The discrete series of semisimple groups

Peter Hochs

September 5, 2019

Abstract

These notes contain some basic facts about discrete series representations of semisimple Lie groups. For a large part, they summarise relevant material from Knapp's book [12]. We discuss the classification of discrete series representations, their characters, their relevance to representation theory, and some explicit realisations of their representation spaces. We also go into classes defined by discrete series representations in K-theory of group C^*-algebras.

Contents

1 Introduction

Let G be a linear, connected, semisimple Lie group. Discrete series representations occur discretely in the Plancherel decomposition of $L^2(G)$, and in the unitary dual $\hat{G}$. See for example Figure 1, where the unitary dual of

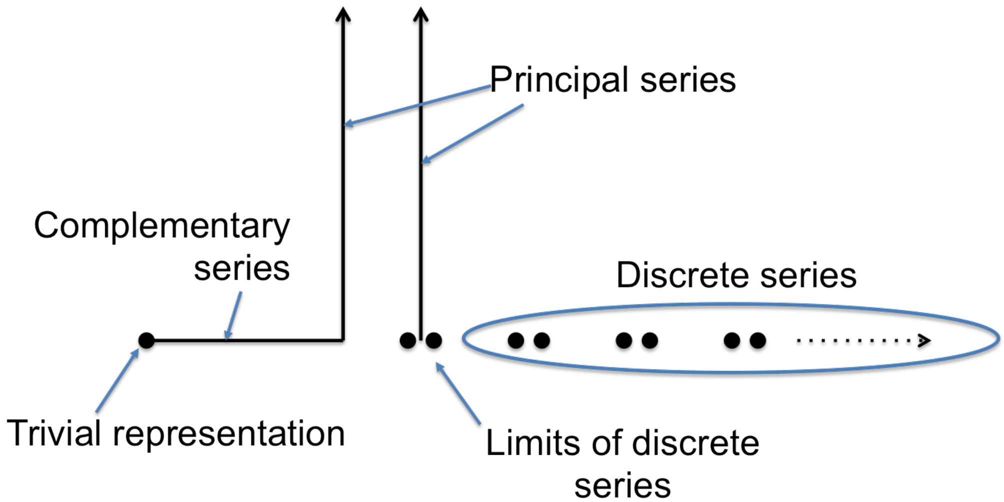

Figure 1: The unitary dual of $\mathrm{SL}(2,\mathbb{R})$

$\mathrm{SL}(2,\mathbb{R})$ is pictured. The topology is as in the diagram, with the exceptions that

1. the left-most two discrete series representations, as well as the trivial representation, are limits as one goes left in the complementary series;

2. both limits of discrete series representations are limits as one goes down in the right hand component of the principal series.

This topology is not Hausdorff, but it is T_1. The discrete series representations are pictured in pairs with the same multiplicity in $L^2(\mathrm{SL}(2,\mathbb{R}))$.

More generally, (almost) all irreducible representations of G that occur in the Plancherel decomposition can be constructed from discrete series representations of subgroups of G. This makes discrete series representations important objects of study in representation theory.

In these notes, some facts about discrete series representations are collected. These mainly summarise parts of Knapp's books [12, 13]. For the proofs of the facts we mention, references are given to these books. As in [12], we will consider *linear* groups, which makes some constructions and arguments simpler. Most statements given are valid slightly more generally, though.

2 Preliminaries

Throughout these notes, G will be a Lie group, with Lie algebra $\mathfrak{g}$. All Lie algebras and Lie groups are assumed to be finite-dimensional. We fix a maximal compact subgroup $K < G$, with Lie algebra $\mathfrak{k}$. We also fix a right Haar measure dg on G.

2.1 Reductive and semisimple groups

Recall that $\mathfrak{g}$ is *reductive* if for every ideal $\mathfrak{a} \subset \mathfrak{g}$ there is an ideal $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$. It is *simple* if it has no nontrivial ideals, and *semisimple* if the equivalent conditions of Proposition 2.1 hold.

Proposition 2.1. *The following conditions on a finite-dimensional Lie algebra* $\mathfrak{g}$ *are equivalent.*

1. $\mathfrak{g}$ *has no nonzero solvable ideals;*

2. $\mathfrak{g}$ *is a direct sum of simple Lie algebras;*

3. *the Killing form* B *on* $\mathfrak{g}$*, defined by*

$$B(X, Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$$

for $X, Y \in \mathfrak{g}$*, is nondegenerate.*

If these conditions hold, $\mathfrak{g}$ *is called* semisimple*.*

Proof. See Theorem 1.42 and 1.51 in [13]. $\square$

A Lie algebra is reductive if and only if it is the direct sum of an abelian and a semisimple Lie algebra. (In particular, semisimple Lie algebras are reductive.)

The group G is called reductive or semisimple if $\mathfrak{g}$ has the corresponding property.

2.2 Admissible and tempered representations

Let π be a continuous representation of G in a Hilbert space $\mathcal{H}$. Let $(-,-)_{\mathcal{H}}$ be the inner product on $\mathcal{H}$. A vector $v \in \mathcal{H}$ is K-*finite* if $\pi(K)v$ spans a finite-dimensional linear subspace of $\mathcal{H}$. A K-*finite matrix coefficient* of π is a function on G of the form

$$g \mapsto (v, \pi(g)w)_{\mathcal{H}},$$

for K-finite vectors $v, w \in \mathcal{H}$.

Definition 2.2. The representation π is

- *admissible* if the restriction $\pi|_K$ is unitary, and decomposes into irreducible representations of K with finite multiplicities;
- if π is admissible, it is *tempered* if all its K-finite matrix coefficients are in $L^{2+\varepsilon}(G)$, for all $\varepsilon > 0$.

If one studies *unitary* irreducible representations for the class of groups we consider, one only needs to consider the addmissible ones.

Theorem 2.3. *If G is linear, connected and reductive, then all unitary irreducible representations of G are admissible.*

Proof. See Theorem 8.1 in [12]. □

2.3 Discrete series representations

Suppose G is linear, connected and reductive. We consider an irreducible representation π of G in a Hilbert space $\mathcal{H}$.

Definition 2.4. The representation π belongs to the *discrete series* of G if all its matrix coefficients are in $L^2(G)$.

Proposition 2.5. *An irreducible unitary representation belongs to the discrete series if and only if it is equivalent to a closed subspace (i.e. a direct summand) of the right regular representation of G in $L^2(G)$.*

Proof. See Theorem 8.51(b) in [12]. If the matrix coefficients of a representation π are in $L^2(G)$, an equivariant isometric embedding $B : \mathcal{H} \to L^2(G)$ can be defined a follows. Fix a nonzero $v_0 \in \mathcal{H}$, and define the map B by

$$\big(B(v)\big)(g) = (\pi(g)v, v_0)_{\mathcal{H}},$$

for $v \in \mathcal{H}$ and $g \in G$. □

Let $\hat{G}$ be the unitary dual of G, i.e. the set of all unitary irreducible representations of G.

The *Plancherel theorem* states that there exists a measure μ on $\hat{G}$, called the *Plancherel measure*, such that, viewed as a representation of $G \times G$ by the left and right regular representations, one has the direct integral decomposition

$$L^2(G) \cong \int_{\hat{G}}^{\oplus} \mathcal{H}_\pi \hat{\otimes} \mathcal{H}_\pi^* \, d\mu(\pi).$$

Proposition 2.5 implies that the discrete series representations are exactly those with positive Plancherel measure. If G is semisimple, the Plancherel measure is supported precisely on the tempered representations.

2.4 Cartan subalgebras and subgroups

Let $\mathfrak{g}$ be a complex Lie algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a nilpotent complex subalgebra. For $\alpha \in \mathfrak{h}^*$, set

(2.1) $\mathfrak{g}_\alpha :=$

$$\{X \in \mathfrak{g}; \text{for all } Y \in \mathfrak{h} \text{ there is an } n \in \mathbb{N} \text{ such that } (\operatorname{ad}(Y) - \alpha(Y))^n X = 0\}.$$

Then one has the decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^* \text{ s.t. } \mathfrak{g}_\alpha \neq 0} \mathfrak{g}_\alpha,$$

and since $\mathfrak{h}$ is nilpotent, $\mathfrak{h} \subset \mathfrak{g}_0$. (See Proposition 2.5 in [13].)

Definition 2.6. The subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a *Cartan subalgebra* if $\mathfrak{h} = \mathfrak{g}_0$. Then the *roots* of $(\mathfrak{g}, \mathfrak{h})$ are the nonzero $\alpha \in \mathfrak{h}^*$ for which $\mathfrak{g}_\alpha \neq 0$. The *root space* associated to a root α is the space $\mathfrak{g}_\alpha$. The *Weyl group* associated to these roots is the subgroup of the orthogonal group of the real span of the roots generated by the reflections in the orthogonal complements of the roots, with respect to some inner product.

