Purification of quantum trajectories

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Abstract: We prove that the quantum trajectory of repeated perfect measurement on a finite quantum system either asymptotically purifies, or hits upon a family of 'dark' subspaces, where the time evolution is unitary.

1. Introduction

A key concept in the modern theory of open quantum systems is the notion of indirect measurement as introduced by Kraus [Kra]. An *indirect measurement* on a quantum system is a (direct) measurement of some quantity in its environment, made after some interaction with the system has taken place.

When we make such a measurement, our description of the quantum system changes in two ways: we account for the flow of time by a unitary transformation (following Schrödinger), and we update our knowledge of the system by conditioning on the measurement outcome (following von Neumann). If we then repeat the indirect measurement indefinitely, we obtain a chain of random outcomes. In the course of time we may keep record of the updated density matrix Θ_t, which at time t reflects our best estimate of all observable quantities of the quantum system, given the observations made up to that time. This information can in its turn be used to predict later measurements outcomes. The stochastic process Θ_t of updated states, is the *quantum trajectory* associated to the repeated measurement process.

By taking the limit of continuous time, we arrive at the modern models of continuous observation: quantum trajectories in continuous time satisfying *stochastic Schrödinger equations* [Dav], [Gis], [Car], [BGM]. These models are employed with great success for calculations and computer simulations of laboratory experiments such as photon counting and homodyne field detection.

In this paper we consider the question, what happens to the quantum trajectory at large times. We do so only for the case of discrete time, not a serious restriction indeed, since asymptotic behaviour remains basically unaltered in the continuous time limit.

We focus on the case of *perfect measurement*, i.e. the situation where no information flows into the system, and all information which leaks out is indeed observed. In classical probability such repeated perfect measurement would lead to a further and further narrowing of the distribution of the system, until it either becomes pure, i.e. an atomic measure, or it remains spread out over some area, thus leaving a certain amount of information 'in the dark' forever. Using a fundamental inequality of Nielsen [Nie] we prove that in quantum mechanics the situation is quite comparable: the density matrix tends to purify, until it hits upon some family of 'dark' subspaces, if such exist, i.e. spaces from which no information can leak out. A crucial difference with the classical case is, however, that even after all available information has been extracted by observation, the state continues to move about in a random fashion between the 'dark' subspaces, thus continuing to produce 'quantum noise'.

The structure of this paper is as follows. In Section 2 we introduce quantum measurement on a finite system, in particular Kraus measurement. In Section 3 repeated

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measurement and the quantum trajectory are introduced, and in Section 4 we prove our main result. Some typical examples of dark subspaces are given in Section 5.

2. A single measurement

Let $\mathcal{A}$ be the algebra of all complex $d \times d$ matrices. By $\mathcal{S}$ we denote the space of $d \times d$ *density matrices*, i.e. positive matrices of trace 1. We think of $\mathcal{A}$ as the observable algebra of some finite quantum system, and of $\mathcal{S}$ as the associated state space.
A measurement on this quantum system is an operation which results in the extraction of information while possibly changing its state. Before the measurement the system is described by a *prior* state $\theta \in \mathcal{S}$, and afterwards we obtain a piece of information, say an outcome $i \in \{1, 2, \ldots, k\}$, and the system reaches some new (or *posterior*) state θ_i':
$$\theta \longrightarrow (i, \theta_i') \ .$$
Now, a probabilistic theory, rather than predicting the outcome i, gives a probability distribution $(\pi_1, \pi_2, \ldots, \pi_k)$ on the possible outcomes. Let

$$T_i : \theta \mapsto \pi_i \theta_i' \ , \qquad (i = 1, \ldots, k) \ . \tag{1}$$

Then the operations T_i, which must be completely positive, code for the probabilities $\pi_i = \mathrm{tr}(T_i \theta)$ of the possible outcomes, as well as for the posterior states $\theta_i' = T_i \theta / \mathrm{tr}(T_i \theta)$, conditioned on these outcomes. The k-tuple $(T_1, \ldots, T_k)$ describes the quantum measurement completely. Its mean effect on the system, averaged over all possible outcomes, is given by the trace-preserving map

$$T : \theta \mapsto \sum_{i=1}^{k} \pi_i \theta_i' = \sum_{i=1}^{k} T_i \theta \ .$$

