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A Few Remarks on $\zeta(3)$

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ABSTRACT. A new proof of the irrationality of the number $\zeta(3)$ is proposed. A new decomposition of this number into a continued fraction is found. Recurrence relations are proved for some sequences of Meyer's G-functions that define a sequence of rational approximations to $\zeta(3)$ at the point 1.

In 1978 R. Apery proved the following assertion.

THEOREM 1 [1]. *The number*

$$\zeta(3)=\sum_{n=1}^{\infty}\frac{1}{n^3}$$

is irrational, and for any constant $\mu > 13.4178202\ldots$ *the inequality*

$$\left|\zeta(3)-\frac{p}{q}\right|<q^{-\mu}$$

has only a finite number of solutions in integers p *and* q.

The exact value of the constant $13.4178202\ldots$ is

$$1+\frac{4\ln\beta+3}{4\ln\beta-3},$$

where $\beta=\sqrt{2}+1$.

As is well known, for Riemann's zeta-function

$$\zeta(s)=\sum_{\nu=1}^{\infty}\nu^{-s}$$

at even points, we have

$$\zeta(2k)=(-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k},$$

where the $B_{2k}\in\mathbb{Q}$ are the Bernoulli numbers satisfying the recurrence relations

$$\sum_{r=0}^{n}\binom{n+1}{r}B_r=0,\qquad n=1,2,\ldots,$$

and the condition $B_0=1$. Thus, for example, $\zeta(2)=\pi^2/6$, $\zeta(4)=\pi^4/90$. These expressions prove the transcendence of the numbers $\zeta(2k)$. Before Apery's result, nothing was known about the arithmetic properties of the zeta-function at odd points.

Apery's proof is based on his discovery of a sequence of rational approximations to the number $\zeta(3)$. For example, this sequence can be constructed as the sequence of convergents of the continued fraction

$$\zeta(3)=\frac{6|}{|5}-\frac{1|}{|117}-\frac{64|}{|535}-\cdots-\frac{n^6|}{|34n^3+51n^2+27n+5}-\cdots \tag{1}$$

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that converges to $\zeta(3)$; or by means of the following recurrence equation

$$(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3 u_{n-1} = 0 \tag{2}$$

that is satisfied by the numerators v_n and denominators u_n of the convergents with the initial conditions

$$v_0 = 0, \quad v_1 = 6, \qquad u_0 = 1, \quad u_1 = 5;$$

or by the explicit representation of the sequences in question

$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \cdot c_{n,k}, \qquad n \geq 1, \tag{3}$$

where

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3} \binom{n}{m}^{-1} \binom{n+m}{m}^{-1}.$$

The greatest difficulty in the proof is the verification of the fact that the sequences (3) satisfy the recurrence equation (2).

Apery himself only published a sketch of the proof. A detailed exposition of this proof can be found in [2–4].

During the 20 years that have passed since, nothing new in principle has been published in this connection. In various articles Apery's phenomenon was interpreted in a number of ways.

F. Beukers (1979, [5]) put forward another interpretation of Apery's rational approximations to $\zeta(3)$. The following relation is valid:

$$J = \int_0^1 \int_0^1 -\frac{\ln(xy)}{1-xy} Q_n(x) Q_n(y)\, dx dy = 2\big(u_n \zeta(3) - v_n\big),$$

where

$$Q_n(x) = \frac{1}{n!} \cdot \left(\frac{d}{dx}\right)^n \big(x^n (1-x)^n\big) \in \mathbb{Z}[x]$$

are Legendre polynomials and the u_n, v_n are Apery's numbers (3).

M. Hata (1990, [6]), modifying the integral J and, consequently, changing the rational approximations to $\zeta(3)$, proved the assertion of Theorem 1 for any $\mu > 8.8302837\ldots$.

L. A. Gutnik (1979, [7]) established that for any rational number q, $q \neq 0$, there is an irrational number among the numbers

$$3\zeta(3) + q \cdot \zeta(2), \qquad \zeta(2) + 2q \cdot \ln 2$$

The polylogarithm

$$L_r(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^r}, \qquad r \geq 1,$$

as well as the so-called Meyer G-functions (see [8]), play an important role in the proof. By means of these functions, the following explicit constructions were established (see [7]):

$$G_n(z) = 2A_n(z) \cdot L_3(z^{-1}) + B_n(z) \cdot L_2(z^{-1}) + C_n(z),$$
$$H_n(z) = A_n(z) \cdot L_2(z^{-1}) + B_n(z) \cdot L_1(z^{-1}) + D_n(z),$$

where A_n, B_n, C_n, D_n are polynomials in z with rational coefficients of degree at most n satisfying the conditions

$$\operatorname{ord}_{z=\infty} G_n(z) \geq n+1, \qquad \operatorname{ord}_{z=\infty} H_n(z) \geq n+1.$$

This functional construction is used in [7] for $z = -1$. For the polynomials A_n, B_n, C_n, D_n, there also exist explicit expressions which, in particular, imply that $A_n(1) = u_n$, $C_n(1) = -2v_n$, and $B_n(1) = 0$.

