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The Universal Coefficient Theorem for C*-algebras with an emphasis on the extensions of C*-algebras

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Abstract

For C*-algebras A and B, the Fredholm picture gives group isomorphisms $\mathrm{KK}_*(\mathbb{C},A) \cong \mathrm{K}_*(A)$ and $\mathrm{KK}_*(\mathbb{C},B) \cong \mathrm{K}_*(B)$. In this way, the Kasparov product induces a group homomorphism $\gamma(A,B)$ from $\mathrm{KK}_*(A,B)$ to $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$. The Universal Coefficient Theorem (UCT) says that for $A \in \mathrm{Ob}(\mathsf{N})$ and B σ-unital, $\gamma(A,B)$ is surjective and $\ker\gamma(A,B) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$, where N is a subcategory of the separable nuclear C*-algebras with postliminal objects. Whenever $\mathrm{K}_*(B)$ is injective, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ vanishes. In that case $\mathrm{KK}_*(A,B)$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ are isomorphic. The UCT is proven in two steps. First, in the case that $\mathrm{K}_*(B)$ is injective. Then the general case is proven by giving an injective resolution of $\mathrm{K}_*(B)$ and applying the injective case to the injective resolution.

Introduction

The goal of this thesis is to give a detailed elaboration of the proof of the Universal Coefficient Theorem (UCT) for C*-algebras. In this thesis N will denote the smallest full subcategory of the category of separable nuclear C*-algebras, which contains the separable type-I (postliminal) C*-algebras and is closed under KK-equivalence, inductive limits, extensions and crossed products by $\mathbb{R}$ and $\mathbb{Z}$. The Universal Coefficient Theorem states that for $A \in \mathrm{Ob}(\mathsf{N})$ and for B a σ-unital C*-algebra, there exists an odd group homomorphism $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) \xrightarrow{\delta(A,B)} \mathrm{KK}_*(A,B)$ (see Definition 17.1) such that the following sequence of $\mathbb{Z}_2$-graded abelian groups is exact:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) \xrightarrow{\delta} \mathrm{KK}_*(A,B) \xrightarrow{\gamma} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) \to 0. \tag{1}$$

In (1) γ denotes the even group homomorphism $\mathrm{KK}_*(A,B) \xrightarrow{\gamma(A,B)} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ (see Definition 12.1) given by application of the Kasparov product to $\mathrm{KK}_*(\mathbb{C},A) = \mathrm{K}_*(A)$ (see Theorem 11.36). To give the reader an idea of what the UCT means; the UCT says that for $A \in \mathrm{Ob}(\mathsf{N})$ and B a σ-unital C*-algebra, every group homomorphism from $\mathrm{K}_*(A)$ to $\mathrm{K}_*(B)$ can be given by taking the Kasparov product with some element of $\mathrm{KK}_*(A,B)$. This element in $\mathrm{KK}_*(A,B)$ need not be unique. The subgroup of $\mathrm{KK}_*(A,B)$ for which the Kasparov product forms the zero map is isomorphic to $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$. The full proof of the UCT for C*-algebras can be found in §15-17. The original proof of the UCT was given by J. Rosenberg and C. Schochet in [27, §2-4]. They also proved that sequence (1) is split. This implies that for $A \in \mathrm{Ob}(\mathsf{N})$ and B a σ-unital C*-algebra the group $\mathrm{KK}_*(A,B)$ can be calculated from the groups $\mathrm{K}_*(A)$ and $\mathrm{K}_*(B)$ alone! The statement of the UCT in this introduction is a generalisation of the UCT as proven by in L.G. Brown, R.G. Douglas and P.A. Filmore in 1977. They proved the UCT for the case that $A = C(X)$ with X a compact Hausdorff space and $B = \mathbb{C}$. This result can be found in [3]. There are other theorems that are also called *Universal Coefficient Theorem*. For example, in algebraic topology, there is a Universal Coefficient Theorem that gives a relation between the homology and cohomology groups of topological spaces. In this thesis, we will only cover the Universal Coefficient Theorem for C*-algebras.

An overview of the proof

We will start the proof of the UCT by showing that $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\_),\mathrm{K}_*(B))$ is a cohomology theory and $\gamma(\_,B)$ is a morphism of cohomology theories (see Definition 8.22). Then we will prove the *injective* case. This is the statement that for a σ-unital C*-algebra B, such that $\mathrm{K}_*(B)$ is injective (see Lemma 4.19), $\gamma(\_, B)$ is an isomorphism of cohomology theories between $\mathrm{KK}_*(\_, B)$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\_), \mathrm{K}_*(B))$, whenever they are restricted to N (see Theorem 15.1). We will prove this statement by showing that $\gamma(A, B)$ is a group isomorphism for each $A \in \mathrm{Ob}(\mathsf{N})$. Then we invoke Lemma 2.23. In the proof of the injective case of the UCT, there are many steps. This is because N is a category whose objects are generated by multiple relations. The order of the steps in the injective case is important. We will start by showing that if $\gamma(\_, B)$ is an isomorphism for two objects of a short exact sequence of C*-algebras, then also for the third. The next step is to show that if $\gamma(\_, B)$ is a group isomorphism for all objects in a cotower in N, then also for the direct limit. Those two steps are necessary for showing that $\gamma(\_, B)$ is a group isomorphism for all commutative C*-algebras. The full proof can be found in §15.

However $\mathrm{K}_*(B)$ need not be injective (with B a σ-unital C*-algebra) for the UCT to hold. In that case $\gamma(A, B)$ need not be a group isomorphism (with $A \in \mathrm{Ob}(\mathsf{N})$). The injective case implies the general case of the UCT. We will deduce the general case of the UCT from the injective case in the following way: A C*-algebra that is KK-equivalent to B (see Definition 11.49) appears in the short exact sequence (60). Its six term sequence with respect to the K-groups splits into two short exact sequences of which the middle and right-hand terms are injective. As a result of Lemma 4.66, applying the $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A), \_)$-functor to these two short exact sequences gives two four term exact sequences like in sequence (14). Such a four term exact sequence can be viewed as a group homomorphism combined with the inclusion of the kernel and the quotient to cokernel. In the injective case (Theorem 15.1) it is shown that $\gamma(A, \_)$ is a morphisms of functors from $\mathrm{KK}_*(A, \_)$ to $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(\_))$ with isomorphic components whenever the K-groups are injective. Like any exact sequence, the six term sequence of (60) with respect to the KK-groups can be split up into multiple short exact sequences containing the cokernel of a previous morphism and the kernel of a following morphism (for inspiration, see sequences (66) and (67)). The isomorphic components of $\gamma(A, \_)$ preserve kernels and cokernels. From this follows sequence (1). The full proof can be found in §17.

The purpose of this thesis

There exist many detailed books and papers about K-theory, KK-theory or the UCT. With regard to K-theory, I personally benefited enormously from [34]. However, it does not cover KK-theory. Books about KK-theory often leave out topics such as classification of extensions of C*-algebras. B. Blackadar covers in [2] both K-theory and KK-theory. This book contains mainly results without much explanation. The proofs in [16] and [27] only give the most important steps, but they leave out the details. As far as I am aware, there is not any source that includes all the definitions and lemmas necessary for understanding the proof of the UCT. In this thesis I tried to provide an elaboration of the proof of the UCT that requires only basic mathematical knowledge from the reader. The added value of this approach is that it makes KK-theory and the Universal Coefficient Theorem understandable for a broader audience. In particular readers with an interest in operator theory, noncommutative geometry, functional analysis, algebraic topology, homological algebra and category theory. In this thesis I tried to emphasise the algebraic and categorical structures that remain more subtle in other sources.

The setup of this thesis

This thesis is divided into four parts. The fourth part contains the proof of the UCT. We will dedicate the third part to KK-theory. In §11 we will define Kasparov modules, several equivalence classes, the KK-functor and the Kasparov product. A Kasparov A-B module (for C*-algebras A and B) is a triple consisting of a Hilbert B-module, a $*$-homomorphism from A to the adjointable operators on the Hilbert module and an odd operator on the Hilbert module under some compactness relation. See Definition 11.1. One equivalence relation of Kasparov A-B-modules is called *homotopy*. The set of equivalence classes under homotopy on Kasparov A-B-modules is denoted by $\mathrm{KK}(A,B)$. The set $\mathrm{KK}(A,B)$ inherits an abelian group structure from the addition of Kasparov modules. In fact KK defines a bifunctor from $\mathsf{C}^* - \mathsf{alg}^{\mathrm{opp}} \times \mathsf{C}^* - \mathsf{alg}$ to Ab. The Kasparov product is a biadditive mapping $\mathrm{KK}(A,B) \times \mathrm{KK}(B,C) \to \mathrm{KK}(A,C)$ for C*-algebras A, B and C. In §13 we will work through the proof of the isomorphism between $\mathrm{Ext}(A,B)$ and $\mathrm{KK}_1(A,B)$, which exists whenever A is a separable nuclear C*-algebra and B is a σ-unital C*-algebra (see Corollary 13.63). The original proof of this isomorphism was given by G.G. Kasparov in [16, §7]. A result of this isomorphism is the half-exactness of certain restrictions of the KK-functors (see Theorems 14.1 and 14.4). The six term sequences for those restricted KK-functors are corollaries of the half-exactness and the stability and homotopy invariance (see Corollaries 14.9 and 14.10). Now the restricted functor $\mathrm{KK}_*(\_,B)$ (with B σ-unital) is a cohomology theory (see Definition 8.19).

In the first part, we introduce the categorical definitions and notation. It can be (partially) skipped by those familiar with category theory. In the second part, we work through all the background information necessary for understanding the proofs. The second part covers a broad variety of topics; classification of extensions of C*-algebras, K-theory, (co)homology functors, Hilbert modules and $\mathbb{Z}_2$-gradings. This setup allows the reader to just read just a selection of (sub)sections. Another asset of this approach is the possibility for the reader to connect to more recent developments in KK-theory, for instance the category KK, which will be the topic of the next subsection. Although we will not use the category KK in the proof of the UCT, Parts I, II and III include sufficient results to give an understanding of the main content of the references in the next subsection.

The category KK

There exists a category with the separable C*-algebras as objects and the KK-classes as morphisms. It is easy to check from the properties of KK-theory in §11 that the morphisms (KK-classes) satisfy Definition 1.1. This category is called KK. KK-equivalent objects are isomorphic in KK. In KK, all finite coproducts exist. The morphism classes (KK-classes) are abelian groups and the composition (Kasparov product) is biadditive. Hence KK is an additive category. The obvious functor from the full category of separable C*-algebras to KK is the identity on the objects and sends the $*$-homomorphisms to the KK-class they represent. See Example 11.5 for this characterisation. In 1987, N. Higson showed that every split-exact homotopy invariant stable functor from the separable C*-algebras to an additive category can be uniquely factored through KK (see [13, Theorem 4.5]). For an overview of all these definitions, see [2, §21].

Higson's factorisation result may have come out of the blue, but it is actually a special case of a more general phenomenon. Half-exact homotopy invariant stable functors (Bott functors) have nice properties. For instance, they are invariant under double suspension and have a six term sequence. See §8.4 for the definition and results. The universal enveloping additive category of an additive category is the category that allows a unique factorisation for every homotopy invariant and half-exact functor to another additive category. In [2, §22.2] we find an algorithm for its construction. The universal enveloping additive category for the category of separable C*-algebras exists, but it is not KK. In fact KK is the universal enveloping additive category of the category $\mathbf{S}$ in [2, p. 225]. The universal enveloping additive category of the separable C*-algebras is called E. In 1989 N. Higson found an explicit expression for E (see [14, Theorem 3.6]). If A is a separable nuclear C*-algebra, then $\mathsf{E}(A, B)$ is group isomorphic to $\mathrm{KK}(A, B)$ for every separable C*-algebra B (in a way that is compatible with the Kasparov product and composition on E). For this result, see [14, Theorem 3.5].

Another feature of KK is its triangulation. There is an isomorphism between extensions of σ-unital C*-algebras by separable nuclear C*-algebras and the KK_1-group (see [16, §7]). A similar result can be obtained when the nuclearity is replaced by the existence of completely positive contractive splitting (this splitting need not be a $*$-homomorphism). Elements of the KK_1-group can be viewed as morphisms in KK by application of Bott-periodicity. An additive category combined with an additive endofunctor is *triangulated* if it satisfies properties TR0-TR4 in [22]. The suspension of C*-algebras (see Definition 6.42) is an endofunctor of the category KK. The suspension of a separable C*-algebra is separable. The suspension of a $*$-homomorphism can be viewed as a KK-class via Example 11.5. In 2008 R. Meyer and R. Nest showed that KK with suspension is triangulated. For this result, see [20].

I hope you will enjoy reading my thesis!

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Part I
Categorical definitions and constructions

The goal of this part is to introduce the reader to category theory and to clarify the definitions that are used in the next parts of this thesis. The first section will go through the properties of categories. We have included a list of categories appearing in the rest of the thesis. See §1.2. The second section will focus on functors and their cones and limits. Many examples will be given in both sections. In the third section we will take a closer look at the pullbacks and tensor products of modules over a ring. The fourth section is about homological algebra and shows the properties of injective modules over a ring.

1 Categories

The definition of a category will be given in this section. The reader has probably heard of several of the examples we will go through. These examples will be helpful in discovering the complex structure of seemingly easy categories. Also some less obvious examples of categories will be given. They will help in understanding the next section in which we go through functors and cones. In the second subsection, there will be an overview of the categories used in the later parts of this thesis.

1.1 The definition and examples

Definition 1.1. A *category* C consists of a class of *objects* $\mathrm{Ob}(\mathsf{C})$ and for every $X, Y \in \mathrm{Ob}(\mathsf{C})$ a class of *morphisms* $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ between X and Y such that for every $X \in \mathrm{Ob}(\mathsf{C})$ there is a morphism $\mathrm{id}_X \in \mathrm{Hom}_{\mathsf{C}}(X, X)$ and for every $X, Y, Z \in \mathrm{Ob}(\mathsf{C})$ there is a *composition* $\mathrm{Hom}_{\mathsf{C}}(X, Y) \times \mathrm{Hom}_{\mathsf{C}}(Y, Z) \xrightarrow{\circ} \mathrm{Hom}_{\mathsf{C}}(X, Z)$ satisfying the following properties:

- $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$ for all $X, Y \in \mathrm{Ob}(\mathsf{C})$ and for all $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(Y, X)$.
- The composition is associative.

Definition 1.2. Let C be a category. A *subcategory* C is a category D for which $\mathrm{Ob}(\mathsf{D})$ is a subclass of $\mathrm{Ob}(\mathsf{C})$ and $\mathrm{Hom}_{\mathsf{D}}(X, Y)$ is a subclass of $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ for every $X, Y \in \mathrm{Ob}(\mathsf{D})$ such that:

- $\mathrm{id}_X \in \mathrm{Hom}_{\mathsf{C}}(X, X)$ is included in $\mathrm{Hom}_{\mathsf{D}}(X, X)$ for every $X \in \mathrm{Ob}(\mathsf{D})$.
- $f \circ g \in \mathrm{Hom}_{\mathsf{C}}(X, Z)$ is included in $\mathrm{Hom}_{\mathsf{D}}(X, Z)$ for every $X, Y, Z \in \mathrm{Ob}(\mathsf{D})$ and every $f \in \mathrm{Hom}_{\mathsf{D}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathsf{D}}(Y, Z)$.

Definition 1.3. A category is *small* if the class of objects and the classes of morphisms between every two objects form a set. A category is *locally small* if for every two objects the class of morphisms between them forms a set.

Remark 1.4. Throughout this thesis, all categories are locally small.

We will go through some easy examples.

Example 1.5. Let k be a field. If we view all vector spaces over k as objects and all linear maps between k-vector spaces as morphisms, then this forms a category, called the *category of vector spaces over* k and denoted by Vec_k. Between every two k-vector spaces there is at least a zero map. The set of morphisms between two k-vector spaces can never be empty! More generally, for a ring R, if we view all (two-sided) modules over R as objects and all R-module homomorphisms as morphisms, then this forms a category, called the *category of R-modules* and denoted by $R-\mathsf{Mod}$. Between every two R-modules there is at least a zero map. Both Vec_k (with k a field) and $R-\mathsf{Mod}$ (with R a ring) are locally small, but not small.

Example 1.6. If we view all groups as objects and all group homomorphisms between groups as morphisms, then this forms a category, called the *category of groups* and denoted by Grp. Between every two groups there is at least the trivial map, which maps everything to the unit element. The set of morphisms between two groups can never be empty! If we view all abelian groups as objects and all group homomorphisms between abelian groups as morphisms, then this forms a category, called the *category of abelian groups* and denoted by Ab. Clearly Ab is a subcategory of Grp. Both are locally small, but not small.

Example 1.7. If we view all rings as objects and all ring homomorphisms between rings as morphisms, then this forms a category, called the *category of rings* and denoted by Ring. If we view all fields as objects and all ring homomorphisms between fields as morphisms, then this forms a category, called the *category of fields* and is denoted by Field. Clearly Field is a subcategory of Ring. No morphisms exist between fields of different characteristics. Let p be a prime number. The category of fields of characteristic p is denoted by Field_p. It is a subcategory of Field.

Example 1.8. If we view all topological spaces as objects and all continuous maps between topological spaces as morphisms, then this forms a category, called the *category of topological spaces* and denoted by Top. One subcategory of Top is the category of Hausdorff spaces. This category has Hausdorff topological spaces as objects and continuous maps between them as morphisms. The category of Hausdorff spaces is denoted by Haus. It is also possible to add extra structure to Top. Let X and Y be topological spaces and let $x \in X$ and $y \in Y$ be points. Let $X \xrightarrow{f} Y$ be a continuous map such that $f(x) = y$. This forms a category; the topological spaces combined with a point form the objects and the continuous maps that send the point of domain to the point of the codomain form the morphisms. It is called *the category of pointed spaces* and denoted by Top_*.

The examples we have gone through so far are all mathematical objects from subjects in mathematics. It is also possible to create categories in an 'artificial' way. These categories do not represent mathematical objects in the way Grp, Ring and Top do. The following examples will clarify that.

Example 1.9. The *empty category* is the category without objects. It satisfies the definition of a category. This category is denoted by $\mathsf{0}$.

Example 1.10. The *trivial category* is the category with one object and the identity as the only morphism between that object and itself. This category is denoted by $\mathsf{1}$.

Example 1.11. The following diagram is a category. It is denoted by 2.

$$\begin{tikzcd} x_1 \arrow[loop left, "\mathrm{id}_{x_1}"] & x_2 \arrow[loop right, "\mathrm{id}_{x_2}"] \end{tikzcd}$$

Example 1.12. The following diagram is a category. It is denoted by $\hat{2}$.

$$\begin{tikzcd}
x_1 \arrow[loop left, "\mathrm{id}_{x_1}"] \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & x_2 \arrow[loop right, "\mathrm{id}_{x_2}"]
\end{tikzcd}$$

Example 1.13. The following diagram is a category. It is denoted by $\bar{3}$.

$$\begin{tikzcd}
 & x_1 \arrow[loop above, "\mathrm{id}_{x_1}"] \arrow[dl, bend left, "f"'] \arrow[dr, bend right, "g"] & \\
x_2 \arrow[loop left, "\mathrm{id}_{x_2}"] & & x_3 \arrow[loop right, "\mathrm{id}_{x_3}"]
\end{tikzcd}$$

Definition 1.14. A *preorder* is a set with a reflexive and transitive binary operation '$\leq$'. In other words, a set $(X, \leq)$ is a preorder if $x \leq x$ for all $x \in X$ and $x \leq y$ and $y \leq z$ implies $x \leq z$ for all $x, y, z \in X$.

Definition 1.15. A *partially ordered set* (*poset* for short) is a preorder for which the $\leq$-relation is anti-symmetric. In other words, a preorder $(X, \leq)$ is a poset if $x \leq y$ and $y \leq x$ implies $x = y$ for all $x, y \in X$.

Example 1.16. Every poset is a preorder, but not every preorder is a partially ordered set. For instance divisibility on $\mathbb{Z}$ defines a preorder structure; being divisible by $a \in \mathbb{Z}$ implies being divisible by all numbers that divide a. This ordering is determined by the unique prime factorisation of integers. Now $a \in \mathbb{Z}$ is divisible by $-a$ and $-a$ is divisible by a. So $a \leq -a$ and $-a \leq a$, while $a \neq -a$.

Example 1.17. $\mathbb{N}$ and $\mathbb{Z}$ are posets under their normal ordering. In fact, for any two numbers in $\mathbb{N}$ or $\mathbb{Z}$, one is smaller than or equal to the other. Such posets are called *completely ordered sets.*

Example 1.18. The open subsets of a topological space form a poset under inclusion.