Example 1: *von Neumann measurement.*
Let $p_1, p_2, \ldots p_k$ be mutually orthogonal projections in $\mathcal{A}$ adding up to $\mathbf{1}$, and let $a \in \mathcal{A}$ be a self-adjoint matrix whose eigenspaces are the ranges of the p_i. Then according to von Neumann's projection postulate a measurement of a is obtained by choosing for T_i the operation

$$T_i(\theta) = p_i \theta p_i \ .$$

Example 2: *Kraus measurement.*
The following indirect measurement procedure was introduced by Karl Kraus [Kra]. It is contains von Neumann's measurement as an ingredient, but is considerably more flexible and realistic.
Our quantum system $\mathcal{A}$ in the state θ is brought into contact with a second system, called the 'ancilla', which is described by a matrix algebra $\mathcal{B}$ in the state β. The two systems interact for a while under Schrödinger's evolution, which results in a rotation over a unitary $u \in \mathcal{B} \otimes \mathcal{A}$. Then the ancilla is decoupled again, and is subjected to a von Neumann measurement given by the orthogonal projections $p_1, \ldots, p_k \in \mathcal{B}$. The outcome of this measurent contains information about the system, since system and ancilla have become correlated during their interaction. In order to assess this information, let us consider an event in our quantum system, described by a projection $q \in \mathcal{A}$. Since each of the projections $p_i \otimes \mathbf{1}$ commutes with

$\mathbf{1} \otimes q$, the events of seeing outcome i and then the occurrence of q are compatible, so according to von Neumann we may express the probability for both of them to happen as:

$$\mathbb{P}[\text{outcome } i \text{ and then event } q] = \text{tr} \otimes \text{tr}\Big(\big(u(\beta \otimes \theta)u^*\big)(p_i \otimes q)\Big) .$$

Therefore the following conditional probability makes physical sense.

$$\mathbb{P}[\text{event } q|\text{outcome } i] = \frac{\text{tr} \otimes \text{tr}\Big(\big(u(\beta \otimes \theta)u^*\big)(p_i \otimes q)\Big)}{\text{tr} \otimes \text{tr}\Big(\big(u(\beta \otimes \theta)u^*\big)(p_i \otimes \mathbf{1})\Big)} .$$

This expression, which describes the posterior probability of any event $q \in \mathcal{A}$, can be considered as the posterior state of our quantum system, conditioned on the measurement of an outcome i on the ancilla, even when no event q is subsequently measured. As above, let us therefore call this state θ_i'. We then have

$$\text{tr}(\theta_i' q) = \frac{\text{tr}\big((T_i\theta)q\big)}{\text{tr}(T_i\theta)} ,$$

where $T_i\theta$ takes the form

$$T_i\theta = \text{tr} \otimes \text{id}\Big(\big(u(\beta \otimes \theta)u^*\big)(p_i \otimes \mathbf{1})\Big) .$$

Here, id denotes the identity map $\mathcal{S} \to \mathcal{S}$.

The expression for T_i takes a simple form in the case which will interest us here, namely when the following three conditions are satisfied:

(i) $\mathcal{B}$ consists of all $k \times k$-matrices for some k;
(ii) the orthogonal projections $p_i \in \mathcal{B}$ are one-dimensional (say p_i is the matrix with i-th diagonal entry 1, and all other entries 0);
(iii) β is a pure state (say with state vector $(\beta_1, \ldots, \beta_k) \in \mathbb{C}^k$).

These conditions have the following physical interpretations.

(i) The ancilla is purely quantummechanical;
(ii) the measurement discriminates maximally;
(iii) no new information is fed into the system.