F. Beukers (1981, [9]) proved the uniqueness of polynomials realizing the joint functional approximations indicated above.

Other problems involving joint functional approximations, which lead to the proof of the irrationality of $\zeta(3)$, were treated by Sorokin in [10].

In what follows (§1), we propose a proof of Theorem 1, inspired by Gutnik's work (see [7]), and then (§2) prove Theorem 2, in which a new representation of $\zeta(3)$ by a continued fraction is given.

THEOREM 2. *The following continued-fraction expansion is valid*:

$$2\zeta(3) = 2 + \frac{1|}{|2} + \frac{2|}{|4} + \frac{1|}{|3} + \frac{4|}{|2} + \frac{2|}{|4} + \frac{6|}{|6} + \frac{4|}{|5} + \frac{9|}{|4} + \frac{6|}{|6} + \frac{12|}{|8} + \frac{9|}{|7} + \frac{16|}{|6} + \cdots, \tag{4}$$

where the numerators a_n, $n \geq 2$, *and the denominators* b_n, $n \geq 1$, *are defined by*

$$\begin{aligned} &b_{4k+1} = 2k+2, \quad && a_{4k+1} = k(k+1), \quad && b_{4k+2} = 2k+4, \quad && a_{4k+2} = (k+1)(k+2), \\ &b_{4k+3} = 2k+3, \quad && a_{4k+3} = (k+1)^2, \quad && b_{4k+4} = 2k+2, \quad && a_{4k+4} = (k+2)^2. \end{aligned}$$

The halved convergents for (4) *associated with the numbers* $n = 4k-2$ *yield the set of convergents of Apery's continued fraction for* $\zeta(3)$.

The formation law for the numerators a_n and the denominators b_n of the continued fraction (4) can also be expressed as

$$b_{n+4} = b_n + 2, \qquad n \geq 1,$$
$$a_{n+4} = a_n + b_n, \quad n \equiv 1, 2, 3 \pmod 4, \qquad a_{n+4} = a_n + b_n + 3, \quad n \equiv 0 \pmod 4, \qquad n \geq 2.$$

The convergents of Apery's continued fraction (1) are all less than $\zeta(3)$ and form an increasing sequence. The numerators and denominators of the continued fraction (4) are positive. Therefore, in accordance with the general theory (see [11, §2.1.1]), we can assert that all the even-numbered convergents of the continued fraction (4) are less than $2\zeta(3)$ and are increasing, while those with odd numbers are greater than $2\zeta(3)$ and decreasing.

It should be noted that in §2, practically, a continued fraction expansion is constructed for some family of numbers depending on two parameters α and β. For $\alpha = \beta = 0$, we obtain the continued fraction of the number $2\zeta(3)$ (see Theorem 2). Probably, some kind of arithmetical results can be obtained for other values of α and β as well. This seems to be an interesting problem.

In §3 explicit expressions for the denominators of the convergents (4) are given, and we also derive a certain recurrence equation of fourth order that, for $z = 1$, degenerates into recurrence equations for the continued fraction (4).

§1. Proof of Apery's theorem

Let n be a natural number assumed to be sufficiently large. The following rational function plays an important role in the proof:

$$R(z) = \frac{(z-1)^2 \cdots (z-n)^2}{z^2(z+1)^2 \cdots (z+n)^2},$$

LEMMA 1. *The following relation is valid*:

$$I = \sum_{\nu=1}^{\infty} R'(\nu) = A_n\zeta(3) + B_n, \tag{5}$$

where $A_n \in \mathbb{Z}$, $B_n \in \mathbb{Q}$. *If* $D_n = \prod_{p \leq n} p^{[\log 2n/\log p]}$, *then* $D_n^3 B_n \in \mathbb{Z}$.

PROOF. Let us expand the function $R(z)$ into the sum of partial fractions:

$$R(z) = \sum_{k=0}^{n} \left(\frac{B_{k2}}{(z+k)^2} + \frac{B_{k1}}{z+k} \right),$$

where

$$B_{k2} = (z+k)^2 R(z)\Big|_{z=-k} = \binom{n+k}{k}^2 \binom{n}{k}^2,$$

$$B_{k1} = \frac{d}{dz}\big((z+k)^2 R(z)\big)\Big|_{z=-k} = -2B_{k2}\Bigg(\sum_{j=1}^{n} \frac{1}{k+j} - \sum_{\substack{j=0\\ j\neq k}}^{n} \frac{1}{k-j}\Bigg).$$

It follows from these expressions that $B_{k2} \in \mathbb{Z}$ and $D_n B_{k1} \in \mathbb{Z}$. Since the function $R(z)$ has a zero of second order at the point $z = \infty$, we have

$$\sum_{k=0}^{n} B_{k1} = -\operatorname{Res}_{z=\infty} R(z) = 0. \tag{6}$$

Using (6), we obtain

$$\sum_{\nu=1}^{\infty} R'(\nu) = \sum_{k=0}^{n}\sum_{\nu=1}^{\infty}\bigg(-2\frac{B_{k2}}{(\nu+k)^3} - \frac{B_{k1}}{(\nu+k)^2}\bigg) = A_n\zeta(3) + B_n,$$

where

$$A_n = -2\sum_{k=0}^{n} B_{k2} = -2u_n, \qquad B_n = \sum_{k=1}^{n}\sum_{r=1}^{k}(2B_{k2}r^{-3} + B_{k1}r^{-2}).$$