Cartan subalgebras of complex Lie algebras are unique up to conjugation.

Theorem 2.7. *If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are Cartan subalgebras of a complex Lie algebra, then there is a $a \in \operatorname{Int}(\mathfrak{g})$, the analytic subgroup of $\operatorname{Aut}_{\mathbb{R}}(\mathfrak{g})$ with Lie algebra $\operatorname{ad}(\mathfrak{g})$, such that*

$$\mathfrak{h}_2 = a(\mathfrak{h}_1).$$

Proof. See Theorem 2.15 in [13]. □

For semisimple Lie algebras, Cartan subalgebras and the associated root spaces have additional properties.

Theorem 2.8. *If* $\mathfrak{g}$ *is a complex semisimple Lie algebra, then*

- *all Cartan subalgebras are abelian;*
- *a subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ *is a Cartan subalgebra if and only if* $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ *diagonalises simultaneously;*
- *all root spaces are one-dimensional, and one may take* $n = 1$ *in* (2.1).

Proof. See Proposition 2.10, Corollary 2.13 and Proposition 2.21 in [13]. □

Definition 2.9. If $\mathfrak{g}$ is a *real* Lie algebra, then a *Cartan subalgebra* of $\mathfrak{g}$ is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ whose complexification $\mathfrak{h}_{\mathbb{C}}$ is Cartan subalgebra of the complexification $\mathfrak{g}_{\mathbb{C}}$. For a reductive group G, the *Cartan subgroup* associated to a Cartan subalgebra $\mathfrak{h}$ of its Lie algbera is the centraliser of $\mathfrak{h}$ in G.

Note that not all Cartan subalgebras of a real Lie algebra need to be conjugate in $\mathfrak{g}$; only their complexifications are conjugate in $\mathfrak{g}_{\mathbb{C}}$. This does imply that all Cartan subalgebras have the same dimension. This dimension is the *rank* of $\mathfrak{g}$.

3 Infintesimal characters

Let G be a linear reductive Lie group, with lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_{\mathbb{C}}$ be its complexification, and $\mathfrak{h}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}$ a Cartan subalgebra. Let $U(\mathfrak{g}_{\mathbb{C}})$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$, and let $Z(\mathfrak{g}_{\mathbb{C}}) \subset U(\mathfrak{g}_{\mathbb{C}})$ be its centre.

If π is an irreducible, admissible representation of G in a Hilbert space $\mathcal{H}$, then the action of every element $Z \in Z(\mathfrak{g}_{\mathbb{C}})$ on $\mathcal{H}$ commutes with the representation, and is hence given by a scalar $\chi_\pi(Z)$ (by Schur's lemma). This way, one gets a homomorphism $\chi_\pi : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$. Such homomorphisms can be classified, which provides information about classifying repreentations of G. This classification involves the *Harish–Chandra homomorphism*.

3.1 The Harish–Chandra homomorphism

Let R be the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, and let $R^+ \subset R$ be a choice of positive roots. Consider the root space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\mathbb{C},\alpha}.$$

Let $E_\alpha \in \mathfrak{g}_{\mathbb{C},\alpha}$ be nonzero, and set

$$\mathcal{P} := \bigoplus_{\alpha \in R^+} U(\mathfrak{g}_{\mathbb{C}}) E_\alpha.$$

Lemma 3.1. *One has* $U(\mathfrak{h}_{\mathbb{C}}) \cap \mathcal{P} = \{0\}$, *and* $Z(\mathfrak{g}_{\mathbb{C}}) \subset U(\mathfrak{h}_{\mathbb{C}}) \oplus \mathcal{P}$.

Proof. See Lemma 8.17 in [12]. □

Let

$$p : Z(\mathfrak{g}_{\mathbb{C}}) \to U(\mathfrak{h}_{\mathbb{C}})$$

be the projection according to the decomposition in Lemma 3.1. Let ρ be half the sum of the positive roots in R^+, and let

$$\sigma : \mathfrak{h}_{\mathbb{C}} \to U(\mathfrak{h}_{\mathbb{C}})$$

be given by $\sigma(X) = X - \rho(X)$. By the universal property of the universal enveloping algebra (Proposition 3.1 in [12]), the homomorphism σ extends to an algebra endomorphism of $U(\mathfrak{h}_{\mathbb{C}})$, which we still denote by σ.

Definition 3.2. The *Harish–Chandra homomorphism* is the map

$$\gamma := \sigma \circ p : Z(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{p} U(\mathfrak{h}_{\mathbb{C}}) \xrightarrow{\sigma} U(\mathfrak{h}_{\mathbb{C}}).$$

Theorem 3.3. *The Harish–Chandra homomorphism is an algebra isomorphism from* $Z(\mathfrak{g}_{\mathbb{C}})$ *onto the algebra* $U(\mathfrak{h}_{\mathbb{C}})^W$ *of Weyl group-invariant elements of* $U(\mathfrak{h}_{\mathbb{C}})$.

Proof. See Theorem 8.18 in [12]. □

3.2 The infinitesimal character of an irreducible, admissible representation

Keeping the notation from the start of this section, let $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$. Then λ is an algebra homomorphism from $\mathfrak{h}_{\mathbb{C}}$ to $\mathbb{C}$, which by the universal property of the universal enevloping algebra extends to an algebra homomorphism

$$\lambda : U(\mathfrak{h}_{\mathbb{C}}) \to \mathbb{C}.$$

Definition 3.4. The *infinitesimal character* $\chi_\lambda : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ is defined as

$$\chi_\lambda := \lambda \circ \gamma : Z(\mathfrak{g}_{\mathbb{C}}) \xrightarrow{\gamma} U(\mathfrak{h}_{\mathbb{C}}) \xrightarrow{\lambda} \mathbb{C}.$$

Theorem 3.5. *Every algebra homomorphism* $\chi : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ *is of the form* $\chi = \chi_\lambda$, *for a* $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$. *Two such homomorphisms* χ_λ *and* $\chi_{\lambda'}$ *are equal if and only if* $\lambda' = w\lambda$ *for a Weyl group element* w.

Proof. See Propositions 8.20 and 8.21 in [12]. □

In particular, the homomorphism $\chi_\pi : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ associated to the irreducible, admissible representation π is of the form

$$\chi_\pi = \chi_\lambda$$

for a $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$, determined up to the action of the Weyl group.

Definition 3.6. In this setting, λ or χ_λ is called the *infinitesimal character* of π.

4 Global characters

Let G be a linear, connected, reductive Lie group.

4.1 Existence of global characters

Let π be an admissible representation of G in a Hilbert space $\mathcal{H}$.

Definition 4.1. A distribution $\Theta \in \mathcal{D}'(G)$ on G is the *global character* of π if for all $f \in C_c^\infty(G)$, the operator

$$\pi(f) := \int_G f(g)\pi(g)\, dg$$

is trace class, and

$$\mathrm{tr}(\pi(f)) = \Theta(f).$$

Theorem 4.2. *Every unitary irreducible representation and every admissible irreducible representation of* G *has a global character.*

Proof. See Theorem 10.2 in [12]. □

4.2 Infinitesimal equivalence

We will see that representation with the same global character are *infinitesimally equivalent*. Let us introduce this type of equivalence.

There is a notion of smooth maps from manifolds to the Hilbert space $\mathcal{H}$. A vector $v \in \mathcal{H}$ is a *smooth vector* if the map $G \to \mathcal{H}$ given by $g \mapsto g \cdot v$ is smooth.

Proposition 4.3. *For an admissible representation, every* K*-finite vector is smooth.*

Proof. See Proposition 8.5 in [12]. □

Hence every admissible representation of G gives a representation of $\mathfrak{g}$ on the space of K-finite vectors.

Definition 4.4. Two admissible representations of G are *infinitesimally equivalent* if the corresponding representations of $\mathfrak{g}$ on the spaces of K-finite vectors are algebraically equivalent.

Here algebraic equivalence means that there is a linear isomorphism betwene the two spaces intertwining the representations. This isomorphism is not required to be bounded, for example.

Theorem 4.5. *Two irreducible unitary representations of* G *that are infinitesimally equivalent, are unitarily equivalent.*

Proof. See Corollary 9.2 in [12]. □

Theorem 4.6. *Consider two admissible representations of* G *with global characters. Then these characters are equal if and only if the representations are infinitesimally equivalent.*

Proof. See Proposition 10.5 and Theorem 10.6 in [12]. $\square$

Combining Theorems 4.2, 4.5 and 4.6, we see that irreducible unitary representations of linear, connected, reductive Lie groups have global characters, which determine them up to unitary equivalence.