If these conditions are satisfied, u can be written as a $k \times k$ matrix (u_{ij}) of $d \times d$ matrices, and T_i may be written

$$T_i\theta = a_i\theta a_i^* , \tag{2}$$

where

$$a_i = \sum_{j=1}^{k} \beta_j u_{ij} .$$

We note that, by construction,

$$\sum_{i=1}^{k} a_i^* a_i = \sum_{i=1}^{k}\sum_{j=1}^{k}\sum_{j'=1}^{k} \overline{\beta_j} u_{ji}^* u_{ij'} \beta_{j'} = \|\beta\|^2 = \mathbf{1} .$$

This basic rule expresses the preservation of the trace by T.

Definition 1 *By a perfect measurement on $\mathcal{A}$ we shall mean a k-tuple $(T_1, \ldots, T_k)$ of operations on $\mathcal{S}$, where $T_i\theta$ is of the form $a_i\theta a_i^*$ with $\sum_{i=1}^{k} a_i^* a_i = \mathbf{1}$.*

Mathematically speaking, the measurement $(T_1, \ldots, T_k)$ is perfect iff the Stinespring decomposition of each T_i consists of a single term.
We note that every perfect Kraus measurement is a perfect measurement in the above sense, and that every perfect measurement can be obtained as the result of a perfect Kraus measurement.

3. Repeated measurement

By repeating a measurement on the quantum system $\mathcal{A}$ indefinitely, we obtain a Markov chain with values in the state space $\mathcal{S}$. This is the quantum trajectory which we study in this paper.
Let Ω be the space of infinite outcome sequences $\omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$, with $\omega_j \in \{1, \ldots, k\}$, and let for $m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \{1, \ldots, k\}$ the *cylinder set* $\Lambda_{i_1, \ldots, i_m} \subset \Omega$ be given by

$$\Lambda_{i_1, \ldots, i_m} := \{\omega \in \Omega \,|\, \omega_1 = i_1, \ldots, \omega_m = i_m\} \,.$$

Denote by Σ_m the Boolean algebra generated by these cylinder sets, and by Σ the σ-algebra generated by all these Σ_m. Let $T_1, \ldots, T_k$ be as in Section 2.
Then for every initial state θ_0 on $\mathcal{A}$ there exists a unique probability measure $\mathbb{P}_{\theta_0}$ on (Ω, Σ) satisfying

$$\mathbb{P}_{\theta_0}(\Lambda_{i_1, \ldots, i_m}) = \operatorname{tr}(T_{i_m} \circ \cdots \circ T_{i_1}(\theta_0)) \,.$$

Indeed, according to the Kolmogorov-Daniell reconstruction theorem we only need to check consistency: since $T = \sum_{i=1}^{k} T_i$ preserves the trace,

$$
\begin{aligned}
\sum_{i=1}^{k} \mathbb{P}_{\theta_0}(\Lambda_{i_1, \ldots, i_m, i}) &= \sum_{i=1}^{k} \operatorname{tr}\big(T_i \circ T_{i_m} \circ \cdots \circ T_{i_1}(\theta_0)\big) = \operatorname{tr}(T \circ T_{i_m} \circ \cdots \circ T_{i_1}(\theta_0)) \\
&= \operatorname{tr}\big(T_{i_m} \circ \cdots \circ T_{i_1}(\theta_0)\big) = \mathbb{P}_{\theta_0}\big(\Lambda_{i_1, \ldots, i_m}\big) \,.
\end{aligned}
$$

On the probability space $(\Omega, \Sigma, \mathbb{P}_{\theta_0})$ we now define the *quantum trajectory* $(\Theta_n)_{n \in \mathbb{N}}$ as the sequence of random variables given by

$$\Theta_n : \Omega \to \mathcal{S} : \omega \mapsto \frac{T_{\omega_m} \circ \cdots \circ T_{\omega_1}(\theta_0)}{\operatorname{tr}\big(T_{\omega_m} \circ \cdots \circ T_{\omega_1}(\theta_0)\big)} \,.$$