Therefore, $A_n \in \mathbb{Z}$ and $D_n^3 B_n \in \mathbb{Z}$. This proves Lemma 1. $\square$

LEMMA 2. *For the sum of the series* (5), *the following expression is valid*:

$$I = \frac{1}{2\pi i}\int_L \left(\frac{\pi}{\sin \pi z}\right)^2 R(z)\,dz, \tag{7}$$

where L is the vertical line $\operatorname{Re} z = C$, $0 < C < n+1$, *oriented from top to bottom.*

PROOF. Let $T = N + 1/2$, where N is a sufficiently large integer ($N > n$). The function $(\pi/\sin \pi z)^2$ is bounded on the boundary of the square with vertices at the points $(\pm T, \pm T)$; moreover, $R(z) = O(T^{-2})$. It follows that the integral indicated in Lemma 2 is equal to the sum of residues of the integrand at the points $n+1, n+2, \dots$. Since in the neighborhood of the integer point $z = k$, $k > 0$, the following relations are valid:

$$\left(\frac{\pi}{\sin \pi z}\right)^2 = \frac{1}{(z-k)^2} + O(1), \qquad R(z) = R(k) + R'(k)(z-k) + O\big((z-k)^2\big),$$

we have

$$\operatorname{Res}_{z=k}\bigg(\left(\frac{\pi}{\sin \pi z}\right)^2 R(z)\bigg) = R'(k);$$

furthermore, in view of $R'(k) = 0$ for $k = 1, 2, \dots, n$, we obtain the required expression for the sum of the series I. $\square$

LEMMA 3. *The following asymptotic formula as $n \to \infty$ is valid*:

$$I = -\frac{\pi^{3/2}2^{3/4}}{n^{3/2}}\big(\sqrt{2}-1\big)^{4n+2}\big(1 + o(1)\big).$$

PROOF. We have

$$\left(\frac{\pi}{\sin \pi z}\right)^2 R(z) = \left(\frac{\Gamma(n+1-z)\Gamma(z)^2}{\Gamma(n+1+z)}\right)^2.$$

Now choose $C = (n+1)/\sqrt{2}$ (the point C belongs to the contour of integration L) and in the integral (7) perform the change of variable $z = (n+1)t$. Since on the contour of integration L the quantities $\operatorname{Re}(n+1+z)$, $\operatorname{Re}(n+1-z)$, $\operatorname{Re} z$ are equivalent to positive constants multiplied by n, then for

$$F(z) = \left(\frac{\Gamma(n+1-z)\Gamma(z)^2}{\Gamma(n+1+z)}\right)^2,$$

on the contour of integration, using the relation

$$\ln \Gamma(z) = \left(z - \frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln 2\pi + r(z), \qquad |r(z)| \le K|\operatorname{Re} z|^{-1},$$

where K is an absolute constant, we obtain

$$\begin{aligned} \ln F(z) &= (2n+1-2z)\ln(n+1-z) - 2(n+1-z) + (4z-2)\ln z - 4z \\ &\quad - (2n+1+2z)\ln(n+1+z) + (2n+2+2z) + 2\ln 2\pi + O(n^{-1}) \\ &= \ln h(t) + 2(n+1)f(t) - 2\ln(n+1) + 2\ln 2\pi + O(n^{-1}). \end{aligned}$$

Here the constant in $O(\cdot)$ is absolute, and

$$h(t) = \frac{1+t}{(1-t)t^2}, \qquad f(t) = (1-t)\ln(1-t) + 2t\ln t - (1+t)\ln(1+t).$$

Thus,

$$I = -\frac{2\pi}{ni} \cdot \int_{c-i\infty}^{c+i\infty} e^{2(n+1)f(t)} \frac{1+t}{(1-t)t^2}\big(1 + O(n^{-1})\big)\,dt,$$

where $c = 1/\sqrt{2}$ and the constant in $O(\cdot)$ is absolute.

The point $t = 1/\sqrt{2}$ is a unique maximum point for $\operatorname{Re} f(t)$ on the contour of integration. Indeed, if $g(u) = \operatorname{Re} f(c+iu)$, then $g'(u) = -\operatorname{Im} f'(t) = \operatorname{Im}\big(\ln(t^{-2}-1)\big)$, and hence $g'(u)$ vanishes only for $u = 0$. In the neighborhood of this point,

$$f(t) = f(c) + 2\sqrt{2}(t-c)^2 + O\big((t-c)^3\big),$$

on the contour, $t = c + iu$, $-\infty < u < +\infty$, and in the neighborhood of the point c, we have