4.3 Regularity of global characters

By conjugation invariance of the trace, a global character Θ of an admissible representation is a conjugation-invariant distribution. The universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of G acts on distributions by differential operators. An additional property of Θ is that the centre $Z(\mathfrak{g}_{\mathbb{C}})$ of $U(\mathfrak{g}_{\mathbb{C}})$ acts on it by scalars.

Lemma 4.7. *Suppose* π *is irreducible and admissible. Let* $\chi_\pi : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ *be its infinitesimal character. Then for all* $Z \in Z(\mathfrak{g}_{\mathbb{C}})$*,*

$$Z\Theta = \chi_\pi(Z)\Theta$$

Proof. See Proposition 10.24 in [12]. $\square$

The two properties of global characters of irreducible, admissible representations just mentioned, are important enough to put in a definition.

Definition 4.8. A distribution on G which is conjugation invariant, and on which $Z(\mathfrak{g}_{\mathbb{C}})$ acts by scalars, is called an *invariant eigendistribution*.

Suppose G is linear, connected and semisimple. Any invariant eigendistribution on G is given by an analytic function on the *regular set* of G. To define the regular set of a semisimple group G, note that all Cartan subalgebras of $\mathfrak{g}$ are abelian (see Theorem 2.8). Hence for every element $g \in G$, the map $\mathrm{Ad}(g)$ is the identity on the Lie algebra of the Cartan subgroup containing g. Therefore, the dimension of the kernel of $\mathrm{Ad}(g) - I_{\mathfrak{g}}$ is at least equal to the rank of G.

Definition 4.9. The *regular set* in G is the set

$$G^{\mathrm{reg}} := \left\{ g \in G; \dim\big(\ker(\mathrm{Ad}(g) - I_{\mathfrak{g}})\big) = \mathrm{rank}(G) \right\}.$$

The regular set is open dense in G.

Theorem 4.10. *The restriction of an invariant eigendistribution on G to the regular set is given by an analytic function.*

Proof. See Theorem 10.25 in [12]. $\square$

A priori, it is possible that an invariant eigendistribution has contributions outside the regular set that mean it is not given by a function on all of G. It is a very deep theorem by Harish–Chandra that this is not the case.

Theorem 4.11. *Any invariant eigendistribution on G is given by a locally integrable function.*

This theorem is stated as Theorem 10.36 in [12], but a proof is omitted. The proof spans five papers by Harish–Chandra [4, 5, 6, 7, 8].

By Theorem 4.10, the restriction of the locally integrable function of Theorem 4.11 to the regular set is analytic.

4.4 Computing global characters

By Theorem 4.10, one knows that on the regular set G^{reg}, the global character of an irreducible admissible representation is given by an analytic function. A general form for such a function is given in Theorem 4.14. The expression given there will be made explicit for discrete series representations in Subection 5.2.

Suppose G is linear, connected and semisimple. Let $H < G$ be a Cartan subgroup, with Lie algebra $\mathfrak{h}$.

The general expression for a global character on the regular set will involve the *Weyl denominator*. This function involves exponentials of *analytically integral* linear forms on $\mathfrak{h}_{\mathbb{C}}$.

Definition 4.12. An element $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ is *analytically integral*, if it maps the kernel of the exponential map of H into $2\pi i\mathbb{Z}$.

This condition is equivalent to the existence of a group homomorphism $\xi_\lambda : H \to \mathbb{C}^\times$ such that

$$\xi_\lambda(\exp(X)) = e^{\lambda(X)}$$

for all $x \in \mathfrak{h}$.

Let R be the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Let $R^+ \subset R$ be a choice of positive roots, and let ρ be half the sum of these positive roots. Then all roots are analytically integral, and the fact that the complexification $G_\mathbb{C}$ of G is simply connected implies that ρ is analytically integral as well. Thus, one has group homomorphisms ξ_α as above for all roots α and also ξ_ρ.

Definition 4.13. The *Weyl denominator* is the function D on H given by

$$D := \xi_\rho \prod_{\alpha \in R^+} (1 - \xi_\alpha^{-1}).$$

Now let π be an irreducible, admissible representation of G. Let χ_λ be its infinitesimal character, for $\lambda \in \mathfrak{h}\mathbb{C}^*$. Let Θ be its global character. By Lemma 4.7, this global character satisfies $Z\Theta = \chi_\lambda(Z)\Theta$ for all $Z \in Z(\mathfrak{g}_\mathbb{C})$, where χ_λ is the infinitesimal character of π.

Fix an element $h_1 \in H$. Let $\mathfrak{h}_1$ be a connected component of the set

$$\{X \in \mathfrak{h}; D(h_1 \exp(X)) \neq 0\}.$$

Set

$$(H^{\mathrm{reg}})^G := \{ghg^{-1}; g \in G, h \in H^{\mathrm{reg}}\},$$

Theorem 4.14. *There are unique polynomial functions p_w on $\mathfrak{h}$, for w in the Weyl group W, such that on $(H^{\mathrm{reg}})^G$, the global character Θ is given by the analytic function $\widetilde{\Theta}$ satisfying*

$$\widetilde{\Theta}(ghg^{-1}) = \frac{\tau(h)}{D(h)},$$

for $g \in G$ and $h \in H^{\mathrm{reg}}$, where for all $X \in \mathfrak{h}_1$,

$$\tau(h_1 \exp X) = \sum_{w \in W} p_w(X) e^{(w\lambda)(X)},$$

for every $s \in W$ stabilising λ and all $w \in W$, one has $p_{ws} = p_w$.

Proof. See Theorem 10.35 in [12]. They key point is that the function τ satisfies the differential equation

$$\gamma(Z)\tau = \chi_\lambda(Z)\tau,$$

for all $Z \in Z(\mathfrak{g}_\mathbb{C})$, where γ is the Harish–Chandra homomorphism. □

Remark 4.15. Every regular element $g \in G^{\mathrm{reg}}$ is in the set $(H^{\mathrm{reg}})^G$ for precisely one Cartan subgroup H, see Theorem 5.22(d) in [12].

Group G	Max. cpt. $K < G$	rank(G)	rank(K)	Discrete series?
$SL(n, \mathbb{C})$	$SU(n)$	$2n-2$	$n-1$	no
$SL(n, \mathbb{R})$	$SO(n)$	$n-1$	$\lfloor \frac{n}{2} \rfloor$	iff $n = 2$
$SL(n, \mathbb{H})$	$Sp^*(n)$	$2n-1$	n	no
$SU(p, q)$	$S(U(p) \times U(q))$	$p+q-1$	$p+q-1$	yes
$SO(n, \mathbb{C})$	$SO(n)$	$2\lfloor \frac{n}{2} \rfloor$	$\lfloor \frac{n}{2} \rfloor$	no
$SO(p, q)$	$S(O(p) \times O(q))$	$\lfloor \frac{p+q}{2} \rfloor$	$\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor$	iff pq even
$O^*(2n)$	$U(n)$	n	n	yes
$Sp(n, \mathbb{C})$	$Sp^*(n)$	$2n$	n	no
$Sp(n, \mathbb{R})$	$U(n)$	n	n	yes
$Sp^*(p, q)$	$Sp^*(p) \times Sp^*(q)$	$p+q$	$p+q$	yes

Table 1: Harish–Chandra's criterion $\mathrm{rank}(G) = \mathrm{rank}(K)$ for the existence of discrete series representations, for the non-exceptional real Lie groups

5 Classification and characters of discrete series representations

Let G be linear, connected and semisimple.

5.1 Classification of discrete series representations

One has the following explicit criterion for the existence of discrete series representations.

Theorem 5.1. *The group G has discrete series representations if and only if* $\mathrm{rank}(G) = \mathrm{rank}(K)$, *i.e. G has a compact Cartan subgroup.*

Proof. See Theorem 12.20 in [12]. □

For the non-exceptional simple real Lie groups, this criterion leads to Table 5.1, which was taken from [3].

Now suppose that there is a maximal torus $T < K$ which is a Cartan subgroup of G, so that G has discrete series representations. Let R be the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Let R_c denote the set of compact roots, i.e. those of $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and let $R_n := R \setminus R_c$ be the set of noncompact roots. Fix an element $\lambda \in i\mathfrak{t}^*$. Suppose λ is nonsingular, in the sense that $(\lambda, \alpha) \neq 0$ for all roots

$\alpha \in R$. Let R^+ be the set of positive roots defined by

$$R^+ := \{\alpha \in R; (\alpha, \lambda) > 0\}. \tag{5.1}$$

Let ρ be half the sum of the roots in R^+, and let ρ_c be half the sum of the positive compact roots in $R_c^+ := R^+ \cap R_c$.

