mathbb{Z}^{2g}$. In particular, for $g = 1$ we obtain the collection of functions (0.1) (where $\vartheta_1(z, q)$ has a constant coefficient). We point out at the very beginning that we shall often think of the set $\mathfrak{K} = \mathbb{Z}^{2g}/2\mathbb{Z}^{2g}$, its quotients and subsets as of systems of $\mathbb{Z}^{2g}$-vectors with components zeros and ones.

Similarly to the one-dimensional case we shall drop the parameter T in our notation for theta functions and define the *thetanulls* as the values of even theta functions (0.15) at the point $\boldsymbol{z} = \boldsymbol{0}$. Since

$$\vartheta_{\boldsymbol{a}}(-\boldsymbol{z}) = (-1)^{|\boldsymbol{a}|}\vartheta_{\boldsymbol{a}}(\boldsymbol{z}), \quad \text{where} \quad |\boldsymbol{a}| = {}^t\boldsymbol{a}'\boldsymbol{a}'' \pmod 2, \quad \boldsymbol{a} = (\boldsymbol{a}', \boldsymbol{a}'') \in \mathbb{Z}^{2g}$$

(see [1]; Teil 2, Kapitel 7, § 1), the function $\vartheta_{\boldsymbol{a}}(\boldsymbol{z})$ is even if and only if the integer $|\boldsymbol{a}|$ is even and it is odd otherwise (see [2]; Chapter V, § 1, Theorem 1). We say that the characteristics of even theta functions are *even* and denote the set of even characteristics by $\mathfrak{K}^*$, and we say that the characteristics of odd theta functions are *odd* (and leave the corresponding set without notation of its own). A simple calculation shows (see [1], Teil 2, Kapitel 7, § 3, Satz IV) that the number of even characteristics is $2^{g-1}(2^g + 1)$.

We shall consider the differential operators

$$\delta_{jj} = \frac{1}{\pi i}\frac{\partial}{\partial\tau_{jj}}, \quad j = 1, \ldots, g, \qquad \delta_{jk} = \frac{1}{2\pi i}\frac{\partial}{\partial\tau_{jk}} = \delta_{kj}, \quad j, k = 1, \ldots, g,\ j \neq k,$$

and various partial 'logarithmic' derivatives of the thetanulls:

$$\psi_{\boldsymbol{a},jk} = \psi_{\boldsymbol{a},jk}(\mathrm{T}) = \frac{\delta_{jk}\vartheta_{\boldsymbol{a}}}{\vartheta_{\boldsymbol{a}}} = \psi_{\boldsymbol{a},kj}, \qquad \boldsymbol{a} \in \mathfrak{K}^*, \quad j,k = 1,\dots,g. \tag{0.16}$$

Theorem 1. *The functions* (0.16) *satisfy the system of differential equations*

$$\begin{aligned}\vartheta_{\boldsymbol{a}}^4 \cdot \delta_{ls}\psi_{\boldsymbol{a},jk} = -\vartheta_{\boldsymbol{a}}^4 \cdot \psi_{\boldsymbol{a},ls}\psi_{\boldsymbol{a},jk} - \frac{1}{N_{jkls}}\vartheta_{\boldsymbol{a}}^4 \cdot \sum_{j_1,k_1,l_1,s_1}^{*} \psi_{\boldsymbol{a},j_1k_1}\psi_{\boldsymbol{a},l_1s_1} \\ + \frac{1}{N_{jkls}\cdot 2^{g-2}} \sum_{\boldsymbol{b}\in\mathfrak{K}^*} (-1)^{|\boldsymbol{a},\boldsymbol{b}|}\vartheta_{\boldsymbol{b}}^4 \cdot \sum_{j_1,k_1,l_1,s_1}^{*} \psi_{\boldsymbol{b},j_1k_1}\psi_{\boldsymbol{b},l_1s_1}, \\ \boldsymbol{a} \in \mathfrak{K}^*, \qquad j,k,l,s = 1,\dots,g, \end{aligned} \tag{0.17}$$

$$|\boldsymbol{a},\boldsymbol{b}| = {}^t\boldsymbol{a}'\boldsymbol{b}'' - {}^t\boldsymbol{b}'\boldsymbol{a}'' \pmod 2, \qquad \boldsymbol{a} = (\boldsymbol{a}',\boldsymbol{a}'') \in \mathbb{Z}^{2g}, \quad \boldsymbol{b} = (\boldsymbol{b}',\boldsymbol{b}'') \in \mathbb{Z}^{2g},$$

where $\sum^*$ *is summation over all possible permutations of the set* $\{j,k,l,s\}$ *and* N_{jkls} *is the number of these permutations.*

The differential closure (with respect to δ_{jk}-differentiations, $j,k = 1,\dots,g$) of the field generated over $\mathbb{C}(\mathrm{T}) = \mathbb{C}(\tau_{jk})_{j,k=1,\dots,g}$ by the thetanulls has, in view of Theorem 1, a finite transcendence degree over $\mathbb{C}(\mathrm{T})$. A precise calculation of the transcendence degree of this differential field has been carried out in a joint paper of Bertrand and the present author.

Theorem [21]**.** *The transcendence degree over* $\mathbb{C}(\mathrm{T})$ *of the differential closure* (*with respect to* δ_{jk}*-differentiations,* $j,k = 1,\dots,g$) *of the field generated over* $\mathbb{C}(\mathrm{T})$ *by the thetanulls is* $2g^2 + g$.

The system (0.17) can be written in a more compact form if one takes into consideration the quadratic (in $\boldsymbol{z} \in \mathbb{C}^g$) forms

$$\psi_{\boldsymbol{a}} = \psi_{\boldsymbol{a}}(\boldsymbol{z}) = {}^t\boldsymbol{z}\Psi_{\boldsymbol{a}}\boldsymbol{z}, \quad \Psi_{\boldsymbol{a}} = (\psi_{\boldsymbol{a},jk})_{j,k=1,\dots,g}, \qquad \boldsymbol{a} \in \mathfrak{K}^*, \tag{0.18}$$

$$\delta = \delta(\boldsymbol{z}) = {}^t\boldsymbol{z}\Delta\boldsymbol{z}, \qquad \Delta = (\delta_{jk})_{j,k=1,\dots,g}. \tag{0.19}$$

Of course, δ in (0.19) has sense only when applied to an appropriate object: a function meromorphic in $\mathrm{T} \in \mathfrak{H}_g$ or a quadratic form in $\boldsymbol{z}$ with coefficients that are T-meromorphic functions (only these cases are of interest for us). We can define in a natural way the operations of multiplication of quadratic forms by a T-meromorphic function and of (tensor) multiplication of two quadratic forms — which produces a homogeneous polynomial of degree 4 in $\boldsymbol{z}$ (a quartic). In this notation the system (0.17) can be written as an equality of quartics:

$$\vartheta_{\boldsymbol{a}}^4 \cdot \delta\psi_{\boldsymbol{a}} = \frac{1}{2^{g-2}} \sum_{\boldsymbol{b}\in\mathfrak{K}^*} (-1)^{|\boldsymbol{a},\boldsymbol{b}|}\vartheta_{\boldsymbol{b}}^4 \cdot \psi_{\boldsymbol{b}}^2 - 2\vartheta_{\boldsymbol{a}}^4 \cdot \psi_{\boldsymbol{a}}^2, \qquad \boldsymbol{a} \in \mathfrak{K}^*. \tag{0.20}$$

Despite the symmetry of (0.20), it is only remotely similar to Halphen's system: first, the functions involved in (0.20) have plenty of algebraic relations between them, and second, the singular locus of this system contains the zero loci of all thetanulls (for information about the latter the reader is advised to address [2]; Chapter 5).

However, it has been shown in [22] and [23] that for $g = 2$ one can write down a system similar to (0.7).

Ohyama's theorem. *Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4 \in \mathbb{Z}^4/2\mathbb{Z}^4$ be even characteristics located 'at the vertices of a non-degenerate parallelogram' (that is, $\boldsymbol{a}_1 - \boldsymbol{a}_2 \equiv \boldsymbol{a}_3 - \boldsymbol{a}_4 \pmod{2\mathbb{Z}^4}$). Then the corresponding functions $\psi_{\boldsymbol{a}_1}, \psi_{\boldsymbol{a}_2}, \psi_{\boldsymbol{a}_3}, \psi_{\boldsymbol{a}_4}$ satisfy the differential equation*

$$\delta(\psi_{\boldsymbol{a}_1}+\psi_{\boldsymbol{a}_2}+\psi_{\boldsymbol{a}_3}+\psi_{\boldsymbol{a}_4}) = (\psi_{\boldsymbol{a}_1}+\psi_{\boldsymbol{a}_2}+\psi_{\boldsymbol{a}_3}+\psi_{\boldsymbol{a}_4})^2 - 2(\psi_{\boldsymbol{a}_1}^2+\psi_{\boldsymbol{a}_2}^2+\psi_{\boldsymbol{a}_3}^2+\psi_{\boldsymbol{a}_4}^2). \quad (0.21)$$

Unfortunately, no system of differential equations for the functions (0.18) similar to Halphen's is known so far for $g > 2$. If $g = 3$, then our results (see § 6) give one a system of differential equations in which the δ-derivatives of the functions (0.18) are expressed as rational functions of these functions, with denominators of degrees at most 9.

Our exposition is organized as follows. In § 1 we present two proofs of Halphen's theorem. Our attention to this result is not accidental: in subsequent sections we draw an analogy between these proofs and the derivation of systems of differential equations in the cases $g = 2$ and $g = 3$. In § 2 and § 3 we list properties of multidimensional theta functions necessary for what follows, and in § 4 we prove Theorem 1. The case $g = 2$ (but not only it) is discussed in § 5 and § 6, where among other things we prove Ohyama's theorem. In § 6 we also discuss the case $g = 3$ and the problems of the derivation of systems of differential equations 'without denominators' for the functions (0.18), and in § 7 we present another example of a system of differential equations, of a distinct 'structure' from (0.20). We devote § 8 to modular properties of the functions (0.16) and multidimensional generalizations of relations (0.8).

The author is grateful to Prof. Yu. V. Nesterenko, who initiated my multidimensional activities in theta functions, for his permanent attention and to Prof. D. Bertrand, fruitful discussions with whom were not only useful for the present work, but have also brought us to the joint paper [21]. This work was carried out as the author was at a post-doctoral position at Institut de Mathématiques, Université Paris VI and Centre É. Borel, Institut H. Poincaré, Paris. The author is obliged to the staff of these institutions for their hospitality and friendly working ambience. Separate thanks are due to the Ostrowski Fellowship for the sponsorship of the work.

§ 1. Two proofs of Halphen's theorem

Heat equation (0.2), which holds for each theta function contains information about the expansions of the functions (0.1) in power series in z.

Lemma 1. *The following expansions hold in the neighbourhood of $z = 0$:*

$$\vartheta_j(z) = \vartheta_j \cdot \Big(1 - 2\psi_j z^2 + \frac{2}{3}(\psi_j^2 + \delta\psi_j)z^4\Big) + O(z^6), \qquad j = 2, 3, 4.$$

Proof. We write down even theta functions (0.1) as the sums of the first terms of their Taylor expansions at $z = 0$:

$$\vartheta_j(z) = \vartheta_j \cdot (1 + a_j z^2 + b_j z^4) + O(z^6), \qquad j = 2, 3, 4,$$

where $a_j = a_j(q)$ and $b_j = b_j(q)$ are functions of the q-variable; after that we use heat equation (0.2):

$$\left.\begin{aligned} -\frac{1}{4}\vartheta_j''(z) &= -\frac{1}{2}\vartheta_j \cdot (a_j + 6b_j z^2) + O(z^4), \\ \delta\vartheta_j(z) &= \delta\vartheta_j + (\delta\vartheta_j \cdot a_j + \vartheta_j \cdot \delta a_j)z^2 + O(z^4), \end{aligned}\right\} \qquad j = 2,3,4;$$

hence

$$-\frac{1}{2}\vartheta_j \cdot (a_j + 6b_j z^2) + o(z^2) = \delta\vartheta_j + (\delta\vartheta_j \cdot a_j + \vartheta_j \cdot \delta a_j)z^2 + O(z^4), \qquad j = 2,3,4.$$

Dividing both parts by ϑ_j, $j = 2,3,4$, we obtain

$$-\frac{1}{2}a_j - 3b_j z^2 + O(z^4) = \psi_j + (\psi_j a_j + \delta a_j)z^2 + O(z^4), \qquad j = 2,3,4.$$

Comparing the coefficients of similar degrees of z we see that

$$a_j = -2\psi_j, \quad b_j = -\frac{1}{3}(\psi_j a_j + \delta a_j) = \frac{2}{3}(\psi_j^2 + \delta\psi_j), \qquad j = 2,3,4, \tag{1.1}$$

as required.

Proof of Halphen's theorem. We shall use only the addition formulae

$$\begin{aligned} \vartheta_2(z+y)\vartheta_2(z-y)\vartheta_2^2 &= \vartheta_3^2(z)\vartheta_3^2(y) - \vartheta_4^2(z)\vartheta_4^2(y), \\ \vartheta_3(z+y)\vartheta_3(z-y)\vartheta_3^2 &= \vartheta_2^2(z)\vartheta_2^2(y) + \vartheta_4^2(z)\vartheta_4^2(y), \\ \vartheta_4(z+y)\vartheta_4(z-y)\vartheta_4^2 &= \vartheta_3^2(z)\vartheta_3^2(y) - \vartheta_2^2(z)\vartheta_2^2(y) \end{aligned} \tag{1.2}$$

(see [3]; Chapter I, § 5, Table 3), or more precisely, the duplication formulae

$$\begin{gathered} \vartheta_2(2z)\vartheta_2^3 = \vartheta_3^4(z) - \vartheta_4^4(z), \qquad \vartheta_3(2z)\vartheta_3^3 = \vartheta_2^4(z) + \vartheta_4^4(z), \\ \vartheta_4(2z)\vartheta_4^3 = \vartheta_3^4(z) - \vartheta_2^4(z), \end{gathered} \tag{1.3}$$

which can be deduced from (1.2) by setting $y = z$.

Taking account of the expansions

$$\left.\begin{aligned} \vartheta_j(2z) &= \vartheta_j \cdot (1 + 4a_j z^2 + 16b_j z^4) + o(z^4), \\ \vartheta_j^4(z) &= \vartheta_j^4 \cdot (1 + 4a_j z^2 + (6a_j^2 + 4b_j)z^4) + o(z^4), \end{aligned}\right\} \qquad j = 2,3,4$$

(the coefficients a_j and b_j are as defined in (1.1)), we set equal the coefficients of z^0, z^2, z^4 in duplication formulae (1.3):

$$\vartheta_2^4 - \vartheta_3^4 + \vartheta_4^4 = 0, \qquad \vartheta_2^4 a_2 - \vartheta_3^4 a_3 + \vartheta_4^4 a_4 = 0, \tag{1.4}$$

$$\begin{aligned} 16\vartheta_2^4 b_2 &= \vartheta_3^4(6a_3^2 + 4b_3) - \vartheta_4^4(6a_4^2 + 4b_4), \\ 16\vartheta_3^4 b_3 &= \vartheta_2^4(6a_2^2 + 4b_2) + \vartheta_4^4(6a_4^2 + 4b_4), \\ 16\vartheta_4^4 b_4 &= \vartheta_3^4(6a_3^2 + 4b_3) - \vartheta_2^4(6a_2^2 + 4b_2). \end{aligned} \tag{1.5}$$

Solving the *linear* system for equations (1.5) with respect to $\vartheta_j^4 b_j$, $j=2,3,4$, we obtain

$$\begin{aligned}\vartheta_2^4 b_2 &= \frac{1}{6}(\vartheta_2^4 a_2^2 + 2\vartheta_3^4 a_3^2 - 2\vartheta_4^4 a_4^2),\\ \vartheta_3^4 b_3 &= \frac{1}{6}(2\vartheta_2^4 a_2^2 + \vartheta_3^4 a_3^2 + 2\vartheta_4^4 a_4^2),\\ \vartheta_4^4 b_4 &= \frac{1}{6}(-2\vartheta_2^4 a_2^2 + 2\vartheta_3^4 a_3^2 + \vartheta_4^4 a_4^2).\end{aligned} \tag{1.6}$$

Substituting in (1.6) equalities (1.1) established in Lemma 1 we obtain a system of differential equations connecting the functions (0.5):

$$\begin{aligned}\frac{2}{3}\vartheta_2^4(\psi_2^2+\delta\psi_2) &= \frac{2}{3}(\vartheta_2^4\psi_2^2 + 2\vartheta_3^4\psi_3^2 - 2\vartheta_4^4\psi_4^2),\\ \frac{2}{3}\vartheta_3^4(\psi_3^2+\delta\psi_3) &= \frac{2}{3}(2\vartheta_2^4\psi_2^2 + \vartheta_3^4\psi_3^2 + 2\vartheta_4^4\psi_4^2),\\ \frac{2}{3}\vartheta_4^4(\psi_4^2+\delta\psi_4) &= \frac{2}{3}(-2\vartheta_2^4\psi_2^2 + 2\vartheta_3^4\psi_3^2 + \vartheta_4^4\psi_4^2);\end{aligned}$$

hence

$$\delta\psi_2 = 2\biggl(\frac{\vartheta_3^4}{\vartheta_2^4}\psi_3^2 - \frac{\vartheta_4^4}{\vartheta_2^4}\psi_4^2\biggr), \quad \delta\psi_3 = 2\biggl(\frac{\vartheta_2^4}{\vartheta_3^4}\psi_2^2 + \frac{\vartheta_4^4}{\vartheta_3^4}\psi_4^2\biggr), \quad \delta\psi_4 = 2\biggl(\frac{\vartheta_3^4}{\vartheta_4^4}\psi_3^2 - \frac{\vartheta_2^4}{\vartheta_4^4}\psi_2^2\biggr). \tag{1.7}$$

We can now write (1.4) as follows:

$$\vartheta_2^4 - \vartheta_3^4 + \vartheta_4^4 = 0, \qquad \vartheta_2^4\psi_2 - \vartheta_3^4\psi_3 + \vartheta_4^4\psi_4 = 0.$$

This gives us, in particular, expressions for the ratios of the fourth powers of the theta functions:

$$\frac{\vartheta_3^4}{\vartheta_2^4} = \frac{\psi_2-\psi_4}{\psi_3-\psi_4}, \qquad \frac{\vartheta_4^4}{\vartheta_2^4} = \frac{\psi_2-\psi_3}{\psi_3-\psi_4}, \tag{1.8}$$

which easily allows us to deduce the system (0.7) from (1.7). For example, for the first equation in (1.7) we have

$$\begin{aligned}\delta\psi_2 &= 2\biggl(\frac{\psi_2-\psi_4}{\psi_3-\psi_4}\psi_3^2 - \frac{\psi_2-\psi_3}{\psi_3-\psi_4}\psi_4^2\biggr) = 2\frac{\psi_2(\psi_3^2-\psi_4^2) - \psi_3\psi_4(\psi_3-\psi_4)}{\psi_3-\psi_4}\\ &= 2(\psi_2\psi_3 + \psi_2\psi_4 - \psi_3\psi_4);\end{aligned}$$

in a similar way we process the second and the third equations in (1.7). To complete the proof of the theorem we point out again at the equivalence of the systems (0.7) and (0.6).

We shall now prove identities (0.8). To this end we write (1.8) as follows:

$$\frac{\psi_3-\psi_4}{\vartheta_2^4} = \frac{\psi_2-\psi_4}{\vartheta_3^4} = \frac{\psi_2-\psi_3}{\vartheta_4^4} = \varkappa(q); \tag{1.9}$$

we shall use the system of differential equations (0.7) just established and the relations

$$\delta(\vartheta_j^{-4}) = -4\vartheta_j^{-5}\cdot\delta\vartheta_j = -4\vartheta_j^{-4}\psi_j, \qquad j=2,3,4,$$

to prove that $\varkappa(q)=\text{const}$. This is a consequence of the following chain of equalities:

$$\begin{aligned}\delta\varkappa &= \delta\bigl(\vartheta_2^{-4}(\psi_3-\psi_4)\bigr) = -\vartheta_2^{-4}(4\psi_2(\psi_3-\psi_4)+\delta\psi_3-\delta\psi_4)\\ &= -2\vartheta_2^{-4}\bigl(2\psi_2(\psi_3-\psi_4)+(\psi_2\psi_3+\psi_3\psi_4-\psi_2\psi_4)-(\psi_2\psi_4+\psi_3\psi_4-\psi_2\psi_3)\bigr)=0.\end{aligned}$$

Note now that $\vartheta_3=1+2q+O(q^2)$, $\vartheta_4=1-2q+O(q^2)$, and therefore $\psi_3=2q+O(q^2)$, $\psi_4=-2q+O(q^2)$, and $\vartheta_2^4=16q+O(q^2)$; hence

$$\varkappa=\frac{\psi_3-\psi_4}{\vartheta_2^4}=\frac{1}{4}+O(q),$$

that is, $\varkappa=1/4$. This proves identities (0.8).

A very simple proof of Halphen's theorem. The general addition formula for theta functions yields the relations

$$\vartheta_j^2(z)\vartheta_k^2(z)=\frac{1}{2}\bigl(\vartheta_j(2z)\vartheta_j(0)\vartheta_k^2(0)+\vartheta_j^2(0)\vartheta_k(2z)\vartheta_k(0)\bigr), \qquad j,k=2,3,4, \quad j\ne k \tag{1.10}$$

(see [6]; Exercises 1 and 2 to Chapter 21).

By Lemma 1,

$$\left.\begin{aligned}\vartheta_j^2(z)\vartheta_k^2(z) &= \vartheta_j^2\vartheta_k^2\cdot\Bigl(1-4(\psi_j+\psi_k)z^2\\ &\qquad+16\psi_j\psi_k z^4+\frac{16}{3}(\psi_j^2+\psi_k^2)z^4+\frac{4}{3}\delta(\psi_j+\psi_k)z^4\Bigr)+O(z^6),\\ \frac{\vartheta_j(2z)\vartheta_j\vartheta_k^2+\vartheta_j^2\vartheta_k(2z)\vartheta_k}{2} &= \vartheta_j^2\vartheta_k^2\cdot\Bigl(1-4(\psi_j+\psi_k)z^2\\ &\qquad+\frac{16}{3}(\psi_j^2+\psi_k^2)z^4+\frac{16}{3}\delta(\psi_j+\psi_k)z^4\Bigr)+O(z^6),\end{aligned}\right\} \quad j,k=2,3,4. \tag{1.11}$$

Substituting the expansions (1.11) in identities (1.10) and comparing the coefficients of z^4 we obtain the system of differential equations (0.6).

Remark. Of course, Halphen himself gives in [12] a simple proof of his theorem. The main advantage of the 'long' proof are identities (0.8) obtained on its basis.

§ 2. Expansions of theta functions

Let g be a fixed dimension, let g-dimensional theta functions (with characteristics) be as defined in (0.15), and let $\mathfrak{K}^*$ be the set of even characteristics. In this section we prove the following multidimensional generalization of Lemma 1.