$$f(t) = f(c) - 2\sqrt{2}u^2 + O(u^3).$$

By Laplace's method, we now obtain

$$I = -2\pi n^{-1} \cdot 2\big(\sqrt{2}+1\big)^2\big(\sqrt{2}-1\big)^{4n+4}\left(\frac{2\pi}{n+1}\big(8\sqrt{2}\big)^{-1}\right)^{1/2}\big(1 + O(n^{-1})\big),$$

which gives the required equivalence. Lemma 3 is thus proved. □

It is easily seen from Lemmas 1 and 3 that $\zeta(3)$ is irrational. Indeed, if $\zeta(3) = b/a \in \mathbb{Q}$, $a > 0$, then $2aD_n^3 \cdot I$ is an integer distinct from zero. Thus,

$$1 \le 2aD_n^3 \cdot |I| = O\Big(D_n^3\big(\sqrt{2}-1\big)^{4n}\Big).$$

The last inequality is a contradiction, so that for any $\varepsilon > 0$ and any sufficiently large n the estimate $D_n < e^{(1+\varepsilon)n}$ holds, and moreover, $e^3\big(\sqrt{2}-1\big)^4 < 1$.

The numerical result can be derived from Lemmas 1 and 3 by traditional methods.

Note that it would be of interest to obtain an elementary proof, based on relation (5), of the irrationality of $\zeta(3)$ similar to the proof by Fourier of the irrationality of e. To do this, it suffices to obtain a "nice" estimate of the left-hand side of (5) by elementary methods and prove that the sum of the series is nonzero.

§2. Recurrence relations and proof of Theorem 2

Let $a, b, c, d \in \mathbb{Q}$, where a and b are distinct from the numbers $1, 2, 3, \dots$, and let

$$G(a, b, c, d \mid z) = \frac{1}{2\pi i} \int_L \Gamma(-s)^4 \frac{\Gamma(1-a+s)}{\Gamma(c-s)} \cdot \frac{\Gamma(1-b+s)}{\Gamma(d-s)} z^s \, ds, \qquad |z| < \pi.$$

Here the contour L divides the poles of the functions $\Gamma(-s)$ and $\Gamma(1-a+s)$; the function $\Gamma(1-b+s)$ varies from $-\infty$ to $+\infty$; the integral is absolutely convergent. The function so defined is a very special case,

$$G_{4,4}^{4,2}\left(\begin{matrix} a, \ b, \ c, \ d \\ 0, \dots, 0 \end{matrix} \,\middle|\, z \right),$$

of the so-called Meyer G-functions (see [8]).

Let us point out some properties of $G(a, b, c, d \mid z)$ (see [8, Chap. 5]). To simplify the notation, we drop the argument z of the functions G. The function $G(a+1, b+1, c+1, d+1)$ satisfies the differential equation

$$[z(\delta - a)(\delta - b)(\delta - c)(\delta - d) - \delta^4]y = 0, \qquad \delta = z\frac{d}{dz}. \tag{8}$$

Moreover, the following identities are valid:

$$\begin{aligned} (\delta - a)G(a+1, b, c, d \mid z) &= G(a, b, c, d \mid z), \\ (\delta - c)G(a, b, c+1, d \mid z) &= -G(a, b, c, d \mid z). \end{aligned} \tag{9}$$

The identities are preserved under the permutation of a and b or c and d under the function sign G.

Lemma 4. 1) *The following identity holds*:

$$(c-a)\delta G(a+1, b, c+1, d) = cG(a, b, c+1, d) + aG(a+1, b, c, d). \tag{10}$$

2) *If* $a + b + c + d = 0$, *then*

$$\begin{aligned} a^3 \delta G(a+1, b+1, c+1, d) &+ bc(d-a)G(a, b+1, c+1, d) \\ &+ d^3 \delta G(a, b+1, c+1, d+1) = (z-1)(d-a)G(a, b, c, d). \end{aligned}$$

Proof. 1) It follows from (9) that

$$\begin{aligned} (\delta - a)G(a+1, b, c+1, d \mid z) &= G(a, b, c+1, d \mid z), \\ (\delta - c)G(a+1, b, c+1, d \mid z) &= -G(a+1, b, c, d \mid z). \end{aligned}$$

Multiplying the first relation by c, the second one by a, and subtracting the second from the first, we obtain relation (10).

2) Since both sides of the relation are continuous with respect to the parameters a and d, it follows that we can assume, without loss of generality, that $a \neq d$. Choose complex numbers λ, μ, ν so that the following identity in δ is valid:

$$\lambda \cdot \delta(\delta - d) + \mu \cdot (\delta - a)(\delta - d) + \nu \cdot \delta(\delta - a) = \delta^4 - (\delta - a)(\delta - b)(\delta - c)(\delta - d). \tag{11}$$

Substituting, in turn, $\delta = a$, $\delta = 0$, and $\delta = d$ into (11), we obtain

$$\lambda = \frac{a^3}{a-d}, \qquad \mu = -bc, \qquad \nu = \frac{d^3}{d-a}.$$

Now let us apply both sides of identity (11) to the function $G(a+1, b+1, c+1, d+1 \mid z)$ and use relations (8) and (9). Then we obtain the desired relation. $\square$