Theorem 5.2. *If $\lambda + \rho$ is analytically integral, there is a discrete series representation π_λ of G such that*

1. *the infinitesimal character of π_λ is χ_λ;*
2. *if $\nu := \lambda + \rho - 2\rho_c$, and π_ν^K is the irreducible representation of K with highest weight ν, then the multiplicity of π_ν^K in $\pi_\lambda|_K$ is one;*
3. *if μ is the highest weight of an irreducible representation of K with nonzero multiplicity in $\pi_\lambda|_K$, then there are nonnegative integers n_α such that*

$$\mu = \nu + \sum_{\alpha \in R^+} n_\alpha \alpha.$$

Two such discrete series representations π_λ and $\pi_{\lambda'}$ are equivalent if and only if there is an element w of the Weyl group of R_c such that $\lambda' = w\lambda$.

Proof. See Theorem 9.20 in [12]. □

In the setting of Theorem 5.2, the element $\lambda \in i\mathfrak{t}^*$ is called the *Harish–Chandra parameter* of π_λ. The representation π_ν^K is the *lowest* K*-type* of π_λ, and ν is the *Blattner parameter* of π_λ.

Theorem 5.3. *Every discrete series representation of G equals one of the representations π_λ of Theorem 5.2.*

Proof. See Theorem 12.21 in [12]. □

Theorems 5.2 and 5.3 give a complete classification of the discrete series representations of G. Explicit realisations of these representations are given in Section 8.

5.2 A character formula

We still suppose that G is linear, connected and semisimple, and that $\operatorname{rank}(G) = \operatorname{rank}(K)$. As before, let $T < K$ be a maximal torus that is a Cartan subgroup of G. Let $\lambda \in i\mathfrak{t}^*$ be as in Theorem 5.2, and let Θ_λ be the global character of the discrete series representation π_λ. Let W_c be the Weyl group of the compact root system R_c. For *any* Cartan subgroup $H < G$, let $\widetilde{\Theta}_\lambda$ be the analytic function describing Θ_λ on $(H^{\mathrm{reg}})^G$, and write

$$\widetilde{\Theta}_\lambda(ghg^{-1}) = \frac{\tau_H(h)}{D(h)},$$

for $h \in H^{\mathrm{reg}}$ and $g \in G$, as in Theorem 4.14.

The general expression for Θ_λ given in Theorem 4.14 can now be made more explicit.

Theorem 5.4. *The global character Θ_λ of π_λ has the following properties.*

1. *On the compact Cartan subgroup T, one has*

$$\tau_T = (-1)^{\frac{1}{2}\dim(G/K)} \sum_{w \in W_c} \det(w)\xi_{w\lambda}.$$

2. *On every Cartan subgroup H, the function τ_H is bounded.*

Furthermore, Θ_λ is the only invariant eigendistribution with these properties and the additional one that for all $Z \in Z(\mathfrak{g}_{\mathbb{C}})$, one has $Z\Theta_\lambda = \chi_\lambda(Z)\Theta_\lambda$. (It has this last property by Lemma 4.7 and the first part of Theorem 5.2.)

Proof. See Theorem 12.7 in [12]. □

By Theorem 12.6 in [12], the fact that the function τ_H is bounded for every Cartan subgroup H implies that it is determined by its values on the compact Cartan subgroup T.

6 Example: $SL(2,\mathbb{R})$

For any $n \in \mathbb{N}$, consider the semisimple Lie group $G = SL(n,\mathbb{R})$. Then $K = SO(n)$ is a maximal subgroup of G. Write $n = 2k$ if n is even, and $n = 2k+1$ if n is odd. Then a maximal torus in $SO(n)$ is isomorphic to

$$\underbrace{SO(2) \times \cdots \times SO(2)}_{k \text{ factors}}.$$

Hence K has rank k. A Cartan subalgebra of the complexified Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ is formed by the diagonal elements, and has complex dimension $n-1$. Hence $\mathrm{rank}(G) = n-1$. By Theorem 5.1, $SL(n, \mathbb{R})$ therefore has discrete series representations if and only if

- $n = 2k$ is even, and $k = n-1$; or
- $n = 2k+1$ is odd, and $k = n-1$.

In other words, $SL(n, \mathbb{R})$ has discrete series representations precisely if $n = 2$.

For the rest of this section, we consider the group $SL(2, \mathbb{R})$.

6.1 Cartan subgroups

The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ has two conjugacy classes of Cartan subalgebras. One is represented by $\mathfrak{t} = \mathbb{R}X$, where

$$X := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The other is represented by $\mathfrak{h} = \mathbb{R}Y$, where

$$Y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The corresponding Cartan subgroups are the compact group

$$T := SO(2),$$

and the noncompact group

$$A := \left\{ \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}; r > 0 \right\}.$$

Since we are going to construct discrete series representations of $SL(2, \mathbb{R})$, we focus on the compact Cartan subgroup T.

The corresponding root space decomposition is

$$\mathfrak{sl}(2, \mathbb{C}) = \mathbb{C}X \oplus \mathbb{C}E_{\alpha} + \mathbb{C}E_{-\alpha},$$

where
$$E_\alpha := \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}; \qquad E_{-\alpha} := \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$
One can compute that
$$[X, E_{\pm\alpha}] = \pm 2iE_{\pm\alpha}.$$
Hence the root system of $(\mathfrak{sl}(2,\mathbb{C}), \mathfrak{t}_{\mathbb{C}})$ is $\{\pm\alpha\}$, with α determined by
$$\alpha(X) = 2i.$$
There are no compact roots, i.e. $R_c = \emptyset$.

6.2 Discrete series representations

Let a nonzero element $\lambda \in i\mathfrak{t}^*$ be given. Write $\lambda = l\alpha$, for an $l \in \mathbb{R}$. The choice of positve roots determined by λ is $R^+ = \{\alpha\}$ if $l > 0$, and $R^+ = \{-\alpha\}$ if $l < 0$. Hence
$$\rho = \mathrm{sign}(l)\frac{1}{2}\alpha; \quad \rho_c = 0.$$
For any $a \in \mathbb{R}$, one has
$$\exp(aX) = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}.$$
Hence $\ker\exp = 2\pi\mathbb{Z}X$. Since $\rho(2\pi X) = \mathrm{sign}(l)2\pi i$, we see that ρ is analytically integral. Hence $\lambda + \rho$ is analytically integral if and only if λ is, which is the case precisely if $\lambda(2\pi X) = 4\pi i l \in 2\pi i\mathbb{Z}$, i.e. if
$$\lambda = \lambda_n := \frac{n}{2}\alpha,$$
for a nonzero integer n. The discrete series representations of $SL(2,\mathbb{R})$ are precisely the representations π_λ given in Theorem 5.2, for these values of λ. Write $\pi_n := \pi_{\lambda_n}$. No two of these are equivalent, since the Weyl group of the compact roots is trivial.

6.3 Characters

For every nonzero integer n, let let Θ_n be the global character of the discrete series representation π_n. In this example, we have

- $\dim(G/K) = 2$;
- $W_c = \{e\}$;
- $T^{\mathrm{reg}} = T \setminus \{I\}$.

Hence Theorem 5.4 implies that for all $a \notin 2\pi\mathbb{Z}$,

$$\begin{aligned}
\Theta_n(\exp(aX)) &= (-1)^{\frac{1}{2}\dim(G/K)} \frac{\sum_{w \in W_c} \det(w) \xi_{w\lambda_n}(\exp(aX))}{\xi_\rho(\exp(aX)) \prod_{\alpha \in R^+}(1 - \xi_\alpha(\exp(aX))^{-1})} \\
&= -\frac{e^{\lambda_n(aX)}}{e^{\rho(aX)}\big(1 - e^{-\alpha(aX)}\big)} \\
&= -\frac{e^{ina}}{e^{\operatorname{sign}(n)ia}\big(1 - e^{-2\operatorname{sign}(n)ia}\big)} \\
&= -\operatorname{sign}(n)\frac{e^{ina}}{e^{ia} - e^{-ia}}.
\end{aligned}$$

(**To do**: there is a shift $n \mapsto n-1$ compared to Proposition 10.14 in [12]?)