We note that Θ_n is Σ_n-measurable. The density matrix $\Theta_n(\omega)$ describes the state of the system at time n under the condition that the outcomes $\omega_1, \ldots, \omega_n$ have been seen.
The quantum trajectory $(\Theta_n)_{n \in \mathbb{N}}$ is a Markov chain with transitions

$$\theta \longrightarrow \theta_i' = \frac{T_i\theta}{\operatorname{tr}(T_i\theta)} \qquad \text{with probability} \quad \operatorname{tr}(T_i\theta) \,. \tag{3}$$

4. Purification

In a perfect measurement, when T_i is of the form $\theta \mapsto a_i \theta a_i^*$, a pure prior state $\theta = |\psi\rangle\langle\psi|$ leads to a pure posterior state:

$$\theta_i' = \frac{a_i|\psi\rangle\langle\psi|a_i^*}{\langle\psi, a_i^*a_i\psi\rangle} = |\psi_i\rangle\langle\psi_i| \,, \quad \text{where} \quad \psi_i = \frac{a_i\psi}{\|a_i\psi\|} \,.$$

Hence in the above Markov chain the pure states form a closed set. Experience with quantum trajectories leads one to believe that in many cases even more is true: along a typical trajectory the density matrix tends to purify: its spectrum approaches the set $\{0, 1\}$. In Markov chain jargon: the pure states form an asymptotically stable set.

There is, however, an obvious counterexample to this statement in general. If every a_i is proportional to a unitary, say $a_i = \sqrt{\lambda_i} u_i$ with $u_i^* u_i = \mathbf{1}$, then

$$\theta_i' = \frac{a_i\theta a_i^*}{\mathrm{tr}(a_i\theta a_i^*)} = u_i\theta u_i^* \sim \theta \,,$$

where $\sim$ denotes unitary equivalence. So in this case the eigenvalues of the density matrix remain unchanged along the trajectory: pure states remain pure and mixed states remain mixed with unchanging weights. In this section we shall show that in dimension 2 this is actually the only exception. (Cf. Corollary 2.) In higher dimensions the situation is more complicated: if the state does not purify, the a_i must be proportional to unitaries on a certain collection of 'dark' subspaces, which they must map into each other. (Cf. Corollary 8.)

In order to study purification we shall consider the *moments* of Θ_n. By the m-th *moment* of a density matrix $\theta \in \mathcal{S}$ we mean $\mathrm{tr}\,(\theta^m)$. We note that two states θ and ρ are unitarily equivalent iff all their moments are equal. In dimension d equality of the moments $m = 1, \ldots, d$ suffices.

Definition 2 *We say that the quantum trajectory* $\big(\Theta_n(\omega)\big)_{n\in\mathbb{N}}$ *purifies when*

$$\forall_{m\in\mathbb{N}}: \quad \lim_{n\to\infty} \mathrm{tr}\big(\Theta_n(\omega)^m\big) = 1 \,.$$

Note that the only density matrices ρ satisfying $\mathrm{tr}(\rho^m) = 1$ are one-dimensional projections, the density matrices of pure states. In fact, it suffices that the second moment be equal to 1.

We now state our main result concerning repeated perfect measurement.

Theorem 1 *Let* $\big(\Theta_n\big)_{n\in\mathbb{N}}$ *be the Markov chain with initial state θ_0 and transition probabilities (3). Then one of the following alternatives holds.*

(i) The paths of $\big(\Theta_n\big)_{n\in\mathbb{N}}$ (the quantum trajectories) purify with probability 1, or:

(ii) there exists a projection $p \in \mathcal{A}$ of dimension at least two such that

$$\forall_{i\in\{1,\ldots,k\}}\exists_{\lambda_i\geq 0}: \quad pa_i^*a_ip = \lambda_i p \,.$$

Condition (ii) says that a_i is proportional to an isometry in restriction to the range of p. Note that this condition trivially holds if p is one-dimensional.