Lemma 1.19. Let $(X, \leq)$ be a preorder. View the elements of X as objects. For $x, y \in X$ if $x \leq y$, then define $\mathrm{Hom}_{(X,\leq)}(x, y) := \{*\}$ and otherwise $\mathrm{Hom}_{(X,\leq)}(x, y) := \varnothing$. Now $(X, \leq)$ is a category.

Proof. The reflexivity and transitivity of the $\leq$-relations make sure the properties of Definition 1.1 are satisfied. □

Definition 1.20. Let C be a category. Two objects $X, Y \in \mathrm{Ob}(\mathsf{C})$ are *isomorphic* (in C) if there exist morphisms $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(Y, X)$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Lemma 1.21. Being isomorphic as objects in a category is an equivalence relation.

Proof. Reflexivity and symmetry are obvious. Transitivity follows from composing the isomorphisms. □

Definition 1.22. Let C be a category. A morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ (with $X, Y \in \mathrm{Ob}(\mathsf{C})$) is a *monomorphism* if for every $Z \in \mathrm{Ob}(\mathsf{C})$ and two morphisms $g_1, g_2 \in \mathrm{Hom}_{\mathsf{C}}(Z, X)$ such that $f \circ g_1 = f \circ g_2$, it follows $g_1 = g_2$. A morphism that is a monomorphism is called *monic.*

Definition 1.23. Let C be a category. A morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ (with $X, Y \in \mathrm{Ob}(\mathsf{C})$) is an *epimorphism* if for every $Z \in \mathrm{Ob}(\mathsf{C})$ and two morphisms $g_1, g_2 \in \mathrm{Hom}_{\mathsf{C}}(Y, Z)$ such that $g_1 \circ f = g_2 \circ f$, it follows $g_1 = g_2$. A morphism that is an epimorphism is called *epic*.

Example 1.24. In Vec_k (for k a field), $R-\mathsf{Mod}$ (for R a ring), $\mathsf{Grp}, \mathsf{Ab}, \mathsf{Ring}$ and Top the monomorphisms are the injective morphisms. In Vec_k (for k a ring), $R-\mathsf{Mod}$ (for R a ring), $\mathsf{Grp}, \mathsf{Ab}$ and Top the epimorphisms are the surjective morphisms. In Vec_k (for k a ring), $R-\mathsf{Mod}$ (for R a ring), $\mathsf{Grp}, \mathsf{Ab}$ and Ring the isomorphisms are the bijective morphisms.

Example 1.25. Not every monic and epic morphism needs to be an isomorphism. One example is Top. A homeomorphism is an open bijective continuous map or equivalently a closed bijective continuous map. Being monic (injective) and epic (surjective) is not enough. This example shows also that there are categories for which a bijection need does not need to be an isomorphism.

Example 1.26. It is also not true that in a category in which injectivity or surjectivity makes sense the monomorphisms are the injective morphisms and the epimorphisms are the surjective morphisms. In Haus the epimorphisms are precisely the morphisms with a dense image. A surjective continuous map has a dense image, but a non-surjective continuous map can be epic in Haus.

Definition 1.27. Let C be a category. An object $X \in \mathrm{Ob}(\mathsf{C})$ is called *initial* (in C) if for every object $Y \in \mathrm{Ob}(\mathsf{C})$, there is a unique morphism in C from X to Y. If C is locally small, this can be rephrased to $\#\mathrm{Hom}_{\mathsf{C}}(X, Y) = 1$ for all $Y \in \mathrm{Ob}(\mathsf{C})$.

Lemma 1.28. In a category, all initial objects are isomorphic.

Proof. Let $X, Y \in \mathrm{Ob}(\mathsf{C})$ be two initial elements. $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ and $\mathrm{Hom}_{\mathsf{C}}(Y, X)$ contain precisely one morphism. The composition of the two must be id_X as $\mathrm{Hom}_{\mathsf{C}}(X, X)$ contains just one morphism. Composition in the opposite order equals id_Y by the same argument. An isomorphism between X and Y is established. $\square$

Definition 1.29. Let C be a category. An object $Y \in \mathrm{Ob}(\mathsf{C})$ is called *terminal* (in C) if for every object $X \in \mathrm{Ob}(\mathsf{C})$, there is a unique morphism in C from X to Y. If C is locally small, this can be rephrased to $\#\mathrm{Hom}_{\mathsf{C}}(X, Y) = 1$ for all $X \in \mathrm{Ob}(\mathsf{C})$.

Lemma 1.30. In a category, all terminal elements are isomorphic.

Proof. Identical to the proof of Lemma 1.28 $\square$

Definition 1.31. An object is called *zero* if it is both an initial and terminal object.

Corollary 1.32. In a category, all zero objects are isomorphic.

Example 1.33. In Vec_k (for k a field), $R-\mathsf{Mod}$ (for R a ring), Grp and Ab there is a zero element.

Example 1.34. In Ring there is an initial object; $\mathbb{Z}$. There is also a terminal object; the zero ring. There is no zero object. In Field there is neither an initial, nor a terminal object (note that the zero ring is not a field). Let p be a prime number. In Field_p there is an initial object; $\mathbb{Z}/p\mathbb{Z}$. There is no terminal object; $\mathbb{Z}/p\mathbb{Z}$ fails to be a terminal object, since there is no ring homomorphism from the field with p^2 elements to $\mathbb{Z}/p\mathbb{Z}$.

Example 1.35. In a poset an initial element is a smallest element and a terminal element is a largest element. A poset has a zero element if and only if it has one element.

Definition 1.36. Let C be a category. The *opposite* of C is the category C^{opp} with $\text{Ob}(\mathsf{C}^{\text{opp}}) := \text{Ob}(\mathsf{C})$ and $\text{Hom}_{\mathsf{C}^{\text{opp}}}(X,Y) := \text{Hom}_{\mathsf{C}}(Y,X)$ for every $X, Y \in \text{Ob}(\mathsf{C})$.

Example 1.37. The categories $\mathsf{0}$, $\mathsf{1}$ and $\mathsf{2}$ are their own opposites. Equivalently, $\mathsf{0}^{\text{opp}} = \mathsf{0}$, $\mathsf{1}^{\text{opp}} = \mathsf{1}$ and $\mathsf{2}^{\text{opp}} = \mathsf{2}$.

Example 1.38. The opposite of $\bar{\mathsf{3}}$ (see Example 1.13) is the following diagram:

$$\begin{tikzcd}
 & x_1 \arrow[loop above, "\text{id}_{x_1}"] & \\
x_2 \arrow[ur, "f"] \arrow[loop left, "\text{id}_{x_2}"] & & x_3 \arrow[ul, "g"'] \arrow[loop right, "\text{id}_{x_3}"]
\end{tikzcd}$$

Lemma 1.39. Let C be a category. An initial element in C becomes a terminal element in C^{opp}.

Lemma 1.40. Let C be a category. A terminal element in C becomes an initial element in C^{opp}.

Lemma 1.41. Let C be a category. An epimorphism in C becomes a monomorphism in C^{opp}.

Lemma 1.42. Let C be a category. A monomorphism in C becomes an epimorphism in C^{opp}.

Proof Lemmas 1.39-1.42. Just reverse the direction of the arrows. $\square$

Definition 1.43. The *dual* of a statement is the statement corresponding to the opposite category.

Example 1.44. An initial object is the dual of a terminal object and vice versa (see Lemma 1.39 and Lemma 1.40). Lemma 1.28 (all initial elements are isomorphic) is the dual of Lemma 1.30 (all terminal elements are isomorphic) and vice versa.

Example 1.45. An epimorphism is the dual of a monomorphism and vice versa (see Lemma 1.41 and Lemma 1.42).

Definition 1.46. Let C and D be categories. The *product category* of C and D is the category for which the objects are pairs of objects (X,Y) with $X \in \text{Ob}(\mathsf{C})$ and $Y \in \text{Ob}(\mathsf{D})$ and the morphisms between (X_1, Y_1) and (X_2, Y_2) are the pairs of morphisms (f,g) with $X_1, X_2 \in \text{Ob}(\mathsf{C})$ and $Y_1, Y_2 \in \text{Ob}(\mathsf{D})$ and $f \in \text{Hom}_{\mathsf{C}}(X_1, X_2)$ and $g \in \text{Hom}_{\mathsf{D}}(Y_1, Y_2)$.

1.2 An overview of categories

In this thesis several categories will be encountered. In most of the literature only explicit categories will be written in sans serif font and arbitrary categories in another (sometimes calligraphic font (mathcal)). We need that font already for other things; for example multiplier algebras. Therefore any category in this thesis will be denoted in sans serif font. Below you will find a table of the categories in the later parts of this thesis and their notation. The reader might not be acquainted with all the categories. The most exotic ones will get a proper introduction in this thesis.

Category List		
Category	Objects	Morphisms
Ab	Abelian groups	Group homomorphisms
$\mathsf{C^*-alg}$	C*-algebras	$*$-homomorphisms
$\mathsf{CommMon}$	Commutative monoids	Unital semi-group homomorphisms
$\mathrm{Fun}(\mathbb{N}, \mathsf{Ab})$	Functors from $\mathbb{N}$ to Ab	Morphisms of functors
$\mathbb{N}$	Natural numbers	$\leq$-relations
$\mathbb{N}^{\mathrm{opp}}$	Natural numbers	$\geq$-relations
N	See Definition 12.4	$*$-homomorphisms
$R-\mathsf{Mod}$	R-modules	R-module homomorphisms
$\sigma\mathsf{C^*}$-alg	σ-unital C*-algebras	$*$-homomorphisms
$\mathsf{SepNucC^*-alg}$	Separable nuclear C*-algebras	$*$-homomorphisms
Top	Topological spaces	Continuous maps
$\mathbb{Z}-\mathsf{Mod}$	$\mathbb{Z}$-modules	$\mathbb{Z}$-module homomorphisms

2 Functors and limits

Just like morphisms between objects exist in categories, it is possible to construct mappings from categories to categories that preserve the structure of the category as defined in Definition 1.1. Such mappings are called functors. In the first subsection we will give the precise definition of a functor and state some results. In the second subsection, we will build a new structure on a functor, called a cone. The cones form a category and we will take special interest in terminal objects in the category of cones. In the third subsection, we will focus on cones of the partial order $\mathbb{N}^{\mathrm{opp}}$. In the last subsection, we will show a property of representable functors.

2.1 Functors; a definition and examples

Definition 2.1. Let C and D be categories. A *(covariant) functor* from C to D is a mapping $F\colon \mathsf{C} \to \mathsf{D}$ that sends an object $X \in \mathrm{Ob}(\mathsf{C})$ to an object $F(X) \in \mathrm{Ob}(\mathsf{D})$ and sends a morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ (with $X, Y \in \mathrm{Ob}(\mathsf{C})$) to a morphism $F(f) \in \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$ such that the following conditions are met:

- Identities are preserved, in other words: $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for $X \in \mathrm{Ob}(\mathsf{C})$.
- Compositions are preserved, in other words $F(g \circ f) = F(g) \circ F(f)$ for $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(Y, Z)$ (with $X, Y, Z \in \mathrm{Ob}(\mathsf{C})$).

A functor of the form $G\colon \mathsf{C}^{\mathrm{opp}} \to \mathsf{D}$ is called a *contravariant* functor (from C to D).
Let C_1 and C_2 be categories. A functor of the form $H\colon \mathsf{C}_1 \times \mathsf{C}_2 \to D$ is called a *bifunctor.*

Remark 2.2. The idea behind the contravariant functor is introducing a reversal of domain and codomain of the morphisms. The distinction between covariant and contravariant is only for the understanding of the reader; every covariant functor becomes a contravariant functor when the domain of the functor is changed to the opposite category and vice versa.

Remark 2.3. A bifunctor is a functor for which the domain is just a product category. It can be turned into a normal functor by fixing the object in C_1 or C_2. This is the same as precomposing with an inclusion functor from one category to the product category.

Remark 2.4. It might be tempting to believe there is a category of categories with the categories as objects and the functors between them as morphisms. This would mean that this category is an object of itself. That would not be logically possible. The small categories form a category however. This category is denoted by Cat. By the same argument Cat is not small.

Example 2.5. Let X be a topological space and let $x \in X$ be a point. The fundamental group $\pi_1(X,x)$ is a group and any continuous map $X \xrightarrow{f} Y$ (with Y a topological space) induces a group homomorphism $\pi_1(X,x) \xrightarrow{\pi_1(f)} \pi_1(Y, f(x))$. This construction preserves identities and compositions. So $\pi_1 \colon \mathsf{Top}_* \to \mathsf{Grp}$ is a functor.

Example 2.6. Another example of a functor is the abelianisation of groups. A group can be abelianised by dividing out the commutator subgroup. Every group homomorphism induces a group homomorphism between the abelianisations. Obviously, identities and compositions are preserved. So abelianisation defines a functor from Grp to Ab.

Example 2.7. A covariant functor between two preorders or posets is a mapping of the elements that preserves the ordering. A contravariant functor between two preorders or posets is a mapping of the elements that reverses the ordering.

Example 2.8. In addition to Example 1.18 and Example 2.7, the pre-image of an open subset under a continuous map is open and pre-images preserve the order of inclusion. So there is a functor $\mathrm{Open} \colon \mathsf{Top}^{\mathrm{opp}} \to \mathsf{Cat}$ that sends a topological space to the partial order of its open subsets.

Example 2.9. Let C be a locally small category. Let $X \in \mathrm{Ob}(\mathsf{C})$. Then $\mathrm{Hom}_{\mathsf{C}}(X, \_) \colon \mathsf{C} \to \mathsf{Set}$ defines a functor. Likewise $\mathrm{Hom}_{\mathsf{C}}(\_, X) \colon \mathsf{C}^{\mathrm{opp}} \to \mathsf{Set}$ defines a functor. Both such functors are called *representable*.

Remark 2.10. It may be that for a locally small category the set of morphisms between two objects has more structure than only that of a set. For example, the set of linear maps between vector spaces over a field can be given a vector space structure as well. Another example is the set of group homomorphisms between two abelian groups, which can be given an abelian group structure. The latter example can be generalised to modules over a commutative ring. For this, see Remark 4.17.

Definition 2.11. Let C and D be categories and let $F \colon \mathsf{C} \to \mathsf{D}$ be a functor. Then F is *faithful* if the mapping $\mathrm{Hom}_{\mathsf{C}}(X,Y) \ni f \xrightarrow{F} F(f) \in \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$ is injective for all objects $X, Y \in \mathrm{Ob}(\mathsf{C})$.

Definition 2.12. Let C and D be categories and let $F \colon \mathsf{C} \to \mathsf{D}$ be a functor. Then F is *full* if the mapping $\mathrm{Hom}_{\mathsf{C}}(X,Y) \ni f \xrightarrow{F} F(f) \in \mathrm{Hom}_{\mathsf{D}}(F(X), F(Y))$ is surjective for all objects $X, Y \in \mathrm{Ob}(\mathsf{C})$.
A subcategory is called *full* if the inclusion functor is full.
A functor that is both full and faithful is *fully faithful*.

Example 2.13. The *forgetful* functor $\mathrm{Forget} \colon \mathsf{Top}_* \to \mathsf{Top}$ that forgets the point of a pointed space is faithful, but not full.

Example 2.14. The functor $\mathrm{Open} \colon \mathsf{Top}^{\mathrm{opp}} \to \mathsf{Cat}$ that sends a topological space to the partial order of open subsets (Example 2.8) is not faithful, just think about continuous maps for which the codomain has trivial topology. Any mapping of elements is continuous then. The pre-image of the whole domain is the whole codomain and the pre-image of the empty set is empty. So only one functor can be in the image of Open in that case. It is not full either. By a similar argumentation, the constant functor to the top element (the entire set) cannot be in the image of Open.

Definition 2.15. Let C and D be categories and let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor. Then F is *essentially surjective* if for every object $Y \in \mathrm{Ob}(\mathsf{D})$, there exists an object $X \in \mathrm{Ob}(\mathsf{C})$ such that $F(X) \cong Y$.

Definition 2.16. Let C and D be categories and let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor. It is said that F *preserves* a property if for every object $X \in \mathrm{Ob}(\mathsf{C})$ that has that property it follows that $F(X)$ has that property too. Or if for every $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ (with $X, Y \in \mathrm{Ob}(\mathsf{C})$) that has that property it follows $F(f)$ has that property too.

Example 2.17. By definition a functor preserves identities. Moreover, a functor preserves isomorphisms. This fact can be used to rule out that two objects are isomorphic. We will establish a functor between two categories. When the images of two objects are not isomorphic in the codomain category, then the two objects cannot be isomorphic in the domain category.

Remark 2.18. The method of Example 2.17 can be used with other properties, such as homotopy or homotopy equivalence. A functor that is invariant under some property will be created. When the images are not isomorphic or equal (for morphisms) in the codomain, then it must follow that the property was not satisfied in the domain.

Definition 2.19. Let C and D be categories and let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor. It is said that F *reflects* a property if for every object $X \in \mathrm{Ob}(\mathsf{C})$ it follows from $F(X)$ having that property that X has that property too. Or if for every $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ (with $X, Y \in \mathrm{Ob}(\mathsf{C})$) it follows from $F(f)$ having that property that f has that property too.

Example 2.20. The following statements are examples of how reflection can be used.

- A fully faithful functor reflects (and preserves) isomorphisms.
- A fully faithful functor reflects initial and terminal elements.
- A faithful functor reflects monomorphisms and epimorphisms.

Definition 2.21. Let C and D be categories and let $F, G\colon \mathsf{C} \to \mathsf{D}$ be functors. A *natural transformation* or *morphism of functors* η from F to G consists of a collection of morphisms $F(X) \xrightarrow{\eta_X} G(X)$ in D indexed by $\mathrm{Ob}(\mathsf{C})$ such that for every $X, Y \in \mathrm{Ob}(\mathsf{C})$ and every $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y). \end{tikzcd} \tag{2}$$

Notation: $\eta\colon F \Rightarrow G$. The morphism $F(X) \xrightarrow{\eta_X} G(X)$ for $X \in \mathrm{Ob}(\mathsf{C})$ is called the *component* of η at X. In case η has an inverse, η is called a *natural isomorphism* or an *isomorphism of functors*.

Remark 2.22. The name morphism of functors comes from the fact that there is a category of functors between two categories. In that category, the morphisms of functors are the morphisms.

Lemma 2.23. A morphism of functors is an isomorphism of functors if and only if all the components are isomorphic.

Proof. Let C and D be categories and let $F, G\colon \mathsf{C} \to \mathsf{D}$ be functors. If there exist natural transformations $F \overset{\eta}{\Longrightarrow} G$ and $G \overset{\theta}{\Longrightarrow} F$ such that $\theta \circ \eta = \mathrm{id}_F$ and $\eta \circ \theta = \mathrm{id}_G$, then in particular $\theta_X \circ \eta_X = (\mathrm{id}_F)_X = \mathrm{id}_{F(X)}$ and $\eta_X \circ \theta_X = (\mathrm{id}_G)_X = \mathrm{id}_{G(X)}$ for every $X \in \mathrm{Ob}(\mathsf{C})$.

Conversely, if $F \overset{\eta}{\Longrightarrow} G$ is a natural transformation for which the components happen to be isomorphism, then define $(\eta^{-1})_X := (\eta_X)^{-1}$ for $X \in \mathrm{Ob}(\mathsf{C})$. The following diagram commutes for every $X, Y \in \mathrm{Ob}(\mathsf{C})$ and every $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$:

$$\begin{tikzcd} G(X) \arrow[r, "\eta_X^{-1}"] \arrow[d, "G(f)"'] & F(X) \arrow[d, "F(f)"] \\ G(Y) \arrow[r, "\eta_Y^{-1}"] & F(Y). \end{tikzcd}$$

So η^{-1} defines a natural transformation from G to F. It is the inverse of η. □

Definition 2.24. Let C and D be categories and let $\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}$ be functors. F and G are called *equivalences of categories* if there exist natural isomorphisms $G \circ F \overset{\eta}{\Longrightarrow} \mathrm{id}_{\mathsf{C}}$ and $F \circ G \overset{\theta}{\Longrightarrow} \mathrm{id}_{\mathsf{D}}$. In case F and G are contravariant, they are called *anti-equivalences*.

Remark 2.25. For understanding the principles of category theory it might be helpful to get the analogy between category theory and topology. In topology, there are topological spaces with continuous maps between them. Two continuous maps with the same domain and codomain might be connected by a homotopy. So a topological space compares to a category, a continuous map to a functor and a natural isomorphism to a homotopy.