Lemma 2. *The following expansions hold in the neighbourhood of* $\boldsymbol{z}=\boldsymbol{0}$:

$$\vartheta_{\boldsymbol{a}}(\boldsymbol{z})=\vartheta_{\boldsymbol{a}}\cdot\Big(1-2\psi_{\boldsymbol{a}}+\frac{2}{3}\psi_{\boldsymbol{a}}^2+\frac{2}{3}\delta\psi_{\boldsymbol{a}}\Big)+O(\boldsymbol{z}^6),\qquad \boldsymbol{a}\in\mathfrak{K}^*,$$

where the quadratic forms $\psi_{\boldsymbol{a}}$*,* $\boldsymbol{a}\in\mathfrak{K}^*$*, and* δ *are defined by* (0.18) *and* (0.19)*. In addition,*

$$\delta_{jk}\psi_{\boldsymbol{a},lm}+\psi_{\boldsymbol{a},jk}\psi_{\boldsymbol{a},lm}=\delta_{jl}\psi_{\boldsymbol{a},km}+\psi_{\boldsymbol{a},jl}\psi_{\boldsymbol{a},km},\qquad \delta_{jk}\psi_{\boldsymbol{a},lm}=\delta_{lm}\psi_{\boldsymbol{a},jk},$$
$$\boldsymbol{a}\in\mathfrak{K}^*,\qquad j,k,l,m=1,\dots,g. \tag{2.1}$$

We use the symbol $O(\boldsymbol{z}^n)$, $n=0,1,2,\dots$, to denote the class of functions $f(\boldsymbol{z})$ holomorphic at $\boldsymbol{z}=\boldsymbol{0}$ with Taylor series starting from power n, that is,

$$\frac{\partial^{m_1+\dots+m_g}f}{\partial z_1^{m_1}\cdots\partial z_g^{m_g}}(\boldsymbol{0})=0,\qquad m_1+\dots+m_g<n.$$

It is easy to verify that $O(\boldsymbol{z}^n)+O(\boldsymbol{z}^k)=O(\boldsymbol{z}^{\min\{n,k\}})$ and $O(\boldsymbol{z}^n)\cdot O(\boldsymbol{z}^k)=O(\boldsymbol{z}^{n+k})$ (here we use the standard notation for O-symbols).

As in the one-dimensional case, for the proof of Lemma 2 we require a multidimensional analogue of the heat equation for the functions (0.15).

Lemma 3 ([1], Teil 1, Kapitel 1, § 5, Satz XIII). *Each function* (0.15) *satisfies the differential equations*

$$\delta_{jk}\vartheta_{\boldsymbol{a}}(\boldsymbol{z},\mathrm{T})=-\frac{1}{4}\frac{\partial^2}{\partial z_j\,\partial z_k}\vartheta_{\boldsymbol{a}}(\boldsymbol{z},\mathrm{T}),\qquad j,k=1,\dots,g,\quad \boldsymbol{a}\in\mathbb{Z}^{2g}. \tag{2.2}$$

This can be verified directly for the series (0.15).

The following technical result has no immediate relation to theta functions.

Lemma 4. *Let* $V(\boldsymbol{z})$ *be a homogeneous polynomial of degree four in the variables* $z_1,\dots,z_g$*; let* $\Phi=(\varphi_{jk})_{j,k=1,\dots,g}$ *and* $\Psi=(\psi_{jk})_{j,k=1,\dots,g}$ *be symmetric square matrices. Assume that*

$$\frac{\partial^2V}{\partial z_j\,\partial z_k}=\varphi_{jk}\cdot{}^t\boldsymbol{z}\Psi\boldsymbol{z},\qquad j,k=1,\dots,g. \tag{2.3}$$

Then

$$V(\boldsymbol{z})=\frac{1}{12}({}^t\boldsymbol{z}\Phi\boldsymbol{z})({}^t\boldsymbol{z}\Psi\boldsymbol{z}) \tag{2.4}$$

and

$$\varphi_{jk}\psi_{lm}=\varphi_{jl}\psi_{km}=\varphi_{jm}\psi_{kl}=\varphi_{kl}\psi_{jm}=\varphi_{km}\psi_{jl}=\varphi_{lm}\psi_{jk},$$
$$j,k,l,m=1,\dots,g. \tag{2.5}$$

Proof. We represent the polynomial $V(\boldsymbol{z})$ in the 'symmetric' form

$$V(\boldsymbol{z})=\sum_{l_1,l_2,l_3,l_4=1}^{g}v_{l_1l_2l_3l_4}z_{l_1}z_{l_2}z_{l_3}z_{l_4}, \tag{2.6}$$

where $v_{l_1l_2l_3l_4} = v_{\sigma(l_1l_2l_3l_4)}$ for each permutation σ of the indices l_1, l_2, l_3, l_4. Then

$$\begin{aligned}
\frac{\partial^2 V}{\partial z_j\,\partial z_k} &= \frac{\partial^2}{\partial z_j\,\partial z_k} \sum_{l_1,l_2,l_3,l_4=1}^{g} v_{l_1l_2l_3l_4} z_{l_1} z_{l_2} z_{l_3} z_{l_4} \\
&= \sum_{l_3,l_4=1}^{g} v_{jkl_3l_4} z_{l_3} z_{l_4} + \sum_{l_2,l_4=1}^{g} v_{jl_2kl_4} z_{l_2} z_{l_4} + \dots + \sum_{l_1,l_2=1}^{g} v_{l_1l_2jk} z_{l_1} z_{l_2} \\
&= \sum_{l_3,l_4=1}^{g} (v_{jkl_3l_4} + v_{jl_3kl_4} + \dots + v_{l_3l_4jk}) z_{l_3} z_{l_4} = 12 \sum_{l_3,l_4=1}^{g} v_{jkl_3l_4} z_{l_3} z_{l_4}.
\end{aligned}$$

Comparing the coefficients of the resulting quadratic forms and the right-hand sides of (2.3) we obtain

$$v_{jkl_3l_4} = \frac{1}{12}\varphi_{jk}\psi_{l_3l_4}, \qquad j,k,l_3,l_4 = 1,\dots,g. \tag{2.7}$$

Hence

$$\begin{aligned}
V(\boldsymbol{z}) &= \frac{1}{12} \sum_{l_1,l_2,l_3,l_4=1}^{g} \varphi_{l_1l_2}\psi_{l_3l_4} z_{l_1} z_{l_2} z_{l_3} z_{l_4} \\
&= \frac{1}{12}\biggl(\sum_{l_1,l_2=1}^{g} \varphi_{l_1l_2} z_{l_1} z_{l_2}\biggr)\biggl(\sum_{l_3,l_4=1}^{g} \psi_{l_3l_4} z_{l_3} z_{l_4}\biggr) = \frac{1}{12}({}^t\boldsymbol{z}\Phi\boldsymbol{z})({}^t\boldsymbol{z}\Psi\boldsymbol{z}),
\end{aligned}$$

which proves (2.4). Equalities (2.5) follow by (2.7) and the symmetry of the coefficients of (2.6).

Proof of Lemma 2. This proof does not depend on the particular even characteristic $\boldsymbol{a}$, therefore we shall not indicate it by the subscript.

The expansion of an even function in a power series has the following form:

$$\vartheta(\boldsymbol{z}) = \vartheta \cdot \bigl(1 + U(\boldsymbol{z}) + V(\boldsymbol{z})\bigr) + O(\boldsymbol{z}^6), \qquad \vartheta = \vartheta(\boldsymbol{0}), \tag{2.8}$$

where $U(\boldsymbol{z})$ is a quadratic form (a homogeneous polynomial of degree 2), $V(\boldsymbol{z})$ is a quartic (a homogeneous polynomial of degree 4), and the coefficients of these forms are meromorphic functions of $\mathrm{T} \in \mathfrak{H}_g$.

We apply Lemma 3 to the expansion (2.8). We obtain

$$\left.\begin{aligned}
\delta_{jk}\vartheta(\boldsymbol{z}) &= \delta_{jk}\vartheta + \delta_{jk}\vartheta \cdot U(\boldsymbol{z}) + \vartheta \cdot \delta_{jk}U(\boldsymbol{z}) + O(\boldsymbol{z}^4), \\
\frac{\partial^2}{\partial z_j\,\partial z_k}\vartheta(\boldsymbol{z}) &= \vartheta \cdot \biggl(\frac{\partial^2 U(\boldsymbol{z})}{\partial z_j\,\partial z_k} + \frac{\partial^2 V(\boldsymbol{z})}{\partial z_j\,\partial z_k}\biggr) + O(\boldsymbol{z}^4),
\end{aligned}\right. \qquad j,k = 1,\dots,g,$$

therefore (we divide both parts of (2.2) by ϑ)

$$\begin{gathered}
\psi_{jk} + \psi_{jk}U(\boldsymbol{z}) + \delta_{jk}U(\boldsymbol{z}) + O(\boldsymbol{z}^4) = -\frac{1}{4}\biggl(\frac{\partial^2 U(\boldsymbol{z})}{\partial z_j\,\partial z_k} + \frac{\partial^2 V(\boldsymbol{z})}{\partial z_j\,\partial z_k}\biggr) + O(\boldsymbol{z}^4), \\
j,k = 1,\dots,g.
\end{gathered} \tag{2.9}$$

Comparing homogeneous components in (2.9) we obtain

$$\frac{\partial^2 U(\boldsymbol{z})}{\partial z_j\,\partial z_k} = -4\psi_{jk}, \qquad j,k = 1,\dots,g, \tag{2.10}$$

$$\frac{\partial^2 V(\boldsymbol{z})}{\partial z_j\,\partial z_k} = -4\bigl(\psi_{jk}U(\boldsymbol{z}) + \delta_{jk}U(\boldsymbol{z})\bigr), \qquad j,k = 1,\dots,g. \tag{2.11}$$

Relations (2.10) mean that

$$U(\boldsymbol{z}) = -2{}^t\boldsymbol{z}\Psi\boldsymbol{z} = -2\psi. \tag{2.12}$$

The substitution of (2.12) in (2.11) yields

$$\frac{\partial^2 V(\boldsymbol{z})}{\partial z_j\,\partial z_k} = 8(\psi_{jk} + \delta_{jk}){}^t\boldsymbol{z}\Psi\boldsymbol{z}, \qquad j,k = 1,\dots,g,$$

therefore by Lemma 4,

$$V(\boldsymbol{z}) = \frac{2}{3}({}^t\boldsymbol{z}\Psi\boldsymbol{z})^2 + \frac{2}{3}({}^t\boldsymbol{z}\Delta\boldsymbol{z})({}^t\boldsymbol{z}\Psi\boldsymbol{z}) = \frac{2}{3}\psi^2 + \frac{2}{3}\delta\psi$$

and

$$(\psi_{jk} + \delta_{jk})\psi_{lm} = (\psi_{jl} + \delta_{jl})\psi_{km} = \cdots = (\psi_{lm} + \delta_{lm})\psi_{jk}, \qquad j,k,l,m = 1,\dots,g.$$

The proof of Lemma 2 is complete.

§ 3. Riemann relations

This is the formula used in most applications.

D. Mumford

The quotation from [3] prefacing this section relates to the *Riemann relations* [3]; Chapter II, § 6, which have the following form for the theta functions (0.15):

$$\begin{aligned}
&(-1)^{{}^t(\boldsymbol{c}'+\boldsymbol{d}')\boldsymbol{a}''}\vartheta_{\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}}\left(\frac{\boldsymbol{z}_1+\boldsymbol{z}_2+\boldsymbol{z}_3+\boldsymbol{z}_4}{2}\right)\vartheta_{\boldsymbol{a}+\boldsymbol{c}}\left(\frac{\boldsymbol{z}_1+\boldsymbol{z}_2-\boldsymbol{z}_3-\boldsymbol{z}_4}{2}\right)\\
&\quad\times\vartheta_{\boldsymbol{a}+\boldsymbol{d}}\left(\frac{\boldsymbol{z}_1-\boldsymbol{z}_2+\boldsymbol{z}_3-\boldsymbol{z}_4}{2}\right)\vartheta_{\boldsymbol{a}}\left(\frac{\boldsymbol{z}_1-\boldsymbol{z}_2-\boldsymbol{z}_3+\boldsymbol{z}_4}{2}\right)\\
&\quad=\frac{1}{2^g}\sum_{\boldsymbol{b}\in\mathfrak{K}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{{}^t(\boldsymbol{c}'+\boldsymbol{d}')\boldsymbol{b}''}\vartheta_{\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}}(\boldsymbol{z}_1)\vartheta_{\boldsymbol{b}+\boldsymbol{c}}(\boldsymbol{z}_2)\vartheta_{\boldsymbol{b}+\boldsymbol{d}}(\boldsymbol{z}_3)\vartheta_{\boldsymbol{b}}(\boldsymbol{z}_4),
\end{aligned} \tag{3.1}$$

$$\boldsymbol{a},\boldsymbol{c},\boldsymbol{d} \in \mathfrak{K} = \mathbb{Z}^{2g}/2\mathbb{Z}^{2g}$$

(see [1]; Teil 2, Kapitel VII, § 10, Satz XXXVIII). In fact our main attention in what follows will be focused on a special case of the Riemann relations:

$$\begin{aligned}
&(-1)^{{}^t(\boldsymbol{c}'+\boldsymbol{d}')\boldsymbol{a}''}\vartheta_{\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}}(\boldsymbol{z})\vartheta_{\boldsymbol{a}+\boldsymbol{c}}(\boldsymbol{z})\vartheta_{\boldsymbol{a}+\boldsymbol{d}}(\boldsymbol{z})\vartheta_{\boldsymbol{a}}(\boldsymbol{z})\\
&\quad=\frac{1}{2^g}\sum_{\boldsymbol{b}\in\mathfrak{K}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{{}^t(\boldsymbol{c}'+\boldsymbol{d}')\boldsymbol{b}''}\vartheta_{\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}}(2\boldsymbol{z})\vartheta_{\boldsymbol{b}+\boldsymbol{c}}(\boldsymbol{0})\vartheta_{\boldsymbol{b}+\boldsymbol{d}}(\boldsymbol{0})\vartheta_{\boldsymbol{b}}(\boldsymbol{0}),
\end{aligned} \tag{3.2}$$

$$\boldsymbol{a},\boldsymbol{c},\boldsymbol{d} \in \mathfrak{K},$$

which can be deduced from (3.1) by setting $\boldsymbol{z}_1 = 2\boldsymbol{z}$, $\boldsymbol{z}_2 = \boldsymbol{z}_3 = \boldsymbol{z}_4 = \boldsymbol{0}$.

Lemma 5. *For each even characteristic $\boldsymbol{a}$ there holds the identity*

$$\vartheta_{\boldsymbol{a}}^4(\boldsymbol{z}) = \frac{1}{2^g} \sum_{\boldsymbol{b}\in\mathfrak{K}^*} (-1)^{|\boldsymbol{a},\boldsymbol{b}|} \vartheta_{\boldsymbol{b}}(2\boldsymbol{z}) \vartheta_{\boldsymbol{b}}^3(\boldsymbol{0}). \tag{3.3}$$

Proof. It suffices to set $\boldsymbol{c} = \boldsymbol{d} = \boldsymbol{0}$ in (3.2) and to observe that all the terms on the right-hand side that correspond to odd characteristics are vanishing.

For the symmetric square matrix of order $2^{g-1}(2^g+1)$ corresponding to relations (3.3) we introduce the following notation:

$$\mathrm{M} = \frac{1}{2^g} \big((-1)^{|\boldsymbol{a},\boldsymbol{b}|}\big)_{\boldsymbol{a},\boldsymbol{b}\in\mathfrak{K}^*}.$$

To study its properties we require two simple auxiliary results (from the theory of character sums).

Lemma 6. *There hold the identities*

$$\sum_{\boldsymbol{b}\in\mathfrak{K}} (-1)^{|\boldsymbol{a},\boldsymbol{b}|} = \begin{cases} 2^{2g} & \text{if } \boldsymbol{a} = \boldsymbol{0}, \\ 0 & \text{otherwise}, \end{cases} \qquad \boldsymbol{a} \in \mathfrak{K}. \tag{3.4}$$

Proof. The case $\boldsymbol{a} = \boldsymbol{0}$ is trivial: the sum involves 2^{2g} terms, each equal to 1. If $\boldsymbol{a} \ne \boldsymbol{0}$ then we choose a characteristic $\boldsymbol{c}$ such that the quantity $|\boldsymbol{a},\boldsymbol{c}|$ is odd (it suffices to consider a characteristic with a single one at the jth place, where $\boldsymbol{a}_{j\pm g} \ne 0$) and multiply the sum on the left-hand side of (3.4) by $(-1)^{|\boldsymbol{a},\boldsymbol{c}|} = -1$:

$$\begin{aligned} -\sum_{\boldsymbol{b}\in\mathfrak{K}} (-1)^{|\boldsymbol{a},\boldsymbol{b}|} &= \sum_{\boldsymbol{b}\in\mathfrak{K}} (-1)^{|\boldsymbol{a},\boldsymbol{b}|+|\boldsymbol{a},\boldsymbol{c}|} = \sum_{\boldsymbol{b}\in\mathfrak{K}} (-1)^{{}^t\boldsymbol{a}'(\boldsymbol{b}''+\boldsymbol{c}'')-{}^t(\boldsymbol{b}'+\boldsymbol{c}')\boldsymbol{a}''} \\ &= \sum_{\boldsymbol{b}\in\mathfrak{K}} (-1)^{|\boldsymbol{a},\boldsymbol{b}+\boldsymbol{c}|} = \sum_{\boldsymbol{b}\in\mathfrak{K}} (-1)^{|\boldsymbol{a},\boldsymbol{b}|} \end{aligned} \tag{3.5}$$

(we change the order of summation $\boldsymbol{b}+\boldsymbol{c} \mapsto \boldsymbol{b}$). The right-hand and the left-hand sides of (3.5) have distinct signs, which completes the proof of (3.4).

Lemma 7. *There hold the identities*

$$\sum_{\boldsymbol{b}\in\mathfrak{K}^*} (-1)^{|\boldsymbol{a},\boldsymbol{b}|} = \begin{cases} 2^{g-1}(2^g+1) & \text{if } \boldsymbol{a} = \boldsymbol{0}, \\ (-1)^{|\boldsymbol{a}|} \cdot 2^{g-1} & \text{otherwise}, \end{cases} \qquad \boldsymbol{a} \in \mathfrak{K}. \tag{3.6}$$

Proof. If $\boldsymbol{a} = \boldsymbol{0}$, then each term on the left-hand side is equal to 1, so that the sum is equal to the number of terms. Assume that $\boldsymbol{a} \ne \boldsymbol{0}$ and consider the sums

$$S_+ = \sum_{\boldsymbol{b}\colon |\boldsymbol{b}| \text{ is even}} (-1)^{|\boldsymbol{a},\boldsymbol{b}|}, \qquad S_- = -\sum_{\boldsymbol{b}\colon |\boldsymbol{b}| \text{ is odd}} (-1)^{|\boldsymbol{a},\boldsymbol{b}|}. \tag{3.7}$$

By Lemma 6, $S_+ - S_- = 0$, that is, $S_+ = S_-$. Note now that

$$\begin{aligned}(-1)^{|\boldsymbol{a},\boldsymbol{b}|} &= (-1)^{{}^t\boldsymbol{a}'\boldsymbol{b}''-{}^t\boldsymbol{b}'\boldsymbol{a}''} = (-1)^{{}^t(\boldsymbol{a}'+\boldsymbol{b}')(\boldsymbol{a}''+\boldsymbol{b}'')}(-1)^{{}^t\boldsymbol{a}'\boldsymbol{a}''}(-1)^{{}^t\boldsymbol{b}'\boldsymbol{b}''}\\ &= (-1)^{|\boldsymbol{a}+\boldsymbol{b}|}(-1)^{|\boldsymbol{a}|}(-1)^{|\boldsymbol{b}|}.\end{aligned}$$

Hence (3.7) can be written as follows:

$$S_+ = (-1)^{|\boldsymbol{a}|}\sum_{\boldsymbol{b}\colon |\boldsymbol{b}| \text{ is even}}(-1)^{|\boldsymbol{a}+\boldsymbol{b}|}, \qquad S_- = (-1)^{|\boldsymbol{a}|}\sum_{\boldsymbol{b}\colon |\boldsymbol{b}| \text{ is odd}}(-1)^{|\boldsymbol{a}+\boldsymbol{b}|}.$$

Consequently,

$$\begin{aligned}S_+ + S_- &= (-1)^{|\boldsymbol{a}|}\sum_{\boldsymbol{b}\in\mathfrak{K}}(-1)^{|\boldsymbol{a}+\boldsymbol{b}|} = (-1)^{|\boldsymbol{a}|}\sum_{\boldsymbol{b}\in\mathfrak{K}}(-1)^{|\boldsymbol{b}|}\\ &= (-1)^{|\boldsymbol{a}|}\Bigl(\sum_{\boldsymbol{b}\colon |\boldsymbol{b}| \text{ is even}} 1 - \sum_{\boldsymbol{b}\colon |\boldsymbol{b}| \text{ is odd}} 1\Bigr)\\ &= (-1)^{|\boldsymbol{a}|}\bigl(2^{g-1}(2^g+1) - 2^{2g} + 2^{g-1}(2^g+1)\bigr) = (-1)^{|\boldsymbol{a}|}\cdot 2^g.\end{aligned}$$

Thus, $S_+ = S_- = (-1)^{|\boldsymbol{a}|}\cdot 2^{g-1}$, which completes the proof of (3.6).

Lemma 8. *The matrix* M *satisfies the relation*

$$\mathrm{M}^2 = \frac{1}{2}(\mathrm{M}+E), \tag{3.8}$$

where E is the identity matrix of order $2^{g-1}(2^g+1)$.

Proof. Using Lemma 7 we consider the scalar product of the lines of the matrix M with indices $\boldsymbol{a}$ and $\boldsymbol{c}$:

$$\begin{aligned}\frac{1}{2^{2g}}\sum_{\boldsymbol{b}\in\mathfrak{K}^*}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\cdot(-1)^{|\boldsymbol{c},\boldsymbol{b}|} &= \frac{1}{2^{2g}}\sum_{\boldsymbol{b}\in\mathfrak{K}^*}(-1)^{|\boldsymbol{a}-\boldsymbol{c},\boldsymbol{b}|}\\ &= \begin{cases} 2^{-g-1}(2^g+1) & \text{if } \boldsymbol{a}=\boldsymbol{c},\\ (-1)^{|\boldsymbol{a}-\boldsymbol{c}|}\cdot 2^{-g-1} & \text{otherwise,}\end{cases}\\ &= \begin{cases} \dfrac{1}{2^{g+1}} + \dfrac{1}{2} & \text{if } \boldsymbol{a}=\boldsymbol{c},\\ \dfrac{1}{2^{g+1}}(-1)^{|\boldsymbol{a},\boldsymbol{c}|} & \text{otherwise.}\end{cases}\end{aligned} \tag{3.9}$$

In view of the symmetry of M (the product of two lines is the same as the product of a line and the corresponding column) relations (3.9) are equivalent to matrix identity (3.8).

Corollary. *There holds the identity*

$$\Bigl(\mathrm{M}-\frac{1}{4}E\Bigr)^{-1} = \frac{16}{9}\Bigl(\mathrm{M}-\frac{1}{4}E\Bigr). \tag{3.10}$$

Proof. In fact, by Lemma 8 we obtain

$$\Bigl(\mathrm{M}-\frac{1}{4}E\Bigr)^2=\mathrm{M}^2-\frac{1}{2}\mathrm{M}+\frac{1}{16}E=\frac{9}{16}E,$$

which yields (3.10).