Corollary 2 *In dimension $d = 2$ the quantum trajectory of a repeated perfect measurement either purifies with probability 1, or all the a_i's are proportional to unitaries.*

If the a_i are all proportional to unitaries, the coupling to the environment is *essentially commutative* in the sense of [KüM].

Our proof starts from an inequality of Michael Nielsen [Nie] to the effect that for all $m \in \mathbb{N}$ and all states θ:

$$\sum_{i=1}^{k} \pi_i \operatorname{tr}\big((\theta_i')^m\big) \geq \operatorname{tr}(\theta^m) \;,$$

where

$$\pi_i := \operatorname{tr}(a_i \theta a_i^*) \quad \text{and} \quad \theta_i' := \frac{a_i \theta a_i^*}{\operatorname{tr}(a_i \theta a_i^*)} \;.$$

Nielsen's inequality says that the expected m-th moment of the posterior state is as least as large as the m-th moment of the prior state. In terms of the associated Markov chain we may express this inequality as

$$\forall_{m,n \in \mathbb{N}}: \qquad \mathbb{E}\left(\operatorname{tr}\big(\Theta_{n+1}^m\big) \,\Big|\, \Sigma_n \right) \geq \operatorname{tr}\big(\Theta_n^m\big) \;,$$

i.e. the moments $M_n^{(m)} := \operatorname{tr}\big(\Theta_n^m\big)_{n \in \mathbb{N}}$ are submartingales. Clearly all moments take values in $[0,1]$. Therefore, by the martingale convergence theorem they must converge almost surely to some random variables $M^{(m)}$.

This suggests the following line of proof for our theorem: Since the moments converge, the eigenvalues of $(\Theta_n)_{n \in \mathbb{N}}$ must converge. Hence along a single trajectory the states eventually become unitarily equivalent, i.e. eventually

$$\forall_i: \quad \Theta_n(\omega) \sim \frac{a_i \Theta_n(\omega) a_i^*}{\operatorname{tr}(a_i \Theta_n(\omega) a_i^*)} \;.$$

But this seems to imply that either Θ_n purifies almost surely, or the a_i's are unitary on the support of Θ_n.

In the following proof of Theorem 1 we shall make this suggestion mathematically precise.

Lemma 3 *In the situation of Theorem 1 one of the following alternatives holds.*

  (i) For all $m \in \mathbb{N}$: $\lim_{n \to \infty} \operatorname{tr}\big(\Theta_n^m\big) = 1$ almost surely;

  (ii) there exists a mixed state $\rho \in \mathcal{S}$ such that

$$\forall_{i=1,\ldots,k} \exists_{\lambda_i \geq 0}: \quad a_i \rho a_i^* \sim \lambda_i \rho \;.$$

Proof. For each $m \in \mathbb{N}$ we consider the continuous function

$$\delta_m : \mathcal{S} \to [0,\infty) : \theta \mapsto \sum_{i=1}^{k} \operatorname{tr}(a_i \theta a_i^*) \left(\operatorname{tr}\left(\left(\frac{a_i \theta a_i^*}{\operatorname{tr}(a_i \theta a_i^*)} \right)^m \right) - \operatorname{tr}(\theta^m) \right)^2 \;.$$

Then, using (2) and (3),

$$\delta_m(\Theta_n) = \mathbb{E}\left(\left(M_{n+1}^{(m)} - M_n^{(m)} \right)^2 \Big| \Sigma_n \right) \;.$$

Since $\big(M_n^{(m)}\big)_{n \in \mathbb{N}}$ is a positive submartingale bounded by 1, its increments must be square summable:

$$\forall_{m \in \mathbb{N}}: \qquad \sum_{n=0}^{\infty} \mathbb{E}\big(\delta_m(\Theta_n)\big) \leq 1 \;.$$

In particular

$$\lim_{n\to\infty} \sum_{m=1}^{d} \mathbb{E}\big(\delta_m(\Theta_n)\big) = 0 \ . \tag{4}$$