We will encounter an anti-equivalence in §5.1.

Lemma 2.26. Equivalence of categories is an equivalence relation.

Proof. See Remark 2.25. The proof is identical to the proof that homotopy equivalence of topological spaces is an equivalence. □

Theorem 2.27. A functor is an equivalence of categories if and only if it is essentially surjective and fully faithful.

Proof. See [18, Proposition 1.3.18]. □

Example 2.28. An equivalence of categories is a strictly weaker statement than an isomorphism of categories. A functor is an isomorphism of categories if it gives a bijection between morphisms **and a bijection between objects**. We will give an example of an equivalence that is not an isomorphism. The category $\mathsf{1}$ (see Example 1.10) is equivalent to the category $\check{\mathsf{2}}$ with objects $\{x_1, x_2\}$ and morphism $\mathrm{Hom}_{\check{\mathsf{2}}}(x_1, x_1) = \{\mathrm{id}_{x_1}\}$, $\mathrm{Hom}_{\check{\mathsf{2}}}(x_1, x_2) = \{f\}$, $\mathrm{Hom}_{\check{\mathsf{2}}}(x_2, x_1) = \{g\}$ and $\mathrm{Hom}_{\check{\mathsf{2}}}(x_2, x_2) = \{\mathrm{id}_{x_2}\}$. It has the following diagram:

$$\begin{tikzcd} x_1 \arrow[loop left, "\mathrm{id}_{x_1}"] \arrow[r, bend left, "f"] & x_2 \arrow[l, bend left, "g"] \arrow[loop right, "\mathrm{id}_{x_2}"] \end{tikzcd}.$$

To be a category, it must hold that $g \circ f = \mathrm{id}_{x_1}$ and $f \circ g = \mathrm{id}_{x_2}$. So x_1 and x_2 are isomorphic and any functor from the $\mathsf{1}$ to $\check{\mathsf{2}}$ is essentially surjective. It is automatically fully faithful (just count). It can easily be verified that $\mathsf{1}$ is the terminal category (in Cat). On the other hand $\check{\mathsf{2}}$ is not terminal and therefore $\check{\mathsf{2}}$ cannot be isomorphic to $\mathsf{1}$. It would be in contradiction with Lemma 1.30.

2.2 Limits

Definition 2.29. Let C and D be categories. Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor. Let $D \in \mathrm{Ob}(\mathsf{D})$ and let $\Delta_D\colon \mathsf{C} \to \mathsf{D}$ be the *constant functor* to D, i.e. the functor for which $\Delta_D(X) = D$ for every $X \in \mathrm{Ob}(\mathsf{C})$ and $\Delta_D(f) = \mathrm{id}_D$ for every $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ with $X, Y \in \mathrm{Ob}(\mathsf{C})$. A *cone for F* is a pair (D, μ), consisting of an object $D \in \mathrm{Ob}(\mathsf{D})$ and a natural transformation $\Delta_D \overset{\mu}{\Longrightarrow} F$. In other words, it is a collection of morphisms $D \xrightarrow{\mu_X} F(X)$ for $X \in \mathrm{Ob}(\mathsf{C})$ such that for every $X, Y \in \mathrm{Ob}(\mathsf{C})$ and every $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ the following diagram commutes:

$$\begin{tikzcd} D \arrow[r, "\mu_X"] \arrow[d, "\mathrm{id}_D"'] & F(X) \arrow[d, "F(f)"] \\ D \arrow[r, "\mu_Y"] & F(Y). \end{tikzcd}$$

The set of cones of F is denoted by $\mathrm{Cone}(F)$. A *morphism of cones* of F from (D_1, μ) and (D_2, ν) is morphism $g \in \mathrm{Hom}_{\mathsf{D}}(D_1, D_2)$ such that $\nu \circ \Delta_g = \mu$, where $\Delta_{D_1} \overset{\Delta_g}{\Longrightarrow} \Delta_{D_2}$ is the natural transformation induced by g.

Lemma 2.30. Let C and D be categories. Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor. Then $\mathrm{Cone}(F)$ is a category with the cones as objects and the morphisms of cones as morphisms.

Definition 2.31. Let C and D be categories. Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor. A *limiting cone* for F is a terminal object in $\mathrm{Cone}(F)$. A limiting cone is *small* if C is small.

Remark 2.32. If a limiting cone exists, it must be unique up to unique isomorphism. It is a terminal object in the category of cones over a functor.

A limiting cone will just be called a limit. There are a few limits with special names. They play an important role in determining the behaviour of a category. In the drawings of categories, identity morphisms will be omitted from now on.

Example 2.33 (terminal element)**.** Let D be a category. There is precisely one functor from $\mathsf{0}$ (see Example 1.9) to D. A cone for this functor is a limit precisely if its object in D is terminal.

Example 2.34 (product)**.** Let D be a category. Let $F\colon \mathsf{2} \to \mathsf{D}$ be a functor. The functor F is nothing more than a choice of two objects $D_1, D_2 \in \mathrm{Ob}(\mathsf{D})$, because there are no non-identity morphisms in $\mathsf{2}$. A cone for F is the same as picking an object $D \in \mathrm{Ob}(\mathsf{D})$ and two morphisms $D \xrightarrow{\mu_1} D_1$ and $D \xrightarrow{\mu_2} D_2$. This cone (D, μ) is limiting precisely if for all $E \in \mathrm{Ob}(\mathsf{D})$ and morphisms $E \xrightarrow{\nu_1} D_1$ and $E \xrightarrow{\nu_2} D_2$ there is a unique morphism $f \in \mathrm{Hom}_{\mathsf{D}}(E, D)$ such that the following diagram commutes:

$$\begin{tikzcd} & E \arrow[dl, "\nu_1"'] \arrow[d, "f"] \arrow[dr, "\nu_2"] & \\ D_1 & D \arrow[l, "\mu_1"] \arrow[r, "\mu_2"'] & D_2. \end{tikzcd}$$

If (D, μ) is a limiting cone for the functor $F\colon \mathsf{2} \to \mathsf{D}$, then D is *the product* of the image of F.

Example 2.35 (equalizer)**.** Let D be a category. Let $F\colon \hat{\mathsf{2}} \to \mathsf{D}$ be a functor. The functor F is nothing more than a choice of two objects $D_1, D_2 \in \mathrm{Ob}(\mathsf{D})$ and two morphisms $f, g \in \mathrm{Hom}_{\mathsf{D}}(D_1, D_2)$. A cone for F is the same as picking an object $D \in \mathrm{Ob}(\mathsf{D})$ and two morphisms $D \xrightarrow{\mu_1} D_1$ and $D \xrightarrow{\mu_2} D_2$ such that $f \circ \mu_1 = \mu_2 = g \circ \mu_1$. This cone (D, μ) is limiting precisely if for all $E \in \mathrm{Ob}(\mathsf{D})$ and morphisms $E \xrightarrow{\nu_1} D_1$ and $E \xrightarrow{\nu_2} D_2$ such that $f \circ \nu_1 = \nu_2 = g \circ \nu_1$ there exists a unique morphism $h \in \mathrm{Hom}_{\mathsf{D}}(E, D)$ such that the following diagram commutes:

$$\begin{tikzcd}
& E \arrow[d, "h"] \arrow[ddl, "\nu_1"'] \arrow[ddr, "\nu_2"] & \\
& D \arrow[dl, "\mu_1"] \arrow[dr, "\mu_2"'] & \\
D_1 \arrow[rr, "f"] \arrow[rr, bend right, "g"'] & & D_2.
\end{tikzcd}$$

If (D, μ) is a limiting cone for the functor $F\colon \hat{\mathsf{2}} \to \mathsf{D}$, then D is *the equalizer* of the image of F.

Example 2.36 (pullback)**.** Let D be a category. Let $F\colon \bar{\mathsf{3}}^{\mathrm{opp}} \to \mathsf{D}$ be a functor. The functor F is nothing more than a choice of three objects $D_1, D_2, D_3 \in \mathrm{Ob}(\mathsf{D})$ and morphisms $f \in \mathrm{Hom}_{\mathsf{D}}(D_2, D_1)$ and $g \in \mathrm{Hom}_{\mathsf{D}}(D_3, D_1)$. A cone for F is the same as picking an object $D \in \mathrm{Ob}(\mathsf{D})$ and three morphisms $D \xrightarrow{\mu_1} D_1$, $D \xrightarrow{\mu_2} D_2$ and $D \xrightarrow{\mu_3} D_3$ such that $f \circ \mu_2 = \mu_1 = g \circ \mu_3$. This cone (D, μ) is limiting precisely if for all $E \in \mathrm{Ob}(\mathsf{D})$ and morphisms $E \xrightarrow{\nu_1} D_1$, $E \xrightarrow{\nu_2} D_2$ and $E \xrightarrow{\nu_3} D_3$ such that $f \circ \nu_2 = \nu_1 = g \circ \nu_3$ there exists a unique morphism $h \in \mathrm{Hom}_{\mathsf{D}}(E, D)$ such that the following diagram commutes:

$$\begin{tikzcd}
E \arrow[dr, "h"] \arrow[drr, bend left, "\nu_3"] \arrow[ddrr, bend left, "\nu_1"] \arrow[ddr, bend right, "\nu_2"'] & & \\
& D \arrow[r, "\mu_3"'] \arrow[d, "\mu_2"'] \arrow[dr, bend right, "\mu_1"] & D_3 \arrow[d, "g"] \\
& D_2 \arrow[r, "f"'] & D_1.
\end{tikzcd}$$

If (D, μ) is a limiting cone for the functor $F\colon \bar{\mathsf{3}}^{\mathrm{opp}} \to \mathsf{D}$, then D is *the pullback* of the image of F.

Remark 2.37. The morphisms μ_3 and ν_3 and the conditions $f \circ \mu_2 = \mu_1 = g \circ \mu_3$ and $f \circ \nu_2 = \nu_1 = g \circ \nu_3$ make the following squares commute:

$$\begin{tikzcd}
D \arrow[r, "\mu_3"] \arrow[d, "\mu_2"'] \arrow[dr, "\mu_1"] & D_3 \arrow[d, "g"] & E \arrow[r, "\nu_3"] \arrow[d, "\nu_2"'] \arrow[dr, "\nu_1"] & D_3 \arrow[d, "g"] \\
D_2 \arrow[r, "f"'] & D_1 & D_2 \arrow[r, "f"'] & D_1.
\end{tikzcd}$$

For clarity reasons they are omitted from the diagrams in the onward parts. Instead it is assumed that the diagrams commute and then automatically both directions equal μ_1 and ν_1.

Remark 2.38. Sometimes the pullback is called the *fiber product*. For example in $R-\mathsf{Mod}$ (for a ring R) it is common practice to call the pullbacks fiber products.

Example 2.39 (inverse limit)**.** Let D be a category and let $F\colon \mathbb{N}^{\mathrm{opp}} \to \mathsf{D}$ be a functor. The functor F is nothing more than a choice of objects $D_i \in \mathrm{Ob}(\mathsf{D})$ for $i \in \mathbb{N}$ and morphisms $D_j \xrightarrow{f_{ij}} D_i$ with $i \le j$ such that $f_{ij} \circ f_{jk} = f_{ik}$ for $i \le j \le k$ (and $f_{ii} = \mathrm{id}_{D_i}$ for all $i \in \mathbb{N}$). A cone of F is the same as picking an object $D \in \mathrm{Ob}(\mathsf{D})$ and morphisms $D \xrightarrow{\mu_i} D_i$ such that $f_{ij} \circ \mu_j = \mu_i$ for $i \le j$. This cone (D,μ) is limiting precisely if for all $E \in \mathrm{Ob}(\mathsf{D})$ and morphisms $E \xrightarrow{\nu_i} D_i$ such that $f_{ij} \circ \nu_j = \nu_i$ for $i \le j$ there exists a unique $h \in \mathrm{Hom}_{\mathsf{D}}(E,D)$ such that the following diagram commutes:

$$
\begin{tikzcd}
 & & E \arrow[d, bend right, "h"'] \arrow[ddl, bend right, "\nu_{i-1}"'] \arrow[dd, bend left, "\nu_i"] \arrow[ddr, bend left, "\nu_{i+1}"] & & \\
 & & D \arrow[dl, "\mu_{i-1}"'] \arrow[d, "\mu_i"] \arrow[dr, "\mu_{i+1}"] & & \\
\cdots & D_{i-1} \arrow[l] & D_i \arrow[l, "f_{i-1,i}"] & D_{i+1} \arrow[l, "f_{i,i+1}"] & \cdots. \arrow[l]
\end{tikzcd}
$$

If (D,μ) is a limiting cone for the functor $F\colon \mathbb{N}^{\mathrm{opp}} \to \mathsf{D}$, then D is the *inverse limit* of the image of F.

The following definitions and examples are dual to the definitions and examples given in this paragraph so far.

Definition 2.40 (Dual to Definition 2.29)**.** Let C and D be categories. Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor. A *cocone for* F is a pair (D,μ) of an object $D \in \mathrm{Ob}(\mathsf{D})$ and a natural transformation $F \overset{\mu}{\Longrightarrow} \Delta_D$. In other words, it is a collection of morphisms $F(X) \xrightarrow{\mu_X} D$ for $X \in \mathrm{Ob}(\mathsf{C})$ such that for every $X, Y \in \mathrm{Ob}(\mathsf{C})$ and every $f \in \mathrm{Hom}_{\mathsf{C}}(X,Y)$ the following diagram commutes:

$$
\begin{tikzcd}
F(X) \arrow[r, "\mu_X"] \arrow[d, "F(f)"'] & D \arrow[d, "\mathrm{id}_D"] \\
F(Y) \arrow[r, "\mu_Y"] & D.
\end{tikzcd}
$$

The set of cocones of F is denoted by $\mathrm{Cocone}(F)$. A *morphism of cocones* of F from (D_1,μ) and (D_2,ν) is morphism $g \in \mathrm{Hom}_{\mathsf{D}}(D_1,D_2)$ such that $\nu = \Delta_g \circ \mu$.

Lemma 2.41. Let C and D be categories. Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor. Then $\mathrm{Cocone}(F)$ is a category with the cocones as objects and the morphisms of cocones as morphisms.

Remark 2.42. The motivation for the dual construction of the category of cocones is the following. A functor $F\colon \mathsf{C} \to \mathsf{D}$ induces a (unique) *dual* functor $F^{\mathrm{opp}}\colon \mathsf{C}^{\mathrm{opp}} \to \mathsf{D}^{\mathrm{opp}}$. A terminal cone of $F^{\mathrm{opp}}\colon \mathsf{C}^{\mathrm{opp}} \to \mathsf{D}^{\mathrm{opp}}$ becomes an initial cocone when we dualise (reverse all arrows of) D^{opp} again. To find the dual statements for the product, equalizer, pullback and inverse limit, we need to work with the opposite categories of $\mathsf{2}$, $\hat{\mathsf{2}}$, $\bar{\mathsf{3}}^{\mathrm{opp}}$ and $\mathbb{N}^{\mathrm{opp}}$. Luckily, $\mathsf{2}$ and $\hat{\mathsf{2}}$ are isomorphic to their opposites. For $\bar{\mathsf{3}}$ and $\mathbb{N}$ this is not the case.

Definition 2.43. Let C and D be categories. Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor. A *colimiting cone* for F is an initial object in $\mathrm{Cocone}(F)$. A colimiting cocone is *small* if C is small.

Remark 2.44. If a colimiting cocone exists, it must be unique up to unique isomorphism. It is an initial object in the category of cocones over a functor.

Example 2.45 (initial element)**.** Let D be a category. There is precisely one functor from $\mathsf{0}$ (see Example 1.9) to D. A cocone for this functor is a colimit precisely if its object in D is initial.

Example 2.46 (coproduct)**.** Let D be a category. Let $F\colon \mathsf{2} \to \mathsf{D}$ be a functor. The functor F is nothing more than a choice of two objects $D_1, D_2 \in \mathrm{Ob}(\mathsf{D})$, because there are no non-identity morphisms in $\mathsf{2}$. A cocone for F is the same as picking an object $D \in \mathrm{Ob}(\mathsf{D})$ and two morphisms $D_1 \xrightarrow{\mu_1} D$ and $D_2 \xrightarrow{\mu_2} D$. This cocone (D, μ) is colimiting precisely if for all $E \in \mathrm{Ob}(\mathsf{D})$ and morphisms $D_1 \xrightarrow{\nu_1} E$ and $D_2 \xrightarrow{\nu_2} E$ there is a unique morphism $f \in \mathrm{Hom}_{\mathsf{D}}(D, E)$ such that the following diagram commutes:

$$\begin{tikzcd} D_1 \arrow[r, "\mu_1"] \arrow[dr, "\nu_1"'] & D \arrow[d, "f"] & D_2. \arrow[l, "\mu_2"'] \arrow[dl, "\nu_2"] \\ & E & \end{tikzcd}$$

If (D, μ) is a colimiting cocone for the functor $F\colon \mathsf{2} \to \mathsf{D}$, then D is *the coproduct* of the image of F.

Example 2.47 (coequalizer)**.** Let D be a category. Let $F\colon \hat{\mathsf{2}} \to \mathsf{D}$ be a functor. The functor F is nothing more than a choice of two objects $D_1, D_2 \in \mathrm{Ob}(\mathsf{D})$ and two morphisms $f, g \in \mathrm{Hom}_{\mathsf{D}}(D_1, D_2)$. A cone for F is the same as picking an object $D \in \mathrm{Ob}(\mathsf{D})$ and two morphisms $D_1 \xrightarrow{\mu_1} D$ and $D_2 \xrightarrow{\mu_2} D$ such that $\mu_2 \circ f = \mu_1 = \mu_2 \circ g$. This cocone (D, μ) is colimiting precisely if for all $E \in \mathrm{Ob}(\mathsf{D})$ and morphisms $D_1 \xrightarrow{\nu_1} E$ and $D_2 \xrightarrow{\nu_2} E$ such that $\nu_2 \circ f = \nu_1 = \nu_2 \circ g$ there exists a unique morphism $h \in \mathrm{Hom}_{\mathsf{D}}(D, E)$ such that the following diagram commutes:

$$\begin{tikzcd} D_1 \arrow[rr, bend left, "f"] \arrow[rr, "g"'] \arrow[dr, "\mu_1"] \arrow[ddr, "\nu_1"'] & & D_2. \arrow[dl, "\mu_2"'] \arrow[ddl, "\nu_2"] \\ & D \arrow[d, "h"] & \\ & E & \end{tikzcd}$$

If (D, μ) is a colimiting cocone for the functor $F\colon \hat{\mathsf{2}} \to \mathsf{D}$, then D is *the coequalizer* of the image of F.

Example 2.48 (pushout)**.** Let D be a category. Let $F\colon \bar{\mathsf{3}} \to \mathsf{D}$ be a functor. The functor F is nothing more than a choice of three objects $D_1, D_2, D_3 \in \mathrm{Ob}(\mathsf{D})$ and morphisms $f \in \mathrm{Hom}_{\mathsf{D}}(D_1, D_2)$ and $g \in \mathrm{Hom}_{\mathsf{D}}(D_1, D_3)$. A cocone for F is the same as picking an object $D \in \mathrm{Ob}(\mathsf{D})$ and three morphisms $D_1 \xrightarrow{\mu_1} D$, $D_2 \xrightarrow{\mu_2} D$ and $D_3 \xrightarrow{\mu_3} D$ such that $\mu_2 \circ f = \mu_1 \circ = \mu_3 \circ g$. This cocone (D, μ) is colimiting precisely if for all $E \in \mathrm{Ob}(\mathsf{D})$ and morphisms $D_1 \xrightarrow{\nu_1} E$, $D_2 \xrightarrow{\nu_2} E$ and $D_3 \xrightarrow{\nu_3} E$ such that $\nu_2 \circ f = \nu_1 = \nu_3 \circ g$ there exists a unique morphism $h \in \mathrm{Hom}_{\mathsf{D}}(D, E)$ such that the following diagram commutes:

$$\begin{tikzcd}
D_1 \arrow[r, "\mu_3"] \arrow[d, "\mu_2"'] \arrow[dr, bend right, "\mu_1"] \arrow[drr, bend left, "\nu_1"] & D_3 \arrow[d, "g"'] \arrow[dr, bend left, "\nu_3"] & \\
D_2 \arrow[r, "f"'] \arrow[rr, bend right, "\nu_2"'] & D \arrow[r, bend right, "h"] & E.
\end{tikzcd}$$

If (D, μ) is a colimiting cocone for the functor $F\colon \bar{\mathsf{3}} \to \mathsf{D}$, then D is *the pushout* of the image of F.