Remark. We shall use the scheme of the proof of Lemma 8 in what follows; this is why we prefer to carry it out in detail, without referring to literature for the ready properties of the matrix M (see, for instance, [24]; § 2).

Since M is a symmetric matrix, it has a diagonal Jordan form. Hence all eigenvalues λ of M satisfy the quadratic equation (cf. (3.8))

$$\lambda^2=\frac{1}{2}(\lambda+1),$$

so that $\lambda=1$ or $\lambda=-\frac{1}{2}$.

Lemma 9. *The multiplicities of the eigenvalues* $\lambda_1=1$ *and* $\lambda_2=-\frac{1}{2}$ *in the characteristic polynomial of* M *are*

$$s_1=\frac{(2^{g-1}+1)(2^g+1)}{3} \quad \text{and} \quad s_2=\frac{(2^g-1)(2^g+1)}{3}, \tag{3.11}$$

respectively.

Proof. This is a simple exercise in linear algebra. If s_1 and s_2 are the multiplicities of the zeros $\lambda_1=1$ and $\lambda_2=-\frac{1}{2}$, respectively (the characteristic polynomial of M has no other zeros), then $s_1+s_2=2^{g-1}(2^g+1)$. The trace of a matrix is preserved by the transition to the Jordan form. The Jordan form of a symmetric matrix M is a diagonal matrix, with diagonal containing s_1 entries equal to λ_1 and s_2 entries equal to λ_2. Hence $\operatorname{tr}\mathrm{M}=s_1\lambda_1+s_2\lambda_2=s_1-\frac{1}{2}s_2$. On the other hand the diagonal entries of M are equal to 2^{-g}, so that $\operatorname{tr}\mathrm{M}=2^{-g}\cdot2^{g-1}(2^g+1)=\frac{1}{2}(2^g+1)$. Solving the system of differential equations

$$s_1+s_2=2^{g-1}(2^g+1), \qquad s_1-\frac{1}{2}s_2=\frac{1}{2}(2^g+1),$$

we obtain the quantities in (3.11), which completes the proof.

§ 4. Proof of Theorem 1

Using Lemma 2 we can write expansions for the theta functions on the right-hand and the left-hand sides of the identities from Lemma 5:

$$\begin{aligned}\vartheta_{\boldsymbol{a}}^4(\boldsymbol{z})&=\vartheta_{\boldsymbol{a}}^4\cdot\Bigl(1-2\psi_{\boldsymbol{a}}+\frac{2}{3}\psi_{\boldsymbol{a}}^2+\frac{2}{3}\delta\psi_{\boldsymbol{a}}\Bigr)^4+O(\boldsymbol{z}^6)\\&=\vartheta_{\boldsymbol{a}}^4\cdot\Bigl(1-8\psi_{\boldsymbol{a}}+\frac{80}{3}\psi_{\boldsymbol{a}}^2+\frac{8}{3}\delta\psi_{\boldsymbol{a}}\Bigr)+O(\boldsymbol{z}^6), \qquad \boldsymbol{a}\in\mathfrak{K}^*,\\\vartheta_{\boldsymbol{b}}(2\boldsymbol{z})&=\vartheta_{\boldsymbol{b}}\cdot\Bigl(1-8\psi_{\boldsymbol{b}}+\frac{32}{3}\psi_{\boldsymbol{b}}^2+\frac{32}{3}\delta\psi_{\boldsymbol{b}}\Bigr)+O(\boldsymbol{z}^6), \qquad \boldsymbol{b}\in\mathfrak{K}^*.\end{aligned} \tag{4.1}$$

Substituting (4.1) in (3.3) and comparing homogeneous components of degree 4 of the resulting expansions we arrive at the formulae

$$\vartheta_{\boldsymbol{a}}^4 \cdot \frac{1}{3}(80\psi_{\boldsymbol{a}}^2 + 8\delta\psi_{\boldsymbol{a}}) = \frac{1}{2^g} \sum_{\boldsymbol{b}\in\mathfrak{K}^*} (-1)^{|\boldsymbol{a},\boldsymbol{b}|} \vartheta_{\boldsymbol{b}}^4 \cdot \frac{1}{3}(32\psi_{\boldsymbol{b}}^2 + 32\delta\psi_{\boldsymbol{b}}), \qquad \boldsymbol{a}\in\mathfrak{K}^*. \quad (4.2)$$

We now write relations (4.2) in the matrix form. To this end we consider the (column) vectors

$$\boldsymbol{X} = (\vartheta_{\boldsymbol{a}}^4 \cdot \psi_{\boldsymbol{a}}^2)_{\boldsymbol{a}\in\mathfrak{K}^*} \quad \text{and} \quad \boldsymbol{Y} = (\vartheta_{\boldsymbol{a}}^4 \cdot \delta\psi_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathfrak{K}^*}. \quad (4.3)$$

In the notation (4.3) relations (4.2) take the following form after simplifications:

$$\frac{5}{2}\boldsymbol{X} + \frac{1}{4}\boldsymbol{Y} = \mathrm{M}(\boldsymbol{X}+\boldsymbol{Y}),$$

therefore

$$\begin{aligned}\boldsymbol{Y} &= -\Big(\mathrm{M} - \frac{1}{4}E\Big)^{-1}\Big(\mathrm{M} - \frac{5}{2}E\Big)\boldsymbol{X} = -\boldsymbol{X} + \frac{9}{4}\Big(\mathrm{M} - \frac{1}{4}E\Big)^{-1}\boldsymbol{X} \\ &= -\boldsymbol{X} + 4\Big(\mathrm{M} - \frac{1}{4}E\Big)\boldsymbol{X} = (4\mathrm{M} - 2E)\boldsymbol{X}, \end{aligned} \quad (4.4)$$

where we use Lemma 8 and the corollary to it.

Relation (4.4), upon recalling (4.3), give one the system of differential equations (0.20). The system (0.17) follows from it and relations (2.1) in Lemma 2.

The proof of Theorem 1 is complete.

§ 5. Further generalization of Theorem 1

Exposing the (not the most simple) proof of Halphen's theorem we established in fact the one-dimensional version of Theorem 1 (the system (1.7)) and, in addition, used relations between thetanulls and the functions (0.18) resulting from the comparison of the free terms and the coefficients of the second-order terms. The last procedure has a multidimensional analogue: we can very well compare the homogeneous components of degrees 0 and 2 after the substitution of the expansions (4.1) in identities (3.3):

$$\vartheta_{\boldsymbol{a}}^4 = \frac{1}{2^g}\sum_{\boldsymbol{b}\in\mathfrak{K}_0}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\vartheta_{\boldsymbol{b}}^4, \quad \vartheta_{\boldsymbol{a}}^4\psi_{\boldsymbol{a}} = \frac{1}{2^g}\sum_{\boldsymbol{b}\in\mathfrak{K}_0}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\vartheta_{\boldsymbol{b}}^4\psi_{\boldsymbol{b}}, \qquad \boldsymbol{a}\in\mathfrak{K}_0. \quad (5.1)$$

However, the ensuing expressions for various ratios of thetanulls do not have a form as simple as (1.8). And what system of equations for the ψ-functions can be obtained if one uses the scheme of the simple proof of Halphen's theorem? This is the question we answer in § 5.

We fix the characteristic $\boldsymbol{c}\in\mathfrak{K}$, $\boldsymbol{c}\ne\boldsymbol{0}$, and return to Riemann relations (3.2), setting for this occasion $\boldsymbol{d}=\boldsymbol{0}$:

$$(-1)^{{}^t\boldsymbol{c}'\boldsymbol{a}''}\vartheta_{\boldsymbol{a}+\boldsymbol{c}}^2(\boldsymbol{z})\vartheta_{\boldsymbol{a}}^2(\boldsymbol{z}) = \frac{1}{2^g}\sum_{\boldsymbol{b}\in\mathfrak{K}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{{}^t\boldsymbol{c}'\boldsymbol{b}''}\vartheta_{\boldsymbol{b}+\boldsymbol{c}}(2\boldsymbol{z})\vartheta_{\boldsymbol{b}+\boldsymbol{c}}(\boldsymbol{0})\vartheta_{\boldsymbol{b}}^2(\boldsymbol{0}), \quad (5.2)$$

$$\boldsymbol{a}\in\mathfrak{K}.$$

Note that summation on the right-hand side of (5.2) proceeds now only over the characteristics $\boldsymbol{b}$ such that $\boldsymbol{b}$ and $\boldsymbol{b}+\boldsymbol{c}$ are even. Associated with a characteristic $\boldsymbol{c}\in\mathfrak{K}\setminus\{\mathbf{0}\}$ is the subgroup $\{\mathbf{0},\boldsymbol{c}\}\subset\mathfrak{K}$ of order 2. Elements of the form $(\boldsymbol{a},\boldsymbol{a}+\boldsymbol{c})$ of the quotient group $\mathfrak{K}_{\boldsymbol{c}}=\mathfrak{K}/\{\boldsymbol{c}\}$ will also be called characteristics; we shall identify them with an arbitrary member of the corresponding pair. Only characteristics with both $\boldsymbol{a}$ and $\boldsymbol{a}+\boldsymbol{c}$ even are themselves even. We denote the set of even characteristics in the group $\mathfrak{K}_{\boldsymbol{c}}$ by $\mathfrak{K}^*_{\boldsymbol{c}}$. In what follows, selecting and fixing one representative in each pair $(\boldsymbol{a},\boldsymbol{a}+\boldsymbol{c})\in\mathfrak{K}_{\boldsymbol{c}}$ we shall identify the group $\mathfrak{K}_{\boldsymbol{c}}$ and its subset $\mathfrak{K}^*_{\boldsymbol{c}}$ with some subsets of $\mathfrak{K}$.

Setting

$$\begin{gathered}\theta_{\boldsymbol{a}}(\boldsymbol{z})=i^{{}^t\boldsymbol{c}'\boldsymbol{a}''}\vartheta_{\boldsymbol{a}+\boldsymbol{c}}(\boldsymbol{z})\vartheta_{\boldsymbol{a}}(\boldsymbol{z}),\\ \widetilde{\theta}_{\boldsymbol{a}}(\boldsymbol{z})=i^{{}^t\boldsymbol{c}'\boldsymbol{a}''}\frac12\bigl(\vartheta_{\boldsymbol{a}+\boldsymbol{c}}(2\boldsymbol{z})\vartheta_{\boldsymbol{a}}(\mathbf{0})+\vartheta_{\boldsymbol{a}+\boldsymbol{c}}(\mathbf{0})\vartheta_{\boldsymbol{a}}(2\boldsymbol{z})\bigr),\qquad \theta_{\boldsymbol{a}}=\theta_{\boldsymbol{a}}(\mathbf{0})=\widetilde{\theta}_{\boldsymbol{a}}(\mathbf{0}),\\ \boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c}},\end{gathered}\tag{5.3}$$

and collecting the terms corresponding to the characteristics $\boldsymbol{b}$ and $\boldsymbol{b}+\boldsymbol{c}$, we can write (5.2) as follows:

$$\theta^2_{\boldsymbol{a}}(\boldsymbol{z})=\frac{1}{2^{g-1}}\sum_{\boldsymbol{b}\in\mathfrak{K}^*_{\boldsymbol{c}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\widetilde{\theta}_{\boldsymbol{b}}(\boldsymbol{z})\theta_{\boldsymbol{b}},\qquad \boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c}}.\tag{5.4}$$

Alongside (5.4) we require another, 'degenerate' special case of the Riemann relations:

$$\theta_{\boldsymbol{a}}(\boldsymbol{z})\theta_{\boldsymbol{a}}=\frac{1}{2^{g-1}}\sum_{\boldsymbol{b}\in\mathfrak{K}^*_{\boldsymbol{c}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\theta_{\boldsymbol{b}}(\boldsymbol{z})\theta_{\boldsymbol{b}},\qquad \boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c}},\tag{5.5}$$

which can be obtained by setting $\boldsymbol{d}=\mathbf{0}$, $\boldsymbol{z}_1=\boldsymbol{z}_3=\boldsymbol{z}$, $\boldsymbol{z}_2=\boldsymbol{z}_4=\mathbf{0}$ in (3.1) and collecting the terms on the right-hand side corresponding to the characteristics $\boldsymbol{b}$ and $\boldsymbol{b}+\boldsymbol{c}$.

As in the preamble to the proof of Theorem 1, we now study the properties of the symmetric matrix

$$\mathrm{M}=\mathrm{M}_{\boldsymbol{c}}=\frac{1}{2^{g-1}}\bigl((-1)^{|\boldsymbol{a},\boldsymbol{b}|}\bigr)_{\boldsymbol{a},\boldsymbol{b}\in\mathfrak{K}^*_{\boldsymbol{c}}}$$

(of yet unknown size) corresponding to identities (5.4) and (5.5).

First of all, we calculate several auxiliary sums.

Lemma 10. *There hold the identities*

$$\sum_{\boldsymbol{b}\in\mathfrak{K}^*:\boldsymbol{b}+\boldsymbol{c}\in\mathfrak{K}^*}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}=(-1)^{|\boldsymbol{a}|}\cdot\begin{cases}2^{g-1}(2^{g-1}+1)&\text{if } \boldsymbol{a}=\mathbf{0}\text{ or }\boldsymbol{a}=\boldsymbol{c},\\ ((-1)^{|\boldsymbol{a},\boldsymbol{c}|}+1)\cdot2^{g-2}&\text{otherwise},\end{cases}\qquad \boldsymbol{a}\in\mathfrak{K}.\tag{5.6}$$

Proof. For a fixed characteristic $\boldsymbol{a}\in\mathfrak{K}$ let S_+ be the sum on the right-hand side of (5.6); in addition, we consider the sum

$$S_-=-\sum_{\boldsymbol{b}\in\mathfrak{K}^*:\boldsymbol{b}+\boldsymbol{c}\notin\mathfrak{K}^*}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}.$$

By Lemma 7,

$$S_+ - S_- = (-1)^{|\boldsymbol{a}|} \cdot \begin{cases} 2^{g-1}(2^g+1) & \text{if } \boldsymbol{a} = \boldsymbol{0}, \\ 2^{g-1} & \text{otherwise.} \end{cases} \tag{5.7}$$

Since

$$(-1)^{|\boldsymbol{a},\boldsymbol{b}|} = (-1)^{|\boldsymbol{c}|}(-1)^{|\boldsymbol{b}|}(-1)^{|\boldsymbol{b}+\boldsymbol{c}|}(-1)^{|\boldsymbol{a}+\boldsymbol{c},\boldsymbol{b}|},$$

the sums S_+ and S_- can be written as follows:

$$S_+ = (-1)^{|\boldsymbol{c}|} \sum_{\boldsymbol{b}\in\mathfrak{K}^*:\boldsymbol{b}+\boldsymbol{c}\in\mathfrak{K}^*} (-1)^{|\boldsymbol{a}+\boldsymbol{c},\boldsymbol{b}|}, \qquad S_- = (-1)^{|\boldsymbol{c}|} \sum_{\boldsymbol{b}\in\mathfrak{K}^*:\boldsymbol{b}+\boldsymbol{c}\notin\mathfrak{K}^*} (-1)^{|\boldsymbol{a}+\boldsymbol{c},\boldsymbol{b}|}.$$

Hence

$$\begin{aligned} S_+ + S_- &= (-1)^{|\boldsymbol{c}|} \sum_{\boldsymbol{b}\in\mathfrak{K}^*} (-1)^{|\boldsymbol{a}+\boldsymbol{c},\boldsymbol{b}|} \\ &= (-1)^{|\boldsymbol{c}|} \cdot \begin{cases} 2^{g-1}(2^g+1) & \text{if } \boldsymbol{a} = \boldsymbol{c}, \\ (-1)^{|\boldsymbol{a}+\boldsymbol{c}|} \cdot 2^{g-1} & \text{otherwise,} \end{cases} \end{aligned} \tag{5.8}$$

where we use Lemma 7 again.

Adding equalities (5.7) and (5.8) we see that S_+ is equal to the right-hand side of (5.6), which completes the proof.

Corollary. *The set* $\mathfrak{K}^*_{\boldsymbol{c}}$ *consists of* $2^{g-2}(2^{g-1}+1)$ *elements.*

Proof. We consider identity (5.6) for $\boldsymbol{a} = \boldsymbol{0}$ and obtain on its right-hand side the number of elements of the set $\{\boldsymbol{b} \in \mathfrak{K}^* : \boldsymbol{b} + \boldsymbol{c} \in \mathfrak{K}^*\}$. It remains to divide this quantity by 2.

Lemma 11. *The matrix* $\mathrm{M} = \mathrm{M}_{\boldsymbol{c}}$ *satisfies the relation*

$$\mathrm{M}^2 = \frac{1}{2}(\mathrm{M} + E),$$

where E *is the identity matrix of size* $2^{g-2}(2^{g-1}+1)$.

Proof. Using Lemma 10 we consider the scalar product of the columns of the matrix M with indices $\boldsymbol{a}, \boldsymbol{d} \in \mathfrak{K}^*_{\boldsymbol{c}}$ (the case $\boldsymbol{a} + \boldsymbol{c} \equiv \boldsymbol{d} \pmod{2\mathbb{Z}^{2g}}$ is impossible by the definition of $\mathfrak{K}^*_{\boldsymbol{c}}$!):

$$\begin{aligned} \sum_{\boldsymbol{b}\in\mathfrak{K}^*_{\boldsymbol{c}}} \frac{1}{2^{g-1}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|} \cdot \frac{1}{2^{g-1}}(-1)^{|\boldsymbol{d},\boldsymbol{b}|} &= \frac{1}{2^{2g-1}} \sum_{\boldsymbol{b}\in\mathfrak{K}^*:\boldsymbol{b}+\boldsymbol{c}\in\mathfrak{K}^*} (-1)^{|\boldsymbol{a}+\boldsymbol{d},\boldsymbol{b}|} \\ = \frac{1}{2^{2g-1}} \cdot \begin{cases} 2^{g-1}(2^{g-1}+1) & \text{if } \boldsymbol{a} = \boldsymbol{d}, \\ (-1)^{|\boldsymbol{a}+\boldsymbol{d}|} \cdot 2^{g-1} & \text{otherwise,} \end{cases} \\ = \frac{1}{2} \cdot \begin{cases} \dfrac{1}{2^{g-1}} + \dfrac{1}{2} & \text{if } \boldsymbol{a} = \boldsymbol{d}, \\ \dfrac{1}{2^{g-1}}(-1)^{|\boldsymbol{a},\boldsymbol{d}|} & \text{otherwise.} \end{cases} \end{aligned}$$

Taking account of the symmetry of M (the product of two lines is equal to the product of a line and the corresponding column) one establishes the result of the lemma.

Corollary 1. *There holds the identity*

$$\Big(\mathrm{M}-\frac14E\Big)^{-1}=\frac{16}{9}\Big(\mathrm{M}-\frac14E\Big).$$

Proof. This can be proved in the same way as the corollary to Lemma 8.

Corollary 2. *The matrix*

$$(\mathrm{M}-E)\Big(\mathrm{M}+\frac12E\Big)$$

is equal to zero.

Proof. This expression is a representation of $\mathrm{M}^2-\frac12(\mathrm{M}+E)$ as a product of linear factors.

Now, following the earlier pattern, one should consider the multiplicities of the eigenvalues 1 and $-\frac12$ in the characteristic polynomial of M. We leave this to the reader as a simple (especially so after the proof of Lemma 9) exercise.

Theorem 2. *For a fixed characteristic* $\boldsymbol{c}\in\mathfrak{K}$, $\boldsymbol{c}\ne\boldsymbol{0}$, *and for the corresponding set* $\mathfrak{K}_{\boldsymbol{c}}^*$ *the functions* (0.18) *satisfy the system of differential equations*

$$\begin{aligned}\vartheta_{\boldsymbol{a}+\boldsymbol{c}}^2&\vartheta_{\boldsymbol{a}}^2\delta(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})\\&=\frac{1}{2^{g-2}}\sum_{\boldsymbol{b}\in\mathfrak{K}_{\boldsymbol{c}}^*}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{{}^t\boldsymbol{c}'(\boldsymbol{a}''+\boldsymbol{b}'')}\vartheta_{\boldsymbol{b}+\boldsymbol{c}}^2\vartheta_{\boldsymbol{b}}^2(\psi_{\boldsymbol{b}+\boldsymbol{c}}+\psi_{\boldsymbol{b}})^2\\&\qquad+4\vartheta_{\boldsymbol{a}+\boldsymbol{c}}^2\vartheta_{\boldsymbol{a}}^2\psi_{\boldsymbol{a}+\boldsymbol{c}}\psi_{\boldsymbol{a}}-2\vartheta_{\boldsymbol{a}+\boldsymbol{c}}^2\vartheta_{\boldsymbol{a}}^2(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})^2,\qquad \boldsymbol{a}\in\mathfrak{K}_{\boldsymbol{c}}^*.\end{aligned}\tag{5.9}$$