6.4 Explicit realisations

Let n be a positive integer. Let $\mathbb{H} \subset \mathbb{C}$ be the upper half plane. For functions f_1, f_2 on $\mathbb{H}$ for which the integral converges, set

$$(f_1, f_2)_n := \int_{\mathbb{H}} f_1(x+iy)\overline{f_2(x+iy)}y^{n-1}\,dx\,dy. \tag{6.1}$$

Let $\|\cdot\|_n$ be the associated norm. Consider the Hilbert space

$$\mathcal{H}_n := \big\{f : \mathbb{H} \to \mathbb{C} \text{ analytic}; \|f\|_n < \infty\big\},$$

equipped with the inner product defined by (6.1). Consider the action by $SL(2,\mathbb{R})$ on $\mathcal{H}_n$ defined as follows. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, $f \in \mathcal{H}_n$, and $x + iy \in \mathbb{H}$, set

$$(g \cdot f)(z) = (-bz+d)^{-n+1} f\left(\frac{az-c}{-bz+d}\right). \tag{6.2}$$

It is shown in Subsection II.5 of [12] that this representation is unitary and irreducible. In Proposition 10.14 in [12], it is shown that the global character of this representation is Θ_n (on the compact Cartan subgroup T, hence everywhere by the comment after Theorem 5.4.) Hence this realises the discrete series representation π_n.

For *negative* integers n, let

$$\mathcal{H}_n := \{\bar{f}; f \in \mathcal{H}_{-n}\},$$

equipped with the same inner product as $\mathcal{H}_{-n}$. The action by $SL(2, \mathbb{R})$ on $\mathcal{H}_n$ given by (6.2) is again irreducible and unitary. In Proposition 10.14 in [12], it is shown that the global character of this representation is Θ_n.

7 Relevance to representation theory

Suppose G is linear, connected and semisimple[1] The relevance of discrete series representations is that (almost) every irreducible tempered representation of G can be obtained using induction from parabolic subgroups $S < G$ of relatively simple classes of representations parametrised by discrete series representations of a reductive subgroup $M < S$. In this section, we will make this statement precise.

7.1 The Cartan decomposition

Proposition 7.1. *There is a* Cartan involution θ *of* $\mathfrak{g}$ *such that the bilinear form*

$$-B(-, \theta -)$$

on $\mathfrak{g}$ *is positive definite, where* B *is the Killing form. All Cartan involutions are conjugate via the adjoint representation.*

Proof. See Corollaries 6.18 and 6.19 in [13]. □

Fix a Cartan involution θ.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into the $+1$ and -1 eigenspaces of θ. The is the *Cartan decomposition* of $\mathfrak{g}$. The potential clash of notation

[1]Much of the material in this section is true for more general reductive groups, such as groups in the Harish–Chandra class (which contains the linear, connected semisimple ones).

with the Lie algebra $\mathfrak{k}$ of a maximal compact subgroup is resolved by the following result.

Theorem 7.2. *Let* K *be the analytic subgroup of* G *with Lie algbera* $\mathfrak{k}$*. Then* K *is a maximal compact subgroup of* G*, and the map* $K \times \mathfrak{p} \to G$ *given by*

$$(k, X) \mapsto k \exp(X)$$

for $k \in K$ *and* $X \in \mathfrak{p}$*, is a diffeomorphism onto* G*.*

Proof. See Theorem 6.31(c),(g) in [13]. (The centre of G is finite since G is linear.) □

We will use the maximal compact subgroup K associated to the Cartan involution θ from now on.

7.2 The Iwasawa decomposition

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra. For $\alpha \in \mathfrak{a}^*$, write

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g}; \forall Y \in \mathfrak{a}, [Y, X] = \alpha(Y)X\}.$$

If $\mathfrak{g}_\alpha \neq \{0\}$ and $\alpha \neq 0$, then α is called a *restricted root* of $(\mathfrak{g}, \mathfrak{a})$. Let Σ be the set of restricted roots. Write $\mathfrak{m} := Z_{\mathfrak{k}}(\mathfrak{a})$.

Proposition 7.3. *One has the decomposition*

$$\mathfrak{g} = (\mathfrak{m} \oplus \mathfrak{a}) \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha.$$

Proof. See Proposition 6.40(a) in [13]. □

Example 7.4. (This is Example 1 on p. 313 of [13].) Let $G = SL(n, \mathbb{R})$ or $G = SL(n, \mathbb{C})$. Then one kan take $\mathfrak{k}$ to be the subalgebra of anti-Hermitian matrices, and $\mathfrak{p}$ the subspace of Hermitian matrices. The space $\mathfrak{a}$ of real diagonal matrices with trace zero is a maximal abelian subspace of $\mathfrak{p}$. For $j = 1, \ldots, n$, let $f_j \in \mathfrak{a}^*$ be evaluation at the j'th diagonal element. Then the restricted roots are

$$\Sigma = \{f_j - f_k; j \neq k\}.$$

The restricted root space $\mathfrak{g}_{f_j - f_k}$ is the space of matrices with the only nonzero entry in place (j, k). The real dimension of $\mathfrak{g}_{f_j - f_k}$ is 1 for $SL(n, \mathbb{R})$ and 2 for $SL(n, \mathbb{C})$.

The subalgebra $\mathfrak{m}$ consists of all anti-Hermitian diagonal matrices. This is zero for $SL(2, \mathbb{R})$, and the algebra of imaginary diagonal matrices with trace zero for $SL(2, \mathbb{C})$

Now fix a set of positive restricted roots $\Sigma^+ \subset \Sigma$ and write

$$\mathfrak{n} := \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha.$$

Then the Iwasawa decomposition of $\mathfrak{g}$ is the following statement.

Theorem 7.5. *One has*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

Proof. See Proposition 6.43 in [13]. □

At the group level, one has the following decomposition.

Theorem 7.6. *Let A and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Then the multiplication map*

$$K \times A \times N \to G$$

is a diffeomorphism onto G.

Proof. See Theorem 6.46 in [13]. □

As an aside, we mention that an Iwasawa decomposition of $\mathfrak{g}$ allows one to find an explicit Cartan subalgebra.

Theorem 7.7. *If $\mathfrak{t} \subset \mathfrak{m}$ is a maximal abelian subalgebra, then $\mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. For this Cartan subalgebra, all roots are real on $\mathfrak{a}$ and imaginary on $\mathfrak{t}$.*

Proof. See Proposition 6.47 and Corollary 6.49 in [13]. □

7.3 Parabolic subalgebras

We change notation now, and write $\mathfrak{m}_0$, $\mathfrak{a}_0$ and $\mathfrak{n}_0$ for the subalgebras $\mathfrak{m}, \mathfrak{a}, \mathfrak{n} \subset \mathfrak{g}$ of Subsection 7.2.

Definition 7.8. A *parabolic subalgebra* of $\mathfrak{g}$ is a subalgebra containing a conjugate of $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$.

We will focus on parabolic subalgebras containing $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$; all others can be obtained via conjugation.

To classify the parabolic subalgebras of $\mathfrak{g}$, let $\Sigma_0 \subset \Sigma^+$ be a set of simple restricted roots. For a subset $\Delta \subset \Sigma_0$, set

$$\Sigma_\Delta := \Sigma^+ \cup \{\alpha \in \Sigma; \alpha \in \operatorname{span}(\Delta)\}.$$

Then

$$\mathfrak{s}_\Delta := \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \bigoplus_{\alpha \in \Sigma_\Delta} \mathfrak{g}_\alpha$$

is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$.

Proposition 7.9. *All parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ are of the form $\mathfrak{s}_\Delta$ as above.*

Proof. See Proposition 7.76 in [13]. □

Every parabolic subalgebra admits a decomposition called the *Langlands decomposition*. Let $\mathfrak{s} = \mathfrak{s}_\Delta \subset \mathfrak{g}$ be a parabolic subalgebra as in Proposition 7.9. Set

- $\mathfrak{a} := \bigcap_{\alpha \in \Sigma_\Delta \cap -\Sigma_\Delta} \ker \alpha \subset \mathfrak{a}_0$;
- $\mathfrak{a}_M := \mathfrak{a}^\perp \subset \mathfrak{a}_0$;
- $\mathfrak{m} := \mathfrak{a}_M \oplus \mathfrak{m}_0 \oplus \bigoplus_{\alpha \in \Sigma_\Delta \cap -\Sigma_\Delta} \mathfrak{g}_\alpha$;
- $\mathfrak{n} := \bigoplus_{\alpha \in \Sigma_\Delta \setminus (-\Sigma_\Delta)} \mathfrak{g}_\alpha$.

Theorem 7.10. *The subspaces $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$ of $\mathfrak{g}$ have the following properties.*

1. *$\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$ are Lie subalgebras of $\mathfrak{s}$; $\mathfrak{n}$ is an ideal.*
2. *$\mathfrak{s}$ decomposes as*
$$\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$
3. *$\mathfrak{a}$ is abelian, $\mathfrak{n}$ is nilpotent.*
4. *$\mathfrak{m} \oplus \mathfrak{a}$ is the centraliser of $\mathfrak{a}$ in $\mathfrak{g}$.*

Proof. See Proposition 7.78 in [13]. □

The decomposition $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the *Langlands decomposition* of $\mathfrak{s}$.