Example 2.49 (direct limit)**.** Let D be a category and let $F\colon \mathbb{N} \to \mathsf{D}$ be a functor. The functor F is nothing more than a choice of objects $D_i \in \mathrm{Ob}(\mathsf{D})$ for $i \in \mathbb{N}$ and morphisms $D_i \xrightarrow{f_{ij}} D_j$ with $i \leq j$ such that $f_{jk} \circ f_{ij} = f_{ik}$ for $i \leq j \leq k$ (and $f_{ii} = \mathrm{id}_{D_i}$ for all $i \in \mathbb{N}$). A cocone of F is the same as picking an object $D \in \mathrm{Ob}(\mathsf{D})$ and morphisms $D_i \xrightarrow{\mu_i} D$ such that $\mu_j \circ f_{ij} = \mu_i$ for $i \leq j$. This cocone (D, μ) is colimiting precisely if for all $E \in \mathrm{Ob}(\mathsf{D})$ and morphisms $D_i \xrightarrow{\nu_i} E$ such that $\nu_j \circ f_{ij} = \nu_i$ for $i \leq j$ there exists a unique $h \in \mathrm{Hom}_{\mathsf{D}}(D, E)$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & D_{i-1} \arrow[r, "f_{i-1,i}"] \arrow[dr, "\mu_{i-1}"'] \arrow[ddr, bend right, "\nu_{i-1}"'] & D_i \arrow[r, "f_{i,i+1}"] \arrow[d, "\mu_i"'] \arrow[dd, bend left, "\nu_i"] & D_{i+1} \arrow[r] \arrow[dl, "\mu_{i+1}"] \arrow[ddl, bend left, "\nu_{i+1}"] & \cdots. \\
& & D \arrow[d, bend right, "h"] & & \\
& & E & &
\end{tikzcd}$$

If (D, μ) is a colimiting cocone for the functor $F\colon \mathbb{N} \to \mathsf{D}$, then D is *the direct limit* of the image of F.

Remark 2.50. A direct limit is a categorical **colimit**. Its dual concept, i.e. a limit over the poset $\mathbb{N}$ is called an *inverse* limit. For this see Example 2.39.

Remark 2.51. There is not really widely-used notation for limits or colimits. Some authors use $\varinjlim$ for any categorical colimit and $\varprojlim$ for any categorical limit. In this thesis $\varinjlim$ and $\varprojlim$ will be reserved for direct and inverse limits. Subscripts will only be used when it is not clear over which index the limit is taken. This occurs for example in Theorem 8.45 and Theorem 8.51

Example 2.52. It is easy to see that Field does not have all small limits nor does Field have all small colimits. To have all small limits, it must at least have a terminal object. It has no terminal object. Dually, to have all small colimits, it must have at least an initial object. It has no initial object. See Example 1.7.

Definition 2.53. A category is *complete* if it has all small limits. A category is *cocomplete* if it has all small colimits.

Theorem 2.54. Equivalences of categories preserve and reflect all limits and colimits.

Corollary 2.55. Let C and D be equivalent categories. If C is (co)complete, then so is D.

2.3 Towers of categories

In this subsection we will elaborate on the categorical structure of inverse and direct limits. We will show that inverse and direct limits actually induce functors between (co)complete categories. Although inverse and direct limits do not always commute with other functors between (co)complete categories, it is possible to form a morphism of functors from the inverse or direct limit composed with a functor to the functor composed with an inverse or direct limit. Many of the constructions work for other (co)limits, but in this thesis we will only need them for inverse and direct limits. We will draw many diagrams. The results of this subsection will be used in the proof of the UCT. For instance, in Proposition 15.5.

Definition 2.56. Let C be a category. A *tower* in C consists of a sequence $\{X_i\}_{i=1}^{\infty}$ of objects in C and morphisms $\{f_{i,i+1}\}_{i=1}^{\infty}$ such that $f_{i,i+1} \in \mathrm{Hom}_{\mathsf{C}}(X_{i+1}, X_i)$. They make the following diagram:

$$X_1 \xleftarrow[f_{1,2}]{} X_2 \longleftarrow \cdots \longleftarrow X_{i-1} \xleftarrow[f_{i-1,i}]{} X_i \xleftarrow[f_{i,i+1}]{} X_{i+1} \longleftarrow \cdots .$$

A *morphisms of towers in* C between $(\{X_i\}_{i=1}^{\infty}, \{f_{i,i+1}\}_{i=1}^{\infty})$ and $(\{Y_i\}_{i=1}^{\infty}, \{g_{i,i+1}\}_{i=1}^{\infty})$ is formed by morphisms $\{d_i\colon X_i \xrightarrow{d_i} Y_i\}_{i=1}^{\infty}$ such that the following diagram commutes:

$$\begin{tikzcd}
X_1 \arrow[d, "d_1"] & X_2 \arrow[l, "f_{1,2}"'] \arrow[d, "d_2"] & \cdots \arrow[l] \arrow[d, "\cdots"] & X_{i-1} \arrow[l] \arrow[d, "d_{i-1}"] & X_i \arrow[l, "f_{i-1,i}"'] \arrow[d, "d_i"] & X_{i+1} \arrow[l, "f_{i,i+1}"'] \arrow[d, "d_{i+1}"] & \cdots \arrow[l] \\
Y_1 & Y_2 \arrow[l, "g_{1,2}"] & \cdots \arrow[l] & Y_{i-1} \arrow[l] & Y_i \arrow[l, "g_{i-1,i}"] & Y_{i+1} \arrow[l, "g_{i,i+1}"] & \cdots . \arrow[l]
\end{tikzcd}$$

Definition 2.57 (Dual to Definition 2.56)**.** Let C be a category. A *cotower* in C consists of a sequence $\{X_i\}_{i=1}^{\infty}$ of objects in C and morphisms $\{f_{i,i+1}\}_{i=1}^{\infty}$ such that $f_{i,i+1} \in \mathrm{Hom}_{\mathsf{C}}(X_i, X_{i+1})$. They make the following diagram:

$$X_1 \xrightarrow{f_{1,2}} X_2 \longrightarrow \cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1,i}} X_i \xrightarrow{f_{i,i+1}} X_{i+1} \longrightarrow \cdots .$$

A *morphisms of cotowers in* C between $(\{X_i\}_{i=1}^{\infty}, \{f_{i,i+1}\}_{i=1}^{\infty})$ and $(\{Y_i\}_{i=1}^{\infty}, \{g_{i,i+1}\}_{i=1}^{\infty})$ is formed by morphisms $\{d_i\colon X_i \xrightarrow{d_i} Y_i\}_{i=1}^{\infty}$ such that the following diagram commutes:

$$\begin{tikzcd}
X_1 \arrow[r, "f_{1,2}"] \arrow[d, "d_1"] & X_2 \arrow[r] \arrow[d, "d_2"] & \cdots \arrow[r] \arrow[d, "\cdots"] & X_{i-1} \arrow[r, "f_{i-1,i}"] \arrow[d, "d_{i-1}"] & X_i \arrow[r, "f_{i,i+1}"] \arrow[d, "d_i"] & X_{i+1} \arrow[r] \arrow[d, "d_{i+1}"] & \cdots \\
Y_1 \arrow[r, "g_{1,2}"] & Y_2 \arrow[r] & \cdots \arrow[r] & Y_{i-1} \arrow[r, "g_{i-1,i}"] & Y_i \arrow[r, "g_{i,i+1}"] & Y_{i+1} \arrow[r] & \cdots .
\end{tikzcd}$$

Lemma 2.58. Towers and cotowers are preserved by functors.

Proof. Just apply the functor to the diagrams. □

Lemma 2.59. Contravariant functors turn towers into cotowers and vice versa.

Proof. Applying a contravariant functor reverses the arrows. □

Proposition 2.60. Let C and D be categories. Let $F, G\colon \mathsf{C} \to \mathsf{D}$ be functors. Let $F \overset{\eta}{\Longrightarrow} G$ be a morphism of functors. Let $(\{X_i\}_{i=1}^{\infty}, \{f_{i,i+1}\}_{i=1}^{\infty})$ be a (co)tower in C. The components of η induce a morphism of (co)towers in D between $(\{F(X_i)\}_{i=1}^{\infty}, \{F(f_{i,i+1})\}_{i=1}^{\infty})$ and $(\{G(X_i)\}_{i=1}^{\infty}, \{G(f_{i,i+1})\}_{i=1}^{\infty})$. Moreover, if the components η_{X_i} are isomorphic (in D) for every $i \in \mathbb{N}$, then the induced morphism of (co)towers in D between $(\{F(X_i)\}_{i=1}^{\infty}, \{F(f_{i,i+1})\}_{i=1}^{\infty})$ and $(\{G(X_i)\}_{i=1}^{\infty}, \{G(f_{i,i+1})\}_{i=1}^{\infty})$ is an isomorphism.

Proof. By Lemma 2.58, $(\{F(X_i)\}_{i=1}^{\infty}, \{F(f_{i,i+1})\}_{i=1}^{\infty})$ and $(\{G(X_i)\}_{i=1}^{\infty}, \{G(f_{i,i+1})\}_{i=1}^{\infty})$ are (co)towers in D. As η is a morphisms of functors, the following diagram commutes in D:

$$\begin{tikzcd}
F(X_1) \arrow[r, "F(f_{1,2})"] \arrow[d, "\eta_{X_1}"] & F(X_2) \arrow[r] \arrow[d, "\eta_{X_2}"] & \cdots \arrow[r] \arrow[d, "\cdots"] & F(X_{i-1}) \arrow[r, "F(f_{i-1,i})"] \arrow[d, "\eta_{X_{i-1}}"] & F(X_i) \arrow[r, "F(f_{i,i+1})"] \arrow[d, "\eta_{X_i}"] & F(X_{i+1}) \arrow[r] \arrow[d, "\eta_{X_{i+1}}"] & \cdots \\
G(X_1) \arrow[r, "G(f_{1,2})"] & G(X_2) \arrow[r] & \cdots \arrow[r] & G(X_{i-1}) \arrow[r, "G(f_{i-1,i})"] & G(X_i) \arrow[r, "G(f_{i,i+1})"] & G(X_{i+1}). \arrow[r] & \cdots.
\end{tikzcd}$$

In particular, the sequence $\{\eta_{X_i}\colon F(X_i) \xrightarrow{\eta_{X_i}} G(X_i)\}_{i=1}^{\infty}$ forms a morphism of (co)towers.

If η_{X_i} is isomorphic for every $i \in \mathbb{N}$, then $\{\eta_{X_i}^{-1}\colon G(X_i) \xrightarrow{\eta_{X_i}^{-1}} F(X_i)\}_{i=1}^{\infty}$ forms the inverse morphism of (co)towers in D. See Lemma 2.23. $\square$

Definition 2.61. The category of towers in C is denoted by $\mathrm{Fun}(\mathbb{N}^{\mathrm{opp}}, \mathsf{C})$. The category of cotowers in C is denoted by $\mathrm{Fun}(\mathbb{N}, \mathsf{C})$.

Remark 2.62. There is a different notation for functor categories that puts the domain category in the superscript. That notation makes less clear that towers in C are just functors from $\mathbb{N}^{\mathrm{opp}}$ to C and morphisms of towers in C are morphisms of functors from $\mathbb{N}^{\mathrm{opp}}$ to C.

Remark 2.63. Applying a functor to a (co)tower is nothing more than postcomposing the (co)tower with the functor. The proof of Proposition 2.60 shows explicitly how a morphism of functors maps postcomposition by one functor to the other.

Now that we have identified (co)towers with functors, the connection with (co)cones and (co)limits can be made. In fact, inverse and direct limits define functors, whenever the category in which the tower exists is (co)complete.

Proposition 2.64. Let C be a complete category. The inverse limit of a tower defines a functor $\varprojlim\colon \mathrm{Fun}(\mathbb{N}^{\mathrm{opp}}, \mathsf{C}) \to \mathsf{C}$.

Proof. Because C is complete, an inverse limit exist. Now we will show a morphism of towers induces a morphism in C between the inverse limits. Let $(\{X_i\}_{i=1}^{\infty}, \{f_{i,i+1}\}_{i=1}^{\infty})$ and $(\{Y_i\}_{i=1}^{\infty}, \{g_{i,i+1}\}_{i=1}^{\infty})$ be a towers in C. Let $\{d_i\colon X_i \xrightarrow{d_i} Y_i\}_{i=1}^{\infty}$ be a morphism of towers between them. Define $\mathrm{Ob}(\mathsf{C}) \ni X \coloneqq$

$\varprojlim X_i$ and $\mathrm{Ob}(\mathsf{C}) \ni Y := \varprojlim Y_i$. Putting everything together gives the following diagram:

$$
\begin{tikzcd}
 & & X \arrow[dl, "\mu_{i-1}"'] \arrow[d, "\mu_i"] \arrow[dr, "\mu_{i+1}"] \arrow[dd, dashed, bend right=80, "\exists!\, d"', start anchor=west, end anchor=west, to path={[out=180,in=180] (\tikztostart) to (\tikztotarget)}] & & \\
\cdots & X_{i-1} \arrow[l] \arrow[d, "d_{i-1}"] & X_i \arrow[l, "f_{i-1,i}"] \arrow[d, "d_i"] & X_{i+1} \arrow[l, "f_{i,i+1}"] \arrow[d, "d_{i+1}"] & \cdots \arrow[l] \\
\cdots & Y_{i-1} \arrow[l] & Y_i \arrow[l, "g_{i-1,i}"] & Y_{i+1} \arrow[l, "g_{i,i+1}"] & \cdots . \arrow[l] \\
 & & Y \arrow[ul, "\nu_{i-1}"] \arrow[u, "\nu_i"] \arrow[ur, "\nu_{i+1}"'] & &
\end{tikzcd}
\tag{3}
$$

The $\{\mu_i\}_{i=1}^{\infty}$ form the structure morphisms for the inverse limit X and the $\{\nu_i\}_{i=1}^{\infty}$ form the structure morphisms for the inverse limit Y. The compositions $\{d_i \circ \mu_i\}_{i=1}^{\infty}$ form a cone of the tower $(\{Y_i\}_{i=1}^{\infty}, \{g_{i,i+1}\}_{i=1}^{\infty})$. There exists a unique morphism $d \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ such that the diagram commutes. This construction preserves identities. This construction also preserves compositions. The composition of morphisms in C between the limits forms a morphisms cones. A limiting cone is terminal. Hence it must coincide with the morphism obtained by first composing morphisms of towers and then factoring through the limit. $\square$

Proposition 2.65 (Dual to Proposition 2.64)**.** Let C be a cocomplete category. The direct limit of a tower defines a functor $\varinjlim : \mathrm{Fun}(\mathbb{N}, \mathsf{C}) \to \mathsf{C}$.

Proof. The proof is the same as the proof of Proposition 2.64. Take cotowers instead of towers and take direct limits instead of inverse limits. Then the composition of a morphism of cotowers with the structure morphisms of the direct limit of the codomain make a cocone. The colimit property of the domain can be used to obtain a morphism in C between the direct limits. $\square$

Functors need not preserve inverse or direct limits. When a functor has a (co)complete domain and codomain, it is possible to establish a morphism in the codomain category connecting the image of the limit under the functor with the limit of the image under the functor. We will show how this is done in the following example and its dual. After that, we will see that this construction has a deeper categorical meaning.

Example 2.66. Let C and D be complete categories. Let $(\{X_i\}_{i=1}^{\infty}, \{f_{i,i+1}\}_{i=1}^{\infty})$ be a tower in C. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. Then there is a unique morphism of cones $(\{F(X_i)\}_{i=1}^{\infty}, \{F(f_{i,i+1})\}_{i=1}^{\infty})$ from $F(\varprojlim X_i)$ to $\varprojlim F(X_i)$. As C is complete, $(\{X_i\}_{i=1}^{\infty}, \{f_{i,i+1}\}_{i=1}^{\infty})$ allows an inverse limit $X = \varprojlim X_i$ (see Example 2.39). The following diagram shows the structure morphisms of the inverse limit:

$$
\begin{tikzcd}
 & & X \arrow[dl, "\mu_{i-1}"'] \arrow[d, "\mu_i"] \arrow[dr, "\mu_{i+1}"] & & \\
\cdots & X_{i-1} \arrow[l] & X_i \arrow[l, "f_{i-1,i}"] & X_{i+1} \arrow[l, "f_{i,i+1}"] & \cdots . \arrow[l]
\end{tikzcd}
$$

Applying F to the above diagram gives the following diagram:

$$
\begin{tikzcd}
 & & F(X) \arrow[dl, "F(\mu_{i-1})"'] \arrow[d, "F(\mu_i)"] \arrow[dr, "F(\mu_{i+1})"] & & \\
\cdots & F(X_{i-1}) \arrow[l] & F(X_i) \arrow[l, "F(f_{i-1,i})"] & F(X_{i+1}) \arrow[l, "F(f_{i,i+1})"] & \cdots . \arrow[l]
\end{tikzcd}
$$

As D is complete. The above diagram allows an inverse limit $Y = \varprojlim F(X_i)$. This means there is unique $h \in \mathrm{Hom}_{\mathsf{D}}(F(X), Y)$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & F(X) \arrow[d, "h"] \arrow[dl, bend right, "F(\mu_{i-1})"'] \arrow[dd, bend left, "F(\mu_i)"] \arrow[ddr, bend left, "F(\mu_{i+1})"] \arrow[ddl, bend right] & & \\
 & & Y \arrow[dl, "\nu_{i-1}"'] \arrow[d, "\nu_i"] \arrow[dr, "\nu_{i+1}"] & & \\
\cdots & F(X_{i-1}) \arrow[l] & F(X_i) \arrow[l, "F(f_{i-1,i})"] & F(X_{i+1}) \arrow[l, "F(f_{i,i+1})"] & \cdots . \arrow[l]
\end{tikzcd}$$

(In the above diagram the sequence $\{\nu_i\}_{i=1}^{\infty}$ forms the structure morphisms for the inverse limit). Because limits are unique up to isomorphisms, h is an isomorphism if and only if F preserves the limit, i.e. $\varprojlim F(X_i) = F(\varprojlim X_i)$.

Example 2.67 (Dual to Example 2.66)**.** Let C and D be cocomplete categories. Let $(\{X_i\}_{i=1}^{\infty}, \{f_{i,i+1}\}_{i=1}^{\infty})$ be a cotower in C. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. Then there is unique morphisms of cocones of $(\{F(X_i)\}_{i=1}^{\infty}, \{F(f_{i,i+1})\}_{i=1}^{\infty})$ from $\varinjlim F(X_i)$ to $F(\varinjlim X_i)$. Because C is cocomplete, $(\{X_i\}_{i=1}^{\infty}, \{f_{i,i+1}\}_{i=1}^{\infty})$ allows a direct limit $X = \varinjlim X_i$ (see Example 2.49). The diagrams are the same as in Example 2.66, but with arrows reversed.

Remark 2.68. Example 2.66 also works if the functor is contravariant. In that case the domain category has to be complete and the codomain category has to be cocomplete or vice versa.

Example 2.66 and its dual tell a lot more than one might think initially!. They actually define a morphism between functors. For the precise statements, see the following theorems.

Theorem 2.69. Let C and D be complete categories. Let $F : \mathsf{C} \to \mathsf{D}$ be a functor. The construction in Example 2.66 defines a morphism of functors from the functor $\mathrm{Fun}(\mathbb{N}^{\mathrm{opp}}, \mathsf{C}) \xrightarrow{\varprojlim} \mathsf{C} \xrightarrow{F} \mathsf{D}$ to the functor $\mathrm{Fun}(\mathbb{N}^{\mathrm{opp}}, \mathsf{C}) \xrightarrow{F\circ\_} \mathrm{Fun}(\mathbb{N}^{\mathrm{opp}}, \mathsf{D}) \xrightarrow{\varprojlim} \mathsf{D}$.

Proof. The morphisms in D constructed in Example 2.66 will form the components at the towers. It remains to show that the morphisms of Example 2.66 commute with images of morphisms in $\mathrm{Fun}(\mathbb{N}^{\mathrm{opp}}, \mathsf{C})$ under the functors, so that a commutative diagram like diagram (2) is formed for every morphisms of towers.