In addition, the thetanulls and the functions (0.18) *are connected by the relations*

$$\vartheta_{\boldsymbol{a}+\boldsymbol{c}}^2\vartheta_{\boldsymbol{a}}^2=\frac{1}{2^{g-1}}\sum_{\boldsymbol{b}\in\mathfrak{K}_{\boldsymbol{c}}^*}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{{}^t\boldsymbol{c}'(\boldsymbol{a}''+\boldsymbol{b}'')}\vartheta_{\boldsymbol{b}+\boldsymbol{c}}^2\vartheta_{\boldsymbol{b}}^2,\qquad \boldsymbol{a}\in\mathfrak{K}_{\boldsymbol{c}}^*,\tag{5.10}$$

$$\vartheta_{\boldsymbol{a}+\boldsymbol{c}}^2\vartheta_{\boldsymbol{a}}^2(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})=\frac{1}{2^{g-1}}\sum_{\boldsymbol{b}\in\mathfrak{K}_{\boldsymbol{c}}^*}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}(-1)^{{}^t\boldsymbol{c}'(\boldsymbol{a}''+\boldsymbol{b}'')}\vartheta_{\boldsymbol{b}+\boldsymbol{c}}^2\vartheta_{\boldsymbol{b}}^2(\psi_{\boldsymbol{b}+\boldsymbol{c}}+\psi_{\boldsymbol{b}}),\tag{5.11}$$

$$\boldsymbol{a}\in\mathfrak{K}_{\boldsymbol{c}}^*.$$

Proof. By Lemma 2 we obtain

$$\begin{aligned}\theta_{\boldsymbol{a}}(\boldsymbol{z})&=i^{{}^t\boldsymbol{c}'\boldsymbol{a}''}\vartheta_{\boldsymbol{a}+\boldsymbol{c}}(\boldsymbol{z})\vartheta_{\boldsymbol{a}}(\boldsymbol{z})=\theta_{\boldsymbol{a}}\cdot\Big(1-2(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})\\&\qquad+\frac83\psi_{\boldsymbol{a}+\boldsymbol{c}}\psi_{\boldsymbol{a}}+\frac23(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})^2+\frac23\delta(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})\Big)+O(\boldsymbol{z}^6),\\\theta_{\boldsymbol{a}}^2(\boldsymbol{z})&=(-1)^{{}^t\boldsymbol{c}'\boldsymbol{a}''}\vartheta_{\boldsymbol{a}+\boldsymbol{c}}^2(\boldsymbol{z})\vartheta_{\boldsymbol{a}}^2(\boldsymbol{z})=\theta_{\boldsymbol{a}}^2\cdot\Big(1-4(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})\\&\qquad+\frac{16}{3}\psi_{\boldsymbol{a}+\boldsymbol{c}}\psi_{\boldsymbol{a}}+\frac{16}{3}(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})^2+\frac43\delta(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})\Big)+O(\boldsymbol{z}^6),\\\widetilde\theta_{\boldsymbol{a}}(\boldsymbol{z})&=i^{{}^t\boldsymbol{c}'\boldsymbol{a}''}\frac12\big(\vartheta_{\boldsymbol{a}+\boldsymbol{c}}(2\boldsymbol{z})\vartheta_{\boldsymbol{a}}+\vartheta_{\boldsymbol{a}+\boldsymbol{c}}\vartheta_{\boldsymbol{a}}(2\boldsymbol{z})\big)=\theta_{\boldsymbol{a}}\cdot\Big(1-4(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})\\&\qquad-\frac{32}{3}\psi_{\boldsymbol{a}+\boldsymbol{c}}\psi_{\boldsymbol{a}}+\frac{16}{3}(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})^2+\frac{16}{3}\delta(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})\Big)+O(\boldsymbol{z}^6),\end{aligned}\tag{5.12}$$

$$\boldsymbol{a}\in\mathfrak{K}_{\boldsymbol{c}}^*.$$

Taking account of the equality $\theta_{\boldsymbol{a}}^2=(-1)^{{}^t\boldsymbol{c}'\boldsymbol{a}''}\vartheta_{\boldsymbol{a}+\boldsymbol{c}}^2\vartheta_{\boldsymbol{a}}^2$, $\boldsymbol{a}\in\mathfrak{K}_{\boldsymbol{c}}^*$, substituting (5.12) in identities (5.4) and (5.5) and comparing the homogeneous components of degrees 0, 2, and 4, we obtain relations (5.10), (5.11), and

$$\begin{aligned}
&\theta_{\boldsymbol{a}}^2\big(4\psi_{\boldsymbol{a}+\boldsymbol{c}}\psi_{\boldsymbol{a}}+4(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})^2+\delta(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})\big)\\
&\qquad=\frac{1}{2^{g-1}}\sum_{\boldsymbol{b}\in\mathfrak{K}_{\boldsymbol{c}}^*}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\cdot 4\theta_{\boldsymbol{b}}^2\big(-2\psi_{\boldsymbol{b}+\boldsymbol{c}}\psi_{\boldsymbol{b}}+(\psi_{\boldsymbol{b}+\boldsymbol{c}}+\psi_{\boldsymbol{b}})^2+\delta(\psi_{\boldsymbol{b}+\boldsymbol{c}}+\psi_{\boldsymbol{b}})\big),\\
&\theta_{\boldsymbol{a}}^2\big(4\psi_{\boldsymbol{a}+\boldsymbol{c}}\psi_{\boldsymbol{a}}+(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})^2+\delta(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})\big)\\
&\qquad=\frac{1}{2^{g-1}}\sum_{\boldsymbol{b}\in\mathfrak{K}_{\boldsymbol{c}}^*}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\theta_{\boldsymbol{b}}^2\big(4\psi_{\boldsymbol{b}+\boldsymbol{c}}\psi_{\boldsymbol{b}}+(\psi_{\boldsymbol{b}+\boldsymbol{c}}+\psi_{\boldsymbol{b}})^2+\delta(\psi_{\boldsymbol{b}+\boldsymbol{c}}+\psi_{\boldsymbol{b}})\big),
\end{aligned}\tag{5.13}$$

$$\boldsymbol{a}\in\mathfrak{K}_{\boldsymbol{c}}^*,$$

respectively. By analogy with the proof of Theorem 1 we consider now the (column) vectors

$$\boldsymbol{W}=(\theta_{\boldsymbol{a}}^2\psi_{\boldsymbol{a}+\boldsymbol{c}}\psi_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathfrak{K}_{\boldsymbol{c}}^*},\qquad \boldsymbol{X}=\big(\theta_{\boldsymbol{a}}^2(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})^2\big)_{\boldsymbol{a}\in\mathfrak{K}_{\boldsymbol{c}}^*},$$
$$\boldsymbol{Y}=\big(\theta_{\boldsymbol{a}}^2\delta(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})\big)_{\boldsymbol{a}\in\mathfrak{K}_{\boldsymbol{c}}^*}.$$

Written in the matrix form relations (5.13) yield

$$4\boldsymbol{W}+4\boldsymbol{X}+\boldsymbol{Y}=4\mathrm{M}(-2\boldsymbol{W}+\boldsymbol{X}+\boldsymbol{Y}),\tag{5.14}$$
$$4\boldsymbol{W}+\boldsymbol{X}+\boldsymbol{Y}=\mathrm{M}(4\boldsymbol{W}+\boldsymbol{X}+\boldsymbol{Y}).\tag{5.15}$$

By (5.14) we obtain

$$\Big(\mathrm{M}-\frac{1}{4}E\Big)\boldsymbol{Y}=2\Big(\mathrm{M}+\frac{1}{2}E\Big)\boldsymbol{W}-(\mathrm{M}-E)\boldsymbol{X}.$$

From Lemma 11 and Corollary 1 to it we see that

$$\boldsymbol{Y}=\frac{8}{3}\Big(\mathrm{M}+\frac{1}{2}E\Big)\boldsymbol{W}+\frac{4}{3}(\mathrm{M}-E)\boldsymbol{X}.\tag{5.16}$$

Since

$$\delta\big(\theta_{\boldsymbol{a}}^2(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})\big)=2\theta_{\boldsymbol{a}}^2(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}})^2+\theta_{\boldsymbol{a}}^2\delta(\psi_{\boldsymbol{a}+\boldsymbol{c}}+\psi_{\boldsymbol{a}}),\quad \boldsymbol{a}\in\mathfrak{K}_{\boldsymbol{c}}^*,$$

δ-differentiation of relations (5.11) brings us to the matrix relation

$$2\boldsymbol{X}+\boldsymbol{Y}=\mathrm{M}(2\boldsymbol{X}+\boldsymbol{Y}).$$

Subtracting it from (5.15) we obtain

$$4\boldsymbol{W}-\boldsymbol{X}=\mathrm{M}(4\boldsymbol{W}-\boldsymbol{X}),\tag{5.17}$$

therefore

$$\begin{aligned}
\Big(\mathrm{M}+\frac{1}{2}E\Big)\boldsymbol{W}&=\frac{1}{4}\mathrm{M}(4\boldsymbol{W}-\boldsymbol{X})+\frac{1}{2}\boldsymbol{W}+\frac{1}{4}\mathrm{M}\boldsymbol{X}=\frac{1}{4}(4\boldsymbol{W}-\boldsymbol{X})+\frac{1}{2}\boldsymbol{W}+\frac{1}{4}\mathrm{M}\boldsymbol{X}\\
&=\frac{3}{2}\boldsymbol{W}+\frac{1}{4}(\mathrm{M}-E)\boldsymbol{X}.
\end{aligned}\tag{5.18}$$

Substituting (5.18) in (5.16) we obtain

$$\boldsymbol{Y} = 4\boldsymbol{W} + 2(\mathrm{M} - E)\boldsymbol{X}, \tag{5.19}$$

which corresponds precisely to the system (5.9). The proof is complete.

Remark. Of course, the system of differential equations (5.9) can be written on the basis of (2.1) in a 'customized form' for the functions (0.16). However, we content ourselves with the customization (0.17) of Theorem 1.

Proof of Ohyama's theorem. We demonstrate now that for $g = 2$ Theorem 2 yields Ohyama's theorem. We fix an arbitrary characteristic $\boldsymbol{c} \in \mathbb{Z}^4/2\mathbb{Z}^4$, $\boldsymbol{c} \neq \boldsymbol{0}$. The corresponding set $\mathfrak{K}^*_{\boldsymbol{c}}$ consists of three elements, which we denote by $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$; for convenience, we order these elements so that the first line of the matrix $\mathrm{M} = \mathrm{M}_{\boldsymbol{c}}$ contains only positive integers:

$$\mathrm{M} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}. \tag{5.20}$$

Setting

$$\theta_j = \theta_{\boldsymbol{a}_j}, \quad S_j = \psi_{\boldsymbol{a}_j+\boldsymbol{c}} + \psi_{\boldsymbol{a}_j}, \quad P_j = \psi_{\boldsymbol{a}_j+\boldsymbol{c}}\psi_{\boldsymbol{a}_j}, \qquad j = 1, 2, 3,$$

we write the system of differential equation (5.9) (or (5.19)) in the following form:

$$\begin{aligned} \theta_1^2\delta S_1 &= 4\theta_1^2 P_1 - (\theta_1^1 S_1^2 - \theta_2^2 S_2^2 - \theta_3^2 S_3^2), \\ \theta_2^2\delta S_2 &= 4\theta_2^2 P_2 + (\theta_1^2 S_1^2 - \theta_2^2 S_2^2 - \theta_3^2 S_3^2), \\ \theta_3^2\delta S_3 &= 4\theta_3^2 P_3 + (\theta_1^2 S_1^2 - \theta_2^2 S_2^2 - \theta_3^2 S_3^2). \end{aligned} \tag{5.21}$$

Relations (5.10) and (5.11) now look as follows:

$$\theta_1^2 - \theta_2^2 - \theta_3^2 = 0, \qquad \theta_1^2 S_1 - \theta_2^2 S_2 - \theta_3^2 S_3 = 0,$$

therefore

$$\frac{S_2 - S_3}{\theta_1^2} = \frac{S_1 - S_3}{\theta_2^2} = \frac{S_2 - S_1}{\theta_3^2}, \tag{5.22}$$

so that

$$\begin{aligned} \theta_1^2 S_1^2 - \theta_2^2 S_2^2 - \theta_3^2 S_3^2 &= \theta_1^2(S_1 - S_2)(S_1 - S_3) \\ &= -\theta_2^2(S_2 - S_1)(S_2 - S_3) \\ &= -\theta_3^2(S_3 - S_1)(S_3 - S_2). \end{aligned} \tag{5.23}$$

Substituting (5.23) in equations (5.21) and canceling out thetanulls we obtain the system

$$\begin{aligned} \delta S_1 &= 4P_1 - (S_1 - S_2)(S_1 - S_3), \\ \delta S_2 &= 4P_2 - (S_2 - S_1)(S_2 - S_3), \\ \delta S_3 &= 4P_3 - (S_3 - S_1)(S_3 - S_2). \end{aligned} \tag{5.24}$$

In our notation one can deduce Ohyama's system (0.21) from (5.24) by the addition of two arbitrary equations:

$$\begin{aligned}\delta(S_1+S_2)&=4P_1+4P_2-(S_1-S_2)^2,\\ \delta(S_1+S_3)&=4P_1+4P_3-(S_1-S_3)^2,\\ \delta(S_2+S_3)&=4P_2+4P_3-(S_2-S_3)^2,\end{aligned}$$

because every two of the three pairs $\{\boldsymbol{a}_1,\boldsymbol{a}_1+\boldsymbol{c}\}$, $\{\boldsymbol{a}_2,\boldsymbol{a}_2+\boldsymbol{c}\}$, and $\{\boldsymbol{a}_3,\boldsymbol{a}_3+\boldsymbol{c}\}$ give one the system of vertices of a parallelogram.

Thus, for $g=2$ Ohyama's theorem is a consequence of Theorem 2. Note that this method of the proof of (0.21) is distinct from the original method of [22] and [23], which we expose in the next section.

What can one say about the quantity (5.22), a two-dimensional analogue of (1.9)? Denoting it by $\varkappa=\varkappa_{\boldsymbol{c}}$, differentiating its first definition in (5.22), and taking account of (5.24) we obtain

$$\delta\varkappa=\frac{\delta S_2-\delta S_3-2S_1(S_2-S_3)}{\theta_1^2}=\frac{(4P_2-S_2^2)-(4P_3-S_3^2)}{\theta_1^2}.$$

Carrying out similar differentiation for the second and the third definitions we conclude that

$$\delta\varkappa=\frac{(4P_1-S_1^2)-(4P_3-S_3^2)}{\theta_2^2}=\frac{(4P_2-S_2^2)-(4P_1-S_1^2)}{\theta_3^2}. \tag{5.25}$$

The fact that the quantities in (5.25) are equal is also, similarly to our derivation of (5.22), a consequence of the equality

$$\theta_1^2(4P_1-S_1^2)-\theta_2^2(4P_2-S_2^2)-\theta_3^2(4P_3-S_3^2)=0,$$

which we established in the proof of Theorem 2 (see (5.17)). Note that

$$4P_j-S_j^2=-(\psi_{\boldsymbol{a}_j+\boldsymbol{c}}-\psi_{\boldsymbol{a}_j})^2,$$

therefore the quantity (5.25) is distinct from zero (for otherwise the four thetanulls are related by the equality $\vartheta_1\vartheta_2=\mathrm{const}\cdot\vartheta_3\vartheta_4$, which is impossible). For δ-differentiation (5.25) we must abandon our 'cozy nook' of six functions and enter the 'large world' of the ten functions (0.18). Wary of boring the reader by lengthy calculations we present the final result:

$$\begin{aligned}\delta^2\varkappa&=\frac{2(4P_2-S_2^2)(\Sigma-2S_2-2S_1)-2(4P_3-S_3^2)(\Sigma-2S_3-2S_1)}{\theta_1^2}\\ &=\frac{2(4P_1-S_1^2)(\Sigma-2S_1-2S_2)-2(4P_3-S_3^2)(\Sigma-2S_3-2S_2)}{\theta_2^2}\\ &=\frac{2(4P_2-S_2^2)(\Sigma-2S_2-2S_3)-2(4P_1-S_1^2)(\Sigma-2S_1-2S_3)}{\theta_3^2},\end{aligned} \tag{5.26}$$

where Σ is the sum of all 10 (in the case $g=2$) functions (0.18). The quantity (5.26) is also distinct from zero in general. We interrupt here the procedure of differentiation of $\varkappa$ and return to the generalization of identities (0.8) for $g=2$ and $g=3$ in § 8.

§ 6. Forwards to new differential equations

We were far from exhausting all the resources of the Riemann relations in our proofs of Theorems 1 and 2; we wish to make up for this in the present section: the object of our study here are identities (3.2). To avoid cumbersome calculations we shall rely upon the scheme tried in § 3 and § 5.

Summation on the right-hand side of (3.2) proceeds only over the characteristics $\boldsymbol{b} \in \mathfrak{K}$ such that

$$\boldsymbol{b}, \boldsymbol{b}+\boldsymbol{c}, \boldsymbol{b}+\boldsymbol{d} \in \mathfrak{K}^*. \tag{6.1}$$

We now fix two characteristics

$$\boldsymbol{c}, \boldsymbol{d} \in \mathfrak{K}, \quad \boldsymbol{c} \neq \boldsymbol{d}, \quad \boldsymbol{c} \neq \boldsymbol{0}, \quad \boldsymbol{d} \neq \boldsymbol{0}, \tag{6.2}$$

satisfying the additional condition

$$|\boldsymbol{c}, \boldsymbol{d}| \equiv 0 \pmod 2). \tag{6.3}$$

Since

$$(-1)^{|\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}|} = (-1)^{|\boldsymbol{c},\boldsymbol{d}|}(-1)^{|\boldsymbol{b}|}(-1)^{|\boldsymbol{b}+\boldsymbol{c}|}(-1)^{|\boldsymbol{b}+\boldsymbol{d}|},$$

it follows by (6.1) and (6.3) that $\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d} \in \mathfrak{K}^*$. As in § 5, we shall identify the quotient group $\mathfrak{K}_{\boldsymbol{c},\boldsymbol{d}} = \mathfrak{K}/\{\boldsymbol{0}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{c}+\boldsymbol{d}\}$ (and its subsets) and subsets of $\mathfrak{K}$ by picking one element in each coset. The set of even characteristics $\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}} \subset \mathfrak{K}_{\boldsymbol{c},\boldsymbol{d}}$ consists of the cosets such that (6.1) holds (and therefore also $\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d} \in \mathfrak{K}^*$).

Setting now

$$\begin{gathered}
\theta_{\boldsymbol{a}}(\boldsymbol{z}) = (-1)^{{}^t(\boldsymbol{c}'+\boldsymbol{d}')\boldsymbol{a}''}\vartheta_{\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}}(\boldsymbol{z})\vartheta_{\boldsymbol{a}+\boldsymbol{c}}(\boldsymbol{z})\vartheta_{\boldsymbol{a}+\boldsymbol{d}}(\boldsymbol{z})\vartheta_{\boldsymbol{a}}(\boldsymbol{z}),\\
\widetilde{\theta}_{\boldsymbol{a}}(\boldsymbol{z}) = \theta_{\boldsymbol{a}}(\boldsymbol{0}) \cdot \frac{1}{4}\biggl(\frac{\vartheta_{\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}}(2\boldsymbol{z})}{\vartheta_{\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}}(\boldsymbol{0})} + \frac{\vartheta_{\boldsymbol{a}+\boldsymbol{c}}(2\boldsymbol{z})}{\vartheta_{\boldsymbol{a}+\boldsymbol{c}}(\boldsymbol{0})} + \frac{\vartheta_{\boldsymbol{a}+\boldsymbol{d}}(2\boldsymbol{z})}{\vartheta_{\boldsymbol{a}+\boldsymbol{d}}(\boldsymbol{0})} + \frac{\vartheta_{\boldsymbol{a}}(2\boldsymbol{z})}{\vartheta_{\boldsymbol{a}}(\boldsymbol{0})}\biggr),\\
\boldsymbol{a} \in \mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}},
\end{gathered}$$

we can write (3.2) as follows:

$$\theta_{\boldsymbol{a}}(\boldsymbol{z}) = \frac{1}{2^{g-2}} \sum_{\boldsymbol{b} \in \mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}} (-1)^{|\boldsymbol{a},\boldsymbol{b}|}\widetilde{\theta}_{\boldsymbol{b}}(\boldsymbol{z}), \qquad \boldsymbol{a} \in \mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}. \tag{6.4}$$

In fact, the consistency of (6.4) (the independence of the sum of one's choice of representatives of cosets in $\mathfrak{K}_{\boldsymbol{c},\boldsymbol{d}}$) is a consequence of the relation

$$\vartheta_{\boldsymbol{a}+2\boldsymbol{e}}(\boldsymbol{z}) = (-1)^{{}^t\boldsymbol{a}'\boldsymbol{e}''}\vartheta_{\boldsymbol{a}}(\boldsymbol{z}), \qquad \boldsymbol{a}, \boldsymbol{e} \in \mathbb{Z}^{2g} \tag{6.5}$$

(see [1]; Teil 2, Kapitel 7, § 1). Hence it is an easy verification that if conditions (6.1) and (6.6) are fulfilled, then the quantity $(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\widetilde{\theta}_{\boldsymbol{b}}(\boldsymbol{z})$ is preserved by the replacement of $\boldsymbol{b}$ by $\boldsymbol{b}+\boldsymbol{c}$, $\boldsymbol{b}+\boldsymbol{d}$, or $\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}$.

As in the proofs of Lemmas 8 and 11 we can show that the symmetric square matrix

$$\mathrm{M} = \mathrm{M}_{\boldsymbol{c},\boldsymbol{d}} = \frac{1}{2^{g-2}}\big((-1)^{|\boldsymbol{a},\boldsymbol{b}|}\big)_{\boldsymbol{a},\boldsymbol{b}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}}$$

corresponding to relations (6.4) satisfies the equality

$$\mathrm{M}^2 = \frac{1}{2}(\mathrm{M}+E)$$

and the set $\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}$ consists of $2^{g-3}(2^{g-2}+1)$ elements.

For the simplicity of notation we set

$$\begin{aligned}
\theta_{\boldsymbol{a}} &= \theta_{\boldsymbol{a}}(\mathbf{0}) = (-1)^{{}^t(\boldsymbol{c}'+\boldsymbol{d}')\boldsymbol{a}''}\vartheta_{\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}}\vartheta_{\boldsymbol{a}+\boldsymbol{c}}\vartheta_{\boldsymbol{a}+\boldsymbol{d}}\vartheta_{\boldsymbol{a}},\\
S_{\boldsymbol{a}} &= \psi_{\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}} + \psi_{\boldsymbol{a}+\boldsymbol{c}} + \psi_{\boldsymbol{a}+\boldsymbol{d}} + \psi_{\boldsymbol{a}}, \qquad\qquad \boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}.\\
P_{\boldsymbol{a}} &= \frac{1}{2}(S^2_{\boldsymbol{a}} - \psi^2_{\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}} - \psi^2_{\boldsymbol{a}+\boldsymbol{c}} - \psi^2_{\boldsymbol{a}+\boldsymbol{d}} - \psi^2_{\boldsymbol{a}}),
\end{aligned} \tag{6.6}$$

Theorem 3. *Assume that the characteristics* (6.2) *satisfy additional condition* (6.3) *and let $\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}$ be the corresponding set of even characteristics. Then there the following system of differential equations holds in the notation* (6.6):

$$\theta_{\boldsymbol{a}}\delta S_{\boldsymbol{a}} = \frac{1}{3\cdot 2^{g-5}}\sum_{\boldsymbol{b}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\theta_{\boldsymbol{b}}P_{\boldsymbol{b}} + \frac{4}{3}\theta_{\boldsymbol{a}}P_{\boldsymbol{a}} - \theta_{\boldsymbol{a}}S^2_{\boldsymbol{a}}, \qquad \boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}, \tag{6.7}$$

and in addition,

$$\begin{aligned}
\theta_{\boldsymbol{a}} &= \frac{1}{2^{g-2}}\sum_{\boldsymbol{b}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\theta_{\boldsymbol{b}}, \qquad \boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}},\\
\theta_{\boldsymbol{a}}S_{\boldsymbol{a}} &= \frac{1}{2^{g-2}}\sum_{\boldsymbol{b}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\theta_{\boldsymbol{b}}S_{\boldsymbol{b}}, \qquad \boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}.
\end{aligned} \tag{6.8}$$

Proof. By Lemma 2 we obtain

$$\begin{aligned}
\theta_{\boldsymbol{a}}(\boldsymbol{z}) &= \theta\cdot\Big(1 - 2S_{\boldsymbol{a}} + 4P_{\boldsymbol{a}} + \frac{2}{3}(S^2_{\boldsymbol{a}} - 2P_{\boldsymbol{a}}) + \frac{2}{3}\delta S_{\boldsymbol{a}}\Big) + O(\boldsymbol{z}^6),\\
\widetilde{\theta}_{\boldsymbol{a}}(\boldsymbol{z}) &= \theta\cdot\Big(1 - 2S_{\boldsymbol{a}} + \frac{8}{3}(S^2_{\boldsymbol{a}} - 2P_{\boldsymbol{a}}) + \frac{8}{3}\delta S_{\boldsymbol{a}}\Big) + O(\boldsymbol{z}^6),
\end{aligned} \qquad \boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}. \tag{6.9}$$

Substituting the expansions (6.9) in (6.4) and comparing the coefficients of the homogeneous (in $\boldsymbol{z}$) components of degrees 0, 2, and 4 we obtain relations (6.8) and the equality

$$\theta_{\boldsymbol{a}}\Big(\frac{8}{3}P_{\boldsymbol{a}} + \frac{2}{3}S^2_{\boldsymbol{a}} + \frac{2}{3}\delta S_{\boldsymbol{a}}\Big) = \frac{1}{2^{g-2}}\sum_{\boldsymbol{b}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\theta_{\boldsymbol{b}}\Big(-\frac{16}{3}P_{\boldsymbol{b}} + \frac{8}{3}S^2_{\boldsymbol{b}} + \frac{8}{3}\delta S_{\boldsymbol{b}}\Big),$$
$$\boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}. \tag{6.10}$$

Setting

$$\boldsymbol{W}=(\theta_{\boldsymbol{a}}P_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}},\qquad \boldsymbol{X}=(\theta_{\boldsymbol{a}}S_{\boldsymbol{a}}^2)_{\boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}},\qquad \boldsymbol{Y}=(\theta_{\boldsymbol{a}}\delta S_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}}$$

and writing (6.10) in the matrix form:

$$\frac{8}{3}\boldsymbol{W}+\frac{2}{3}\boldsymbol{X}+\frac{2}{3}\boldsymbol{Y}=-\frac{16}{3}\mathrm{M}\boldsymbol{W}+\frac{8}{3}\mathrm{M}\boldsymbol{X}+\frac{8}{3}\mathrm{M}\boldsymbol{Y},$$

we see that

$$\Bigl(\mathrm{M}-\frac{1}{4}E\Bigr)(\boldsymbol{X}+\boldsymbol{Y})=2\Bigl(\mathrm{M}+\frac{1}{2}E\Bigr)\boldsymbol{W},$$

therefore

$$\boldsymbol{Y}=\frac{8}{3}\Bigl(\mathrm{M}+\frac{1}{2}E\Bigr)\boldsymbol{W}-\boldsymbol{X}. \tag{6.11}$$

The return to the earlier notation in (6.11) completes the proof.