Example 7.11. (This is Example 1 on p. 413 of [13].) Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ or $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Then a minimal parabolic subalgebra $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ is the subalgebra of upper-triangular matrices. The other parabolic subalgebras are the block-upper triangular subalgebras.

7.4 Parabolic subgroups

Fix a parabolic subalgebra $\mathfrak{s}$ containing $\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$, and let $\mathfrak{s} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be the Langlands decomposition of $\mathfrak{s}$. Let A and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively. Let M_0 be the analytic subgroup of G with Lie algebra $\mathfrak{m}$, and set[2] $M := Z_K(\mathfrak{a})M_0$. Set

$$S := MAN.$$

Theorem 7.12. *1. The subgroup M is reductive, and has Lie algebra $\mathfrak{m}$.*

2. The set S equals

$$S = N_G(\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n})$$

and is hence a closed subgroup of G. Its Lie algebra is $\mathfrak{s}$.

3. The multiplication map

$$M \times A \times N \to S$$

is a diffeomorphism onto S.

Proof. See Propositions 7.82(a) and 7.83(b),(c),(d) in [13]. □

Definition 7.13. The closed subgroup $S = MAN < G$ is the *parabolic subgroup* associated to the parabolic subalgebra $\mathfrak{s}$.

Example 7.14. (This is the example on p. 421 of [13].) Let $G = SL(3, \mathbb{R})$. Let $\mathfrak{a}_0 < \mathfrak{g}$ be the diagonal subalgebra. In the notation of Exampe 7.4, consider the set of positive restricted roots

$$\Sigma^+ := \{f_1 - f_2, f_2 - f_3, f_1 - f_3\}.$$

If one takes $\Delta := \{f_1 - f_2\}$, then the associated parabolic subgroup S is the subgroup of block-upper triangular matrices with block sizes 2 and 1. Now $\mathfrak{a}$ consists of the diagonal matrices with diagonal entries $(r, r, -2r)$, for real r.

[2]Another definition of M is used in [13], this one is from [12], p. 133.

7.5 Induced representations

Let $S = MAN < G$ be a parabolic subgroup of G. Let

$$\sigma : M \to U(\mathcal{H})$$

be a unitary irreducible representation. Let $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. Write

$$\rho_N := \frac{1}{2} \sum_{\alpha \in \Sigma_\Delta \setminus (-\Sigma_\Delta)} \alpha.$$

Definition 7.15. The (normalised or unitary) *induced representation* $\mathrm{Ind}_S^G(\sigma \otimes e^\nu \otimes 1)$ is the representation of G defined as follows. Consider the space of continuous functions $f : G \to \mathcal{H}$ such that for all $g \in G$, $m \in M$, $X \in \mathfrak{a}$ and $n \in N$,

$$f(gm \exp(X)n) = e^{-(\nu+\rho_N)(X)} \sigma(m)^{-1} f(g).$$

Let $\mathcal{H}_{S,\sigma,\nu}$ be the completion of this space in the norm defined by

$$\|f\| := \|f|_K\|_{L^2(K)}.$$

Then $\mathrm{Ind}_S^G(\sigma \otimes e^\nu \otimes 1)$ is the representation of G on $\mathcal{H}_{S,\sigma,\nu}$ given by

$$(g \cdot f)(g') = f(g^{-1}g'),$$

for $g, g' \in G$.

7.6 The classification of tempered representations

The statement about relevance of discrete series representations to the general representation theory of G made at the start of this section can now be made more precise. A parabolic subgroup $S = MAN < G$ is called *cuspidal* if M has discrete series representations.

Theorem 7.16. *Every irreducible tempered representation* π *of* G *can be obtained by induction from a cuspidal parabolic subgroup* $S = MAN < G$ *as*

$$\pi = \mathrm{Ind}_S^G(\sigma \otimes e^{i\nu} \otimes 1),$$

where $\nu \in \mathfrak{a}^*$*, and* σ *is a discrete series representation of* M*, or a* limit of discrete series representations *of* M.

Proof. See Theorem 14.76 in [12]. □

Knapp and Zuckerman also determined which P, σ and ν occur in Theorem 7.16, completing the classification of tempered representations. See Theorem 14.2 in [15].

The definition of limits of the discrete series is given in Section XII.7 of [12]. However, even if one needs a limit of the discrete series in the setting of Theorem 7.16, then π is still *contained* in a representation induced from a discrete series representation of M.

Theorem 7.17. *If $S = MAN$ is a cuspidal parabolic subgroup, $\nu \in i\mathfrak{a}^*$, and σ is a limit of discrete series representations of M, then there are a parabolic subgroup $S' = M'A'N' \subset G$, a discrete series representation σ' of M' and a $\nu' \in i\mathfrak{a}'^*$, such that $\pi := \mathrm{Ind}_S^G(\sigma \otimes e^{i\nu} \otimes 1)$ is contained in $\pi' := \mathrm{Ind}_{S'}^G(\sigma' \otimes e^{\nu'} \otimes 1)$, in the sense that the global character of π equals the sum of the global character of π' plus another global character.*

Proof. See Corollary 14.72 in [12]. □

Combining Theorems 7.16 and 7.17, see also Corollary 8.8 in [14], we obtain the following result.

Corollary 7.18. *Every tempered representation of G is contained in a representation of the form $\mathrm{Ind}_S^G(\sigma \otimes e^{i\nu} \otimes 1_N)$ for a cuspidal parabolic subgroup $S = MAN < G$, a discrete series representation σ of M and $\nu \in \mathfrak{a}^*$.*

8 Explicit realisations of the discrete series

The constructions of discrete series representations given in the proof of Theorem 5.2 in [12] are not very explicit. There are more concrete realisations, just like the Borel–Weil(–Bott) theorem gives explicit realisations of irreducible representations of compact groups.

Let G be linear, connected and semisimple, and let $K < G$ be maximal compact. Suppose that there is a maximal torus $T < K$ which is a Cartan subgroup of G, i.e. that G has discrete series representations.

8.1 Dolbeault cohomology

In [18], Schmid proved a conjecture of Langlands about realising discrete series representation in the L^2-Dolbeault cohomology of G/T. The invariant complex structures on this manifold correspond to choices of positive roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

Theorem 8.1. *For every choice of positive roots R^+ of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, there is precisely one G-invariant complex structure on G/T such that, under the identification*

$$T_{eT}(G/T)_{\mathbb{C}} = (\mathfrak{g}/\mathfrak{t})_{\mathbb{C}} = \bigoplus_{\alpha \in R} (\mathfrak{g}_{\mathbb{C}})_{\alpha},$$

the subspace $T^{0,1}_{eT}(G/T)$ corresponds to

$$\bigoplus_{\alpha \in R^+} (\mathfrak{g}_{\mathbb{C}})_{\alpha}.$$

Fix a set R^+ of positive roots and corresponding G-invariant complex structure on G/T. Let R^+_c and R^+_n be the sets of compact and noncompact positive roots, respectively.

Let $\lambda \in i\mathfrak{t}^*$, and suppose $\lambda + \rho$ is analytically integral. Then so is $\lambda - \rho$. Consider the line bundle

$$L_{\lambda-\rho} := G \times_T \mathbb{C}_{\lambda-\rho} \to G/T,$$

where T acts on $\mathbb{C}_{\lambda-\rho}$ via $\xi_{\lambda-\rho}$. It has the structure of a holomorphic G-line bundle.

Let $H^p(G/T; L_{\lambda-\rho})$ be the p'th L^2-Dolbeault cohomology group of G/T with coefficients in $L_{\lambda-\rho}$.

Theorem 8.2. *If λ is singular, then $H^p(G/T; L_{\lambda-\rho}) = 0$ for all p. If λ is nonsingular, set*[3]

$$k := \#\{\alpha \in R^+_c; (\lambda, \alpha) < 0\} + \#\{\alpha \in R^+_n; (\lambda, \alpha) > 0\}.$$

Then $H^p(G/T; L_{\lambda-\rho}) = 0$ if $p \neq k$, while the representation of G in $H^k(G/T; L_{\lambda-\rho})$ is irreducible, and is equivalent to the discrete series representation π_λ of Theorem 5.2.

Proof. See Theorem 1.5 in [18]. □

[3]Note that in this setting, the set R^+ of positive roots is fixed, whereas in Theorem 5.2 it depended on λ.

8.2 Dirac operators

Another realisation of discrete series representations was given by Parthasarathy [17] and Atiyah and Schmid [1]. They realised these representations in L^2-kernels of Spin-Dirac operators on G/K. Parthasarathy needed a condition that Atiyah and Schmid were able to omit. In addition, Atiyah and Schmid actually reproved the classification of discrete series representation given in Theorem 5.2 and the character formula in Theorem 5.4.