Let α and β be complex numbers. Set $\alpha_0 = \alpha$, $\alpha_1 = \beta$, and let

$$\alpha_{n+2} = \alpha_n + 1, \qquad n \geq 0.$$

For $n \geq 0$, also denote

$$\begin{aligned} r_{2n} &= G(-\alpha_{n+1}, -\alpha_{n+2}, \alpha_{n+1} + 1, \alpha_{n+2} + 1 \mid 1), \\ r_{2n+1} &= \frac{\alpha_{n+3}^2}{\alpha_{n+2}\alpha_{n+4}} \delta G(-\alpha_{n+1}, -\alpha_{n+2}, \alpha_{n+1} + 2, \alpha_{n+2} + 1 \mid 1). \end{aligned}$$

LEMMA 5. *The following relations are valid*:

$$\alpha_{n+3}^2 r_{2n+2} = 2\alpha_{n+2}\alpha_{n+4} r_{2n+1} + \alpha_{n+3}^2 r_{2n}, \qquad n \geq 0,$$
$$\alpha_{n+4} r_{2n+1} = (\alpha_{n+2} + \alpha_{n+3}) r_{2n} + \alpha_{n+1} r_{2n-1}, \qquad n \geq 1.$$

Given

$$r_{-1} = \frac{\alpha_2^2}{\alpha_3} \delta G(-\alpha_0, -\alpha_1, \alpha_0 + 2, \alpha_1 + 1 \mid 1),$$

the following relation will also hold:

$$\alpha_4 r_1 = (\alpha_2 + \alpha_3) r_0 + r_{-1}. \tag{12}$$

PROOF. To prove the first relation, we use the first identity of Lemma 4 with

$$a = -\alpha_{n+3}, \quad b = -\alpha_{n+2}, \quad c = \alpha_{n+3}, \quad d = \alpha_{n+2} + 1.$$

To prove the second relation and (12), we use the identity from item 2) of Lemma 4 with

$$a = -\alpha_{n+2}, \quad b = -\alpha_{n+3}, \quad c = \alpha_{n+2}, \quad d = \alpha_{n+3}. \quad \square$$

PROOF OF THEOREM 2. Let us choose $\alpha_0 = \alpha_1 = 0$; hence $\alpha_n = [n/2]$. In this case the sequence α_n is increasing and α_{n+3} equals either α_{n+2} or α_{n+4}, so that a cancellation occurs in the first relation. As a result, the recurrence equations for r_n take the form

$$\begin{aligned} 2r_1 &= 2r_0 + r_{-1}, & \\ (k+2)r_{4k+1} &= (2k+2)r_{4k} + kr_{4k-1}, & k \geq 1, \\ (k+1)r_{4k+2} &= (2k+4)r_{4k+1} + (k+1)r_{4k}, & k \geq 0, \\ (k+2)r_{4k+3} &= (2k+3)r_{4k+2} + (k+1)r_{4k+1}, & k \geq 0, \\ (k+2)r_{4k+4} &= (2k+2)r_{4k+3} + (k+2)r_{4k+2}, & k \geq 0. \end{aligned} \tag{13}$$

It is easily verified by a straightforward calculation that

$$r_{-1} = -1, \qquad r_0 = 2\zeta(3) - 1,$$

Let us define the sequences $g_n, h_n,\ n \geq -1$ by means of the recurrence equations (13) and the initial conditions

$$g_{-1} = 0, \quad g_0 = 1, \qquad h_{-1} = 1, \quad h_0 = 1.$$

Hence, we have the relations

$$r_n = g_n \cdot 2\zeta(3) - h_n, \qquad n \geq -1. \tag{14}$$

This fact is readily proved by induction via the recurrence relations.

The recurrence equations (13) (see also Lemma 5) define a continued fraction with the elements

$$\begin{aligned} a_1' &= \frac{1}{\alpha_4}, & b_1' &= \frac{\alpha_2 + \alpha_3}{\alpha_4}, & \\ a_{2n+1}' &= \frac{\alpha_{n+1}}{\alpha_{n+4}}, & b_{2n+1}' &= \frac{\alpha_{n+2} + \alpha_{n+3}}{\alpha_{n+4}}, & n \geq 1, \\ a_{2n+2}' &= 1, & b_{2n+2}' &= 2\frac{\alpha_{n+2}\alpha_{n+4}}{\alpha_{n+3}^2}, & n \geq 0. \end{aligned}$$

Define the numbers ρ_j, $j \geq 0$, to be

$$\rho_0 = 1, \quad \rho_{2n+1} = \alpha_{n+4}, \quad \rho_{2n+2} = \alpha_{n+3}, \qquad n \geq 0.$$

The continued fraction obtained above will have the same sequence of convergents as the fraction with elements $b_n = b'_n \rho_n$, $a_n = a'_n \rho_n \rho_{n-1}$, i.e.,

$$\begin{aligned} a_1 &= 1, & b_1 &= \alpha_2 + \alpha_3, & \\ a_{2n+1} &= \alpha_{n+1}\alpha_{n+2}, & b_{2n+1} &= \alpha_{n+2} + \alpha_{n+3}, & n \geq 1, \\ a_{2n+2} &= \alpha_{n+3}\alpha_{n+4}, & b_{2n+2} &= 2\frac{\alpha_{n+2}\alpha_{n+4}}{\alpha_{n+3}}, & n \geq 0 \end{aligned}$$

(see [11, Chap. 2, Theorem 2.6]). However, this continued fraction coincides with the fraction appearing in Theorem 2. Therefore, the ratios h_n/g_n are the convergents of the continued fraction indicated in Theorem 2, and to prove the convergence of the continued fraction (4) to $2\zeta(3)$, it suffices to establish that the sequence h_n/g_n tends to $2\zeta(3)$.