One ‘deficiency’ of (6.7) is the following feature: the sum on the right-hand side contains *sums of pairwise products* of functions (0.18) (cf. the systems (0.20) and (5.9)). It is this feature that obstructs the derivation in the case $g=3$ of a system without denominators, similarly to the proofs of Halphen’s and Ohyama’s theorems. Unfortunately, using other special cases of the Riemann relations one can improve upon the system (6.7) only superficially, distorting its inner symmetry.

If $g=2$ then the set $\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}$ consists of a single element, therefore the matrix M is a scalar: $\mathrm{M}=1$. Hence all theta-coefficients in equations (6.7) cancel out and we obtain (0.21):

$$\delta S_{\boldsymbol{a}}=4P_{\boldsymbol{a}}-S_{\boldsymbol{a}}^2. \tag{6.12}$$

This gives us yet another proof of Ohyama’s theorem.

We demonstrate now how, for $g=2$, one can derive from the system (0.21) differential equations for each function (0.18). Associating with the pairs $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$ the integers 0, 1, 2, and 3, respectively, we shall express a characteristic $\boldsymbol{a}=(\boldsymbol{a}',\boldsymbol{a}'')\in\mathbb{Z}^4/2\mathbb{Z}^4$ as a pair of integers, the first of which corresponds to $\boldsymbol{a}'$ and the second to $\boldsymbol{a}''$.

For $g=2$,

$$\mathfrak{K}^*=\{00,01,02,03,10,12,20,21,30,33\}.$$

It is an easy calculation that for various choices of the characteristics (6.2) one can obtain precisely 15 distinct equations of the form (6.12), which correspond to the collections of characteristics

$$\begin{gathered}\{00,01,02,03\},\ \{00,01,20,21\},\ \{02,03,20,21\},\ \{00,02,10,12\},\ \{01,03,10,12\},\\ \{00,03,30,33\},\ \{01,02,30,33\},\ \{00,10,20,30\},\ \{02,12,20,30\},\ \{01,10,21,30\},\\ \{03,12,21,30\},\ \{00,12,21,33\},\ \{02,10,21,33\},\ \{01,12,20,33\},\ \{03,10,20,33\};\end{gathered}$$

moreover, each of the 10 thetanulls is involved in precisely 6 of them. Summing all these equations we obtain

$$\delta\sum_{\boldsymbol{a}\in\mathfrak{K}^*}\psi_{\boldsymbol{a}}=-\sum_{\boldsymbol{a}\in\mathfrak{K}^*}\psi_{\boldsymbol{a}}^2+\frac{1}{3}\sum_{\substack{\boldsymbol{a},\boldsymbol{b}\in\mathfrak{K}^*\\ \boldsymbol{a}\ne\boldsymbol{b}}}\psi_{\boldsymbol{a}}\psi_{\boldsymbol{b}}. \tag{6.13}$$

Each characteristic $\boldsymbol{a}_0 \in \mathfrak{K}^*$ can be grouped together with the remaining 9 even characteristics into three 'parallelograms':

$$\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \qquad \boldsymbol{a}_0, \boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6, \qquad \boldsymbol{a}_0, \boldsymbol{a}_7, \boldsymbol{a}_8, \boldsymbol{a}_9 \tag{6.14}$$

(note that there exist just two ways to form such quadruples). Each quadruple in (6.14) satisfies equation (6.12). Adding these equations we obtain

$$\begin{aligned} 2\delta\psi_{\boldsymbol{a}_0} &+ \delta \sum_{\boldsymbol{a}\in\mathfrak{K}^*} \psi_{\boldsymbol{a}} \\ &= -2\psi_{\boldsymbol{a}_0}^2 - \sum_{\boldsymbol{a}\in\mathfrak{K}^*} \psi_{\boldsymbol{a}}^2 + 2\psi_{\boldsymbol{a}_0} \sum_{\substack{\boldsymbol{a}\in\mathfrak{K}^* \\ \boldsymbol{a}\neq\boldsymbol{a}_0}} \psi_{\boldsymbol{a}} + 2(\psi_{\boldsymbol{a}_1}\psi_{\boldsymbol{a}_2} + \psi_{\boldsymbol{a}_1}\psi_{\boldsymbol{a}_3} + \psi_{\boldsymbol{a}_2}\psi_{\boldsymbol{a}_3}) \\ &\quad + 2(\psi_{\boldsymbol{a}_4}\psi_{\boldsymbol{a}_5} + \psi_{\boldsymbol{a}_4}\psi_{\boldsymbol{a}_6} + \psi_{\boldsymbol{a}_5}\psi_{\boldsymbol{a}_6}) + 2(\psi_{\boldsymbol{a}_7}\psi_{\boldsymbol{a}_8} + \psi_{\boldsymbol{a}_7}\psi_{\boldsymbol{a}_9} + \psi_{\boldsymbol{a}_8}\psi_{\boldsymbol{a}_9}). \end{aligned} \tag{6.15}$$

Now, subtracting equation (6.13) from (6.15) we see that

$$\begin{aligned} \delta\psi_{\boldsymbol{a}_0} &= -\psi_{\boldsymbol{a}_0}^2 + \psi_{\boldsymbol{a}_0} \sum_{\substack{\boldsymbol{a}\in\mathfrak{K}^* \\ \boldsymbol{a}\neq\boldsymbol{a}_0}} \psi_{\boldsymbol{a}} - \frac{1}{6} \sum_{\substack{\boldsymbol{a},\boldsymbol{b}\in\mathfrak{K}^* \\ \boldsymbol{a}\neq\boldsymbol{b}}} \psi_{\boldsymbol{a}}\psi_{\boldsymbol{b}} + (\psi_{\boldsymbol{a}_1}\psi_{\boldsymbol{a}_2} + \psi_{\boldsymbol{a}_1}\psi_{\boldsymbol{a}_3} + \psi_{\boldsymbol{a}_2}\psi_{\boldsymbol{a}_3}) \\ &\quad + (\psi_{\boldsymbol{a}_4}\psi_{\boldsymbol{a}_5} + \psi_{\boldsymbol{a}_4}\psi_{\boldsymbol{a}_6} + \psi_{\boldsymbol{a}_5}\psi_{\boldsymbol{a}_6}) + (\psi_{\boldsymbol{a}_7}\psi_{\boldsymbol{a}_8} + \psi_{\boldsymbol{a}_7}\psi_{\boldsymbol{a}_9} + \psi_{\boldsymbol{a}_8}\psi_{\boldsymbol{a}_9}) \\ &= -2\psi_{\boldsymbol{a}_0}^2 - \frac{1}{3}(\psi_{\boldsymbol{a}_0}^2 + \psi_{\boldsymbol{a}_1}^2 + \cdots + \psi_{\boldsymbol{a}_9}^2) - \frac{1}{6}(\psi_{\boldsymbol{a}_0} + \psi_{\boldsymbol{a}_1} + \cdots + \psi_{\boldsymbol{a}_9})^2 \\ &\quad + \frac{1}{2}(\psi_{\boldsymbol{a}_0} + \psi_{\boldsymbol{a}_1} + \psi_{\boldsymbol{a}_2} + \psi_{\boldsymbol{a}_3})^2 + \frac{1}{2}(\psi_{\boldsymbol{a}_0} + \psi_{\boldsymbol{a}_4} + \psi_{\boldsymbol{a}_5} + \psi_{\boldsymbol{a}_6})^2 \\ &\quad + \frac{1}{2}(\psi_{\boldsymbol{a}_0} + \psi_{\boldsymbol{a}_7} + \psi_{\boldsymbol{a}_8} + \psi_{\boldsymbol{a}_9})^2. \end{aligned} \tag{6.16}$$

To complete the 'customization' of Ohyama's system of equations we present here all possible partitionings into quadruples (6.14) for each even characteristic $\boldsymbol{a}_0$:

$\boldsymbol{a}_0$	$\{\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$	$\{\boldsymbol{a}_0, \boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6\}$	$\{\boldsymbol{a}_0, \boldsymbol{a}_7, \boldsymbol{a}_8, \boldsymbol{a}_9\}$
00	$\{00, 01, 02, 03\}$	$\{00, 10, 20, 30\}$	$\{00, 12, 21, 33\}$
	$\{00, 01, 20, 21\}$	$\{00, 02, 10, 12\}$	$\{00, 03, 30, 33\}$
01	$\{00, 01, 02, 03\}$	$\{01, 10, 21, 30\}$	$\{01, 12, 20, 33\}$
	$\{00, 01, 20, 21\}$	$\{01, 03, 10, 12\}$	$\{01, 02, 30, 33\}$
02	$\{00, 01, 02, 03\}$	$\{02, 12, 20, 30\}$	$\{02, 10, 21, 33\}$
	$\{02, 03, 20, 21\}$	$\{00, 02, 10, 12\}$	$\{01, 02, 30, 33\}$
03	$\{00, 01, 02, 03\}$	$\{03, 12, 21, 30\}$	$\{03, 10, 20, 33\}$
	$\{02, 03, 20, 21\}$	$\{01, 03, 10, 12\}$	$\{00, 03, 30, 33\}$
10	$\{00, 02, 10, 12\}$	$\{01, 10, 21, 30\}$	$\{03, 10, 20, 33\}$
	$\{01, 03, 10, 12\}$	$\{00, 10, 20, 30\}$	$\{02, 10, 21, 33\}$

12	$\{00,02,10,12\}$ $\{01,03,10,12\}$	$\{03,12,21,30\}$ $\{02,12,20,30\}$	$\{01,12,20,33\}$ $\{00,12,21,33\}$
20	$\{00,01,20,21\}$ $\{02,03,20,21\}$	$\{02,12,20,30\}$ $\{00,10,20,30\}$	$\{03,10,20,33\}$ $\{01,12,20,33\}$
21	$\{00,01,20,21\}$ $\{02,03,20,21\}$	$\{03,12,21,30\}$ $\{01,10,21,30\}$	$\{02,10,21,33\}$ $\{00,12,21,33\}$
30	$\{00,03,30,33\}$ $\{01,02,30,33\}$	$\{02,12,20,30\}$ $\{00,10,20,30\}$	$\{01,10,21,30\}$ $\{03,12,21,30\}$
33	$\{00,03,30,33\}$ $\{01,02,30,33\}$	$\{02,10,21,33\}$ $\{00,12,21,33\}$	$\{01,12,20,33\}$ $\{03,10,20,33\}$

Each partitioning into quadruples (6.14) gives rise to an equation (6.16), and for each even characteristic $\boldsymbol{a}_0$ the choice of partitionings can be made in two ways. Hence equations (6.16) produce 10 quadratic relations of the form

$$\begin{aligned}&(\psi_{\boldsymbol{a}_1}+\psi_{\boldsymbol{a}_2}+\psi_{\boldsymbol{a}_3})^2+(\psi_{\boldsymbol{a}_4}+\psi_{\boldsymbol{a}_5}+\psi_{\boldsymbol{a}_6})^2+(\psi_{\boldsymbol{a}_7}+\psi_{\boldsymbol{a}_8}+\psi_{\boldsymbol{a}_9})^2\\&\qquad=(\psi_{\boldsymbol{a}_1^*}+\psi_{\boldsymbol{a}_2^*}+\psi_{\boldsymbol{a}_3^*})^2+(\psi_{\boldsymbol{a}_4^*}+\psi_{\boldsymbol{a}_5^*}+\psi_{\boldsymbol{a}_6^*})^2+(\psi_{\boldsymbol{a}_7^*}+\psi_{\boldsymbol{a}_8^*}+\psi_{\boldsymbol{a}_9^*})^2\end{aligned}$$

(we denote by asterisks elements from the second partitioning). Calculations show (see also [22]; Proposition 4.1) that for $g=2$ the ideal I of the ring $\mathbb{Q}[x_{\boldsymbol{a}}]_{\boldsymbol{a}\in\mathfrak{K}^*}$ generated (in accordance with the above table) by ten homogeneous polynomials of the second degree

$$\begin{aligned}y_{\boldsymbol{a}_0}&=(x_{\boldsymbol{a}_1}+x_{\boldsymbol{a}_2}+x_{\boldsymbol{a}_3})^2+(x_{\boldsymbol{a}_4}+x_{\boldsymbol{a}_5}+x_{\boldsymbol{a}_6})^2+(x_{\boldsymbol{a}_7}+x_{\boldsymbol{a}_8}+x_{\boldsymbol{a}_9})^2\\&\quad-(x_{\boldsymbol{a}_1^*}+x_{\boldsymbol{a}_2^*}+x_{\boldsymbol{a}_3^*})^2-(x_{\boldsymbol{a}_4^*}+x_{\boldsymbol{a}_5^*}+x_{\boldsymbol{a}_6^*})^2-(x_{\boldsymbol{a}_7^*}+x_{\boldsymbol{a}_8^*}+x_{\boldsymbol{a}_9^*})^2,\qquad \boldsymbol{a}_0\in\mathfrak{K}^*,\end{aligned}$$

has dimension 6.

In fact, using the two partitioning into parallelograms one can write the system (6.16) in the following compact form:

$$\delta\psi_{\boldsymbol{a}}=-2\psi_{\boldsymbol{a}}^2-\frac13\sum_{\boldsymbol{b}\in\mathfrak{K}^*}\psi_{\boldsymbol{b}}^2-\frac16\biggl(\sum_{\boldsymbol{b}\in\mathfrak{K}^*}\psi_{\boldsymbol{b}}\biggr)^2+\frac14\sum_{\mathfrak{G}\ni\boldsymbol{a}}\biggl(\sum_{\boldsymbol{b}\in\mathfrak{G}}\psi_{\boldsymbol{b}}\biggr)^2,\qquad \boldsymbol{a}\in\mathfrak{K}^*,\tag{6.17}$$

where summation $\sum_{\mathfrak{G}\ni\boldsymbol{a}}$ proceeds over all six parallelograms containing the characteristic $\boldsymbol{a}\in\mathfrak{K}^*$.

Theorem 4. *Let $g=3$ and let $\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}=\{\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3\}$ be the set corresponding to a fixed pair of characteristics* (6.2) *satisfying* (6.3)*. Then the following system of differential equations holds in the notation* (6.6):

$$\begin{aligned}\delta S_{\boldsymbol{a}_1}&=4P_{\boldsymbol{a}_1}-S_{\boldsymbol{a}_1}^2-\frac43\,\frac{(S_{\boldsymbol{a}_2}-S_{\boldsymbol{a}_3})P_{\boldsymbol{a}_1}+(S_{\boldsymbol{a}_3}-S_{\boldsymbol{a}_1})P_{\boldsymbol{a}_2}+(S_{\boldsymbol{a}_1}-S_{\boldsymbol{a}_2})P_{\boldsymbol{a}_3}}{S_{\boldsymbol{a}_2}-S_{\boldsymbol{a}_3}},\\\delta S_{\boldsymbol{a}_2}&=4P_{\boldsymbol{a}_2}-S_{\boldsymbol{a}_2}^2-\frac43\,\frac{(S_{\boldsymbol{a}_2}-S_{\boldsymbol{a}_3})P_{\boldsymbol{a}_1}+(S_{\boldsymbol{a}_3}-S_{\boldsymbol{a}_1})P_{\boldsymbol{a}_2}+(S_{\boldsymbol{a}_1}-S_{\boldsymbol{a}_2})P_{\boldsymbol{a}_3}}{S_{\boldsymbol{a}_3}-S_{\boldsymbol{a}_1}},\\\delta S_{\boldsymbol{a}_3}&=4P_{\boldsymbol{a}_3}-S_{\boldsymbol{a}_3}^2-\frac43\,\frac{(S_{\boldsymbol{a}_2}-S_{\boldsymbol{a}_3})P_{\boldsymbol{a}_1}+(S_{\boldsymbol{a}_3}-S_{\boldsymbol{a}_1})P_{\boldsymbol{a}_2}+(S_{\boldsymbol{a}_1}-S_{\boldsymbol{a}_2})P_{\boldsymbol{a}_3}}{S_{\boldsymbol{a}_1}-S_{\boldsymbol{a}_2}}.\end{aligned}\tag{6.18}$$

Proof. Simple calculations show that if $g = 3$ then the set $\mathfrak{K}^*_{\boldsymbol{c},\boldsymbol{d}}$ consists of three elements. Without loss of generality we can assume that the elements $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ are ordered so that the matrix $\mathrm{M} = \mathrm{M}_{\boldsymbol{c},\boldsymbol{d}}$ has the form (5.20). By Theorem 3 we now obtain the system of differential equations

$$\begin{aligned} \theta_{\boldsymbol{a}_1}\delta S_{\boldsymbol{a}_1} &= 4\theta_{\boldsymbol{a}_1}P_{\boldsymbol{a}_1} - \theta_{\boldsymbol{a}_1}S^2_{\boldsymbol{a}_1} - \frac{4}{3}(\theta_{\boldsymbol{a}_1}P_{\boldsymbol{a}_1} - \theta_{\boldsymbol{a}_2}P_{\boldsymbol{a}_2} - \theta_{\boldsymbol{a}_3}P_{\boldsymbol{a}_3}), \\ \theta_{\boldsymbol{a}_2}\delta S_{\boldsymbol{a}_2} &= 4\theta_{\boldsymbol{a}_2}P_{\boldsymbol{a}_2} - \theta_{\boldsymbol{a}_2}S^2_{\boldsymbol{a}_2} + \frac{4}{3}(\theta_{\boldsymbol{a}_1}P_{\boldsymbol{a}_1} - \theta_{\boldsymbol{a}_2}P_{\boldsymbol{a}_2} - \theta_{\boldsymbol{a}_3}P_{\boldsymbol{a}_3}), \\ \theta_{\boldsymbol{a}_3}\delta S_{\boldsymbol{a}_3} &= 4\theta_{\boldsymbol{a}_3}P_{\boldsymbol{a}_3} - \theta_{\boldsymbol{a}_3}S^2_{\boldsymbol{a}_3} + \frac{4}{3}(\theta_{\boldsymbol{a}_1}P_{\boldsymbol{a}_1} - \theta_{\boldsymbol{a}_2}P_{\boldsymbol{a}_2} - \theta_{\boldsymbol{a}_3}P_{\boldsymbol{a}_3}); \end{aligned} \tag{6.19}$$

relations (6.8) can be transformed in this case to the following form:

$$\frac{S_{\boldsymbol{a}_2} - S_{\boldsymbol{a}_3}}{\theta_{\boldsymbol{a}_1}} = \frac{S_{\boldsymbol{a}_1} - S_{\boldsymbol{a}_3}}{\theta_{\boldsymbol{a}_2}} = \frac{S_{\boldsymbol{a}_2} - S_{\boldsymbol{a}_1}}{\theta_{\boldsymbol{a}_3}}. \tag{6.20}$$

Relations (6.20) show that

$$\begin{aligned} &\theta_{\boldsymbol{a}_1}P_{\boldsymbol{a}_1} - \theta_{\boldsymbol{a}_2}P_{\boldsymbol{a}_2} - \theta_{\boldsymbol{a}_3}P_{\boldsymbol{a}_3} \\ &\quad= \quad\theta_{\boldsymbol{a}_1}\frac{(S_{\boldsymbol{a}_2} - S_{\boldsymbol{a}_3})P_{\boldsymbol{a}_1} + (S_{\boldsymbol{a}_3} - S_{\boldsymbol{a}_1})P_{\boldsymbol{a}_2} + (S_{\boldsymbol{a}_1} - S_{\boldsymbol{a}_2})P_{\boldsymbol{a}_3}}{S_{\boldsymbol{a}_2} - S_{\boldsymbol{a}_3}} \\ &\quad= -\theta_{\boldsymbol{a}_2}\frac{(S_{\boldsymbol{a}_2} - S_{\boldsymbol{a}_3})P_{\boldsymbol{a}_1} + (S_{\boldsymbol{a}_3} - S_{\boldsymbol{a}_1})P_{\boldsymbol{a}_2} + (S_{\boldsymbol{a}_1} - S_{\boldsymbol{a}_2})P_{\boldsymbol{a}_3}}{S_{\boldsymbol{a}_3} - S_{\boldsymbol{a}_1}} \\ &\quad= -\theta_{\boldsymbol{a}_3}\frac{(S_{\boldsymbol{a}_2} - S_{\boldsymbol{a}_3})P_{\boldsymbol{a}_1} + (S_{\boldsymbol{a}_3} - S_{\boldsymbol{a}_1})P_{\boldsymbol{a}_2} + (S_{\boldsymbol{a}_1} - S_{\boldsymbol{a}_2})P_{\boldsymbol{a}_3}}{S_{\boldsymbol{a}_1} - S_{\boldsymbol{a}_2}}. \end{aligned} \tag{6.21}$$

Dividing both parts of (6.19) by appropriate thetanulls and using (6.21) we obtain the system of differential equations (6.18). The proof is complete.