Let $(\{X_i\}_{i=1}^{\infty}, \{f_{i,i+1}\}_{i=1}^{\infty})$ and $(\{Y_i\}_{i=1}^{\infty}, \{g_{i,i+1}\}_{i=1}^{\infty})$ be a towers in C. Let $\{d_i : X_i \xrightarrow{d_i} Y_i\}_{i=1}^{\infty}$ be a morphism of towers between them. Define $\mathrm{Ob}(\mathsf{C}) \ni X := \varprojlim X_i$ and $\mathrm{Ob}(\mathsf{C}) \ni Y := \varprojlim Y_i$. And apply F to all the objects and morphisms. Diagram (3) shows what we have right now. If we take limits in D, then we get the following situation:

$$
\begin{tikzcd}
 & & F(X) \arrow[dl, "F(\mu_{i-1})"'] \arrow[d, "F(\mu_i)"] \arrow[dr, "F(\mu_{i+1})"] \arrow[rrr, dotted] \arrow[dddd, dashed, bend right=90, "F(d)"'] & & & \varprojlim F(X_i) \arrow[dllll] \arrow[dlll] \arrow[dll] \arrow[ddd, dotted, bend left] \\
\cdots & F(X_{i-1}) \arrow[l] \arrow[d, "F(d_{i-1})"] & F(X_i) \arrow[l, "F(f_{i-1,i})"] \arrow[d, "F(d_i)"] & F(X_{i+1}) \arrow[l, "F(f_{i,i+1})"] \arrow[d, "F(d_{i+1})"] & \cdots \arrow[l] & \\
\cdots & F(Y_{i-1}) \arrow[l] & F(Y_i) \arrow[l, "F(g_{i-1,i})"'] & F(Y_{i+1}) \arrow[l, "F(g_{i,i+1})"'] & \cdots \arrow[l] & \\
 & & & & & \varprojlim F(Y_i). \arrow[ullll] \arrow[ulll] \arrow[ull] \\
 & & F(Y) \arrow[uul, "F(\nu_{i-1})"] \arrow[uu, "F(\nu_i)"] \arrow[uur, "F(\nu_{i+1})"'] \arrow[urrr, dashed] & & &
\end{tikzcd}
\tag{4}
$$

The compositions $\{d_i \circ \mu_i\}_{i=1}^{\infty}$ form a cone of $(\{F(Y_i)\}_{i=1}^{\infty}, \{F(g_{i,i+1})\}_{i=1}^{\infty})$. Both the dashed morphism $F(X) \xrightarrow{F(d)} F(Y) \to \varprojlim F(Y_i)$ and the dotted morphism $F(X) \to \varprojlim F(X_i) \to \varprojlim F(Y_i)$ form morphisms of cones of $(\{F(Y_i)\}_{i=1}^{\infty}, \{F(g_{i,i+1})\}_{i=1}^{\infty})$. As $\varprojlim F(Y_i)$ is terminal, the two morphisms must coincide. $\square$

Theorem 2.70 (Dual to Theorem2.69)**.** Let C and D be cocomplete categories. Let $F\colon \mathsf{C} \to \mathsf{D}$ be a functor. The construction in Example 2.67 defines a morphism of functors from the functor $\mathrm{Fun}(\mathbb{N}, \mathsf{C}) \xrightarrow{F \circ \_} \mathrm{Fun}(\mathbb{N}, \mathsf{D}) \xrightarrow{\varinjlim} \mathsf{D}$ to the functor $\mathrm{Fun}(\mathbb{N}, \mathsf{C}) \xrightarrow{\varinjlim} \mathsf{C} \xrightarrow{F} \mathsf{D}$.

Proof. The proof is the same as the proof of Theorem 2.69. Take cotowers instead of towers and take direct limits instead of inverse limits. The cones become cocones. The dashed and dotted arrows of diagram (4) commute, because a colimiting cone is an initial one. $\square$

2.4 Representable functors

In this subsection special attention is dedicated to the *representable* functors. They preserve limits (or turn colimits into limits in the contravariant case). This is certainly not a property that all functors possess. For an explicit example, think about the inclusion of $\mathsf{1}$ (Example 1.10) into a category where the object is sent to a non-terminal or non-initial object. See Examples 2.33 and 2.45. We expand the definition of representable before we give a proof. See the following remark.

Remark 2.71. In Example 2.9, a definition of representable covariant and contravariant functors is given. From this point the definition is extended to any functor *isomorphic* to functors of the form in Example 2.9.

Remark 2.72. Some authors call covariant representable functors *corepresentable* and reserve representable for contravariant functors. In this thesis, it will always be explicitly stated if a functor is contravariant.

Theorem 2.73. Covariant representable functors preserve limits.

Proof. The proof works exactly the same over any limit (see [18, Proposition 6.2.2]). To give an idea of the proof, we will work with inverse limits. Let D be a category. Take $\mathrm{Ob}(\mathsf{D}) \ni X = \varprojlim X_i$. Applying $\mathrm{Hom}_{\mathsf{D}}(Y, \_)$ to the inverse limit structure gives the following diagram:

$$\begin{tikzcd}
 & & \mathrm{Hom}_{\mathsf{D}}(Y,X) \arrow[dl, "\mu_{i-1}\circ_"'] \arrow[d, "\mu_i\circ_"] \arrow[dr, "\mu_{i+1}\circ_"] & & \\
\cdots & \mathrm{Hom}_{\mathsf{D}}(Y,X_{i-1}) \arrow[l] & \mathrm{Hom}_{\mathsf{D}}(Y,X_i) \arrow[l, "f_{i-1,i}\circ_"] & \mathrm{Hom}_{\mathsf{D}}(Y,X_{i+1}) \arrow[l, "f_{i,i+1}\circ_"] & \cdots . \arrow[l]
\end{tikzcd}$$

Let $Z \in \mathrm{Ob}(\mathsf{Set})$. Let $\nu_i \in \mathrm{Hom}_{\mathsf{Set}}(Z, \mathrm{Hom}_{\mathsf{D}}(Y, X_i))$ (with $i \in \mathbb{N}$) be such that $f_{ij} \circ \nu_j = \nu_i$ for $i \leq j$. Let $z \in Z$. Now the $\nu_i(z) \in \mathrm{Hom}_{\mathsf{D}}(Y, X_i)$ (with $i \in \mathbb{N}$) form a cone. There exists a unique $h_z \in \mathrm{Hom}_{\mathsf{D}}(Y, X)$ such that $\nu_i(z) = \mu_i \circ h_z$ (with $i \in \mathbb{N}$). Define $g \in \mathrm{Hom}_{\mathsf{Set}}(Z, \mathrm{Hom}_{\mathsf{D}}(Y, X))$ to be the morphism for which $g(z) = h_z$. Now $g \in \mathrm{Hom}_{\mathsf{Set}}(Z, \mathrm{Hom}_{\mathsf{D}}(Y, X))$ is the unique morphism to make the following diagram commute:

$$\begin{tikzcd}
 & & Z \arrow[d, bend right, "g"'] \arrow[ddl, bend right, "\nu_{i-1}"'] \arrow[dd, bend left] \arrow[ddr, bend left, "\nu_{i+1}"] & & \\
 & & \mathrm{Hom}_{\mathsf{D}}(Y,X) \arrow[dl, "\mu_{i-1}\circ_"'] \arrow[d, "\mu_i\circ_"'] \arrow[dr, "\mu_{i+1}\circ_"] & & \\
\cdots & \mathrm{Hom}_{\mathsf{D}}(Y,X_{i-1}) \arrow[l] & \mathrm{Hom}_{\mathsf{D}}(Y,X_i) \arrow[l, "f_{i-1,i}\circ_"] & \mathrm{Hom}_{\mathsf{D}}(Y,X_{i+1}) \arrow[l, "f_{i,i+1}\circ_"] & \cdots . \arrow[l]
\end{tikzcd}$$

Hence $\mathrm{Hom}_{\mathsf{D}}(Y, X) = \varprojlim \mathrm{Hom}_{\mathsf{D}}(Y, X_i)$. □

Theorem 2.74 (Dual to Theorem 2.73)**.** Contravariant representable functors turn colimits into limits.

3 The category of R-modules

The examples given so far have been useful in showing the properties of categories and functors. But they are not necessarily the most prominent ones in this thesis. The category $R - \mathsf{Mod}$ for a ring R is one of the more prominent. Its structure allows many constructions. We will encounter those in §4. In this section we will do some preparation by taking a closer look at the pullbacks in $R - \mathsf{Mod}$ and the tensor product.

3.1 Pullbacks of R-modules

Limits in $R - \mathsf{Mod}$ will appear frequently throughout this thesis. In this subsection, we will first guarantee the reader of the existence of small (co)limits in $R - \mathsf{Mod}$. Finally, we will find an explicit description of the pullbacks of modules over a ring. Recall that pullbacks in $R - \mathsf{Mod}$ are called fiber products.

Theorem 3.1. The category of R-modules is complete and cocomplete.

Proof. See [33, Corollary 5.7]. $\square$

Lemma 3.2. Let R be a ring and let E, F and M be R-modules. Let $E \xrightarrow{p} M$ and $F \xrightarrow{q} M$ be R-module morphisms. Define the set $E \times_M F := \{(e,f) \in E \oplus F : p(e) = q(f)\}$. Then $E \times_M F$ is an R-module under the R-action $r \cdot (e,f) := (re, rf)$.

Proof. We will first show that $E \times_M F$ is an abelian group. Let $(e,f),(e',f') \in E \times_M F$. $p(e+e') = p(e) + p(e')$ and $q(f+f') = q(f) + q(f')$, so $p(e+e') = q(f+f')$, so $(e+e', f+f') \in E \times_M N$.
We will now show that the R-action is well-defined. Let $r \in R$ and $(e,f) \in E \times_M F$, then $p(re) = rp(e) = rq(f) = q(rf)$, so $(re, rf) \in E \times_M F$. $\square$

Proposition 3.3. Let R be a ring and let E, F and M be R-modules. Let $E \xrightarrow{p} M$ and $F \xrightarrow{q} M$ be R-module morphisms. Then $E \times_M F$ is the pullback of p and q (via the projections $E \times_M F \xrightarrow{\pi_E} E$ and $E \times_M F \xrightarrow{\pi_F} F$) in $R-\mathsf{Mod}$.

Proof. Let N be an R-module and let $N \xrightarrow{f} E$ and $N \xrightarrow{g} F$ be R-module morphisms such that $p \circ f = q \circ g$. Let $n \in N$, by construction $p(f(n)) = q(g(n))$. There is an R-module morphism $N \xrightarrow{h} E \times_M F$ such that $h(n) = (f(n), g(n)) \in E \times_M F$. This is the only way to make the following diagram commute:

$$\begin{tikzcd}
N \arrow[dr, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& E \times_M F \arrow[r, "\pi_F"'] \arrow[d, "\pi_E"'] & F \arrow[d, "q"] \\
& E \arrow[r, "p"'] & M.
\end{tikzcd}$$

$\square$

Remark 3.4. Although the underlying morphisms are omitted in the notation, they certainly matter. Different morphisms give a different fiber product.

3.2 Tensor products

The goal of this section is to introduce the tensor product on modules over a ring. It contains only the construction and a basic property. Those familiar with tensor products can skip this section.

Definition 3.5. Let R be a commutative ring. Let M and N be R-modules. The *(algebraic) tensor product* of M and N is the R-linear span of $M \times N$ (Cartesian product) under the relations:

1. $(m_1, n) + (m_2, n) = (m_1 + m_2, n)$ for $m_1, m_2 \in M$, $n \in N$
2. $(m, n_1) + (m, n_2) = (m, n_1 + n_2)$ for $m \in M$, $n_1, n_2 \in N$
3. $(rm, n) = (m, rn)$ for $m \in M$, $n \in N$ and $r \in R$.

The (algebraic) tensor product of M and N is denoted by $M \otimes_R N$. The element in $M \otimes_R N$ represented by $(m,n) \in M \times N$ is denoted by $m \otimes n$.

Remark 3.6. Elements of a tensor product are informally called *tensors*.

Lemma 3.7. Let R be a commutative ring. Let M and N be R-modules. Then $M \otimes_R N$ is an R-module under the action $r(m \otimes n) = (rm \otimes n) = (m \otimes rn)$.

Proof. Follows immediately from the R-module structure of M and N. □

Remark 3.8. Take $R = \mathbb{Z}$ to get the tensor products of abelian groups. Take R a field to get the tensor products of R-vector spaces. In both cases, for free modules of finite dimension (for fields all modules are free), the tensor product is free of dimension the product of dimensions. More generally, the tensor product is distributive with the direct sum.

Remark 3.9. Tensor products can also be defined for non-commutative rings. In that case the left- and right-actions need not agree. In a commutative ring, the left- and right-action agree.

Definition 3.10. Let R be a ring. Let M and N be R-modules. Let $m \in M$ and $n \in N$. Elements of the form $m \otimes n \in M \otimes_R N$ are called *pure* tensors.

Remark 3.11. Not all tensors need to be pure. Take $R = \mathbb{R}$ and $e_1 \otimes e_2 + e_2 \otimes e_1 \in \mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2 \cong \mathbb{R}^4$ for example, where e_1, e_2 denote the standard basis elements of $\mathbb{R}^2$.

Lemma 3.12. Let R be a ring. Let M, N and K be R-modules. For an R-bilinear map $M \times N \xrightarrow{f} K$, there exists a unique R-bilinear map $\tilde{f}\colon M \otimes_R N \to K$ such that the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "-\otimes-"] \arrow[d, "f"'] & M \otimes_R N. \arrow[dl, "\tilde{f}"] \\ K & \end{tikzcd}$$

Proof. See [1, Proposition 2.12] for the biadditivity and uniqueness. The R-bilinearity is immediate from the relations. □

4 Prerequisites on modules and homological algebra

In this section exactness of sequences is defined for modules over a ring. The definition of injectivity will be given and it will be shown that every module over a ring allows an injective resolution. Using an injective resolution, the Ext-functors for modules over a ring are constructed. Finally, a correspondence between Ext-groups and extensions will be proven. This correspondence will give important information about the behaviour of the Ext-functors, which is necessary for the eventual goal of this thesis; the understanding of the proof of the UCT. In §4.5 we dualise the most important result from the previous subsections. The dual of injective is projective. In the final subsection, we will show how to turn a projective resolution into an injective resolution.

4.1 Exactness, splitness and five lemma

Definition 4.1 (Exactness)**.** Let R be a ring. Let $I \subset \mathbb{Z}$ be a subset such that for any $n_1, n_2, n_3\, \mathbb{Z}$, if $n_1 \leq n_2 \leq n_3$ and $n_1, n_3 \in I$, then $n_2 \in I$. A sequence $\dots \xrightarrow{f_{i-2}} M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \dots$ of R-modules and R-module morphisms indexed by I is *exact* if $\operatorname{im} f_i = \ker f_{i+1}$ for all $i \in I$ (except the maximal element in I, if there exist one).

Example 4.2. The sequence $0 \to A \xrightarrow{f} B$ is exact if and only $\ker f = \{0\}$, i.e. f is injective.

Example 4.3. The sequence $B \xrightarrow{g} C \to 0$ is exact if and only if $\operatorname{im} g = C$, i.e. g is surjective.

Example 4.4. The sequence $0 \to A \to 0$ is exact if and only if $A = \{0\}$.

Example 4.5. Let R be a commutative ring. An R-module morphism $M \xrightarrow{f} N$ induces the following exact sequence:

$$0 \to \ker f \to M \xrightarrow{f} N \to N/f(M) \to 0. \tag{5}$$

Definition 4.6. An exact sequence of the form $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is called a *short exact sequence.*

Remark 4.7. In this section we will focus short exact sequences of modules over a ring. In §6 we will focus on short exact sequences of C*-algebras.

Lemma 4.8. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. The following are equivalent:

1. There exists an R-module morphism $s\colon C \to B$ such that $g \circ s = \mathrm{id}_C$. Such an R-module morphism is a *section.*
2. There exists an R-module morphism $r\colon B \to A$ such that $r \circ f = \mathrm{id}_A$. Such an R-module morphism is a *retraction.*
3. There is an isomorphism of R-modules $\phi\colon B \to A \oplus C$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "g"] \arrow[d, "\phi"'] & C \arrow[r] \arrow[d, "\mathrm{id}_C"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & A \oplus C \arrow[r, "p"] & C \arrow[r] & 0. \end{tikzcd} \tag{6}$$

   i is the inclusion; $i(a) = (a, 0)$ for $a \in A$. p is the projection; $p(a, c) = c$ for $a \in A$ and $c \in C$.

Proof. 1 induces 2: Let $s\colon C \to B$ be a section. The sequence is exact, so g is surjective. For every $b \in B$ there exists a $c \in C$ such that $b \in g^{-1}(c)$. If $b \in g^{-1}(c)$, then $b - s(c) \in \ker(g)$. The exactness also implies $\operatorname{im} f = \ker g$ and f is injective. Hence there exists a unique $a \in A$ such that $f(a) = b - s(c)$. Define $r\colon B \to A$ to be the map that sends $b \in B$ to this $a \in A$ such that $f(a) = b - s(c)$. We will show r is an R-module morphism. Let $b_1, b_2 \in B$, if $b_1 \in g^{-1}(c_1)$ and $b_2 \in g^{-1}(c_2)$ for $c_1, c_2 \in C$, then $b_1 + b_2 \in g^{-1}(c_1 + c_2)$. Now $s(c_1 + c_2) = s(c_1) + s(c_2)$. If $b_1 - s(c_1) = f(a_1)$ and $b_2 - s(c_2) = f(a_2)$, then $b_1 + b_2 - g(c_1 + c_2) = f(a_1) + f(a_2) = f(a_1 + a_2)$. Hence $r(b_1 + b_2) = r(b_1) + r(b_2)$. Let $t \in R$, $b \in B$, if $b \in g^{-1}(c)$ for $c \in C$, then $tb \in g^{-1}(tc)$. Then $tb - s(tc) = tb - ts(c) = t(b - s(c)) = tf(a) = f(ta)$. Hence $r(tb) = tr(b)$. The proof for the right-action is identical. If $b = f(a)$ for some $a \in A$, then $b \in g^{-1}(0)$ and so $r(b) = r(f(a)) = a$. So r is a retraction.
2 induces 3: Let $r\colon B \to A$ be a retraction. Define $\phi\colon B \to A \oplus C$ by $\phi(b) = (r(b), g(b))$. Then ϕ is an R-module morphism. Let $a \in A$, now $\phi(f(a)) = (r(f(a)), g(f(a))) = (a, 0)$ as $g \circ f = 0$ and $r \circ f = \mathrm{id}_A$. Let $b \in B$, then $p(\phi(b)) = p(r(b), g(b)) = g(b)$. We conclude diagram (6) commutes. Let $b \in B$, if $\phi(b) = (r(b), g(b)) = (0, 0)$, then $b \in \operatorname{im} f$. Combined with $r(b) = 0$, this means $b = f(0) = 0$. So ϕ is injective. Let $a \in A$ and $c \in C$. There exists a $b \in B$ such that $g(b) = c$. By exactness, $\ker g = \operatorname{im} f$. So $g^{-1}(c) = \{b + f(a) \colon a \in A\}$. So there is an element $b_1 \in g^{-1}(c)$ such that $r(b_1) = a$. So ϕ is surjective.
3 induces 1: The map $j\colon C \to A \oplus C$ with $j(c) = (0, c)$ for $c \in C$ is an R-module morphism. As $p \circ \phi = g$ by commutativity and $p \circ j = \mathrm{id}_C$, it follows $g \circ \phi^{-1} \circ j = \mathrm{id}_C$. So $\phi^{-1} \circ j$ forms a section. ☐

Definition 4.9. A short exact sequence of R-modules is *split* if the statements of Lemma 4.8 hold.

Lemma 4.10 (Five lemma)**.** Let R be a ring and let the following diagram be a commutative diagram of R-modules for which the horizontal rows are exact:

$$\begin{tikzcd} A_1 \arrow[r, "\alpha_1"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "f_2"'] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "f_3"'] & A_4 \arrow[r, "\alpha_4"] \arrow[d, "f_4"'] & A_5 \arrow[d, "f_5"'] \\ B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r, "\beta_4"] & B_5. \end{tikzcd}$$

1. If f_2 and f_4 are injective and f_1 is surjective, then f_3 is injective.
2. If f_2 and f_4 are surjective and f_5 is injective, then f_3 is surjective.
3. If f_2 and f_4 are bijective, f_1 is surjective and f_5 is injective, then f_3 is bijective.

Proof.