For a given irreducible representation V of K, the Dirac operator D^V used by Parthasarathy and Atiyah–Schmid is defined as follows. Consider the inner product on $\mathfrak{p}$ given by the restriction of the Killing form. The adjoint representation

$$\mathrm{Ad} : K \to GL(\mathfrak{p})$$

of K on $\mathfrak{p}$ takes values in $SO(\mathfrak{p})$, because the Killing form is $\mathrm{Ad}(K)$-invariant, and K is connected. We suppose that it has a lift $\widetilde{\mathrm{Ad}}$ to the double cover $\mathrm{Spin}(\mathfrak{p})$ of $SO(\mathfrak{p})$. It may be necessary to replace G and K by double covers for this lift to exist. Then the homogeneous space G/K has a G-equivariant Spin-structure

$$P^{G/K} := G \times_K \mathrm{Spin}(\mathfrak{p}) \to G/K.$$

Here $G \times_K \mathrm{Spin}(\mathfrak{p})$ is the quotient of $G \times \mathrm{Spin}(\mathfrak{p})$ by the action of K defined by

$$k(g, a) = (gk^{-1}, \widetilde{\mathrm{Ad}}(k)a),$$

for $k \in K$, $g \in G$ and $a \in \mathrm{Spin}(\mathfrak{p})$.

Set $d := \dim(\mathfrak{p}) = \dim(G/K)$. Note that d equals the number of noncompact roots, which is twice the number of positive noncompact roots, and hence even. Fix an orthonormal basis $\{X_1, \ldots, X_d\}$ of $\mathfrak{p}$. Using this basis, we identify $\mathrm{Spin}(\mathfrak{p}) \cong \mathrm{Spin}(d)$. Let Δ_d be the canonical $2^{\frac{d}{2}}$-dimensional representation of $\mathrm{Spin}(d)$. Because $\mathfrak{p}$ is even-dimensional, Δ_d splits into two irreducible subrepresentations Δ_d^+ and Δ_d^-. Consider the G-vector bundles

$$E_V^{\pm} := G \times_K (\Delta_d^{\pm} \otimes V) \to G/K.$$

Note that

$$\Gamma^\infty(G/K, E_V^{\pm}) \cong \left(C^\infty(G) \otimes \Delta_d^{\pm} \otimes V\right)^K, \tag{8.1}$$

where K acts on $C^\infty(G) \otimes \Delta_d^{\pm} \otimes V$ by

$$k \cdot (f \otimes \delta \otimes v) = (f \circ l_{k^{-1}} \otimes \widetilde{\mathrm{Ad}}(k)\delta \otimes k \cdot v) \tag{8.2}$$

for all $k \in K$, $f \in C^\infty(G)$, $\delta \in \Delta_d$ and $v \in V$. Here $l_{k^{-1}}$ denotes left multiplication by k^{-1}.

Using the basis $\{X_1, \ldots, X_d\}$ of $\mathfrak{p}$ and the isomorphism (8.1), define the differential operator

$$D^V : \Gamma^\infty(E_V^+) \to \Gamma^\infty(E_V^-) \tag{8.3}$$

by the formula

$$D^V := \sum_{j=1}^{d} X_j \otimes c(X_j) \otimes 1_V. \tag{8.4}$$

Here in the first factor, X_j is viewed as a left invariant vector field on G, and in the second factor, $c : \mathfrak{p} \to \mathrm{End}(\Delta_d)$ is the Clifford action. This action is odd with respect to the grading on Δ_d. The operator (8.3) is the Spin-Dirac operator on G/K (see e.g. [17], Proposition 1.1.

Let $\lambda \in i\mathfrak{t}^*$ and suppose $\lambda + \rho$ is analytically integral, ρ is half the sum of a choice[4] of positive roots having nonnegative inner products with λ. Let V be the irreducible representation of K with highest weight $\lambda - \rho_c$.

Theorem 8.3. *If λ is singular, then the L^2-kernel of D^V is zero. If λ is nonsingular, then the representation of G in the L^2-kernel of D^V is equivalent to the discrete series representation π_λ of Theorem 5.2.*

Proof. See Theorem 9.3 in [1]. □

9 K-theory of group C^*-algebras

This section is a modified version of Subsection 1.4 in [11].

For any locally compact topological group G, the *reduced group C^*-algebra* $C^*_r G$ of G is the completion of the convolution algebra $C_c(G)$ in the norm $\|\cdot\|_{C^*_r G}$, defined by

$$\|\varphi\|_{C^*_r G} := \|\varphi * -\|_{\mathcal{B}(L^2(G))},$$

the operator norm of convolution by $\varphi \in C_c(G)$. A class in the *even K-theory* $K_0(A)$ of a C^*-algebra A (e.g. $A = C^*_r G$) is defined by a projection matrix $p \in M_n(A)$ for some n. I.e. $p^2 = p$ and $p^* = p$.

[4]If λ is singular, this does not determine the positive root system uniquely.

In [16], V. Lafforgue reproves some classical results about discrete series representations by Harish-Chandra [9, 10], analogous to the results by Atiyah and Schmid [1] and Parthasarathy [17], using group C^*-algebras, K-homology, K-theory and the analytic assembly map that features in the Baum–Connes conjecture.

9.1 Dirac induction

Let V be an irreducible representation of K. Lafforgue (see also Wassermann [19]) uses the Dirac operator D^V defined in (8.4) to define a *Dirac induction* map

$$\text{D-Ind}_K^G : R(K) \to K_0(C_r^*(G)) \tag{9.1}$$

by

$$\text{D-Ind}_K^G[V] := \left[\left(C_r^*(G) \otimes \Delta_d \otimes V\right)^K, b\left(D^V\right)\right], \tag{9.2}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a normalising function, e.g. $b(x) = \frac{x}{\sqrt{1+x^2}}$. The expression on the right hand side defines a class in Kasparov's KK-group $KK_0(\mathbb{C}, C_r^*(G))$, which is isomorphic to the K-theory group $K_0(C_r^*(G))$. In [19], Wassermann proves the Connes–Kasparov conjecture, which states that this Dirac induction map is a bijection, for linear reductive groups. The case for general almost connected Lie groups is proved in [2].

9.2 Reduction at discrete series representations

The relation between the Dirac induction map and the work of Atiyah and Schmid and of Parthasarathy can be seen by embedding the discrete series of G into $K_0(C_r^*(G))$ via the map

$$\mathcal{H} \mapsto [\mathcal{H}] := [d_{\mathcal{H}} c_{\mathcal{H}}],$$

where $\mathcal{H}$ is a Hilbert space with inner product $(-,-)_{\mathcal{H}}$, equipped with a discrete series representation of G, $c_{\mathcal{H}} \in C(G)$ is the function

$$c_{\mathcal{H}}(g) = (v, g \cdot v)_{\mathcal{H}},$$

for a fixed $v \in \mathcal{H}$ of norm 1, and $d_{\mathcal{H}}$ is the inverse of the L^2-norm of $c_{\mathcal{H}}$ (so that the function $d_{\mathcal{H}}c_{\mathcal{H}}$ has L^2-norm 1). Because $d_{\mathcal{H}}c_{\mathcal{H}}$ is a projection in $C^*_r(G)$, it indeed defines a class in $K_0(C^*_r(G))$.

Next, Lafforgue defines a map[5]

$$R^{\mathcal{H}}_G : K_0(C^*_r(G)) \to \mathbb{Z} \tag{9.3}$$

that amounts to taking the multiplicity of the irreducible discrete series representation $\mathcal{H}$, as follows. Consider the map

$$C^*_r(G) \to \mathcal{K}(\mathcal{H})$$

(the C^*-algebra of compact operators on $\mathcal{H}$), given on $C_c(G) \subset C^*_r(G)$ by

$$f \mapsto \pi(f) := \int_G f(g)\,\pi(g)\,dg. \tag{9.4}$$

Here π is the representation of G in $\mathcal{H}$. For all $f \in C^\infty_c(G)$, the operator $\pi(f)$ is trace class, and hence compact, by Theorem 4.2. Since $K_0(\mathcal{K}(\mathcal{H})) \cong \mathbb{Z}$, this map induces a map $K_0(C^*_r(G)) \to \mathbb{Z}$ on K-theory, which by definition is (9.3).

The map $R^{\mathcal{H}}_G$ has the property that for all irreducible discrete series representations $\mathcal{H}$ and $\mathcal{H}'$ of G, one has

$$R^{\mathcal{H}}_G([\mathcal{H}']) = \begin{cases} 1 & \text{if } \mathcal{H} \cong \mathcal{H}' \\ 0 & \text{if } \mathcal{H} \not\cong \mathcal{H}'. \end{cases}$$

Hence it can indeed be interpreted as a multiplicity function. For compact groups, it follows from Schur orthogonality that this is indeed the usual multiplicity.