Without enlarging on differential equations for each function (0.18) in the case $g = 3$ we indicate a method of their derivation from (6.18). Fixing the pair of even characteristics $\boldsymbol{a}_1, \boldsymbol{a}_2$ and the corresponding characteristic $\boldsymbol{c} = \boldsymbol{a}_2 - \boldsymbol{a}_1$ in the pair (6.2) we consider even characteristics $\boldsymbol{a}_3$, $\boldsymbol{a}_4 = \boldsymbol{a}_3 + \boldsymbol{c}$ and $\boldsymbol{a}_5$, $\boldsymbol{a}_6 = \boldsymbol{a}_5 + \boldsymbol{c}$. Each of the quadruples

$$\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4\}, \quad \{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_5, \boldsymbol{a}_6\}, \quad \{\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6\}$$

corresponds to the vertices of a parallelogram, so that the system (6.18) enables us to write down differential equations for the quantities

$$\delta(\psi_{\boldsymbol{a}_1} + \psi_{\boldsymbol{a}_2} + \psi_{\boldsymbol{a}_3} + \psi_{\boldsymbol{a}_4}), \quad \delta(\psi_{\boldsymbol{a}_1} + \psi_{\boldsymbol{a}_2} + \psi_{\boldsymbol{a}_5} + \psi_{\boldsymbol{a}_6}), \quad \delta(\psi_{\boldsymbol{a}_3} + \psi_{\boldsymbol{a}_4} + \psi_{\boldsymbol{a}_5} + \psi_{\boldsymbol{a}_6}),$$

which bring us to differential equations for $\delta(\psi_{\boldsymbol{a}_1} + \psi_{\boldsymbol{a}_2})$ and $\delta(\psi_{\boldsymbol{a}_3} + \psi_{\boldsymbol{a}_4})$ with denominator of degree 3. In particular, we obtain an equation for

$$\delta(2\psi_{\boldsymbol{a}_1} + 2\psi_{\boldsymbol{a}_2} - \psi_{\boldsymbol{a}_3} - \psi_{\boldsymbol{a}_4}) \tag{6.22}$$

with the same denominator on the right-hand side. In a similar way we deduce equations for

$$\delta(2\psi_{\boldsymbol{a}_1}+2\psi_{\boldsymbol{a}_3}-\psi_{\boldsymbol{a}_2}-\psi_{\boldsymbol{a}_4}),\quad \delta(2\psi_{\boldsymbol{a}_1}+2\psi_{\boldsymbol{a}_4}-\psi_{\boldsymbol{a}_2}-\psi_{\boldsymbol{a}_3}), \tag{6.23}$$

each with denominator of degree 3 (depending on the equation). Adding these differential equations for the quantities in (6.22) and (6.23) we obtain an equation for $\delta\psi_{\boldsymbol{a}_1}$ with denominator of degree 9 on the right-hand side. Of course, the freedom in our choice of $\boldsymbol{a}_2,\boldsymbol{a}_3,\boldsymbol{a}_5,\dots$ means that we can obtain several equations for the function $\psi_{\boldsymbol{a}_1}$, which gives us many algebraic relations between the functions (0.18). Recall, however, that there exist 36 even characteristics for $g=3$; for this reason the combinatorial problem of the description of all these algebraic relations lies beyond the scope of this work.

§ 7. Systems of characteristics and theta-relations

In this section we perform an audit of the special cases of Riemann relations (3.1) encountered before. First of all, we give legal status to additive systems of characteristics, of which a special case in dimension $g=2$ we have called a parallelogram.

Characteristics $\boldsymbol{a},\boldsymbol{b}\in\mathfrak{K}=\mathbb{Z}^{2g}/2\mathbb{Z}^{2g}$ are said to be *syzygetic* if $|\boldsymbol{a},\boldsymbol{b}|$ is an even integer. For a pair of syzygetic characteristics we can define the quantity $\boldsymbol{ab}={}^t\boldsymbol{a}'\boldsymbol{b}''$ $\pmod 2$; since $\boldsymbol{ab}+\boldsymbol{ba}\equiv|\boldsymbol{a},\boldsymbol{b}|\pmod 2$, it follows that $\boldsymbol{ab}=\boldsymbol{ba}$. In addition, $\boldsymbol{aa}=|\boldsymbol{a}|$.

By an *additive group of characteristics* $\mathfrak{A}\subset\mathfrak{K}$ we mean an additive subgroup of $\mathfrak{K}$ with pairwise syzygetic elements. In such a group one can find l basic vectors $\boldsymbol{a}_1,\dots,\boldsymbol{a}_l$ (in more than one way) such that $\mathfrak{A}=\{\alpha_1\boldsymbol{a}_1+\dots+\alpha_l\boldsymbol{a}_l:\alpha_1,\dots,\alpha_l=0,1\}$; in that case $(-1)^{|\boldsymbol{a}_j,\boldsymbol{a}_k|}=1$ for all $j,k=1,\dots,l$. The number of elements of the additive group $\mathfrak{A}$ is 2^l, we call l its *dimension* and write $\dim\mathfrak{A}=l$. We point out straight away that $\dim\mathfrak{A}\leqslant g$ (see [1]; Teil 2, Kapitel 7, § 8). The above-defined *commutative* operation of multiplication of syzygetic characteristics is distributive over addition in the additive group $\mathfrak{A}$.

An *additive system of characteristics* $\mathfrak{G}$ is a set of *even* characteristics such that their various pairwise sums make up an additive group; such a system can be represented in the form $\mathfrak{G}=\boldsymbol{a}+\mathfrak{A}\subset\mathfrak{K}^*$, where $\boldsymbol{a}$ is an arbitrary element of $\mathfrak{G}$ and $\mathfrak{A}$ is the corresponding (uniquely defined) additive group. The *dimension of an additive system* is by definition the dimension of the corresponding additive group ($\dim\mathfrak{G}=\dim\mathfrak{A}$), and its cardinality is $2^{\dim\mathfrak{G}}$. Each even characteristic is an additive system of dimension 0. An additive system of maximum dimension g is called a *Göpel system*; examples of such systems in dimension $g=2$ were discovered in connection with Ohyama's theorem.

Lemma 12. *The number of additive systems of dimension $l\leqslant g$ in $\mathbb{Z}^{2g}/2\mathbb{Z}^{2g}$ is*

$$2^{g-l-1}(2^{g-l}+1)\cdot\frac{(2^{2g}-1)(2^{2g-2}-1)\cdots(2^{2(g-l+1)}-1)}{(2^l-1)(2^{l-1}-1)\cdots(2-1)}.$$

In particular, the number of Göpel systems is

$$\frac{(2^{2g}-1)(2^{2g-2}-1)\cdots(2^2-1)}{(2^g-1)(2^{g-1}-1)\cdots(2-1)}=(2^g+1)(2^{g-1}+1)\cdots(2+1).$$

Proof. This is in effect a purely combinatorial fact: the number of additive groups of characteristics of dimension l in $\mathfrak{K}$ is

$$\frac{(2^{2g}-1)(2^{2g-2}-1)\cdots(2^{2(g-l+1)}-1)}{(2^l-1)(2^{l-1}-1)\cdots(2-1)}$$

(see [1]; Teil 2, Kapitel 7, § 8). For such a group $\mathfrak{A}$ the quotient group $\mathfrak{K}/\mathfrak{A}$ is isomorphic to $\mathbb{Z}^{2g-2l}/2\mathbb{Z}^{2g-2l}$, therefore the number of the corresponding additive systems is equal to the number of even characteristics in the last group, which is $2^{g-l-1}(2^{g-l}+1)$. This completes the proof.

Corollary. *The number of additive systems of dimension $l \leqslant g$ in $\mathbb{Z}^{2g}/2\mathbb{Z}^{2g}$ containing a fixed even characteristic $\boldsymbol{a}$ is*

$$(2^{g-l}+1)(2^g-1)\cdot\frac{(2^{2g-2}-1)\cdots(2^{2(g-l+1)}-1)}{(2^l-1)(2^{l-1}-1)\cdots(2-1)}.$$

Proof. We point out first that the number $\varkappa_{\boldsymbol{a}}$ of additive systems in question does not depend on one's choice of $\boldsymbol{a} \in \mathfrak{K}^*$. Since the cardinality of $\mathfrak{K}^*$ is $2^{g-1}(2^g+1)$ and each additive system consists of 2^l characteristics, it follows that

$$\varkappa_{\boldsymbol{a}} = \frac{2^l\cdot\varkappa}{2^{g-1}(2^g+1)} = \frac{\varkappa}{2^{g-l-1}(2^g+1)}, \qquad \boldsymbol{a}\in\mathfrak{K}^*,$$

where $\varkappa$ is the total number of additive systems of dimension l. Using now the precise value of $\varkappa$ calculated in Lemma 12 we arrive at the required result.

All arguments below are valid only for $g \geqslant 3$. We fix an element $\boldsymbol{c}\in\mathfrak{K}$ distinct from zero (and therefore we fix also the quotient $\mathfrak{K}^*_{\boldsymbol{c}} = \mathfrak{K}/\{\boldsymbol{0},\boldsymbol{c}\}$). Returning to the notation (5.3) in § 5 we can write relations (3.2) (or (6.4)) as follows:

$$\theta_{\boldsymbol{a}+\boldsymbol{d}}(\boldsymbol{z})\theta_{\boldsymbol{a}}(\boldsymbol{z}) = \frac{1}{2^{g-2}}\sum_{\boldsymbol{b}\in\mathfrak{K}^*_{\boldsymbol{c}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|+{}^t\boldsymbol{d}'(\boldsymbol{a}''+\boldsymbol{b}'')}\frac{1}{2}\big(\widetilde{\theta}_{\boldsymbol{b}+\boldsymbol{d}}(\boldsymbol{z})\theta_{\boldsymbol{b}}+\theta_{\boldsymbol{b}+\boldsymbol{d}}\widetilde{\theta}_{\boldsymbol{b}}(\boldsymbol{z})\big), \tag{7.1}$$
$$\boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c}},\quad \boldsymbol{d}\in\mathfrak{D},$$

where $\mathfrak{D} = \mathfrak{D}_{\boldsymbol{c}} = \mathfrak{K}^*_{\boldsymbol{c}} + \boldsymbol{a} \subset \mathfrak{K}_{\boldsymbol{c}}$ for some (any) $\boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c}}$.

We now transform (7.1) into the following form:

$$\begin{aligned}\theta_{\boldsymbol{a}+\boldsymbol{d}}(\boldsymbol{z})\theta_{\boldsymbol{a}}(\boldsymbol{z}) &- \frac{1}{2^{g-1}}\big(\widetilde{\theta}_{\boldsymbol{a}+\boldsymbol{d}}(\boldsymbol{z})\theta_{\boldsymbol{a}}+\theta_{\boldsymbol{a}+\boldsymbol{d}}\widetilde{\theta}_{\boldsymbol{a}}(\boldsymbol{z})\big)\\ &= \frac{1}{2^{g-2}}\sum_{\substack{\boldsymbol{b}\in\mathfrak{K}^*_{\boldsymbol{c}}\\ \boldsymbol{b}\neq\boldsymbol{a},\boldsymbol{a}+\boldsymbol{d}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|+{}^t\boldsymbol{d}'(\boldsymbol{a}''+\boldsymbol{b}'')}\frac{1}{2}\big(\widetilde{\theta}_{\boldsymbol{b}+\boldsymbol{d}}(\boldsymbol{z})\theta_{\boldsymbol{b}}+\theta_{\boldsymbol{b}+\boldsymbol{d}}\widetilde{\theta}_{\boldsymbol{b}}(\boldsymbol{z})\big),\end{aligned} \tag{7.2}$$
$$\boldsymbol{a}\in\mathfrak{K}^*_{\boldsymbol{c}},\quad \boldsymbol{d}\in\mathfrak{D};$$

next, we multiply both parts of (7.2) by $(-1)^{|\boldsymbol{a},\boldsymbol{d}|}\theta_{\boldsymbol{a}+\boldsymbol{d}}$ and sum over $\boldsymbol{d}\in\mathfrak{D}\setminus\{\boldsymbol{0}\}$, using (only on the left-hand side) formulae (5.4) and (5.5):

$$\begin{aligned}2(2^{g-2}-1)\biggl(\theta_{\boldsymbol{a}}^2(\boldsymbol{z})\theta_{\boldsymbol{a}}-\frac{1}{2^{g-1}}\widetilde{\theta}_{\boldsymbol{a}}(\boldsymbol{z})\theta_{\boldsymbol{a}}^2\biggr)\\ =\frac{1}{2^{g-2}}\sum_{\mathfrak{G}=\{\boldsymbol{a},\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3\}}(-1)^{|\mathfrak{G}|}\bigl(\widetilde{\theta}_{\boldsymbol{a}_1}(\boldsymbol{z})\theta_{\boldsymbol{a}_2}\theta_{\boldsymbol{a}_3}+\theta_{\boldsymbol{a}_1}\widetilde{\theta}_{\boldsymbol{a}_2}(\boldsymbol{z})\theta_{\boldsymbol{a}_3}+\theta_{\boldsymbol{a}_1}\theta_{\boldsymbol{a}_2}\widetilde{\theta}_{\boldsymbol{a}_3}(\boldsymbol{z})\bigr),\\ \boldsymbol{a}\in\mathfrak{K}_{\boldsymbol{c}}^*,\end{aligned}\tag{7.3}$$

where summation on the right-hand side proceeds over all 2-dimensional additive systems $\mathfrak{G}=\{\boldsymbol{a},\boldsymbol{a}+\boldsymbol{d},\boldsymbol{a}+\boldsymbol{e},\boldsymbol{a}+\boldsymbol{d}+\boldsymbol{e}\}$ in $\mathfrak{K}_{\boldsymbol{c}}^*$ and the quantity

$$|\mathfrak{G}|=|\boldsymbol{d}|+|\boldsymbol{e}|+\boldsymbol{d}\boldsymbol{e}\pmod 2$$

is independent on one's choice of an element $\boldsymbol{a}\in\mathfrak{G}$ and generators $\boldsymbol{d},\boldsymbol{e}$ of the corresponding additive group $\mathfrak{G}+\boldsymbol{a}\subset\mathfrak{K}_{\boldsymbol{c}}$.

Setting $\boldsymbol{z}=\boldsymbol{0}$ in (7.3) we obtain

$$\frac{1}{3}\cdot\frac{2(2^{g-2}-1)(2^{g-1}-1)}{2^{g-1}}\theta_{\boldsymbol{a}}^3=\frac{1}{2^{g-2}}\sum_{\mathfrak{G}=\{\boldsymbol{a},\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3\}}(-1)^{|\mathfrak{G}|}\theta_{\boldsymbol{a}_1}\theta_{\boldsymbol{a}_2}\theta_{\boldsymbol{a}_3},\qquad \boldsymbol{a}\in\mathfrak{K}_{\boldsymbol{c}}^*.\tag{7.4}$$

Multiplying both parts of (7.3) by $\theta_{\boldsymbol{a}}$, both parts of (7.4) by $\widetilde{\theta}_{\boldsymbol{a}}(\boldsymbol{z})$, and adding the resulting relations we obtain

$$2(2^{g-2}-1)\biggl(\theta_{\boldsymbol{a}}^2(\boldsymbol{z})\theta_{\boldsymbol{a}}^2+\frac{1}{3}\Bigl(1-\frac{1}{2^{g-3}}\Bigr)\widetilde{\theta}_{\boldsymbol{a}}(\boldsymbol{z})\theta_{\boldsymbol{a}}^3\biggr)=\frac{1}{2^{g-2}}\sum_{\mathfrak{G}\ni\boldsymbol{a}}\widetilde{\theta}_{\mathfrak{G}}(\boldsymbol{z}),\qquad \boldsymbol{a}\in\mathfrak{K}_{\boldsymbol{c}}^*,\tag{7.5}$$

where

$$\widetilde{\theta}_{\mathfrak{G}}(\boldsymbol{z})=(-1)^{|\mathfrak{G}|}\theta_{\boldsymbol{a}}\theta_{\boldsymbol{a}_1}\theta_{\boldsymbol{a}_2}\theta_{\boldsymbol{a}_3}\cdot\biggl(\frac{\widetilde{\theta}_{\boldsymbol{a}}(\boldsymbol{z})}{\theta_{\boldsymbol{a}}}+\frac{\widetilde{\theta}_{\boldsymbol{a}_1}(\boldsymbol{z})}{\theta_{\boldsymbol{a}_1}}+\frac{\widetilde{\theta}_{\boldsymbol{a}_2}(\boldsymbol{z})}{\theta_{\boldsymbol{a}_2}}+\frac{\widetilde{\theta}_{\boldsymbol{a}_3}(\boldsymbol{z})}{\theta_{\boldsymbol{a}_3}}\biggr),$$
$$\mathfrak{G}=\{\boldsymbol{a},\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3\}\ni\boldsymbol{a}.$$

Recalling our definitions (5.3) and proceeding from two-dimensional additive systems $\mathfrak{G}$ in $\mathfrak{K}_{\boldsymbol{c}}$ to three-dimensional systems $\mathfrak{F}=\{\boldsymbol{a},\boldsymbol{a}+\boldsymbol{c}:\boldsymbol{a}\in\mathfrak{G}\}$ in $\mathfrak{K}$, we can rewrite (7.5):

$$\begin{aligned}(2^{g-2}-1)(-1)^{|\boldsymbol{a},\boldsymbol{c}|}\vartheta_{\boldsymbol{a}+\boldsymbol{c}}^2(\boldsymbol{z})\vartheta_{\boldsymbol{a}+\boldsymbol{c}}^2\vartheta_{\boldsymbol{a}}^2(\boldsymbol{z})\vartheta_{\boldsymbol{a}}^2\\ +\frac{(2^{g-3}-1)(2^{g-2}-1)}{3\cdot 2^{g-2}}(-1)^{|\boldsymbol{a},\boldsymbol{c}|}\bigl(\vartheta_{\boldsymbol{a}+\boldsymbol{c}}(2\boldsymbol{z})\vartheta_{\boldsymbol{a}}+\vartheta_{\boldsymbol{a}+\boldsymbol{c}}\vartheta_{\boldsymbol{a}}(2\boldsymbol{z})\bigr)\vartheta_{\boldsymbol{a}+\boldsymbol{c}}^3\vartheta_{\boldsymbol{a}}^3\\ =\frac{1}{2^{g-3}}\sum_{\mathfrak{F}\supset\{\boldsymbol{a},\boldsymbol{a}+\boldsymbol{c}\}}\vartheta_{\mathfrak{F}}(2\boldsymbol{z}),\qquad \boldsymbol{a}\in\mathfrak{K}^*,\quad \boldsymbol{c}\ne\boldsymbol{0},\end{aligned}\tag{7.6}$$

where on the right-hand side we have summation proceeding over all additive systems $\mathfrak{F}=\{\boldsymbol{a},\boldsymbol{a}+\boldsymbol{c},\boldsymbol{a}+\boldsymbol{d},\boldsymbol{a}+\boldsymbol{e},\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d},\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{e},\boldsymbol{a}+\boldsymbol{d}+\boldsymbol{e},\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}+\boldsymbol{e}\}$ in $\mathfrak{K}$ and

$$\vartheta_{\mathfrak{F}}(\boldsymbol{z})=(-1)^{|\boldsymbol{c}|+|\boldsymbol{d}|+|\boldsymbol{e}|+\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{e}+\boldsymbol{e}\boldsymbol{c}}\prod_{\boldsymbol{b}\in\mathfrak{F}}\vartheta_{\boldsymbol{b}}\cdot\frac{1}{8}\sum_{\boldsymbol{b}\in\mathfrak{F}}\frac{\vartheta_{\boldsymbol{b}}(\boldsymbol{z})}{\vartheta_{\boldsymbol{b}}}.\tag{7.7}$$

Lemma 13. *The quantity*

$$\begin{aligned}\vartheta_{\mathfrak{F}} = \vartheta_{\mathfrak{F}}(\mathbf{0}) = (-1)^{|\boldsymbol{c}|+|\boldsymbol{d}|+|\boldsymbol{e}|+\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{e}+\boldsymbol{e}\boldsymbol{c}}\vartheta_{\boldsymbol{a}}\vartheta_{\boldsymbol{a}+\boldsymbol{c}}\vartheta_{\boldsymbol{a}+\boldsymbol{d}}\vartheta_{\boldsymbol{a}+\boldsymbol{e}}\\ \times \vartheta_{\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}}\vartheta_{\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{e}}\vartheta_{\boldsymbol{a}+\boldsymbol{d}+\boldsymbol{e}}\vartheta_{\boldsymbol{a}+\boldsymbol{c}+\boldsymbol{d}+\boldsymbol{e}}\end{aligned} \tag{7.8}$$

is independent on one's choice of an element $\boldsymbol{a}$ of the additive system $\mathfrak{F}$ and generators $\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ of the corresponding additive group $\mathfrak{F} + \boldsymbol{a}$.

Proof. In view of the symmetry of the quantity (7.8) relative to the generators $\boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ of the additive group $\mathfrak{F} + \boldsymbol{a}$, it suffices to verify the invariance of $\vartheta_{\mathfrak{F}}$ after the replacements of $\boldsymbol{a} = \boldsymbol{b} + \boldsymbol{c}$ by $\boldsymbol{b}$ (the independence of one's choice of an element of the additive system $\mathfrak{F}$) and $\boldsymbol{c} = \boldsymbol{b} + \boldsymbol{d}$ by $\boldsymbol{b}$ (the independence of one's choice of generators in the additive group $\mathfrak{F} + \boldsymbol{a}$). This invariance can be established by direct calculations using formula (6.5). We leave the details to the reader.

Lemma 13 shows that the quantity (7.7) is well-defined. We have thus proved the following result.

Lemma 14. *There hold the relations*

$$\begin{aligned}&(2^{g-2} - 1)(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\vartheta^2_{\boldsymbol{a}}(\boldsymbol{z})\vartheta^2_{\boldsymbol{a}}\vartheta^2_{\boldsymbol{b}}(\boldsymbol{z})\vartheta^2_{\boldsymbol{b}}\\ &\quad + \frac{(2^{g-3} - 1)(2^{g-2} - 1)}{3 \cdot 2^{g-2}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\big(\vartheta_{\boldsymbol{a}}(2\boldsymbol{z})\vartheta_{\boldsymbol{b}} + \vartheta_{\boldsymbol{a}}\vartheta_{\boldsymbol{b}}(2\boldsymbol{z})\big)\vartheta^3_{\boldsymbol{a}}\vartheta^3_{\boldsymbol{b}}\\ &\quad = \frac{1}{2^{g-3}}\sum_{\mathfrak{F}\supset\{\boldsymbol{a},\boldsymbol{b}\}}\vartheta_{\mathfrak{F}}(2\boldsymbol{z}), \qquad \boldsymbol{a}, \boldsymbol{b} \in \mathfrak{K}^*, \quad \boldsymbol{a} \neq \boldsymbol{b},\end{aligned} \tag{7.9}$$

where summation on the right-hand side proceeds over all additive systems $\mathfrak{F} \supset \{\boldsymbol{a}, \boldsymbol{b}\}$ of dimension 3 in $\mathfrak{K}$ and the functions $\vartheta_{\mathfrak{F}}(\boldsymbol{z})$ are defined by (7.7). In particular, for $g = 3$ there hold the relations

$$(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\vartheta^2_{\boldsymbol{a}}(\boldsymbol{z})\vartheta^2_{\boldsymbol{a}}\vartheta^2_{\boldsymbol{b}}(\boldsymbol{z})\vartheta^2_{\boldsymbol{b}} = \sum_{\mathfrak{F}\supset\{\boldsymbol{a},\boldsymbol{b}\}}\vartheta_{\mathfrak{F}}(2\boldsymbol{z}), \qquad \boldsymbol{a}, \boldsymbol{b} \in \mathfrak{K}^*, \quad \boldsymbol{a} \neq \boldsymbol{b}.$$

Proof. It suffices to set $\boldsymbol{b} = \boldsymbol{a} + \boldsymbol{c}$ in (7.6).