Now let us assume that (i) is not the case, i.e. for some (and hence for all) $m \geq 2$ the expectation $\mathbb{E}(M^{(m)}) =: \mu_m$ is strictly less than 1. For any $n \in \mathbb{N}$ consider the event

$$A_n := \left\{ \omega \in \Omega \ \Big| \ M_n^{(2)} \leq \frac{\mu_2 + 1}{2} \right\} \ .$$

Then, since $\mathbb{E}\left(M_n^{(2)}\right)$ is increasing in n, we have for all $n \in \mathbb{N}$:

$$\mathbb{E}(M_n^{(2)}) \leq \mathbb{E}(M^{(2)}) = \mu_2 < 1 \ .$$

Therefore for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\mu_2 &\geq \mathbb{E}\left(M_n^{(2)} \cdot 1_{[M_n^{(2)} > \frac{\mu_2+1}{2}]} \right) \\
&\geq \frac{\mu_2 + 1}{2} \, \mathbb{P}\left[M_n^{(2)} > \frac{\mu_2 + 1}{2} \right] \\
&= \frac{\mu_2 + 1}{2} \left(1 - \mathbb{P}(A_n) \right) \ ,
\end{aligned}
$$

so that

$$\mathbb{P}(A_n) \geq \frac{1 - \mu_2}{1 + \mu_2} \ . \tag{5}$$

On the other hand, A_n is Σ_n-measureable and therefore it is a union of sets of the from $\Lambda_{i_1,\dots,i_n}$. Since Θ_n is Σ_n-measureable, Θ_n is constant on such sets; let us call the constant $\Theta_n(i_1,\dots,i_n)$. We have the following inequality:

$$\frac{1}{\mathbb{P}(A_n)} \sum_{\Lambda_{i_1,\dots,i_n} \subset A_n} \mathbb{P}\left(\Lambda_{i_1,\dots,i_n}\right) \left(\sum_{m=1}^{d} \delta_m\big(\Theta_n(i_1,\dots,i_n)\big) \right) \leq \frac{1}{\mathbb{P}(A_n)} \sum_{m=1}^{d} \mathbb{E}\big(\delta_m(\Theta_n)\big) \ .$$

On the left hand side we have an average of numbers which are each of the form $\sum_{m=1}^{d} \delta_m(\Theta_n(i_1,\dots,i_n))$, hence we can choose $(i_1,\dots,i_n)$ such that $\rho_n := \Theta_n(i_1,\dots,i_n)$ satisfies, by (5),

$$\sum_{m=1}^{d} \delta_m(\rho_n) \leq \frac{\mu_2 + 1}{\mu_2 - 1} \sum_{m=1}^{d} \mathbb{E}\big(\delta_m(\Theta_n)\big) \ .$$

Since $\Lambda_{i_1,\dots,i_n} \subset A_n$, the sequence $(\rho_n)_{n\in\mathbb{N}}$ lies entirely in the compact set

$$\left\{ \theta \in \mathcal{S} \ \Big| \ \mathrm{tr}(\theta^2) \leq \frac{\mu_2 + 1}{2} \right\} \ .$$

Let ρ be a cluster point of this sequence. Then, since $\mathbb{E}(\delta_m(\Theta_n))$ tends to 0 as $n \to \infty$, and δ_m is continuous, we may conclude that for $m = 1,\dots,d$:

$$\delta_m(\rho) = 0 \ , \quad \text{and} \quad \mathrm{tr}(\rho^2) \leq \frac{\mu_2 + 1}{2} < 1 \ .$$

So ρ is a mixed state, and, by the definition of δ_m,

$$\mathrm{tr}(a_i\rho a_i^*)\left(\mathrm{tr}\left(\left(\frac{a_i\rho a_i^*}{\mathrm{tr}(a_i\rho a_i^*)}\right)^m\right) - \mathrm{tr}(\rho^m)\right)^2 = 0$$

for all $m = 1, 2, 3, \ldots, d$ and all $i = 1, \ldots, k$. Therefore either $\mathrm{tr}(a_i\rho a_i^*) = 0$, i.e. $a_i\rho a_i^* = 0$, proving our statement (ii) with $\lambda_i = 0$; or $\mathrm{tr}(a_i\rho a_i^*) > 0$, in which case $\rho_i' := a_i\rho a_i^*/\mathrm{tr}(a_i\rho a_i^*)$ and ρ itself have the same moments of orders $m = 1, 2, \ldots, d$, so that they are unitarily equivalent. This proves (ii).