Let us now prove that the sequence r_n is bounded. As in (7), for positive integers a and b, the following representation holds:

$$G(-a, -b, a+1, b+1 \mid 1) = \frac{1}{2\pi i} \int_L \left(\frac{\pi}{\sin \pi z} \right)^2 R_{a,b}(z)\, dz,$$

where

$$R_{a,b}(z) = z^{-2} \prod_{k=1}^{a} \frac{z+k}{z-k} \cdot \prod_{k=1}^{b} \frac{z+k}{z-k},$$

and the contour of integration L is the vertical line $\operatorname{Re} z = -1$. Using the fact that on this curve $|z+k| < |z-k|$, we conclude that the function $R_{a,b}(z)$ is bounded on L. The integral of the function $(\pi/\sin \pi z)^2$ along this contour is absolutely convergent. This proves that the set $G(-a, -b, a+1, b+1 \mid 1)$ is bounded for nonnegative integers a and b, and hence that the set of numbers r_{2n} is bounded. The boundedness property for r_{2n+1} can be established in a similar way or by means of the first of the recurrence equations mentioned in Lemma 5.

The recurrence equations (13) can be rewritten in the form

$$u_{n+1} = a(n) u_n + b(n) u_{n-1}.$$

The limiting recurrence equation for them is

$$x^2 - 2x - 1 = 0,$$

and its roots are $1 \pm \sqrt{2}$. Hence, since $g_n > 0$ for $n \geq 1$, it is easily shown that the sequence g_n approaches $+\infty$. In view of (14) and by the boundedness of r_n, this leads to $h_n/g_n \to 2\zeta(3)$, i.e., to the convergence of the continued fraction (4) to $2\zeta(3)$.

It should be noted that, using Poincaré's theorem (see [12, Chap 5, §5]), we can obtain the estimates $g_n = O(\mu^n)$, $r_n = O(\lambda^n)$ for any constants λ and μ satisfying the inequalities

$$\mu > 1 + \sqrt{2}, \qquad \lambda > \sqrt{2} - 1.$$

With the help of (13), we can easily write out the recurrence equation relating the terms of the sequence r_{4n-2}, $n \geq 1$, and check that this equation is identical with (2). Since the two sequences g_n and h_n satisfy Eqs. (13), it follows that each of the sequences g_{4n-2} and h_{4n-2} also satisfy Eq. (2). Comparing the initial terms of these sequences, we see that

$$g_{4n-2} = u_n, \quad h_{4n-2} = 2v_n, \qquad n \geq 1,$$

and this completes the proof of Theorem 2. $\square$

§3. Recurrence equations and explicit representations

In this section, we shall find explicit representations for g_n. This can be done in the same way as in the proof of Theorem 1, simultaneously obtaining representations for h_n. However, here we shall use other considerations.

The function

$$H(a_1, a_2, b_1, b_2 \mid z) = \frac{\Gamma(1-a_1)\Gamma(1-a_2)}{\Gamma(b_1)\Gamma(b_2)} \cdot {}_3F_4\left(\begin{matrix} 1-a_1, & 1-a_2, & 1-b_1, & 1-b_2 \\ 1, & 1, & 1 & \end{matrix} \,\middle|\, z\right)$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma(1+n-a_1)\Gamma(1+n-a_2)}{\Gamma(b_1-n)\Gamma(b_2-n)} \cdot \frac{z^n}{(n!)^4}$$

satisfies the same differential equation as the function $G(a_1, a_2, b_1, b_2 \mid z)$, and moreover, it satisfies the identities (9) (see [8]). Therefore, it satisfies the identities from Lemma 4, and the sequences

$$C_{2n} = H(-\alpha_{n+1}, -\alpha_{n+2}, \alpha_{n+1}+1, \alpha_{n+2}+1 \mid 1),$$
$$C_{2n+1} = \frac{\alpha_{n+3}^2}{\alpha_{n+2}\alpha_{n+4}} \delta H(-\alpha_{n+1}, -\alpha_{n+2}, \alpha_{n+1}+2, \alpha_{n+2}+1 \mid 1),$$

satisfy the recurrence equations of Lemma 5. Comparing the initial terms, we can easily verify that $C_n = g_n$, $n \geq 0$.