1. Let $x \in A_3$ such that $f_3(x) = 0$, then $\beta_3(f_3(x)) = \beta_3(0) = 0$. Since $\beta_3 \circ f_3 = f_4 \circ \alpha_3$ and $f_4(\alpha_3(x)) = 0$. The injectivity of f_4 implies $\alpha_3(x) = 0$. By exactness, there exists an $y \in A_2$ such that $\alpha_2(y) = x$. As $0 = f_3(x) = f_3(\alpha_2(y)) = \beta_2(f_2(y))$, there exists a $z \in B_1$ such that $\beta_1(z) = f_2(y)$ by exactness. f_1 is surjective, so there exists a $w \in A_1$ such that $f_1(w) = z$. Now $f_2(y) = \beta_1(z) = \beta_1(f_1(w)) = f_2(\alpha_2(w))$. As f_2 is injective, this implies $y = \alpha_1(w)$. Since $y = \alpha_1(w)$, we get $x = \alpha_2(y) = \alpha_2(\alpha_1(w)) = 0$. So $x = 0$ and f_3 is injective.

2. Let $x \in B_3$, then since f_4 is surjective, there exists a $y \in A_4$ such that $f_4(y) = \beta_3(x)$. Now $f_5(\alpha_4(y)) = \beta_4(f_4(y)) = \beta_4(\beta_3(y)) = 0$. The injectivity of f_5 implies $\alpha_4(y) = 0$. Therefore there is a $z \in A_3$ such that $\alpha_3(z) = y$. Now $f_4(y) = f_4(\alpha_3(z)) = \beta_3(f_3(z))$. So $\beta_3(f_3(z)) = \beta_3(x)$. So there exists a $w \in B_2$ such that $\beta_2(w) = x - f_3(z)$. f_2 is surjective, so there exists a $v \in A_2$ such that $f_2(v) = w$. Now $f_3(\alpha_2(v)) = \beta_2(f_2(v)) = \beta_2(w) = x - f_3(z)$. So $f_3(\alpha_2(v)) + f_3(z) = x$. So f_3 is surjective.

3. Follows immediately from the two previous statements. □

Remark 4.11. In practice a weaker version is used; if f_1, f_2, f_4 and f_5 are bijective, then so is f_3.

Example 4.12. Lemma 4.10 shows that the map ϕ in diagram (6) is an isomorphism. In retrospect it was not necessary to show it!

4.2 Injectivity and injective resolutions

Definition 4.13 (Exact functor)**.** Let R be a ring. A functor $F: R-\mathsf{Mod} \to R-\mathsf{Mod}$ is *exact* if it preserves short exact sequences.

Remark 4.14. Also contravariant functors will be called exact. The definition of exact functors extends to other categories for which exactness is defined.

Remark 4.15. Similarly, left-exactness, right-exactness and half-exactness can defined. See [25, Proposition 4.5.10] and [34, Definition 11.1.1]. For extra clarity; a contravariant functor F is left-exact if for every exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the sequence $0 \to F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A)$ is exact. It is right-exact if for every exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the sequence $F(C) \xrightarrow{F(g)} F(B) \xrightarrow{F(f)} F(A) \to 0$ is exact.

Lemma 4.16. Let R be a commutative ring. Let M be an R-module. The contravariant functor $\operatorname{Hom}_R(\_, M)$ is left-exact. In other words, let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of R-modules. Applying $\operatorname{Hom}_R(\_, M)$ gives an exact sequence $0 \to \operatorname{Hom}_R(C, M) \xrightarrow{\_\circ g} \operatorname{Hom}_R(B, M) \xrightarrow{\_\circ f} \operatorname{Hom}_R(A, M)$.

Proof. Let $h_1, h_2 \in \operatorname{Hom}_R(C, M)$ be such that $h_1 \circ g = h_2 \circ g$, then $h_1 = h_2$, since g is surjective (g is an epimorphism). By exactness $g \circ f = 0$. Let $k \in \operatorname{Hom}_R(B, M)$ be such that $k \circ f = 0$. Let $x, y \in B$ be such that $g(x) = g(y)$, then $g(x - y) = 0$, so $x - y \in \operatorname{im}(f)$. So $k(x - y) = 0$ and $k(x) = k(y)$. It follows k is factored through g and comes from $\operatorname{Hom}_R(C, M)$. $\square$

Remark 4.17. Let R be a ring and let M and N be R-modules. In order for $\operatorname{Hom}_R(M, N)$ to have an R-module structure, the ring R needs to be commutative. The natural R-action is the following: $(r \cdot f)(m) := rf(m) = f(rm)$ with $r \in R$ and $f \in \operatorname{Hom}_R(M, N)$. Now let $M \ni m' = sm$ for some $s \in R$. This equation only holds if R is commutative:

$$(rf)(m') = rf(m') = f(rm') = f(r(sm)) = f((rs)m) = (rs)f(m) = srf(m) = s((rf)(m)).$$

If R fails to be commutative, then $\operatorname{Hom}_R(M, N)$ is only an abelian group. Henceforth, ring will mean commutative ring.

Example 4.18. Let R be a commutative ring. Let M be an R-module. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence of R-modules. The mapping $\operatorname{Hom}_R(B, M) \xrightarrow{\_\circ f} \operatorname{Hom}_R(A, M)$ need not be surjective. Take $R = \mathbb{Z}$, $M = \mathbb{Z}/2\mathbb{Z}$ and the exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$. The induced map $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ cannot be surjective. This is because $\# \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 2$ and $\# \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 2$. In a short exact sequence of finite R-modules the cardinality of the middle term must be the product of the left-hand and the right-hand.

Lemma 4.19. Let R be a commutative ring. Let M be an R-module. The following are equivalent:

1. Let N be an R-module such that $M \subset N$ is an R-submodule, then there exists an R-module K such that $N \cong M \oplus K$.
2. Any short exact sequence of R-modules $0 \to M \to N \to K \to 0$ splits.
3. Let $f: X \to Y$ be an injective morphism (monomorphism) of R-modules and let $g: X \to M$ be an arbitrary morphism of R-modules, then there exists an R-module morphism $h: Y \to M$ such that $h \circ f = g$.
4. The contravariant functor $\operatorname{Hom}_R(\_, M)$ is exact.

Proof. We prove $4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1 \Rightarrow 4$.
4 induces 3: Let $f: X \to Y$ be an injective R-module morphism. Then this induces a short exact sequence; $0 \to X \xrightarrow{f} Y \xrightarrow{q} \operatorname{coker} f \to 0$, where $Y \xrightarrow{q} \operatorname{coker} f$ is the projection. As $\operatorname{Hom}_R(\_, M)$ is exact, it follows $0 \to \operatorname{Hom}_R(\operatorname{coker} f, M) \xrightarrow{\_\circ q} \operatorname{Hom}_R(Y, M) \xrightarrow{\_\circ f} \operatorname{Hom}_R(X, M) \to 0$ is exact. In particular, for any $g \in \operatorname{Hom}_R(X, M)$, there exists a $h \in \operatorname{Hom}_R(Y, M)$ such that $h \circ f = g$.

3 induces 2: Let $0 \to M \xrightarrow{f} N \xrightarrow{g} K \xrightarrow{0}$ be an exact sequence. Then $f: M \to N$ is injective. Take $\mathrm{id}_M: M \to M$. There exists a $g \in \operatorname{Hom}_R(N, M)$ such that $g \circ f = \mathrm{id}_M$, so there exists a retraction to f. The exact sequence is split.

2 induces 1: Let N be an R-module such that $M \subset N$ is an R-submodule. Let $M \xrightarrow{i} N$ be the inclusion. There is an exact sequence $0 \to M \xrightarrow{i} N \xrightarrow{q} \operatorname{coker} i \to 0$, where $N \xrightarrow{q} \operatorname{coker} i$ is the projection. The exact sequence is split, so $N \cong M \oplus \operatorname{coker} i$. Take $K = \operatorname{coker} i$.
1 induces 4: We will use the fact that every R-module M is a submodule of an R-module N for which the contravariant $\operatorname{Hom}_R(\_, N)$ is exact [5, Theorem 6]. Assume statement 1 and this fact. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence. Let $M \xrightarrow{i} N$ be the inclusion. Statement 1 says there is an R-module K such that $N \cong M \oplus K$. Equivalently, the exact sequence $0 \to M \xrightarrow{i} N \xrightarrow{q} K \to 0$ splits, where $N \xrightarrow{q} K$ is the projection. The sequence $0 \to \operatorname{Hom}_R(A, M) \xrightarrow{i} \operatorname{Hom}_R(A, N) \xrightarrow{q} \operatorname{Hom}_R(A, K) \to 0$ is exact, as $\operatorname{Hom}_R(A, N) = \operatorname{Hom}_R(A, M \oplus K) \cong \operatorname{Hom}_R(A, M) \oplus \operatorname{Hom}_R(A, K)$. Similar statements are true, when A is replaced by B or C. This all results in the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Hom}_R(C,M) \arrow[r, "_\circ g"] \arrow[d, "i\circ_"'] & \operatorname{Hom}_R(B,M) \arrow[r, "_\circ f"] \arrow[d, "i\circ_"'] & \operatorname{Hom}_R(A,M) \arrow[d, "i\circ_"'] & \\
0 \arrow[r] & \operatorname{Hom}_R(C,N) \arrow[r, "_\circ g"] \arrow[d, "q\circ_"'] & \operatorname{Hom}_R(B,N) \arrow[r, "_\circ f"] \arrow[d, "q\circ_"'] & \operatorname{Hom}_R(A,N) \arrow[r] \arrow[d, "q\circ_"'] & 0. \\
0 \arrow[r] & \operatorname{Hom}_R(C,K) \arrow[r, "_\circ g"] \arrow[d] & \operatorname{Hom}_R(B,K) \arrow[r, "_\circ f"] \arrow[d] & \operatorname{Hom}_R(A,K) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{7}$$

Now it is a matter of diagram chasing: let $h_1 \in \operatorname{Hom}_R(A, M)$. By construction, $\operatorname{Hom}_R(\_, N)$ is exact. Let $h_2 = i \circ h_1$. There exists a $h_3 \in \operatorname{Hom}_R(B, N)$ such that $h_3 \circ f = h_2$. Now $q \circ h_3 \circ f = q \circ h_2 = q \circ i \circ h_1 = 0$. So $q \circ h_3$ comes from $\operatorname{Hom}_R(C, K)$; there exists a $h_4 \in \operatorname{Hom}_R(C, K)$ such that $h_4 \circ g = q \circ h_3$. There exists a $h_5 \in \operatorname{Hom}_R(C, N)$ such that $q \circ h_5 = h_4$. Now $q \circ h_3 = h_4 \circ g = q \circ h_5 \circ g$. So $q \circ (h_3 - h_5 \circ g) = 0$ and $h_3 - h_5 \circ g$ comes from $\operatorname{Hom}_R(B, M)$; there exists a $h_6 \in \operatorname{Hom}_R(B, M)$ such that $i \circ h_6 = h_3 - h_5 \circ g$. Now $i \circ h_6 \circ f = (h_3 - h_5 \circ g) \circ f = h_3 \circ f = h_2 = i \circ h_1$. Composition by i is injective, so $i \circ (h_6 \circ f) = i \circ h_1$ implies $h_6 \circ f = h_1$, so $\operatorname{Hom}_R(B, M) \xrightarrow{\_\circ f} \operatorname{Hom}_R(A, M)$ is surjective. So $\operatorname{Hom}_R(\_, M)$ is exact. $\square$

Definition 4.20. Let R be a commutative ring. Let M be an R-module. The R-module M is *injective* if the statements of Lemma 4.19 hold.

Definition 4.21. Let R be a ring and let M be an R-module. The R-module M is *divisible* if for every $x \in M$ and non-zero $r \in R$ there exists a $y \in M$ such that $x = ry$.

Definition 4.22 (Equivalent to Definition 4.21)**.** Let R be a ring and let M be an R-module. The R-module M is divisible if the R-module morphism $M \xrightarrow{r\cdot\_} M$ is surjective for all $r \in R \setminus \{0\}$.

Lemma 4.23. Let R be an integral domain, then every injective R-module is divisible.

Proof. Let M be an injective R-module. There is always an isomorphism $\operatorname{Hom}_R(R, M) \cong M$. This follows from the fact that every R-module morphism from R to M is completely determined by the image of $1 \in R$. Let $x \in M$ and let $f \in \operatorname{Hom}_R(R, M)$ be such that $f(1) = x$. As R is an integral domain, the left-multiplication by $r \in R \smallsetminus \{0\}$ is injective. As M is injective, there exists an element $g \in \operatorname{Hom}_R(R, M)$ such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "r\cdot\_"] \arrow[d, "f"'] & R. \arrow[dl, "g"] \\ M & \end{tikzcd}$$

If $g(1) = y \in M$, then $x = f(1) = g(r) = ry$. So M is divisible. □

Lemma 4.24. Let R be a principal ideal domain (PID). Then every divisible R-module is injective.

Proof. Every ideal in R is principal. Let $\{0\} \neq \mathfrak{a} \subset R$ be an ideal and write $\mathfrak{a} = (r)$. Now $\operatorname{Hom}_R(\mathfrak{a}, M) \cong M$. Every R-module morphism from $\mathfrak{a}$ to M is completely determined by the image of $r \in \mathfrak{a}$. So let $x \in M$ and $f \in \operatorname{Hom}_R(\mathfrak{a}, M)$ such that $f(r) = x$. As M is divisible, there exists a $y \in R$ such that $x = ry$. Let $g \in \operatorname{Hom}_R(R, M)$ be the morphism such that $g(1) = y$. This makes the following diagram commute:

$$\begin{tikzcd} \mathfrak{a} \arrow[r, hook] \arrow[d, "f"'] & R. \arrow[dl, "g"] \\ M & \end{tikzcd}$$

Now Baer's lemma says that M is injective [5, Theorem 2]. □

Corollary 4.25. Let R be a PID. A quotient of an injective R-module is an injective R-module.

Proof. From Lemmas 4.23 and 4.24 follows that an R-module is injective if and only if it is divisible. Let M be an injective R-module and let $N \subset M$ be an R-submodule. Now M/N is divisible, since every class in M/N can be represented by an element in M. The divisibility says that for $x \in M$ and $r \in R \smallsetminus \{0\}$there exists an $y \in M$ such that $x = ry$. So now $[x] = [ry] = r[y]$ in M/N. So M/N is divisible and therefore injective. □

Example 4.26. $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules.

Example 4.27. $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module. In general, free modules need not be injective.

The proof of the equivalences of Lemma 4.19 used the fact that for a ring R, every R-module is the submodule of an injective module [5, Theorem 6]. For an R-module M, there is an injective R-module N_0 such that $M \subset N_0$. Using this fact again for N_0/M, an injective R-module N_1 exists such that $(N_0/M) \subset N_1$. Again an injective R-module N_2 can be found for which $N_1/(N_0/M)$ is a submodule. Generalise this for $j \geq 1$; define $d_j\colon N_j \to N_{j+1}$ to be the composition of the quotient $N_j \to N_j/\operatorname{im} d_{j-1}$ and the inclusion $N_j/\operatorname{im} d_{j-1} \to N_{j+1}$. Inclusions are injective. By construction, the kernel of the quotient $N_j \to N_j/\operatorname{im} d_{j-1}$ is $\operatorname{im} d_{j-1}$. It follows $0 \to M \xrightarrow{i} N_0 \xrightarrow{d_0} N_1 \xrightarrow{d_1} N_2 \to \ldots$ is exact!

Definition 4.28 (Resolution)**.** Let R be a ring and let M be an R-module. A *resolution* of M is an exact sequence $0 \to M \xrightarrow{i} N_0 \xrightarrow{d_0} N_1 \xrightarrow{d_1} N_2 \to \ldots$ with R-modules N_i for $i \geq 0$.
If the R-modules N_i are injective for all $i \geq 0$, then this exact sequence is an *injective resolution*.

Proposition 4.29. Let R be a ring. Every R-module allows an injective resolution.

Proof. Construction above Definition 4.28. $\square$

Let R be a ring and let M be an R-module. A resolution $0 \to M \xrightarrow{i} N_0 \xrightarrow{d_0} N_1 \xrightarrow{d_1} N_2 \to \ldots$ forms a cochain complex $(N^\bullet, d)$, when i and d_0 are composed. The sequence $0 \to N_0 \xrightarrow{d_0} N_1 \xrightarrow{d_1} N_2 \to \ldots$ is exact everywhere except at N_0. From the exactness of the injective resolution, it follows $\ker d_0 = M$. So $H^i((N^\bullet, d)) = M$ for $i = 0$ and $H^i((N^\bullet, d)) = 0$ for $i \geq 1$.

Definition 4.30 (Chain homotopy)**.** Let R be a ring. Let $(C^\bullet, d)$, $(D^\bullet, e)$ be cochain complexes and let $f, g: (C^\bullet, d) \to (D^\bullet, e)$ be a cochain maps. The cochain maps f and g are *cochain homotopic* if there exists a family of R-module morphisms $\{h_i\}_{i\geq 1}$ with $h_i: C^i \to D^{i-1}$ such that $h_{i+1} \circ d_i + e_{i-1} \circ h_i = f_i - g_i$ for all $i \geq 1$ and $h_1 \circ d_0 = f_0 - g_0$. Notation: $f \simeq g$.

Definition 4.31 (Chain homotopy equivalence)**.** Let R be a ring. Let $(C^\bullet, d)$, $(D^\bullet, e)$ be cochain complexes. The cochain complexes $(C^\bullet, d)$ and $(D^\bullet, e)$ are *cochain homotopy equivalent* if there exist cochain maps $f: (C^\bullet, d) \to (D^\bullet, e)$ and $g: (D^\bullet, e) \to (C^\bullet, d)$ such that $g \circ f \simeq \mathrm{id}_{(C^\bullet, d)}$ and $f \circ g \simeq \mathrm{id}_{(D^\bullet, e)}$

Lemma 4.32 (Chain homotopy invariance)**.** Let R be a ring. Let $(C^\bullet, d)$, $(D^\bullet, e)$ be cochain complexes and let $f, g: (C^\bullet, d) \to (D^\bullet, e)$ be cochain homotopic maps. The induced maps $H^i(f), H^i(g): H^i(C^\bullet, d) \to H^i(D^\bullet, e)$ agree for all $i \geq 0$.

Proof. The cochain maps f and g are cochain homotopic, so there exists a family $\{h_i\}_{i\geq 1}$ with $h_i: C^i \to D^{i-1}$ such that $h_{i+1} \circ d_i + e_{i-1} \circ h_i = f_i - g_i$ for all $i \geq 1$. For $x \in \ker d_i$, we get $f_i(x) - g_i(x) = h_{i+1}(d_i(x)) + e_{i-1}(h_i(x)) = 0 + e_{i-1}(h_i(x))$. On the cocycles, f and g differ by a coboundary. Hence $H^i(f), H^i(g): H^i((C^\bullet, d)) \to H^i(D^\bullet, e)$ agree for all $i \geq 0$. $\square$

Corollary 4.33. The i-th cohomology groups of cochain equivalent cochain complexes agree for all $i \geq 0$. This follows from the fact that the i-th homology is functorial for every $i \geq 0$ and therefore an identity morphism induces an identity map on the homology groups.

Remark 4.34. Similar definitions exist for chain complexes. A chain homotopy consists of arrows going a degree up instead of down. Chain homotopic maps induce the same R-module morphisms on the homology groups, similar to Lemma 4.32.

Proposition 4.35. (Co)chain homotopy is an equivalence relation on (co)chain maps. (Co)chain homotopy equivalence is an equivalence relation on (co)chain complexes.

Theorem 4.36. Let R be a ring, let M, N be R-modules and let $f: M \to N$ be a morphism of R-modules. Let $(P^\bullet, d)$ be an injective resolution of M and let $(Q^\bullet, e)$ be an injective resolution of N. There exists a cochain map $g: (P^\bullet, d) \to (Q^\bullet, e)$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "f"] & P_0 \arrow[r, "d_0"] \arrow[d, "g_0"] & P_1 \arrow[r, "d_1"] \arrow[d, "g_1"] & \cdots \\ 0 \arrow[r] & N \arrow[r, "j"] & Q_0 \arrow[r, "e_0"] & Q_1 \arrow[r, "e_1"] & \cdots. \end{tikzcd} \tag{8}$$

This cochain map g is unique up to cochain homotopy.