$\square$

From Lemma 3 to Theorem 1 is an exercise in linear algebra:

Lemma 4 *Let* $a_1, \ldots, a_k \in M_d$ *be such that* $\sum_{i=1}^k a_i^* a_i = \mathbf{1}$. *Suppose that there exists a density matrix* $\rho \in M_d$ *such that for* $i = 1, \ldots, k$

$$a_i\rho a_i^* \sim \lambda_i\rho \ .$$

Let p *denote the support of* ρ. *Then for all* $i = 1, \ldots, k$:

$$pa_i^* a_i p = \lambda_i p \ .$$

Proof. Let us define, for a nonnegative matrix x, the *positive determinant* $\det_{\mathrm{pos}}(x)$ to be the product of all its strictly positive eigenvalues (counted with their multiplicities). Then, if p denotes the support projection of x, we have the implication

$$\det_{\mathrm{pos}}(x) = \det_{\mathrm{pos}}(\lambda p) \implies \mathrm{tr}(xp) \geq \mathrm{tr}(\lambda p) \tag{6}$$

with equality iff $x = \lambda p$. (This follows from the fact that the sum of a set of positive numbers with given product is minimal iff these numbers are equal.)

Now let p be the support of ρ as in the Lemma. Let $v_i\sqrt{pa_i^* a_i p}$ denote the polar decomposition of $a_i p$. Then we have by assumption,

$$\begin{aligned}
\det_{\mathrm{pos}}(\lambda_i\rho) &= \det_{\mathrm{pos}}(a_i\rho a_i^*) \\
&= \det_{\mathrm{pos}}(a_i p\rho p a_i^*) \\
&= \det_{\mathrm{pos}}(v_i\sqrt{pa_i^* a_i p}\rho\sqrt{pa_i^* a_i p}v_i^*) \\
&= \det_{\mathrm{pos}}(\sqrt{pa_i^* a_i p}\rho\sqrt{pa_i^* a_i p}) \\
&= \det_{\mathrm{pos}}(pa_i^* a_i p)\det_{\mathrm{pos}}(\rho) \ .
\end{aligned}$$

Now, since $\det_{\mathrm{pos}}(\lambda_i\rho) = \det_{\mathrm{pos}}(\lambda_i p) \cdot \det_{\mathrm{pos}}(\rho)$ and $\det_{\mathrm{pos}}(\rho) > 0$, it follows that

$$\det_{\mathrm{pos}}(\lambda_i p) = \det_{\mathrm{pos}}(pa_i^* a_i p) \ .$$

By the implication (6) we may conclude that

$$\mathrm{tr}(pa_i^* a_i p) \geq \mathrm{tr}\lambda_i p) \ . \tag{7}$$

On the other hand,

$$\sum_{i=1}^k \lambda_i = \sum_{i=1}^k \mathrm{tr}(\lambda_i\rho) = \sum_{i=1}^k \mathrm{tr}(a_i\rho a_i^*) = \mathrm{tr}\left(\rho\left(\sum_{i=1}^k a_i a_i^*\right)\right) = \mathrm{tr}\rho = 1 \ ,$$

where in the second equality sign the assumption was used again. Then, by (7),

$$\operatorname{tr} p = \sum_{i=1}^{k} \operatorname{tr}(p a_i^* a_i p) \geq \sum_{i=1}^{k} \operatorname{tr}(\lambda_i p) = \left(\sum_{i=1}^{k} \lambda_i \right) \operatorname{tr} p = \operatorname{tr} p \ .$$

So apparently, in this chain, we have equality. But then, since equality is reached in (6), we find that

$$p a_i^* a_i p = \lambda_i p \ .$$

$\square$

5. Dark subspaces

By considering more than one step at a time the following stronger conclusion can be drawn.