9.3 Reduction and Dirac induction

Dirac induction links the reduction map $R^{\mathcal{H}}_G$ to multiplicities of irreducible representation of K in the following way.

Let R be the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, let $R_c \subset R$ be the subset of compact roots, and let $R_n := R \setminus R_c$ be the set of noncompact roots. Let $R^+_c \subset R_c$ be a choice of positive compact roots, and let $\Lambda^{\mathfrak{k}}_+$ be the set of dominant integral weights of $(\mathfrak{k}, \mathfrak{t})$ with respect to R^+_c.

[5]In Laffforgues's notation, $R^{\mathcal{H}}_G(x) = \langle \mathcal{H}, x \rangle$.

Let $\mathcal{H}$ be a discrete series representation of G. Let λ be the Harish–Chandra parameter of $\mathcal{H}$ such that $(\alpha, \lambda) > 0$ for all $\alpha \in R_c^+$. Let $R^+ \subset R$ be the positive root system defined by (5.1). Then $R_c^+ \subset R^+$, and we denote by $R_n^+ := R^+ \setminus R_c^+$ the set of noncompact positive roots. We will write $\rho := \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ and $\rho_c := \frac{1}{2}\sum_{\alpha \in R_c^+} \alpha$. We will use the fact that $\lambda - \rho_c$ lies on the dominant weight lattice $\Lambda_+^{\mathfrak{k}}$, since $\lambda \in \Lambda_+^{\mathfrak{k}} + \rho$. As before, we set $d := \dim(G/K)$.

Lemma 9.1. *Let $\mu \in \Lambda_+^{\mathfrak{k}}$ be given. Let V_μ be the irreducible representation of K with highest weight μ. We have*

$$R_G^{\mathcal{H}}\big(\text{D-Ind}_K^G[V_\mu]\big) = \begin{cases} (-1)^{d/2} & \text{if } \mu = \lambda - \rho_c \\ 0 & \text{otherwise.} \end{cases} \tag{9.5}$$

Proof. According to Lafforgue [16], Lemma 2.1.1, we have

$$\begin{aligned} R_G^{\mathcal{H}}\big(\text{D-Ind}_K^G[V_\mu]\big) &= \dim\big(V_\mu^* \otimes \Delta_d^* \otimes \mathcal{H}\big)^K \\ &= \big[\Delta_d^* \otimes \mathcal{H}|_K : V_\mu\big], \end{aligned} \tag{9.6}$$

the multiplicity of V_μ in $\Delta_d^* \otimes \mathcal{H}|_K$. Let us compute this multiplicity.

By Theorem 5.4, the character Θ_λ of $\mathcal{H}$ satisfies

$$\Theta_\lambda|_{T^{\text{reg}}} = (-1)^{d/2}\frac{\sum_{w \in W_c} \det(w) e^{w\lambda}}{\prod_{\alpha \in R^+}\big(e^{\alpha/2} - e^{-\alpha/2}\big).}$$

The character χ_{Δ_d} of the representation

$$K \xrightarrow{\widetilde{\text{Ad}}} \text{Spin}(\mathfrak{p}) \to GL(\Delta_d), \tag{9.7}$$

on the other hand, is given by (Parthasarathy [17], Remark 2.2)

$$\chi_{\Delta_d}|_{T^{\text{reg}}} := \big(\chi_{\Delta_d^+} - \chi_{\Delta_d^-}\big)|_{T^{\text{reg}}} = \prod_{\alpha \in R_n^+}\big(e^{\alpha/2} - e^{-\alpha/2}\big).$$

It follows from this formula that for all $t \in T^{\text{reg}}$,

$$\chi_{\Delta_d^*}(t) = \overline{\chi_{\Delta_d}(t^{-1})} = \chi_{\Delta_d}(t),$$

and hence

$$\begin{aligned} \big(\Theta_\lambda \chi_{\Delta_d^*}\big)|_{T^{\text{reg}}} &= (-1)^{d/2}\frac{\sum_{w \in W(\mathfrak{k},\mathfrak{t})} \varepsilon(w) e^{w\lambda}}{\prod_{\alpha \in R_c^+}\big(e^{\alpha/2} - e^{-\alpha/2}\big)} \\ &= (-1)^{d/2}\chi_{\lambda-\rho_c}^K, \end{aligned}$$

by Weyl's character formula. Here $\chi^K_{\lambda-\rho_c}$ is the character of the irreducible representation of K with highest weight $\lambda - \rho_c$.

Therefore, by (9.6),

$$\begin{aligned} R^{\mathcal{H}}_G\big(\text{D-Ind}^G_K[V_\mu]\big) &= \big[\Delta^*_d \otimes \mathcal{H}|_K : V_\mu\big] \\ &= (-1)^{d/2}[V_{\lambda-\rho_c} : V_\mu] \\ &= \begin{cases} (-1)^{d/2} & \text{if } \mu = \lambda - \rho_c \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

□

Corollary 9.2. *Let $\mathcal{H}$ be a Hilbert space carrying a discrete series representation of G, with Harish–Chandra parameter λ. Let V be the irreducible representation of K with highest weight $\mu - \rho_c$. Then the class in $K_0(C^*_rG)$ defined by $\mathcal{H}$ equals*

$$[\mathcal{H}] = (-1)^{d/2}\,\text{D-Ind}^G_K[V]$$

Proof. The comment below Lemma 2.2.1 in [16] implies that the class $[\mathcal{H}]$ is of the form

$$[\mathcal{H}] = \pm\,\text{D-Ind}^G_K[V]$$

for an irreducible representation V of K. Given this relation, Lemma 9.1 yields the more explicit expression

$$[\mathcal{H}] = (-1)^{d/2}\,\text{D-Ind}^G_K[V],$$

where V has highest weight $\lambda - \rho_c$. □

References

[1] M.F. Atiyah and W. Schmid, 'A geometric construction of the discrete series for semisimple Lie groups', *Invent. Math.* 42 (1977) 1–62.

[2] J. Chabert, S. Echterhoff R. Nest, 'The Connes–Kasparov conjecture for almost connected groups and for linear p-adic groups', *Publ. Math. Inst. Hautes Études Sci.* 97 (2003) 239–278.

[3] P. Garrett, 'Some facts about discrete series', informal note (2004).

[4] Harish–Chandra, 'Invariant distributions on Lie algebras', *American J. Math.* 86 (1964) 271–309

[5] Harish–Chandra, 'Invariant differential operators and distributions on a semisimple Lie algebra', *American J. Math.* 86 (1964) 534–564.

[6] Harish–Chandra, 'Some results on an invariant integral on a semisimple Lie algebra', *Ann. Math.* 80 (1964) 551–593.

[7] Harish–Chandra, 'Invariant eigendistributions on a semisimple Lie algebra', *Publ. Math. Inst. Hautes Études Sci.* 27 (1965) 5–54.

[8] Harish–Chandra (1965b), 'Invariant eigendistributions on a semisimple Lie group', *Trans. Amer. Math. Soc.*119 (1965) 457508

[9] Harish–Chandra, 'Discrete series for semsimple Lie groups I', *Acta Math.* 112 (1965) 241–318.

[10] Harish–Chandra, 'Discrete series for semsimple Lie groups II', *Acta Math.* 116 (1966) 1–111.

[11] P. Hochs, *Quantisation commutes with reduction at discrete series representations of semisimple groups*, Adv. Math. 222 (2009), no. 3, 862–919.

[12] A. Knapp, *Representation theory of semisimple groups*, Princeton landmarks in mathematics, Princeton University Press, 2001.

[13] A. Knapp, *Lie groups beyond an introduction*, Progress in mathematics, vol. 140, Birkhäuser, 2001.

[14] A.W. Knapp and G.J. Zuckerman, 'Classification of irreducible tempered representations of semisimple groups', *Ann. Math.* 116(2) (1982) 389–455.

[15] A.W. Knapp and G.J. Zuckerman, 'Classification of irreducible tempered representations of semisimple groups II', *Ann. Math.* 116(3) (1982) 457–501.

[16] V. Lafforgue, 'Banach KK-theory and the Baum-Connes conjecture', *Proc. ICM Beijing* vol. 2 (2002) 795–812.

[17] R. Parthasarathy, 'Dirac operator and the discrete series', *Ann. Math.* (2) 96 (1972) 1–30.

[18] W. Schmid, 'L^2-cohomology and the discrete series', *Ann. Math.* vol. 103, no. 3 (1976) 375–394.

[19] A. Wassermann, 'Une démonstration de la conjecture de Connes–Kasparov pour les groupes de Lie connexes reductifs', *C. R. Acad. Sci. Paris* 304 (1987) 559–562.