Proof. Existence is proven by induction. As Q_0 is injective, it satisfies statement 3 of Lemma 4.19. As $i: M \to P_0$ is injective, there exists a $g_0: P_0 \to Q_0$ such that $g_0 \circ i = j \circ f$. Now d_0 induces an embedding $P_0/M \to P_1$. By commutativity of the diagram, $e_0 \circ g_0 \circ i = e_0 \circ j \circ f = 0$, as $e_0 \circ j = 0$. So $e_0 \circ g_0$ induces an R-module morphism $P_0/M \xrightarrow{e_0 \circ g_0} Q_1$. Similarly, as Q_1 is injective, there exists a $g_1: P_1 \to Q_1$ that connects the inclusion $P_0/M \to P_1$ with the R-module morphism $P_0/M \xrightarrow{e_0 \circ g_0} Q_1$. Identically, there is a $g_i: P_i \to Q_i$ that connects the inclusion of $P_{i-1}/\operatorname{im} d_{i-2} \to P_i$ and the induced R-module morphism $P_{i-1}/\operatorname{im} d_{i-2} \xrightarrow{e_{i-1} \circ g_{i-1}} Q_i$ for $i \geq 2$.

Assume there exist two cochain maps $g, h: (P^\bullet, d) \to (Q^\bullet, e)$ that make the diagram commute. A cochain homotopy of g and h will be constructed. For $m \in M$ holds $g_0(i(m)) = j(f(m)) = h_0(i(m))$ by commutativity. So g_0 and h_0 agree on M and $g_0 - h_0$ defines an R-module morphism $P_0/M \to Q_0$. Now Q_0 is injective, so there exists an R-module morphism $k_1: P_1 \to Q_0$ that connects the inclusion $P_0/M \to P_1$ with $P_0/M \xrightarrow{g_0 - h_0} Q_0$. By construction $k_1 \circ d_0 = g_0 - h_0$. Assume k_1 is known, do the same thing for $g_1 - h_1 - e_0 \circ k_1$. Let $x \in P_0$, use $e_0 \circ g_0 = g_1 \circ d_0$ and $e_0 \circ h_0 = h_1 \circ d_0$ and $k_1 \circ d_0 = g_0 - h_0$ to get

$$g_1(d_0(x)) - h_1(d_0(x)) - e_0(k_1(d_0(x))) = e_0(g_0(x)) - e_0(h_0(x)) - e_0((g_0 - h_0)(x) = 0.$$

So $g_1 - h_1 - e_0 \circ k_1$ induces an R-module morphism $P_1/\operatorname{im} d_0 \xrightarrow{g_1 - h_1 - e_0 \circ k_1} Q_1$. Now $P_1/\operatorname{im} d_0$ is embedded in P_2 and Q_1 is an injective R-module. So there exists an R-module morphism $k_2: P_2 \to Q_1$ that connects $P_1/\operatorname{im} d_0 \to P_2$ with $g_1 - h_1 - e_0 \circ k_1$. Identically, Q_{i-1} is injective, so there exists an R module morphism $k_i: P_i \to Q_{i-1}$ that connects the inclusion $P_{i-1}/\operatorname{im} d_{i-2} \to P_i$ with $P_{i-1}/\operatorname{im} d_{i-2} \xrightarrow{g_{i-1} - h_{i-1} - e_{i-2} \circ k_{i-1}} Q_{i-1}$. □

Corollary 4.37. Let R be a ring. Let M be an R-module. Proposition 4.29 says there exists an injective resolution of M. If there exists multiple injective resolutions of M, then these resolutions are cochain homotopy equivalent.

Proof. Let $(P^\bullet, d)$ and $(Q^\bullet, e)$ be injective resolutions of M. Apply Theorem 4.36 to id_M. Now there exists a cochain map $g: (P^\bullet, d) \to (Q^\bullet, e)$ and there exists a cochain map $h: (Q^\bullet, e) \to (P^\bullet, d)$. There is a cochain map $(h \circ g): (P^\bullet, d) \to (P^\bullet, d)$. Theorem 4.36 says that $h \circ g$ and $\mathrm{id}_{(P^\bullet, d)}$ are cochain homotopic. The same goes for $g \circ f$ and $\mathrm{id}_{(Q^\bullet, e)}$. So $h \circ g \simeq \mathrm{id}_{(P^\bullet, d)}$ and $g \circ h \simeq \mathrm{id}_{(Q^\bullet, e)}$. So $(P^\bullet, d)$ and $(Q^\bullet, e)$ are cochain homotopy equivalent. □

4.3 Ext-functors and the induced long sequence

Definition 4.38 (Ext-functor)**.** Let R be a ring. Let N and M be R-modules. Take an injective resolution $(P^\bullet, d)$ of M. Apply $\operatorname{Hom}_R(N, \_)$ to $(P^\bullet, d)$ to get a cochain complex $\operatorname{Hom}_R(N, (P^\bullet, d))$ and take the i-th cohomology of it. This R-module is denoted by $\operatorname{Ext}^i_R(N, M)$.

Remark 4.39. Let R be a ring and let N be an R-module. The functor $\operatorname{Hom}_R(N, \_)$ is covariant and left-exact. The proof is identical to Lemma 4.16.

Lemma 4.40. Let R be a ring. Let N and M be R-modules. The R-module $\operatorname{Ext}^i_R(N, M)$ as defined in Definition 4.38 is independent of the choice of injective resolutions.

Proof. Let $(P^\bullet, d)$ and $(Q^\bullet, e)$ be two injective resolutions of M. $(P^\bullet, d)$ and $(Q^\bullet, e)$ are cochain homotopy equivalent by Corollary 4.37. The codifferentials of $(\mathrm{Hom}_R(N, (P^\bullet, d))$ and $(\mathrm{Hom}_R(N, (Q^\bullet, e))$ are just the pushforwards of the codifferentials of $(P^\bullet, d)$ and $(Q^\bullet, e)$. The cochain homotopies that turn $(P^\bullet, d)$ and $(Q^\bullet, e)$ into a cochain homotopy equivalence also turn $\mathrm{Hom}_R(N, (P^\bullet, d))$ and $\mathrm{Hom}_R(N, (Q^\bullet, e))$ into a cochain homotopy equivalence by pushing forward. It follows $H^i(\mathrm{Hom}_R(N, (P^\bullet, d)))$ and $H^i(\mathrm{Hom}_R(N, (Q^\bullet, e)))$ are isomorphic $i \geq 0$ by Lemma 4.32. □

Theorem 4.41. Let R be a ring. Let N_1, N_2 and M_1, M_2 be R-modules. Let $N_2 \xrightarrow{f} N_1$ and $M_1 \xrightarrow{g} M_2$ be R-module morphisms. Take an injective resolution $(P^\bullet, d)$ of M_1 and an injective resolution $(Q^\bullet, e)$ of M_2. Apply Theorem 4.36 to $(P^\bullet, d)$, $(Q^\bullet, e)$ and $M_1 \xrightarrow{g} M_2$ to get a cochain map as in diagram (8). Apply $\mathrm{Hom}_R(N_1, \_)$ to diagram (8). Pull $\mathrm{Hom}_R(N_1, (Q^\bullet, e))$ back on f. This defines an R-module morphism $\mathrm{Ext}^i_R(N_1, M_1) \xrightarrow{(f,g)} \mathrm{Ext}^i_R(N_2, M_2)$.

Proof. On the right-hand side, the cochain map induced by the injective resolutions is unique up to cochain homotopy by Theorem 4.36. This cochain homotopy is preserved by the Hom_R-functor, since the codifferentials of the cochain complex under the Hom_R-functor are just the pushforwards of the injective resolutions. Again Lemma 4.32 says that cochain homotopic maps give identical morphisms between the cohomology groups. It is obvious that the pullback by f defines group homomorphisms $\mathrm{Hom}_R(N_1, Q_i)$ to $\mathrm{Hom}_R(N_2, Q_i)$ for $i \geq 0$. Those group homomorphisms give well-defined group homomorphisms between the cohomology groups $H^i(\mathrm{Hom}_R(N_1, (Q^\bullet, e)))$ and $H^i(\mathrm{Hom}_R(N_2, (Q^\bullet, e)))$ for $i \geq 0$. □

Proposition 4.42. Let R be a ring. Let $\mathrm{Ext}^i_R\colon R-\mathsf{Mod}^{\mathrm{opp}} \times R-\mathsf{Mod} \to R-\mathsf{Mod}$ be the mapping of objects in Definition 4.38 and the mapping of morphisms in Theorem 4.41. Then Ext^i_R defines a functor for $i \geq 0$.

Proof. Ext^i_R is independent of the choice of injective resolutions for every $i \geq 0$ on objects (Lemma 4.40). The construction in Theorem 4.41 preserves identities and compositions in both terms (with order reversal in the left-hand side). □

Remark 4.43. For R-modules M and N, $\mathrm{Ext}^1_R(M, N)$ only has an R-module structure if R is commutative. See Remark 4.17. In case R is not commutative, the Ext-functors can still be defined. They will map to Ab instead.

Remark 4.44. Take cochain complexes as objects and homotopy classes of cochain maps as morphisms. They define a category. The injective resolution construction, described in Theorem 4.36 induces an injective resolution functor from R-Mod to that category. Now Ext^i_R is the composition of an injective resolution functor, a covariant Hom_R-functor and a cohomology functor in some degree.

For certain rings and certain modules, there exist *easy* injective resolutions. As a result of that they give *easy* Ext-functors for higher degrees.

Example 4.45. Let R be a ring. Let M be an injective R-module. Now $0 \to M \to M \to 0$ is an injective resolution of M. So $\mathrm{Ext}^i_R(N, M) = 0$ for every R-module N and every $i \geq 1$.

Example 4.46. Let M be a $\mathbb{Z}$-module. There exists an injective $\mathbb{Z}$-module P such that $M \subset P$ is a $\mathbb{Z}$-submodule. By Corollary 4.25, P/M is the quotient of an injective $\mathbb{Z}$-module and therefore injective itself. So $0 \to P \to P/M \to 0$ is an injective resolution of M. Therefore $\mathrm{Ext}^i_{\mathbb{Z}}(N, M) = 0$ for every $\mathbb{Z}$-module N and every $i \geq 2$.

Remark 4.47. $\mathbb{Z}-\mathsf{Mod}$ and Ab are isomorphic. A $\mathbb{Z}$-module is an abelian group (just as any module over a ring is). An abelian group can be endowed with a $\mathbb{Z}$-module structure. Let M be an abelian group and let $x \in M$ and $n \in \mathbb{Z}$, then $n \cdot m := \underbrace{x + ... + x}_{\text{n-times}}$ (If $n < 0$, take $n \cdot m := \underbrace{-x - ... - x}_{\text{-n-times}}$). A $\mathbb{Z}$-module morphism is a group homomorphism. A group homomorphism between abelian groups preserves the $\mathbb{Z}$-action. Therefore $\mathbb{Z}-\mathsf{Mod}$ or Ab are used, depending on what is most intuitive.

Lemma 4.48. $\mathrm{Ext}^0_R(\_,\_)$ and $\mathrm{Hom}_R(\_,\_)$ are isomorphic as bifunctors.

Proof. Let R be a ring and let N and M be R-modules. Let $(P^\bullet, d)$ be an injective resolution of M. Now $0 \to M \to P_0 \xrightarrow{d_0} P_1 \to \dots$ is exact. $\mathrm{Hom}_R(N,\_)$ is a left-exact functor. So $0 \to \mathrm{Hom}_R(N,M) \to \mathrm{Hom}_R(N,P_0) \xrightarrow{d_0 \circ \_} \mathrm{Hom}_R(N,P_1)$ is exact.
Clearly $\mathrm{Ext}^0_R(N,M) := H^0(\mathrm{Hom}_R(N,(P^\bullet,d)) = \ker(d_0 \circ \_) = \mathrm{Hom}_R(N,M)$.

A similar approach works with regards to the morphisms. For the left-hand side, both $\mathrm{Ext}^0_R(\_,\_)$ and $\mathrm{Hom}_R(\_,\_)$ let a morphism of R-modules act as a pullback. For the right-hand side, the diagram of Theorem 4.36 shows that a morphism of R-modules induces a cochain map. In this case, let the morphism of R-modules act as a pushforward. The induced map on the 0-th homology of the $\mathrm{Hom}_R(\_,\_)$-modules clearly is the original R-module morphism. ☐

Remark 4.49. In the proof the morphisms $\{\mathrm{id}_{\mathrm{Hom}_R(N,M)}\}_{N,M\in \mathrm{Ob}(R-\mathsf{Mod})}$ define a natural transformation from $\mathrm{Ext}^0_R(\_,\_)$ to $\mathrm{Hom}_R(\_,\_)$. It is level-wise isomorphic and therefore an isomorphism of functors.

Remark 4.50. $\mathrm{Hom}_R(N,\_)$ is a left-exact functor. It might be tempting to believe that the induced sequence of the injective resolution of M under this functor is exact from $\mathrm{Hom}_R(N,P^1)$ onward (and that as a result of this $\mathrm{Ext}^i_R(N,M) := H^i(\mathrm{Hom}_R(N,(P^\bullet,d)) = 0$ for $i \geq 0$). This is not true! A left-exact functor guarantees left-exactness for short exact sequences only, see Definition 4.6. In general $\mathrm{Hom}_R(N,\_)$ does not preserve exact sequences. Consider the following sequences:

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{2} \dots \tag{9}$$

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \dots . \tag{10}$$

Sequence (9) is an exact sequence of $\mathbb{Z}/4\mathbb{Z}$-modules. Notice $\mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/4\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Applying $\mathrm{Hom}_{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\_)$ to sequence (9) gives sequence (10), which is not exact. It is exact up to its first three terms; $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z}$ is exact.

Remark 4.51. $\mathrm{Hom}_R(N,(P^\bullet,d))$ may not be exact, it certainly is a cochain complex, as $d_{i+1} \circ d_i = 0$ for all $i \geq 0$ (and this also holds with respect to the pushforwards).

In a short exact sequence of (co)chain complexes [19, p. 41], there exists a long exact sequence of (co)homology groups [19, Theorem 4.1]. $\mathrm{Ext}^i_R(\_,\_)$ is the cohomology functor over $\mathrm{Hom}_R(\_,\_)$.

Theorem 4.52 (Long exact sequence)**.** Let R be a ring. Let A, B, C, M be R-modules and let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. There exists a long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(C,M) \arrow[r, "-\circ g"] & \mathrm{Hom}_R(B,M) \arrow[r, "-\circ f"] & \mathrm{Hom}_R(A,M) \arrow[dll, "\delta_0"] \\
& \mathrm{Ext}^1_R(C,M) \arrow[r, "-\circ g"] & \mathrm{Ext}^1_R(B,M) \arrow[r, "-\circ f"] & \mathrm{Ext}^1_R(A,M) \arrow[dll, "\delta_1"] \\
& \mathrm{Ext}^2_R(C,M) \arrow[r, "-\circ g"] & \mathrm{Ext}^2_R(B,M) \arrow[r, "-\circ f"] & \mathrm{Ext}^2_R(A,M) \arrow[dll, "\delta_2"] \\
& \cdots & &
\end{tikzcd}$$

Proof. Let $(P^\bullet, d)$ be an injective resolution of M. Now apply $\mathrm{Hom}_R(\_, P_i)$ to $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$. $0 \to \mathrm{Hom}_R(C, P_i) \xrightarrow{-\circ g} \mathrm{Hom}_R(B, P_i) \xrightarrow{-\circ f} \mathrm{Hom}_R(A, P_i) \to 0$ is exact for $i \geq 0$ as P_i is injective (Lemma 4.19, statement 4). Pushing forward the codifferentials gives the following commutative diagram:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_R(C,P_0) \arrow[r, "-\circ g"] \arrow[d, "d_0\circ_"] & \mathrm{Hom}_R(B,P_0) \arrow[r, "-\circ f"] \arrow[d, "d_0\circ_"] & \mathrm{Hom}_R(A,P_0) \arrow[r] \arrow[d, "d_0\circ_"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(C,P_1) \arrow[r, "-\circ g"] \arrow[d, "d_1\circ_"] & \mathrm{Hom}_R(B,P_1) \arrow[r, "-\circ f"] \arrow[d, "d_1\circ_"] & \mathrm{Hom}_R(A,P_1) \arrow[r] \arrow[d, "d_1\circ_"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(C,P_2) \arrow[r, "-\circ g"] \arrow[d] & \mathrm{Hom}_R(B,P_2) \arrow[r, "-\circ f"] \arrow[d] & \mathrm{Hom}_R(A,P_2) \arrow[r] \arrow[d] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd} \tag{11}$$

The horizontal rows of diagram (11) are exact. It is an exact sequence of cochain complexes. The exact sequence of cochain complexes induces the following long exact sequence. The construction is given in [19, §II.4].

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathrm{Hom}_R(C,(P^\bullet,d))) \arrow[r, "-\circ g"] & H^0(\mathrm{Hom}_R(B,(P^\bullet,d))) \arrow[r, "-\circ f"] & H^0(\mathrm{Hom}_R(A,(P^\bullet,d))) \arrow[dll, "\delta_0"] \\
& H^1(\mathrm{Hom}_R(C,(P^\bullet,d))) \arrow[r, "-\circ g"] & H^1(\mathrm{Hom}_R(B,(P^\bullet,d))) \arrow[r, "-\circ f"] & H^1(\mathrm{Hom}_R(A,(P^\bullet,d))) \arrow[dll, "\delta_1"] \\
& H^2(\mathrm{Hom}_R(C,(P^\bullet,d))) \arrow[r, "-\circ g"] & H^2(\mathrm{Hom}_R(B,(P^\bullet,d))) \arrow[r, "-\circ f"] & H^2(\mathrm{Hom}_R(A,(P^\bullet,d))) \arrow[dll, "\delta_2"] \\
& \cdots & &
\end{tikzcd}$$

Application of Definition 4.38 and Lemma 4.48 finalises the proof. □

4.4 Extensions of R-modules and Baer sum

Definition 4.53 (Extension)**.** Let R be a ring and let M and N be R-modules. An *extension of M by N* is a short exact sequence $0 \to N \xrightarrow{i} E \xrightarrow{p} M \to 0$.

Definition 4.54 (Extension equivalence)**.** Let R be a ring. Let M and N be R-modules. Extensions $0 \to N \xrightarrow{i} E \xrightarrow{p} M \to 0$ and $0 \to N \xrightarrow{j} F \xrightarrow{q} M \to 0$ of M by N are *equivalent* if there exists an R-module morphism $\phi: E \to F$ such the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "i"] \arrow[d, "\mathrm{id}_N"'] & E \arrow[r, "p"] \arrow[d, "\phi"'] & M \arrow[r] \arrow[d, "\mathrm{id}_M"'] & 0 \\
0 \arrow[r] & N \arrow[r, "j"] & F \arrow[r, "q"] & M \arrow[r] & 0.
\end{tikzcd} \tag{12}$$

The set of equivalence classes of extensions of M by N is denoted by $\mathrm{Ext}_R(M,N)$

Remark 4.55. The middle terms of equivalent extensions are always isomorphic by Lemma 4.10. But the fact that there exists an isomorphism between two middle terms, does not necessarily mean that the extensions are equivalent. Note the following diagram of $\mathbb{Z}$-modules:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/3\mathbb{Z} \arrow[r, "\cdot 3"] \arrow[d, "\mathrm{id}_{\mathbb{Z}/3\mathbb{Z}}"'] & \mathbb{Z}/9\mathbb{Z} \arrow[r] & \mathbb{Z}/3\mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}_{\mathbb{Z}/3\mathbb{Z}}"'] & 0 \\
0 \arrow[r] & \mathbb{Z}/3\mathbb{Z} \arrow[r, "\cdot 6"] & \mathbb{Z}/9\mathbb{Z} \arrow[r] & \mathbb{Z}/3\mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

There exists no group homomorphism $\mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/9\mathbb{Z}$ that makes the diagram commute. This means that one cannot identify (an equivalence class of) extensions by their middle object. The morphisms of the exact sequence matter!

Proposition 4.56. Equivalence of extensions of modules is an equivalence relation.

Proof. This follows from the fact that the vertical morphism in the diagram of Definition 4.54 is an isomorphism. It has an inverse. Compositions of isomorphisms are isomorphisms. ☐

Corollary 4.57. Let R be a ring. Let M and N be R-modules. All split extensions of M by N are equivalent. Moreover, the split extensions form their own class in $\mathrm{Ext}_R(M,N)$.

Proof. The isomorphism of statement 3 of Lemma 4.8 creates an equivalence. Transitivity implies equivalence between any two split extensions of M by N. ☐

$\mathrm{Ext}_R(M,N)$ can be endowed with an abelian group structure. This requires some preparation.

Lemma 4.58. Let R be a ring. Let M and N be R-modules. Let $0 \to N \xrightarrow{i} E \xrightarrow{p} M \to 0$ and $0 \to N \xrightarrow{j} F \xrightarrow{q} M \to 0$ be extensions of M by N. Define $Y := (E \times_M F)/\{(i(n), -j(n)) : n \in N\}$ (see Proposition 3.3). Then $0 \to N \xrightarrow{(i,0)} Y \xrightarrow{p} M \to 0$ defines an extension of M by N.