Corollary 5 *In the situation of Theorem 1, either the quantum trajectory purifies with probability 1 or there exists a projection p of dimension at least 2 such that for all $l \in \mathbb{N}$ and all $i_1, \ldots, i_l$ there is $\lambda_{i_1, \ldots, i_l} \geq 0$ with*

$$p a_{i_1}^* \cdots a_{i_l}^* a_{i_l} \cdots a_{i_1} p = \lambda_{i_1, \ldots, i_l} p \ . \tag{8}$$

We shall call a projection p satisfying (8) a *dark* projection, and its range a *dark subspace.*
Let p be a dark projection, and let $v_i \sqrt{p a_i^* a_i} = \sqrt{\lambda_i} v_i p$ be the polar decomposition of $a_i p$. Then the projection $p_i' := v_i p v_i^*$ satisfies:

$$
\begin{aligned}
\lambda_i p_i' a_{i_1}^* \cdots a_{i_m}^* a_{i_m} \cdots a_{i_1} p_i' &= \lambda_i (v_i p v_i^*) a_{i_1}^* \cdots a_{i_m}^* a_{i_m} \cdots a_{i_1} (v_i p v_i^*) \\
&= v_i p a_i^* a_{i_1}^* \cdots a_{i_m}^* a_{i_m} \cdots a_{i_1} a_i p v_i^* \\
&= \lambda_{i, i_1, \ldots, i_m} \cdot p_i' \ .
\end{aligned}
$$

Hence if p is dark, and $\lambda_i \neq 0$ then also p_i' is dark with constants

$$\lambda'_{i_1, \ldots, i_m} = \lambda_{i, i_1, \ldots, i_m} / \lambda_i \ .$$

We conclude that asymptotically the quantum trajectory performs a random walk between dark subspaces of the same dimension, with transition probabilities $p \longrightarrow p_i'$ equal to λ_i, the scalar value in $p a_i^* a_i p = \lambda_i p$. In the trivial case that the dimension of p is 1, purification has occurred.
Inspection of the a_i should reveal the existence of nontrivial dark subspaces. If none exist, then purification is certain.
We end this Section with two examples where nontrivial dark subspaces occur.

Example 1. Let $d = l \cdot e$ and let $\mathcal{H}_1, \ldots, \mathcal{H}_l$ be mutually orthogonal e-dimensional subspaces of $\mathcal{H} = \mathbb{C}^d$. Let (π_{ij}) be an $l \times l$ matrix of transition probabilities. Define $a_{ij} \in \mathcal{A}$ by

$$a_{ij} := \sqrt{\pi_{ij}} v_{ij} \ ,$$

where the maps $v_{ij} : \mathcal{H}_i \to \mathcal{H}_j$ are isometric. Then the matrices a_{ij}, $i, j = 1, \ldots, l$ define a perfect measurement whose dark subspaces are $\mathcal{H}_1, \ldots, \mathcal{H}_l$.

Example 2.

The following example makes clear that nontrivial dark subspaces need not be orthogonal.

Let $\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{D}$, where $\mathcal{D}$ is some finite dimensional Hilbert space, and for $i = 1, \ldots, k$ let $a_i := b_i \otimes u_i$, where the 2×2-matrices b_i satisfy the usual equality

$$\sum_{i=1}^{k} b_i^* b_i = \mathbf{1} \ ,$$

and the u_i are unitaries $\mathcal{D} \to \mathcal{D}$. Suppose that the b_i are not all proportional to unitaries. Then the quantum trajectory defined by the a_i has dark subspaces $\psi \otimes \mathcal{D}$, with ψ running through the unit vectors in $\mathbb{C}^2$. Physically this example describes a pair of systems without any interaction between them, one of which is coupled to the environment in an essentially commutative way, whereas the other purifies.

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