We proceed now to summation of relations (7.9) over all $\boldsymbol{b} \in \mathfrak{K}^* \setminus \{\boldsymbol{a}\}$. For the left-hand sides we use the following consequence of Riemann relations (3.1):

$$\sum_{\substack{\boldsymbol{b}\in\mathfrak{K}^*\\ \boldsymbol{b}\neq\boldsymbol{a}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\vartheta^2_{\boldsymbol{b}}(\boldsymbol{z})\vartheta^2_{\boldsymbol{b}} = (2^g - 1)\vartheta^2_{\boldsymbol{a}}(\boldsymbol{z})\vartheta^2_{\boldsymbol{a}}, \qquad \boldsymbol{a} \in \mathfrak{K}^*,$$

$$\sum_{\substack{\boldsymbol{b}\in\mathfrak{K}^*\\ \boldsymbol{b}\neq\boldsymbol{a}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\vartheta^4_{\boldsymbol{b}} = (2^g - 1)\vartheta^4_{\boldsymbol{a}}, \qquad \boldsymbol{a} \in \mathfrak{K}^*,$$

$$\sum_{\substack{\boldsymbol{b}\in\mathfrak{K}^*\\ \boldsymbol{b}\neq\boldsymbol{a}}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\vartheta_{\boldsymbol{b}}(2\boldsymbol{z})\vartheta^3_{\boldsymbol{b}} = 2^g\vartheta^4_{\boldsymbol{a}}(\boldsymbol{z}) - \vartheta_{\boldsymbol{a}}(2\boldsymbol{z})\vartheta^3_{\boldsymbol{a}}, \qquad \boldsymbol{a} \in \mathfrak{K}^*.$$

Hence

$$\sum_{\substack{\boldsymbol{b}\in\mathfrak{K}^*\\ \boldsymbol{b}\neq\boldsymbol{a}}}\biggl((2^{g-2}-1)(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\vartheta_{\boldsymbol{a}}^2(\boldsymbol{z})\vartheta_{\boldsymbol{a}}^2\vartheta_{\boldsymbol{b}}^2(\boldsymbol{z})\vartheta_{\boldsymbol{b}}^2 + \frac{(2^{g-2}-1)(2^{g-3}-1)}{3\cdot 2^{g-2}}(-1)^{|\boldsymbol{a},\boldsymbol{b}|}\bigl(\vartheta_{\boldsymbol{a}}(2\boldsymbol{z})\vartheta_{\boldsymbol{b}}+\vartheta_{\boldsymbol{a}}\vartheta_{\boldsymbol{b}}(2\boldsymbol{z})\bigr)\vartheta_{\boldsymbol{a}}^3\vartheta_{\boldsymbol{b}}^3\biggr)$$
$$= \frac{7(2^{g-2}-1)(2^{g-1}-1)}{3}\vartheta_{\boldsymbol{a}}^4(\boldsymbol{z})\vartheta_{\boldsymbol{a}}^4 + \frac{(2^{g-3}-1)(2^{g-2}-1)(2^{g-1}-1)}{3\cdot 2^{g-3}}\vartheta_{\boldsymbol{a}}(2\boldsymbol{z})\vartheta_{\boldsymbol{a}}^7, \quad \boldsymbol{a}\in\mathfrak{K}^*. \tag{7.10}$$

The number of additive systems $\mathfrak{F}\supset\{\boldsymbol{a},\boldsymbol{b}\}$ of dimension 3 in $\mathfrak{K}=\mathbb{Z}^{2g}/2\mathbb{Z}^{2g}$ is equal to the number of 2-dimensional additive systems $\mathfrak{G}\ni\boldsymbol{a}$ in $\mathfrak{K}_{\boldsymbol{b}-\boldsymbol{a}}\simeq\mathbb{Z}^{2g-2}/2\mathbb{Z}^{2g-2}$, which by the corollary to Lemma 12 is

$$(2^{g-3}+1)(2^{g-1}-1)\frac{2^{2g-4}-1}{(2^2-1)(2-1)}.$$

The cardinality of the set $\mathfrak{K}^*\setminus\{\boldsymbol{a}\}$ is

$$2^{g-1}(2^g+1)-1=(2^{g-1}+1)(2^g-1).$$

Hence summing the right-hand sides of (7.9) for $\boldsymbol{b}\in\mathfrak{K}^*\setminus\{\boldsymbol{a}\}$ we obtain

$$\begin{aligned}\varkappa_1 &= (2^{g-3}+1)(2^{g-1}+1)(2^{g-1}-1)(2^g-1)\frac{2^{2g-4}-1}{(2^2-1)(2-1)}\\ &= (2^{g-3}+1)(2^g-1)\frac{(2^{2g-2}-1)(2^{2g-4}-1)}{(2^2-1)(2-1)}\end{aligned}$$

terms, each corresponding to some additive system $\mathfrak{F}\ni\boldsymbol{a}$ of dimension 3 in $\mathfrak{K}$. The number of such additive systems is by Corollary to Lemma 12 equal to

$$\varkappa_2=(2^{g-3}+1)(2^g-1)\frac{(2^{2g-2}-1)(2^{2g-4}-1)}{(2^3-1)(2^2-1)(2-1)}.$$

Consequently,

$$\frac{1}{2^{g-3}}\sum_{\substack{\boldsymbol{b}\in\mathfrak{K}^*\\ \boldsymbol{b}\neq\boldsymbol{a}}}\sum_{\mathfrak{F}\supset\{\boldsymbol{a},\boldsymbol{b}\}}\vartheta_{\mathfrak{F}}(2\boldsymbol{z})=\frac{1}{2^{g-3}}\cdot\frac{\varkappa_1}{\varkappa_2}\sum_{\mathfrak{F}\ni\boldsymbol{a}}\vartheta_{\mathfrak{F}}(2\boldsymbol{z})=\frac{7}{2^{g-3}}\sum_{\mathfrak{F}\ni\boldsymbol{a}}\vartheta_{\mathfrak{F}}(2\boldsymbol{z}), \qquad \boldsymbol{a}\in\mathfrak{K}^*. \tag{7.11}$$

Combining (7.10) and (7.11) we obtain the following result.

Lemma 15. *The following relations hold*:

$$\frac{(2^{g-2}-1)(2^{g-1}-1)}{3}\vartheta_{\boldsymbol{a}}^4(\boldsymbol{z})\vartheta_{\boldsymbol{a}}^4+\frac{(2^{g-3}-1)(2^{g-2}-1)(2^{g-1}-1)}{3\cdot 7\cdot 2^{g-3}}\vartheta_{\boldsymbol{a}}(2\boldsymbol{z})\vartheta_{\boldsymbol{a}}^7 = \frac{1}{2^{g-3}}\sum_{\mathfrak{F}\ni\boldsymbol{a}}\vartheta_{\mathfrak{F}}(2\boldsymbol{z}), \qquad \boldsymbol{a}\in\mathfrak{K}^*, \tag{7.12}$$

where summation on the right-hand side proceeds over all additive systems $\mathfrak{F} \ni \boldsymbol{a}$ of dimension 3 *in* $\mathfrak{K}$ *and the functions* $\vartheta_{\mathfrak{F}}(\boldsymbol{z})$ *are defined by* (7.7). *In particular, for* $g = 3$ *there hold the relations*

$$\vartheta_{\boldsymbol{a}}^4(\boldsymbol{z})\vartheta_{\boldsymbol{a}}^4 = \sum_{\mathfrak{F}\ni\boldsymbol{a}} \vartheta_{\mathfrak{F}}(2\boldsymbol{z}), \qquad \boldsymbol{a} \in \mathfrak{K}^*.$$

For an additive system $\mathfrak{F}$ in $\mathfrak{K}$ of dimension $\dim \mathfrak{F} = 3$ let $S_{\mathfrak{F}}$ and $P_{\mathfrak{F}}$ be the sum of the functions $\psi_{\boldsymbol{b}}$, $\boldsymbol{b} \in \mathfrak{F}$, and the sum of all their pairwise products, respectively. Relations (7.12) enable one to deduce another system of differential equations for the functions (0.18).

Theorem 5. *The functions* (0.18) *satisfy the system of differential equations*

$$\vartheta_{\boldsymbol{a}}^8 \delta\psi_{\boldsymbol{a}} = -\frac{2}{3}\left(13 - \frac{1}{2^{g-4}}\right)\vartheta_{\boldsymbol{a}}^8\psi_{\boldsymbol{a}}^2 + \frac{1}{2^{g-3}(2^{g-2}-1)(2^{g-1}-1)} \sum_{\mathfrak{F}\ni\boldsymbol{a}} \vartheta_{\mathfrak{F}} P_{\mathfrak{F}}, \quad \boldsymbol{a} \in \mathfrak{K}^*, \tag{7.13}$$

where summation on the right-hand side proceeds over all additive systems $\mathfrak{F} \ni \boldsymbol{a}$ *of dimension* 3 *in* $\mathfrak{K}$ *and the functions* $\vartheta_{\mathfrak{F}}$ *are defined by equalities* (7.8). *In addition,*

$$\begin{aligned}\vartheta_{\boldsymbol{a}}^8 &= \frac{21}{(2^{g-2}-1)(2^{g-1}-1)(2^g-1)} \sum_{\mathfrak{F}\ni\boldsymbol{a}} \vartheta_{\mathfrak{F}}, \qquad \boldsymbol{a} \in \mathfrak{K}^*,\\ \vartheta_{\boldsymbol{a}}^8\psi_{\boldsymbol{a}} &= \frac{21}{8(2^{g-2}-1)(2^{g-1}-1)(2^g-1)} \sum_{\mathfrak{F}\ni\boldsymbol{a}} \vartheta_{\mathfrak{F}} S_{\mathfrak{F}}, \qquad \boldsymbol{a} \in \mathfrak{K}^*.\end{aligned} \tag{7.14}$$

Proof. For each additive system $\mathfrak{F}$ of dimension 3 in $\mathfrak{K}$ we set

$$(S_{\mathfrak{F}})^2 = \left(\sum_{\boldsymbol{a}\in\mathfrak{F}} \psi_{\boldsymbol{a}}\right)^2, \qquad S_{\mathfrak{F}}^2 = \sum_{\boldsymbol{a}\in\mathfrak{F}} \psi_{\boldsymbol{a}}^2$$

and use in both parts of (7.12) the expansions (4.1) with remainders in $O(\boldsymbol{z}^6)$:

$$\begin{aligned}&\frac{(2^{g-2}-1)(2^{g-1}-1)}{3}\vartheta_{\boldsymbol{a}}^8 \cdot \left(1 - 8\psi_{\boldsymbol{a}} + \frac{80}{3}\psi_{\boldsymbol{a}}^2 + \frac{8}{3}\delta\psi_{\boldsymbol{a}}\right)\\ &\quad + \frac{(2^{g-3}-1)(2^{g-2}-1)(2^{g-1}-1)}{3\cdot 7\cdot 2^{g-3}}\vartheta_{\boldsymbol{a}}^8 \cdot \left(1 - 8\psi_{\boldsymbol{a}} + \frac{32}{3}\psi_{\boldsymbol{a}}^2 + \frac{32}{3}\delta\psi_{\boldsymbol{a}}\right)\\ &\quad = \frac{1}{2^{g-3}} \sum_{\mathfrak{F}\ni\boldsymbol{a}} \vartheta_{\mathfrak{F}} \cdot \left(1 - S_{\mathfrak{F}} + \frac{4}{3}S_{\mathfrak{F}}^2 + \frac{4}{3}\delta S_{\mathfrak{F}}\right), \qquad \boldsymbol{a} \in \mathfrak{K}^*.\end{aligned}$$

Hence

$$\frac{(2^{g-2}-1)(2^{g-1}-1)(2^g-1)}{3\cdot 7}\vartheta_{\boldsymbol{a}}^8 = \sum_{\mathfrak{F}\ni\boldsymbol{a}} \vartheta_{\mathfrak{F}}, \qquad \boldsymbol{a} \in \mathfrak{K}^*, \tag{7.15}$$

$$\frac{(2^{g-2}-1)(2^{g-1}-1)(2^g-1)}{3\cdot 7}\vartheta_{\boldsymbol{a}}^8\psi_{\boldsymbol{a}} = \frac{1}{8} \sum_{\mathfrak{F}\ni\boldsymbol{a}} \vartheta_{\mathfrak{F}} S_{\mathfrak{F}}, \qquad \boldsymbol{a} \in \mathfrak{K}^*, \tag{7.16}$$

$$\begin{aligned}&\frac{(2^{g-3}-1)(2^{g-2}-1)(2^{g-1}-1)}{3\cdot 7}\vartheta_{\boldsymbol{a}}^8(74\psi_{\boldsymbol{a}}^2 + 11\delta\psi_{\boldsymbol{a}})\\ &\quad + \frac{(2^{g-2}-1)(2^{g-1}-1)}{3}\vartheta_{\boldsymbol{a}}^8(10\psi_{\boldsymbol{a}}^2 + \delta\psi_{\boldsymbol{a}}) = \frac{1}{2}\sum_{\mathfrak{F}\ni\boldsymbol{a}} \vartheta_{\mathfrak{F}}(S_{\mathfrak{F}}^2 + \delta S_{\mathfrak{F}}), \qquad \boldsymbol{a} \in \mathfrak{K}^*.\end{aligned} \tag{7.17}$$

Relations (7.15) and (7.16) bring us to (7.14); δ-differentiating (7.16) we obtain

$$\frac{(2^{g-2}-1)(2^{g-1}-1)(2^g-1)}{3\cdot 7}\vartheta_{\boldsymbol a}^8(8\psi_{\boldsymbol a}^2+\delta\psi_{\boldsymbol a})=\frac{1}{8}\sum_{\mathfrak F\ni\boldsymbol a}\vartheta_{\mathfrak F}\bigl((S_{\mathfrak F})^2+\delta S_{\mathfrak F}\bigr),\qquad \boldsymbol a\in\mathfrak K^*. \tag{7.18}$$

Multiplying (7.18) by 4 and subtracting (7.17) we see that

$$\begin{aligned}\frac{2}{3}(13\cdot 2^{g-3}-2)(2^{g-2}-1)(2^{g-1}-1)\vartheta_{\boldsymbol a}^8\psi_{\boldsymbol a}^2+2^{g-3}(2^{g-2}-1)(2^{g-1}-1)\vartheta_{\boldsymbol a}^8\delta\psi_{\boldsymbol a}\\ =\frac{1}{2}\sum_{\mathfrak F\ni\boldsymbol a}\vartheta_{\mathfrak F}\bigl((S_{\mathfrak F})^2-S_{\mathfrak F}^2\bigr),\qquad \boldsymbol a\in\mathfrak K^*,\end{aligned}$$

which brings us to the system of differential equations (7.13).

Corollary. *For $g=3$ the functions* (0.18) *satisfy the system of differential equations*

$$\vartheta_{\boldsymbol a}^8\delta\psi_{\boldsymbol a}=-\frac{22}{3}\vartheta_{\boldsymbol a}^8\psi_{\boldsymbol a}^2+\frac{1}{3}\sum_{\mathfrak F\ni\boldsymbol a}\vartheta_{\mathfrak F}P_{\mathfrak F},\qquad \boldsymbol a\in\mathfrak K^*, \tag{7.19}$$

where summation on the right-hand side proceeds over all Göpel systems $\mathfrak F\ni\boldsymbol a$, the functions $\vartheta_{\mathfrak F}$ are defined by (7.8), *and $P_{\mathfrak F}$ is the sum of all pairwise products of the functions $\psi_{\boldsymbol b}$, $\boldsymbol b\in\mathfrak F$. Moreover,*

$$\vartheta_{\boldsymbol a}^8=\sum_{\mathfrak F\ni\boldsymbol a}\vartheta_{\mathfrak F},\quad \vartheta_{\boldsymbol a}^8\psi_{\boldsymbol a}=\frac{1}{8}\sum_{\mathfrak F\ni\boldsymbol a}\vartheta_{\mathfrak F}S_{\mathfrak F},\qquad \boldsymbol a\in\mathfrak K^*. \tag{7.20}$$

Note that, according to the corollary to Lemma 12, the number of the Göpel systems involved in summation in (7.19) and (7.20) is 30.

§ 8. Modular nature

Thus far, we took no account of the *modular properties* of the thetanulls and their logarithmic derivatives. This was an affordable luxury because we were interested only in differential equations for these functions. In addition, we left open the question on the existence of relations expressing thetanulls in terms of their logarithmic derivatives (for $g=1$ examples of such relations are formulae (0.8) established in § 1).

We consider the one-dimensional case first. The action of the group $SL_2(\mathbb R)$, the group $SL_2(\mathbb Z)$, and each its *congruence subgroup*

$$\Gamma\subset\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}\pmod L\right\}\subset SL_2(\mathbb Z)$$

(*of level* L) on the upper half-plane $\mathfrak H_1=\{\tau:\operatorname{Im}\tau>0\}$ is defined by the formula

$$\tau\mapsto\gamma\tau=\frac{a\tau+b}{c\tau+d},\qquad \gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb R).$$

A holomorphic function $F(\tau)$ on $\mathfrak{H}_1$ is called a (*holomorphic*) *modular form of weight* w *with respect to* Γ if

$$F(\gamma\tau) = (c\tau + d)^w F(\tau) \qquad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \tag{8.1}$$

and for each $\gamma \in SL_2(\mathbb{Z})$ the function $(c\tau+d)^{-w}F(\gamma\tau)$ can be expanded in a Fourier series

$$\sum_{n=0}^{\infty} f_n e^{2\pi i n\tau/L}.$$

The set of modular forms of weight w with respect to a fixed congruence subgroup Γ is a vector space, which we denote by $\mathrm{Mod}_w(\Gamma)$. Calculating the logarithmic derivative of both parts of (8.1) for an arbitrary form $F \in \mathrm{Mod}_w(\Gamma)$ we obtain

$$\frac{dF/d\tau}{F}\left(\frac{a\tau+b}{c\tau+d}\right) = wc(c\tau+d) + (c\tau+d)^2\frac{dF/d\tau}{F}\left(\frac{a\tau+b}{c\tau+d}\right), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{8.2}$$

It is easy to deduce from (8.2) that the differential operator

$$\mathcal{D}\colon F \mapsto \delta\left(\frac{\delta F}{F}\right) - \frac{1}{w}\left(\frac{\delta F}{F}\right)^2, \qquad \delta = \frac{1}{\pi i}\frac{d}{d\tau}, \tag{8.3}$$

maps the space $\mathrm{Mod}_w(\Gamma)$ into $\mathrm{Mod}_4(\Gamma)$, that is, the function $\mathcal{D}F$ is a modular form of weight 4 with respect to Γ.

For a pair of forms $F_1, F_2 \in \mathrm{Mod}_w(\Gamma)$ it follows by equalities (8.2) that the function

$$\frac{\delta F_1}{F_1} - \frac{\delta F_2}{F_2} \tag{8.4}$$

is a modular form of weight 2 with respect to Γ. This fact and the functional equation for the thetanulls

$$\vartheta_j\left(\frac{a\tau+b}{c\tau+d}\right) = \xi(c\tau+d)^{1/2}\vartheta_j(\tau), \qquad j = 2,3,4, \quad \xi^8 = 1,$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{1,2} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z},\ ad-bc = 1,\ ab \in 2\mathbb{Z},\ cd \in 2\mathbb{Z}\right\}$$

(the appropriate choice of $\xi = \xi(\gamma)$ and a branch of the root function is described in Theorem 7.1 in [3]; Chapter I) enable one to give another proof of identities (0.8).

For $g > 1$ on the Siegel upper half-space $\mathfrak{H}_g \subset Sym_g(\mathbb{C})$ we have the action of the *symplectic group*

$$Sp_{2g}(\mathbb{R}) = \left\{\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : {}^t\gamma\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}\gamma = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}\right\}, \tag{8.5}$$

where A, B, C, D are real square matrices of order g and E is the identity matrix. This action is described by the formula

$$\mathrm{T} \mapsto \gamma\mathrm{T} = (A\mathrm{T} + B)(C\mathrm{T} + D)^{-1}, \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{R}).$$

The definition of the *Siegel modular group* $Sp_{2g}(\mathbb{Z})$ is the same as (8.5), except that the entries of the matrices A, B, C, D must now be integers.

Lemma 16 (see [25]; unproved formula (4.2)). *Let $F(\mathrm{T})$, $F\colon \mathfrak{H}_g \to \mathbb{R}$, be a meromorphic function and let Δ be the matrix differential operator defined by formula* (0.19) (*that is, ΔF is a symmetric square matrix of order g with entries that are the corresponding partial δ_{jk}-derivatives of the function F*)*. Then*

$$(\Delta F)(\gamma\mathrm{T}) = (C\mathrm{T}+D)\cdot\Delta F(\gamma\mathrm{T})\cdot{}^{t}(C\mathrm{T}+D) \tag{8.6}$$
$$\textit{for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{Z}).$$

Proof. Let τ_{jk} and τ'_{jk}, $j,k=1,\dots,g$, be the entries of the matrices $\mathrm{T}\in\mathfrak{H}_g$ and $\mathrm{T}'=\gamma\mathrm{T}\in\mathfrak{H}_g$, where $\gamma\in Sp_{2g}(\mathbb{Z})$; we denote the corresponding δ-differentiations by δ_{jk} and δ'_{jk}, $j,k=1,\dots,g$. In view of the symmetry of the matrices T and T', only the entries with indices j,k such that $1\leqslant j\leqslant k\leqslant g$, are independent. By the rules of differentiation of composite functions we obtain

$$\delta_{jk}F(\mathrm{T}') = \biggl(\sum_{l,m=1}^{g}\frac{\partial\tau'_{lm}}{\partial\tau_{jk}}\delta'_{lm}F\biggr)(\mathrm{T}'), \qquad j,k=1,\dots,g. \tag{8.7}$$

For we have

$$\begin{aligned}
\delta_{jj}F(\mathrm{T}') &= \frac{1}{\pi i}\frac{\partial F(\mathrm{T}')}{\partial\tau_{jj}} = \frac{1}{\pi i}\biggl(\sum_{1\leqslant l\leqslant m\leqslant g}\frac{\partial\tau'_{lm}}{\partial\tau_{jj}}\frac{\partial F}{\partial\tau'_{lm}}\biggr)(\mathrm{T}')\\
&= \biggl(\sum_{l,m=1}^{g}\frac{\partial\tau'_{lm}}{\partial\tau_{jj}}\delta'_{lm}F\biggr)(\mathrm{T}'), \qquad j=1,\dots,g,\\
\delta_{jk}F(\mathrm{T}') &= \frac{1}{2\pi i}\frac{\partial F(\mathrm{T}')}{\partial\tau_{jk}} = \frac{1}{2\pi i}\biggl(\sum_{1\leqslant l\leqslant m\leqslant g}\frac{\partial\tau'_{lm}}{\partial\tau_{jk}}\frac{\partial F}{\partial\tau'_{lm}}\biggr)(\mathrm{T}')\\
&= \biggl(\sum_{l,m=1}^{g}\frac{\partial\tau'_{lm}}{\partial\tau_{jk}}\delta'_{lm}F\biggr)(\mathrm{T}'), \qquad j,k=1,\dots,g.
\end{aligned}$$

It suffices to verify equality (8.6) for the generators

$$\gamma_1 = \begin{pmatrix} E & B \\ 0 & E \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} {}^{t}A & 0 \\ 0 & A^{-1} \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \tag{8.8}$$

of the group $Sp_{2g}(\mathbb{Z})$, where the matrix $A\in GL_g(\mathbb{Z})$ and the symmetric integer matrix B can be arbitrary (see [3]; Chapter II, Appendix to § 5, Proposition A.5). Performing direct calculating with matrices it is easy to show that if (8.6) holds for $\gamma,\gamma'\in Sp_{2g}(\mathbb{Z})$, then this relation holds also for the product $\gamma\gamma'$.