We have

$$H(-a, -b, c+1, d+1 \mid z) = \sum_{i=0}^{\infty} \frac{\Gamma(1+a+i)\Gamma(1+b+i)}{\Gamma(1+c-i)\Gamma(1+d-i)} \cdot \frac{z^i}{(i!)^4},$$

and, if a, b, c, d are nonnegative integers, $d \geq c$, then

$$H(-a, -b, c+1, d+1 \mid z) = \frac{a!b!}{c!d!} \sum_{i=0}^{c} \binom{a+i}{i} \binom{b+i}{i} \binom{c}{i} \binom{d}{i} z^i.$$

Hence, we also obtain the following two representations:

$$\frac{(a+1)^2}{b(b+1)} \delta H(-a, -b, a+2, b+1 \mid z) = \frac{a+1}{b+1} \sum \binom{a+i}{i} \binom{b+i}{i} \binom{a+1}{i} \binom{b-1}{i-1} z^i$$
$$= \frac{a+1}{b} \sum \binom{a+i}{i} \binom{b+i}{i-1} \binom{a+1}{i} \binom{b}{i} z^i,$$

where the summation index ranges over the interval $1 \leq i \leq \min(a+1, b)$.

Using these relations, we now find that for $k \geq 1$ the following relations are valid:

$$g_{4k} = \sum_{i=0}^{k} \binom{k+i}{i} \binom{k+1+i}{i} \binom{k}{i} \binom{k+1}{i},$$
$$g_{4k-1} = \sum_{i=1}^{k} \binom{k+i}{i} \binom{k+i}{i} \binom{k+1}{i} \binom{k-1}{i-1},$$
$$g_{4k-2} = \sum_{i=0}^{k} \binom{k+i}{i} \binom{k+i}{i} \binom{k}{i} \binom{k}{i},$$
$$g_{4k-3} = \sum_{i=0}^{k} \binom{k+i}{i-1} \binom{k-1+i}{i} \binom{k}{i} \binom{k}{i},$$

i.e., $g_n \in \mathbb{Z}$. The formulas just obtained furnish yet another proof of the relation $g_{4k-2} = u_k$.

Consider the sequence of functions

$$f_n(z) = G(-\alpha_{n+1}, -\alpha_{n+2}, \alpha_{n+1}+1, \alpha_{n+2}+1 \mid z), \qquad n \geq 0.$$

An argument similar to that used in the proof of Lemma 4 enables us to write out the recurrence equations of fourth order that are satisfied by this sequence.

Choose constants x_0, x_1, x_2, x_3, x_4 so that the following identity is satisfied:

$$\begin{aligned}\delta^4 = {} & x_0(\delta-c)(\delta-c-1)(\delta-d)(\delta-d-1) + x_1(\delta-b-1)(\delta-c)(\delta-c-1)(\delta-d-1) \\ & + x_2(\delta-a-1)(\delta-b-1)(\delta-c-1)(\delta-d-1) + x_3(\delta-a-1)(\delta-b)(\delta-b-1)(\delta-c-1) \\ & + x_4(\delta-a)(\delta-a-1)(\delta-b)(\delta-b-1). \end{aligned} \tag{15}$$

To do this, we must solve a system of five equations in five unknowns. Substituting $\delta = b+1$ in the identity (15), we obtain

$$x_0 = \frac{(b+1)^4}{(b-c)(b-c+1)(b-d)(b-d+1)}.$$

Further, the parameters a, b, c, d will be replaced by specific numbers so as to determine x_0 as well as $x_1, \ldots, x_4$. Substituting $\delta = c+1$ in identity (15), we find that

$$x_4 = \frac{(c+1)^4}{(c-a)(c-a+1)(c-b)(c-b+1)}.$$

Substituting $\delta = a+1$ and then, in the resulting equation, the derived value of x_0, we obtain

$$x_1 = \frac{a+1-d}{a-b} \cdot \left(\frac{(a+1)^4}{(a-c)(a-c+1)(a-d)(a-d+1)} - \frac{(b+1)^4}{(b-c)(b-c+1)(b-d)(b-d+1)} \right).$$

Similarly, substituting $\delta = d+1$, we find that

$$x_3 = \frac{d+1-a}{d-c} \cdot \left(\frac{(d+1)^4}{(d-a)(d-a+1)(d-b)(d-b+1)} - \frac{(c+1)^4}{(c-a)(c-a+1)(c-b)(c-b+1)} \right).$$

Finally, we have

$$x_2 = 1 - x_0 - x_1 - x_3 - x_4.$$

Applying this identity to the function $G(a+2, b+2, c+2, d+2)$, we obtain

$$\begin{aligned} zG(a+1, b+1, c+1, d+1) = {} & x_0 G(a+2, b+2, c, d) - x_1 G(a+2, b+1, c, d+1) \\ & + x_2 G(a+1, b+1, c+1, d+1) - x_3 G(a+1, b, c+1, d+2) \\ & + x_4 G(a, b, c+2, d+2). \end{aligned}$$