For $\mathrm{T}'=\gamma_1\mathrm{T}=\mathrm{T}+B$ we have $\tau'_{jk}=\tau_{jk}+b_{jk}$, $j,k=1,\dots,g$, therefore

$$\delta_{jk}F(\mathrm{T}') = (\delta_{jk}F)(\mathrm{T}'), \qquad j,k=1,\dots,g.$$

Hence

$$\Delta F(\gamma_1\mathrm{T}) = (\Delta F)(\gamma_1\mathrm{T}). \tag{8.9}$$

If $\mathrm{T}' = \gamma_2\mathrm{T} = {}^t A\mathrm{T}A$, where $A = (a_{jl})_{j,l=1,\ldots,g}$, then

$$\tau'_{lm} = \sum_{j,k=1}^{g} a_{jl}a_{km}\tau_{jk}, \qquad l,m = 1,\ldots,g,$$

and therefore

$$\frac{\partial\tau'_{lm}}{\partial\tau_{jk}} = a_{jl}a_{km}, \qquad j,k = 1,\ldots,g, \quad l,m = 1,\ldots,g.$$

By formula (8.7),

$$\delta_{jk}F(\mathrm{T}') = \Bigl(\sum_{l,m=1}^{g} a_{jl}a_{km}\delta'_{lm}F\Bigr)(\mathrm{T}'), \qquad j,k = 1,\ldots,g,$$

so that

$$\Delta F(\gamma_2\mathrm{T}) = A\cdot(\Delta F)(\gamma_2\mathrm{T})\cdot {}^t A. \tag{8.10}$$

For the last generator γ_3 we have $\mathrm{T}' = \gamma_3\mathrm{T} = -\mathrm{T}^{-1}$; in other words, $\mathrm{T}'\mathrm{T} = -E$. Hence

$$0 = \frac{\partial}{\partial\tau_{jk}}(\mathrm{T}'\mathrm{T}) = \frac{\partial}{\partial\tau_{jk}}\mathrm{T}'\cdot\mathrm{T} + \mathrm{T}'\cdot\frac{\partial}{\partial\tau_{jk}}\mathrm{T}, \qquad j,k = 1,\ldots,g$$

(we have the matrix zero on the left-hand side), and

$$\frac{\partial}{\partial\tau_{jk}}\mathrm{T}' = -\mathrm{T}'\cdot\frac{\partial}{\partial\tau_{jk}}\mathrm{T}\cdot\mathrm{T}^{-1} = \mathrm{T}'\cdot\frac{\partial}{\partial\tau_{jk}}\mathrm{T}\cdot\mathrm{T}', \qquad j,k = 1,\ldots,g,$$

therefore

$$\frac{\partial\tau'_{lm}}{\partial\tau_{jk}} = \tau'_{lj}\tau'_{km}, \qquad j,k = 1,\ldots,g, \quad l,m = 1,\ldots,g.$$

Substituting this in relations (8.7) we obtain

$$\delta_{jk}F(\mathrm{T}') = \Bigl(\sum_{l,m=1}^{g} \tau'_{lj}\tau'_{km}\delta'_{lm}F\Bigr)(\mathrm{T}'), \qquad j,k = 1,\ldots,g,$$

so that

$$\Delta F(\gamma_3\mathrm{T}) = \mathrm{T}'\cdot(\Delta F)(\gamma_3\mathrm{T})\cdot\mathrm{T}' = \mathrm{T}^{-1}\cdot(\Delta F)(\gamma_3\mathrm{T})\cdot\mathrm{T}^{-1}. \tag{8.11}$$

Combining equalities (8.9)–(8.11), for the generators (8.8) of the group $Sp_{2g}(\mathbb{Z})$ we obtain the equality

$$\Delta F(\gamma\mathrm{T}) = (C\mathrm{T}+D)^{-1}\cdot(\Delta F)(\gamma\mathrm{T})\cdot {}^t(C\mathrm{T}+D)^{-1} \tag{8.12}$$

(the matrix T is symmetric). Equality (8.6) follows from (8.12). Hence (8.6) holds for all $\gamma \in Sp_{2g}(\mathbb{Z})$, which completes the proof.

A holomorphic (meromorphic) function $F(\mathrm{T})$ on $\mathfrak{H}_g$ is called a *holomorphic* (respectively, *meromorphic*) *modular form of weight w with respect to* a congruence subgroup

$$\Gamma \subset \left\{ \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \pmod L \right\} \subset Sp_{2g}(\mathbb{Z})$$

if

$$F(\gamma\mathrm{T}) = \det{}^w(C\mathrm{T}+D)\cdot F(\mathrm{T}) \qquad \text{for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma. \tag{8.13}$$

The additional conditions constraining the growth of $F(\mathrm{T})$ at the 'vertices' hold automatically for $g>1$: this is the so-called *Koecher principle* (see [2]; Chapter V, Lemma 19).

In the perfect accordance with the one-dimensional case, the set of modular forms of weight w with respect to a fixed congruence subgroup Γ is a vector space, which we denote by $\mathrm{Mod}_w(\Gamma) = \mathrm{Mod}_w^{(g)}(\Gamma)$.

Lemma 17. *Let $F\colon \mathfrak{H}_g \to \mathbb{R}$ be a congruence form of weight w with respect to a congruence subgroup Γ of $Sp_{2g}(\mathbb{Z})$. Then the (matrix-valued) function*

$$\Psi = \frac{\Delta F}{F}\colon \mathfrak{H}_g \to Sym_g(\mathbb{C}) \tag{8.14}$$

satisfies the functional equation

$$\Psi(\gamma\mathrm{T}) = \pi i w(C\mathrm{T}+D)\cdot {}^tC + (C\mathrm{T}+D)\cdot\Psi(\mathrm{T})\cdot {}^t(C\mathrm{T}+D), \qquad \gamma\in\Gamma. \tag{8.15}$$

Proof. We point out straight away the simple equality

$$\frac{\Delta\det(C\mathrm{T}+D)}{\det(C\mathrm{T}+D)} = \pi i {}^tC\cdot {}^t(C\mathrm{T}+D)^{-1} \tag{8.16}$$

holding for arbitrary matrices C, D. Its verification is immediate (cf. [25]; formula (4.3)).

Taking the logarithm of both sides of (8.13) and applying the differential operator Δ, in view of (8.16), we obtain

$$\frac{\Delta F(\gamma\mathrm{T})}{F(\gamma\mathrm{T})} = \pi i w {}^tC\cdot {}^t(C\mathrm{T}+D)^{-1} + \frac{\Delta F(\mathrm{T})}{F(\mathrm{T})}, \qquad \gamma\in\Gamma,$$

and therefore

$$\begin{aligned}(C\mathrm{T}+D)&\cdot\frac{\Delta F(\gamma\mathrm{T})}{F(\gamma\mathrm{T})}\cdot {}^t(C\mathrm{T}+D)\\ &= \pi i w(C\mathrm{T}+D)\cdot {}^tC + (C\mathrm{T}+D)\cdot\frac{\Delta F(\mathrm{T})}{F(\mathrm{T})}\cdot {}^t(C\mathrm{T}+D), \quad \gamma\in\Gamma.\end{aligned} \tag{8.17}$$

By Lemma 16,

$$(C\mathrm{T}+D)\cdot\frac{\Delta F(\gamma\mathrm{T})}{F(\gamma\mathrm{T})}\cdot {}^t(C\mathrm{T}+D) = \left(\frac{\Delta F}{F}\right)(\gamma\mathrm{T}), \qquad \gamma\in Sp_{2g}(\mathbb{Z}). \tag{8.18}$$

Comparing (8.17) and (8.18), we obtain (8.15).

Corollary 1. *Let F_1 and F_2 be arbitrary modular forms of weight w with respect to a congruence subgroup Γ of $Sp_{2g}(\mathbb{Z})$. Then the function*

$$\det\left(\frac{\Delta F_1}{F_1} - \frac{\Delta F_2}{F_2}\right)(\mathrm{T}) \tag{8.19}$$

is a meromorphic modular form of weight 2 with respect to the same subgroup Γ.

Proof. Applying Lemma 17 to the functions F_1 and F_2, subtracting one of the resulting equalities from the other, and calculating the determinants we obtain

$$\det\left(\frac{\Delta F_1}{F_1} - \frac{\Delta F_2}{F_2}\right)(\gamma\mathrm{T}) = \det{}^2(C\mathrm{T}+D)\cdot\det\left(\frac{\Delta F_1}{F_1} - \frac{\Delta F_2}{F_2}\right)(\mathrm{T}), \qquad \gamma\in\Gamma.$$

This relation means precisely that the function (8.19) is a modular form of weight 2 with respect to the congruence subgroup Γ.

Remark. By contrast to the case $g = 1$, the entries of the matrix-valued function (8.14) are not holomorphic functions on $\mathfrak{H}_g$ therefore we must emphasize that the function (8.19) is a *meromorphic* modular form. In the case of holomorphic modular forms F_1 and F_2 their Δ-derivatives have holomorphic entries, therefore we can refine the last result as follows.

Corollary 2. *Let F_1 and F_2 be arbitrary holomorphic modular forms of weight w with respect to the congruence subgroup $\Gamma \subset Sp_{2g}(\mathbb{Z})$. Then the function*

$$(F_1F_2)^g\cdot\det\left(\frac{\Delta F_1}{F_1} - \frac{\Delta F_2}{F_2}\right)(\mathrm{T}) \tag{8.20}$$

is a holomorphic modular form of weight $2wg+2$ with respect to the same subgroup Γ.

Proof. This can be established by the calculation of the weight of the holomorphic modular form (8.20), because the fact that it is a modular form is a consequence of Corollary 1.

Corollaries to Lemma 17 generalize in a certain sense the differential operation (8.4) to the case of arbitrary dimension. Note that the differential operator (8.3) also has multidimensional generalizations (see [26]).

As in the one-dimensional case, for $g > 1$ there exists a functional equation holding for all thetanulls:

$$\vartheta_{\boldsymbol{a}}\big((A\mathrm{T}+B)(C\mathrm{T}+D)^{-1}\big) = \xi\cdot\det{}^{1/2}(C\mathrm{T}+D)\cdot\vartheta_{\boldsymbol{a}}(\mathrm{T}), \qquad \boldsymbol{a}\in\mathfrak{K}^*, \quad \xi^8 = 1, \tag{8.21}$$

$$\begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \Gamma_{1,2} = \Gamma_{1,2}^{(g)}$$

(see [3]; Chapter II, § 5). The congruence subgroup $\Gamma_{1,2}$ of $Sp_{2g}(\mathbb{Z})$ (of level 2) is generated by the elements (8.8), where the matrix $A \in GL_g(\mathbb{Z})$ and the symmetric integer matrix B with even main diagonal are arbitrary (see [3]; Chapter II, Appendix to § 5, Proposition A.4). The definition (8.13) of modular forms can be extended by letting in unity roots as coefficients in the right-hand side of (8.13). Without concentrating on this we observe that the presence of a constant coefficient ξ does not influence logarithmic Δ-differentiation of (8.21), therefore Lemma 17 brings us to the following result.

Lemma 18. *For each even characteristic* $\boldsymbol{a}$ *the matrix-valued function*

$$\Psi_{\boldsymbol{a}} = \frac{\Delta\vartheta_{\boldsymbol{a}}}{\vartheta_{\boldsymbol{a}}} : \mathfrak{H}_g \to Sym_g(\mathbb{C})$$

with entries that are logarithmic derivatives of thetanulls (*see* (0.16)) *satisfies the functional equation*

$$\Psi_{\boldsymbol{a}}(\gamma \mathrm{T}) = \frac{\pi i}{2}(C\mathrm{T}+D)\cdot {}^tC + (C\mathrm{T}+D)\cdot \Psi_{\boldsymbol{a}}(\mathrm{T})\cdot {}^t(C\mathrm{T}+D), \qquad \gamma\in\Gamma_{1,2}.$$

Corollary 1. *For all even characteristics* $\boldsymbol{a},\boldsymbol{b}\in\mathfrak{K}$ *the function*

$$(\vartheta_{\boldsymbol{a}}\vartheta_{\boldsymbol{b}})^g\cdot\det(\Psi_{\boldsymbol{a}}-\Psi_{\boldsymbol{b}})(\mathrm{T})$$

is a holomorphic modular form of weight $g+2$ *with respect to* $\Gamma_{1,2}$.

Of course, Corollary 1 is just *one* consequence of Lemma 18. We now present a more general result in the spirit of § 7.

Corollary 2. *Let* $\mathfrak{G}$ *be an additive system of characteristics in* $\mathfrak{K}$ *of dimension* $l\leqslant g$ *that is partitioned into disjoint additive subsystems* $\mathfrak{F}+\boldsymbol{c}$ *and* $\mathfrak{F}$ *of dimension* $l-1$: $\mathfrak{G}=(\mathfrak{F}+\boldsymbol{c})\cup\mathfrak{F}$, *where* $\boldsymbol{c}\in\mathfrak{K}$ *is a characteristic distinct from zero in the additive group corresponding to* $\mathfrak{G}$. *Then the function*

$$\biggl(\prod_{\boldsymbol{a}\in\mathfrak{F}}\vartheta_{\boldsymbol{a}+\boldsymbol{c}}\vartheta_{\boldsymbol{a}}\biggr)^g\cdot\det\biggl(\sum_{\boldsymbol{a}\in\mathfrak{F}}(\Psi_{\boldsymbol{a}+\boldsymbol{c}}-\Psi_{\boldsymbol{a}})\biggr) \tag{8.22}$$

is a holomorphic modular form of weight $2^{l-1}g+2$ *with respect to* $\Gamma_{1,2}$.

For $g=2$ and $g=3$ these results can be put into a quantitative form. For an arbitrary subset $\mathfrak{G}$ of $\mathfrak{K}^*$ let

$$\vartheta_{\mathfrak{G}}=\prod_{\boldsymbol{a}\in\mathfrak{G}}\vartheta_{\boldsymbol{a}}, \qquad \Psi_{\mathfrak{G}}=\sum_{\boldsymbol{a}\in\mathfrak{G}}\Psi_{\boldsymbol{a}}.$$

We set

$$\chi_g=\begin{cases}\vartheta^4_{\mathfrak{K}^*} & \text{for } g=1,\\ \vartheta^2_{\mathfrak{K}^*} & \text{for } g=2,\\ \vartheta_{\mathfrak{K}^*} & \text{for } g=3\end{cases} \tag{8.23}$$

(the minimal *parabolic* modular forms with respect to $Sp_{2g}(\mathbb{Z})$ in dimensions 1, 2, and 3, respectively; see [15]).

Lemma 19. *For $g \leqslant 3$ let $\mathfrak{G}$ be a Göpel system of characteristics in $\mathfrak{K}$ partitioned into disjoint additive subsystems $\mathfrak{F} + \boldsymbol{c}$ and $\mathfrak{F}$ of dimension $g-1$. Then*

$$\det(\Psi_{\mathfrak{F}+\boldsymbol{c}} - \Psi_{\mathfrak{F}}) = \pm\frac{1}{2^{2g}}\frac{\chi_g}{(\vartheta_{\mathfrak{F}+\boldsymbol{c}}\vartheta_{\mathfrak{F}})^4}. \tag{8.24}$$

Proof. For $g = 1$ relations (8.24) become formulae (0.8) (although the latter contain no $\pm$-ambiguities). By the main result of § 5 in [2] the holomorphic forms (8.22) can be expressed as polynomials of the squares of thetanulls for $g = 2$ and of thetanulls for $g = 3$. Hence the proof reduces to the calculation of the first coefficients of the Fourier expansions for the holomorphic functions (8.22), for monomials of degree 6 of the squares of thetanulls for $g = 2$, and for monomials of degree 28 of thetanulls for $g = 3$. The details of these calculations are beyond the scope of this work.

Theorem 6. *For $g \leqslant 3$ the thetanulls are algebraic over the field generated by their δ-logarithmic derivatives* (0.16).

Proof. For $g = 1$ formulae (0.8) show that the fourth degrees of thetanulls are (up to constant coefficients) differences of δ-logarithmic derivatives (0.5).

Formulae (8.24) are independent (up to a sign) of the partitioning $\mathfrak{G} = (\mathfrak{F}+\boldsymbol{c})\cup\mathfrak{F}$ of a Göpel system into two disjoint additive systems. For that reason we fix one such partitioning for each Göpel system by setting

$$\lambda_{\mathfrak{G}} = \det(\Psi_{\mathfrak{F}+\boldsymbol{c}} - \Psi_{\mathfrak{F}}), \tag{8.25}$$

and write relations (8.24) as follows:

$$\pm 2^{2g}\lambda_{\mathfrak{G}} = \chi_g \vartheta_{\mathfrak{G}}^{-4}. \tag{8.26}$$

Multiplying equalities (8.26) for all Göpel systems in $\mathfrak{K}$, of which the number is 15 for $g = 2$ and 135 for $g = 3$ by Lemma 12, we obtain

$$\begin{aligned} \pm 2^{60}\prod_{\mathfrak{G}}\lambda_{\mathfrak{G}} &= \chi_2^{15}\prod_{\mathfrak{G}}\vartheta_{\mathfrak{G}}^{-4} = \chi_2^{15}\prod_{\boldsymbol{a}\in\mathfrak{K}^*}\vartheta_{\boldsymbol{a}}^{-24} = \chi_2^{3} && \text{for } g = 2,\\ \pm 2^{6\cdot 135}\prod_{\mathfrak{G}}\lambda_{\mathfrak{G}} &= \chi_3^{135}\prod_{\mathfrak{G}}\vartheta_{\mathfrak{G}}^{-4} = \chi_3^{135}\prod_{\boldsymbol{a}\in\mathfrak{K}^*}\vartheta_{\boldsymbol{a}}^{-120} = \chi_3^{15} && \text{for } g = 3. \end{aligned} \tag{8.27}$$

We used here the definition (8.23) of the quantities χ_2 and χ_3 and the following consequence of Lemma 12: each even characteristic participates in 6 Göpel systems for $g = 2$ and in 30 systems for $g = 3$.

Fixing an even characteristic $\boldsymbol{a}$ we now multiply equalities (8.26) for all Göpel systems $\mathfrak{G}$ containing $\boldsymbol{a}$:

$$\pm 2^{24}\prod_{\mathfrak{G}\ni\boldsymbol{a}}\lambda_{\mathfrak{G}} = \chi_2^{6}\prod_{\mathfrak{G}\ni\boldsymbol{a}}\vartheta_{\mathfrak{G}}^{-4} = \chi_2^{6}\cdot\vartheta_{\boldsymbol{a}}^{-16}\prod_{\boldsymbol{b}\in\mathfrak{K}^*}\vartheta_{\boldsymbol{b}}^{-8} = \chi_2^{2}\cdot\vartheta_{\boldsymbol{a}}^{-16} \qquad \text{for } g = 2, \tag{8.28}$$

$$\pm 2^{6\cdot 30}\prod_{\mathfrak{G}\ni\boldsymbol{a}}\lambda_{\mathfrak{G}} = \chi_3^{30}\prod_{\mathfrak{G}\ni\boldsymbol{a}}\vartheta_{\mathfrak{G}}^{-4} = \chi_3^{30}\cdot\vartheta_{\boldsymbol{a}}^{-96}\prod_{\boldsymbol{b}\in\mathfrak{K}^*}\vartheta_{\boldsymbol{b}}^{-24} = \chi_3^{6}\cdot\vartheta_{\boldsymbol{a}}^{-96} \qquad \text{for } g = 3. \tag{8.29}$$

Raising equalities (8.28) to the third and equalities (8.29) to the fifth degree and substituting (8.27) we obtain

$$
\begin{aligned}
\vartheta_{\boldsymbol{a}}^{48} &= \pm\chi_2^6 \cdot 2^{72} \prod_{\mathfrak{G}\ni\boldsymbol{a}} \lambda_{\mathfrak{G}}^{-3} = \pm 2^{120} \prod_{\mathfrak{G}} \lambda_{\mathfrak{G}}^2 \cdot 2^{72} \prod_{\mathfrak{G}\ni\boldsymbol{a}} \lambda_{\mathfrak{G}}^{-3} \\
&= \pm 2^{192} \prod_{\mathfrak{G}\not\ni\boldsymbol{a}} \lambda_{\mathfrak{G}}^2 \cdot \prod_{\mathfrak{G}\ni\boldsymbol{a}} \lambda_{\mathfrak{G}}^{-1} \qquad \text{for } g=2, \\
\vartheta_{\boldsymbol{a}}^{480} &= \pm\chi_2^{30} \cdot 2^{900} \prod_{\mathfrak{G}\ni\boldsymbol{a}} \lambda_{\mathfrak{G}}^{-5} = \pm 2^{12\cdot 135} \prod_{\mathfrak{G}} \lambda_{\mathfrak{G}}^2 \cdot 2^{900} \prod_{\mathfrak{G}\ni\boldsymbol{a}} \lambda_{\mathfrak{G}}^{-5} \\
&= \pm 2^{2520} \prod_{\mathfrak{G}\not\ni\boldsymbol{a}} \lambda_{\mathfrak{G}}^2 \cdot \prod_{\mathfrak{G}\ni\boldsymbol{a}} \lambda_{\mathfrak{G}}^{-3} \qquad \text{for } g=3.
\end{aligned}
\tag{8.30}
$$

Relations (8.30) show that for $g=2$ and $g=3$ the thetanulls are algebraic over the field generated by the functions (8.24), which are polynomials of the logarithmic derivatives (0.16) of thetanulls. The proof is complete.

One may regard both (8.24) and (8.30) as generalizations of formulae (0.8) to $g=2$ and $g=3$. We point out other formulae of this kind, the proof of which is 'the same' as the proof of Lemma 19 and is left out for this reason.

Lemma 20. (a) *Let $g=2$. Then for all even characteristics $\boldsymbol{a}$ and $\boldsymbol{b}$ there holds the relation*

$$\det(\Psi_{\boldsymbol{a}} - \Psi_{\boldsymbol{b}}) = \pm\frac{1}{2^4}\chi_2 \cdot \prod_{\mathfrak{G}\supset\{\boldsymbol{a},\boldsymbol{b}\}} \vartheta_{\mathfrak{G}}^{-2},$$

where the product $\prod_{\mathfrak{G}\supset\{\boldsymbol{a},\boldsymbol{b}\}}$ is taken over all (=both) Göpel system containing $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) *Let $g=3$ and let $\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3,\boldsymbol{a}_4$ be even characteristics making up an additive system $\mathfrak{A}$ (of dimension 2). Then*

$$\det(\Psi_{\boldsymbol{a}_1} + \Psi_{\boldsymbol{a}_2} - \Psi_{\boldsymbol{a}_3} - \Psi_{\boldsymbol{a}_4}) = \pm\frac{1}{2^6}\chi_3 \cdot \prod_{\mathfrak{G}\supset\mathfrak{A}} \vartheta_{\mathfrak{G}}^{-2},$$

where the product $\prod_{\mathfrak{G}\supset\mathfrak{A}}$ is taken over all (=both) Göpel systems containing the system $\mathfrak{A}$.

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Moscow State University
E-mail address: `wadim@ips.ras.ru`

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