The parameters of the function G in all the summands are of the form $(a+i, b+j, c+k, d+l)$, where $i, j, k, l \geq 0$, $i+k = 2$, $j+l = 2$, $j+1 \geq i \geq j$. The substitution of

$$a = -\alpha_{n+4}, \quad b = -\alpha_{n+3}, \quad c = \alpha_{n+2}, \quad d = \alpha_{n+1}, \tag{16}$$

in the relation just mentioned leads to the following recurrence equation of fourth order:

$$zf_n = x_0 f_{n-2} - x_1 f_{n-1} + x_2 f_n - x_3 f_{n+1} + x_4 f_{n+2}. \tag{17}$$

The simplest way to obtain its limiting equation is to make the substitution $\delta = nt$ in identity (15) for the parameter values (16) and pass to the limit as $n \to \infty$. The identity then becomes

$$t^4 = x_0\left(t-\frac{1}{2}\right)^4 - x_1\left(t+\frac{1}{2}\right)\left(t-\frac{1}{2}\right)^3 + x_2\left(t+\frac{1}{2}\right)^2\left(t-\frac{1}{2}\right)^2 - x_3\left(t+\frac{1}{2}\right)^3\left(t-\frac{1}{2}\right) + x_4\left(t+\frac{1}{2}\right)^4.$$

Manipulations similar to the ones described above enable us to find

$$x_0 = x_4 = \frac{1}{16}, \quad x_1 = x_3 = \frac{1}{4}, \quad x_2 = \frac{3}{8};$$

hence, the limiting equation is of the form

$$t^4 - 4t^3 + (6-16z)t^2 - 4t + 1 = 0, \qquad \text{or} \qquad (t-1)^4 - 16zt^2 = 0.$$

For $z = 1$ (Apery's case), the limiting equation can be reduced to the form

$$t^4 - 4t^3 - 10t^2 - 4t + 1 = (t+1)^2(t^2 - 6t + 1) = 0,$$

and for $z = -1$ (the case considered in [7]), it can be reduced to the following form:

$$t^4 - 4t^3 + 22t^2 - 4t + 1 = (t-1)^4 + 16t^2 = 0.$$

The sequence $f_n(1)$ satisfies a recurrence equation of second order. The latter can be obtained from the recurrence equations of Lemma 5 or directly from the identity of the following lemma.

LEMMA 6. *If $a+b+c+d=0$, then*

$$\frac{d^4}{b-d}G(a,b,c+1,d+1) + \left(\frac{bd^3}{b-d} + \frac{ca^3}{a-c} + bc(a-d)\right)G(a,b+1,c+1,d) + \frac{a^4}{a-c}G(a+1,b+1,c,d) = (z-1)(a-d)G(a,b,c,d). \tag{18}$$

PROOF. Define the unknowns λ, μ, ν so that

$$\lambda\cdot(\delta-a)(\delta-b) + \mu\cdot(\delta-a)(\delta-d) + \nu\cdot(\delta-c)(\delta-d) = \delta^4 - (\delta-a)(\delta-b)(\delta-c)(\delta-d).$$

Next, applying this identity to $G(a+1,b+1,c+1,d+1)$, we obtain (18).

It follows from (18) for $a = -\alpha_{n+2}$, $b = -\alpha_{n+3}$, $c = \alpha_{n+2}$, $d = \alpha_{n+3}$ that

$$\alpha_{n+3}^3 f_{n+1}(z) - (\alpha_{n+3}^3 + 2\alpha_{n+3}^2\alpha_{n+2} + 2\alpha_{n+3}\alpha_{n+2}^2 + \alpha_{n+2}^3)f_n(z) + \alpha_{n+2}^3 f_{n-1}(z) = 2(z-1)(\alpha_{n+2}+\alpha_{n+3})G(-\alpha_{n+2}, -\alpha_{n+3}, \alpha_{n+2}, \alpha_{n+3}). \tag{19}$$

Now substitute $z = 1$ in this relation. As a result, we find the required recurrence equation

$$\alpha_{n+3}^3 r_{2n+2} - (\alpha_{n+3}^3 + 2\alpha_{n+3}^2\alpha_{n+2} + 2\alpha_{n+3}\alpha_{n+2}^2 + \alpha_{n+2}^3)r_{2n} + \alpha_{n+2}^3 r_{2n-2} = 0.$$

Its limiting characteristic equation is of the form

$$t^2 - 6t + 1 = 0.$$

We can write out identities similar to (19) if we replace n by $n-1$ and by $n+1$. On the right-hand sides of these three identities, the following functions will appear:

$$G(-\alpha_{n+i}, -\alpha_{n+i+1}, \alpha_{n+i}, \alpha_{n+i+1}), \qquad i = 1, 2, 3. \tag{20}$$

Now choose numbers $\lambda, \mu, \nu, \gamma$ so that the following identity holds:

$$\lambda \cdot (\delta - d)(\delta - c) + \mu \cdot (\delta - c)(\delta - b) + \nu \cdot (\delta - b)(\delta - a) + \gamma = 0.$$

Applying this identity to the function $G(a+1, b+1, c+1, d+1 \mid z)$ and choosing $a = -\alpha_{n+3}$, $b = -\alpha_{n+2}$, $c = \alpha_{n+1}$, $d = \alpha_n$, we obtain a relation between the functions (20). This furnishes yet another proof of (17). □

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