Proof. $\{(i(n), -j(n)) : n \in N\}$ is an R-submodule of $E \times_M F$, since $\{(i(n), -j(n)) : n \in N\}$ is the image of $N \xrightarrow{(i,-j)} E \oplus F$. Notice $p(i(n)) = 0 = q(-j(n))$ for all $n \in N$. Therefore $\{(i(n), -j(n)) : n \in N\} \subset E \times_M F$. Hence $(E \times_M F)/\{(i(n), -j(n)) : n \in N\}$ is an R-module.

$N \xrightarrow{(i,0)} Y$ is well-defined and injective. It follows from the exactness of $0 \to N \xrightarrow{i} E \xrightarrow{p} M \to 0$.

As $p \circ i = 0 = q \circ j$ by exactness, $Y \xrightarrow{p} M$ is independent of representative. Let $m \in M$, then there exist $e \in E$ and $f \in F$ such that $p(e) = m = q(f)$. So $[(e,f)] \in Y$ is in the pre-image of m of $Y \xrightarrow{p} M$.

$\operatorname{im}(i,0) \subset \ker p$, since $\operatorname{im} i = \ker p$. Conversely, let $[(e,f)] \in Y$ such that $p([(e,f)]) = 0$, then $p(e) = 0 = q(f)$, so $e \in \operatorname{im} i$ and $f \in \operatorname{im} j$. Write $e = i(n)$ and $f = j(n')$ for some $n, n' \in N$. Now $[(e,f)] = [(i(n), j(n'))] = [(i(n) + i(n'), 0)] = [(i(n+n'), 0)]$. So $[(e,f)] \in \operatorname{im}(i,0)$. So $\ker p \subset \operatorname{im}(i,0)$. The resulting sequence is exact; it is an extension of M by N. □

Definition 4.59 (Baer sum)**.** Let R be a ring and let M and N be R-modules. The *(Baer) sum* of extensions $0 \to N \xrightarrow{i} E \xrightarrow{p} M \to 0$ and $0 \to N \xrightarrow{j} F \xrightarrow{q} M \to 0$ of M by N is the extension $0 \to N \xrightarrow{(i,0)} Y \xrightarrow{p} M \to 0$ defined in Lemma 4.58.

Theorem 4.60. The Baer sum (Definition 4.59) endows $\operatorname{Ext}_R(M,N)$ with a group structure.

Proof. We will show that the Baer sum is independent of representative in $\operatorname{Ext}_R(M,N)$: Take equivalent extensions as in diagram (12). Take an arbitrary extension $0 \to M \xrightarrow{k} G \xrightarrow{r} N \to 0$. A commutative diagram arises:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "{(i,0)}"] \arrow[d, "\mathrm{id}_N"'] & Y \arrow[r, "p"] \arrow[d, "{(\phi,\mathrm{id}_G)}"'] & M \arrow[r] \arrow[d, "\mathrm{id}_M"'] & 0 \\
0 \arrow[r] & N \arrow[r, "{(j,0)}"] & Z \arrow[r, "q"] & M \arrow[r] & 0,
\end{tikzcd}$$

with $Y := (E \times_M G)/\{(i(n), -k(n)) : n \in N\}$ and $Z := (F \times_M G)/\{(j(n), -k(n)) : n \in N\}$.
Now $Y \xrightarrow{(\phi,\mathrm{id}_G)} Z$ defines a morphism as $(\phi, \mathrm{id}_F)(i(n), -k(n)) = (j(n), -k(n))$.
It follows $0 \to N \xrightarrow{i} Y \xrightarrow{p} M \to 0$ and $0 \to N \xrightarrow{j} Z \xrightarrow{p'} M \to 0$ are equivalent. They represent the same class in $\operatorname{Ext}_R(M,N)$.

We will show that the class of split extensions forms the unit. Let $0 \to N \xrightarrow{i} E \xrightarrow{p} M \to 0$ be an extension. Let $N \oplus M \xrightarrow{\pi} M$ be the projection. Define $E \times_M (N \oplus M) := \{(e,(n,m)) \in E \oplus (N \oplus M) : p(e) = \pi(n,m)\}$. As $\pi(n,m) = m$, it follows $(e,(n,m)) \in E \oplus (N \oplus M)$ is in $E \times_M (N \oplus M)$ if and only if $p(e) = m$. So $E \times_M (N \oplus M) \cong E \oplus N$.
Now $(E \times_M (N \oplus M))/\{(i(n),(-n,0)) : n \in N\} \cong (E \oplus N)/\{(i(n),-n) : n \in N\} \cong E$. This isomorphism between $(E \times_M (N \oplus M))/\{(i(n),(-n,0)) : n \in N\}$ and E commutes with the morphisms.

We will show that the Baer sum is associative. Let $0 \to N \xrightarrow{i} D \xrightarrow{p} M \to 0$, $0 \to N \xrightarrow{j} E \xrightarrow{q} M \to 0$ and $0 \to N \xrightarrow{k} F \xrightarrow{r} M \to 0$ be extensions. There are isomorphisms between the following three R-modules

1. $$(((D \times_M E)/\{(i(n), -j(n)) : n \in N\}) \times_M F)/\{((i(n),0), -k(n)) : n \in N\}$$
2. $\{(d,e,f) \in D \oplus E \oplus F : p(d) = q(e) = r(f)\}/(\{(i(n), -j(n), 0) : n \in N\} \oplus \{(i(n), 0, -k(n)) : n \in N\})$

$$(D \times_M ((E \times_M F)/\{(j(n), -k(n)) : n \in N\}))/\{((i(n), 0), -j(n)) : n \in N\}.$$

3.

We will give an explicit inverse. Let $0 \to N \xrightarrow{i} E \xrightarrow{p} M \to 0$ be an extension. Add $0 \to N \xrightarrow{-i} E \xrightarrow{p} M \to 0$ to it. $E \times_M E := \{(e, f) \in E \oplus E : p(e) = p(f)\}$. So $(e, f) \in E \oplus E$ is in $E \times_M E$ if and only if $p(e - f) = 0$, i.e. $e - f \in \operatorname{im} i$. Now i is injective and therefore there exists a unique $n \in N$ such that $e - f = i(n)$. Define $Y := (E \times_M E)/\{(i(n), i(n)) : n \in N\}$. Now Y is isomorphic to $N \oplus M$ via the morphism $(E \times_M E)/\{(i(n), i(n)) : n \in N\} \ni [(e, f)] \xrightarrow{\phi} (i^{-1}(e - f), p(e)) \in N \oplus M$. ☐

Remark 4.61. The isomorphisms for the associativity are obtained by picking elements $d \in D$, $e \in E$, $f \in F$ such that $p(d) = q(e) = r(f)$ and dividing out the subgroup generated by images of N.

Lemma 4.62. The Baer sum of Definition 4.59 is commutative. $\operatorname{Ext}_R(M, N)$ is an abelian group.

Proof. $E \times_M F := \{(e, f) \in E \oplus F : p(e) = q(f)\}$ and $F \times_M E := \{(f, e) \in F \oplus E : q(f) = p(e)\}$ are isomorphic. The morphisms $N \xrightarrow{(i,0)} Y$ and $N \xrightarrow{(0,j)} Y$ are identical, since $[(i(n), 0)] = [(0, j(n))]$ in Y. Likewise the morphisms $Y \xrightarrow{p} M$ and $Y \xrightarrow{q} M$ are identical by construction (as $p(e) = q(f)$). ☐

Remark 4.63. From now on $\operatorname{Ext}_R(M, N)$ will always mean the abelian group under the Baer sum.

Theorem 4.64. Let R be a ring. Let M and N be R-modules. Let $0 \to N \xrightarrow{i} E \xrightarrow{p} M \to 0$ be an extension of M by N. This short exact sequence induces the long exact sequence from diagram (14). For now only this part is necessary:

$$0 \to \operatorname{Hom}_R(M, N) \xrightarrow{i\circ\_} \operatorname{Hom}_R(M, E) \xrightarrow{p\circ\_} \operatorname{Hom}_R(M, M) \xrightarrow{\delta_0} \operatorname{Ext}^1_R(M, N) \to \dots.$$

View $\delta_0(\mathrm{id}_M) \in \operatorname{Ext}^1_R(M, N)$ as the image of the class $0 \to N \xrightarrow{i} E \xrightarrow{p} M \to 0$ in $\operatorname{Ext}_R(M, N)$. This mapping defines an isomorphism of groups between $\operatorname{Ext}_R(M, N)$ and $\operatorname{Ext}^1_R(M, N)$.

Proof. We will show the mapping is independent of representative of equivalence class. For an equivalent extension $0 \to N \xrightarrow{j} F \xrightarrow{q} M \to 0$, there exists an R-module isomorphism $E \xrightarrow{\phi} F$ as in diagram (12). The following commutative diagram arises by the functoriality of the Ext-functors and naturality of the connecting maps δ_0:

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(M,N) \arrow[r, "i\circ_"] \arrow[d, "\mathrm{id}_{\operatorname{Hom}_R(M,N)}"'] & \operatorname{Hom}_R(M,E) \arrow[r, "p\circ_"] \arrow[d, "(\mathrm{id}_M,\phi)"'] & \operatorname{Hom}_R(M,M) \arrow[r, "\delta_0"] \arrow[d, "\mathrm{id}_{\operatorname{Hom}_R(M,M)}"'] & \operatorname{Ext}^1_R(M,N) \arrow[r] \arrow[d, "\mathrm{id}_{\operatorname{Ext}^1_R(M,N)}"'] & \dots \\
0 \arrow[r] & \operatorname{Hom}_R(M,N) \arrow[r, "j\circ_"] & \operatorname{Hom}_R(M,F) \arrow[r, "q\circ_"] & \operatorname{Hom}_R(M,M) \arrow[r, "\delta_0'"] & \operatorname{Ext}^1_R(M,N) \arrow[r] & \dots.
\end{tikzcd}$$

It follows $\delta_0(\mathrm{id}_M) = \delta_0'(\mathrm{id}_M)$. So this construction is independent of the choice of representatives.

We will show injectivity. If $\delta_0(\mathrm{id}_M) = 0$, then by exactness, $\mathrm{id}_M \in \operatorname{im}(\operatorname{Hom}_R(M, E) \xrightarrow{p\circ\_} \operatorname{Hom}_R(M, M))$. So there exists an element $s \in \operatorname{Hom}_R(M, E)$ such that $p \circ s = \mathrm{id}_M$. There exists a section of p. Statement 1 of Lemma 4.8 is satisfied and the extension is split. It represents the trivial class in $\operatorname{Ext}_R(M, N)$. The kernel of this map is trivial.

We will show surjectivity. The R-module N allows an injective resolution $(P^\bullet, d)$(see Proposition 4.29). In particular, N can be embedded in some injective R-module [5, Theorem 6]. Pick the P_0 from the injective resolution. The exact sequence $0 \to N \xrightarrow{i} P_0 \xrightarrow{p} P_0/N \to 0$ induces the following long exact sequence:

$$0 \to \mathrm{Hom}_R(M,N) \xrightarrow{i\circ\_} \mathrm{Hom}_R(M,P_0) \xrightarrow{p\circ\_} \mathrm{Hom}_R(M,P_0/N) \xrightarrow{\delta_0} \mathrm{Ext}^1_R(M,N) \to 0. \qquad (13)$$

P_0 is injective and therefore $\mathrm{Ext}^1_R(M,P_0) = 0$ (see Example 4.45). The long exact sequence is cut short. The connecting map δ_0 is surjective. For $x \in \mathrm{Ext}^1_R(M,N)$, there exists an element $f \in \mathrm{Hom}_R(M,P_0/N)$ such that $\delta_0(f) = x$. Now define $P_0 \times_{P_0/N} M := \{(e,m) \in P_0 \oplus M : p(e) = f(m)\}$. The following commutative diagram arises:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "{(i,0)}"] \arrow[d, "\mathrm{id}_N"'] & P_0 \times_{P_0/N} M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & N \arrow[r, "i"] & P_0 \arrow[r, "p"] & P_0/N \arrow[r] & 0.
\end{tikzcd}$$

The upper row is exact; $(i,0)$ is injective, since i is injective. p is surjective, so for every $m \in M$ there exists a $e \in P_0$ such that $p(e) = f(m)$. So $P_0 \times_{P_0/N} M \to M$ is surjective. Clearly $\mathrm{im}(i,0) \subset \ker(P_0 \times_{P_0/N} M \to M)$. Conversely, if $(e,m) \in \ker(P_0 \times_{P_0/N} M \to M)$, then $m = 0$. As$f(0) = 0$, it follows $p(e) = 0$. By exactness $e \in \mathrm{im}\, i$ and $(e,0) \in \mathrm{im}(i,0)$. The functoriality of the Ext-functors and the naturality of the connecting map δ_0 will induce the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M,N) \arrow[r, "{(i,0)\circ_}"] \arrow[d, "\mathrm{id}_{\mathrm{Hom}_R(M,N)}"'] & \mathrm{Hom}_R(M,P_0 \times_{P_0/N} M) \arrow[r] \arrow[d] & \mathrm{Hom}_R(M,M) \arrow[r, "\delta_0'"] \arrow[d, "f\circ_"'] & \mathrm{Ext}^1_R(M,N) \arrow[r] \arrow[d, "\mathrm{id}_{\mathrm{Ext}^1_R(M,N)}"'] & \dots \\
0 \arrow[r] & \mathrm{Hom}_R(M,N) \arrow[r, "i\circ_"] & \mathrm{Hom}_R(M,P_0) \arrow[r, "p\circ_"] & \mathrm{Hom}_R(M,P_0/N) \arrow[r, "\delta_0"] & \mathrm{Ext}^1_R(M,N) \arrow[r] & 0.
\end{tikzcd}$$

Now $\delta_0' = \delta_0 \circ f$ and so $\delta_0'(\mathrm{id}_M) = \delta_0(f \circ \mathrm{id}_M) = \delta_0(f) = x$. So there exists a class of extensions for which id_M is mapped to x under the connecting map. Therefore the mapping is surjective.

To prove the group action is preserved, go back to the long exact sequence cut short, induced by $0 \to N \xrightarrow{i} P_0 \xrightarrow{p} P_0/N \to 0$, see diagram (13). Let $x, y \in \mathrm{Ext}^1_R(M,N)$ and let $f, g \in \mathrm{Hom}_R(M,P_0/N)$ such that $\delta_0(f) = x$ and $\delta_0(g) = y$. Now $\delta_0(f+g) = x+y$. The extension $0 \to N \to P_0 \times^f_{P_0/n} M \to M \to 0$ was sent to x and the extension $0 \to N \to P_0 \times^g_{P_0/n} M \to M \to 0$ was sent to y. Apply the Baer sum to these extensions:

$(P_0 \times^f_{P_0/n} M) \times_M (P_0 \times^g_{P_0/n} M) = \{(d,m,e,n) \in P_0 \oplus M \oplus P_0 \oplus M : p(d) = f(m),\ p(e) = g(n),\ m = n\}$.

$(P_0 \times^f_{P_0/n} M) \times_M (P_0 \times^g_{P_0/n} M)$ is isomorphic to $\{(d,e,m) \in P_0 \oplus P_0 \oplus M : p(e) = f(m),\ p(f) = g(m)\}$. Define $Y := \{(d,e,m) \in P_0 \oplus P_0 \oplus M : p(d) = f(m),\ p(e) = g(m)\}/\{(i(n),-i(n),0) : n \in N\}$.

Define the R-module morphism $Y \xrightarrow{\phi} P_0 \times^{f+g}_{P_0/N} M$ such that $\phi(e,f,m) = (e+f,m)$. Notice that $\phi(i(n),-i(n),0) = (0,0)$ for all $n \in N$. $p(e+f) = p(e)+p(f) = f(m)+g(m)+(f+g)(m)$ as $p(e) = f(m)$ and $p(f) = g(m)$. The following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "{(i,0,0)}"] \arrow[d, "\mathrm{id}_N"'] & Y \arrow[r] \arrow[d, "\phi"'] & M \arrow[r] \arrow[d, "\mathrm{id}_M"'] & 0 \\
0 \arrow[r] & N \arrow[r, "{(i,0)}"] & P_0 \times^{f+g}_{P_0/N} M \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

The upper extension (the Baer sum of the extensions induced by f and g) is equivalent to the lower extension (induced by $f+g$). The mapping preserves the group action. A group homomorphism with trivial kernel is injective. Combined with surjectivity, this makes the mapping bijective. $\square$

Remark 4.65. $\mathrm{Ext}_R(M,N)$ is an abelian group, but it has no R-module structure (unless $R=\mathbb{Z}$). If R is commutative, then $\mathrm{Ext}^1_R(M,N)$ is an R-module (see Remark 4.43). $\mathrm{Ext}_R(M,N)$ will be endowed with an R-module structure based on this isomorphism between $\mathrm{Ext}^1_R(M,N)$ and $\mathrm{Ext}_R(M,N)$. This construction copies the R-action of $\mathrm{Ext}^1_R(M,N)$ to the isomorphic abelian group $\mathrm{Ext}_R(M,N)$. The R-module structure of $\mathrm{Ext}_R(M,N)$ depends on the isomorphism of Theorem 4.64. Another isomorphism, will give another R-module structure on $\mathrm{Ext}_R(M,N)$. Henceforth, it will be assumed that the R-module structure of $\mathrm{Ext}_R(M,N)$ comes from Theorem 4.64.

Lemma 4.66. Let R be a ring. Let M be an R-module. The following statements are equivalent:

1. M is injective, i.e. the statements of Lemma 4.19 hold.
2. $\mathrm{Ext}^1_R(N,M)=0$ for all R-modules N.

Proof. $1\Rightarrow 2$: See Example 4.45.
$2\Rightarrow 1$: If $\mathrm{Ext}^1_R(N,M)=0$ for all R-modules N, then by Theorem 4.64 $\mathrm{Ext}_R(N,M)=0$. Every exact sequence $0\to M\xrightarrow{i}E\xrightarrow{p}N\to 0$ is split. Statement 2 of Lemma 4.19 is satisfied. $\square$

4.5 Dual statements: projective modules

This section is solely for the purpose of guaranteeing the existence of a projective resolution and verifying it is a dual construction. This subsection can be skipped completely if you believe me (or know that the statements about duality are true). I will work with index sets of arbitrary cardinality. Assumption of the axiom of choice is necessary.

Lemma 4.67 (Dual to Lemma 4.19)**.** Let R be a commutative ring. Let M be an R-module. The following are equivalent:

1. There exists an R-module K such that $M\oplus K$ is free.
2. Any short exact sequence of R-modules $0\to K\to N\to M\to 0$ splits.
3. Let $f\colon X\to Y$ be a surjective morphism (epimorphism) of R-modules and let $g\colon M\to Y$ be an arbitrary morphism of R-modules, then there exists an R-module morphism $h\colon M\to X$ such that $f\circ h=g$.
4. The covariant functor $\mathrm{Hom}_R(M,\_)$ is exact.

Proof. We prove $2\Rightarrow 1\Rightarrow 4$. The proof of $4\Rightarrow 3$ is the same as in Lemma 4.19 (replace coker by ker). The proof of $3\Rightarrow 2$ is also the same as in Lemma 4.19 (there is a section instead of a retraction).
2 induces 1: Let $\{m_i\}_{i\in I}\subset M$ be a set of generators of M, where I is the index set. Define $R^I\ni e_i:=\underbrace{(\dots,0,1,0,\dots)}_{\text{i-th position}}$. The R-module morphism $R^I\xrightarrow{\phi}M$ for which $\phi(e_i)=m_i$ $(i\in I)$ is surjective. The sequence $0\to\ker\phi\to R^I\xrightarrow{\phi}M\to 0$ is exact. It is split. So $M\oplus\ker\phi\cong R^I$.

1 induces 4: There exists an R-module K such that $M \oplus K$ is free. Say $M \oplus K \cong R^I$ for some index set I. The exact sequence $0 \to K \xrightarrow{i} R^I \xrightarrow{q} M \to 0$ with i the inclusion and q the quotient, is split. Now $\mathrm{Hom}_R(R^I, \_)$ is exact. This is because $\mathrm{Hom}_R(R, A) \cong A$ for any R-module A. The element $1 \in R$ can be sent to any element of A. This determines the whole R-module morphism from R to A. So $\mathrm{Hom}_R(R^I, A) \cong A^I$ and therefore $\mathrm{Hom}_R(R^I, \_)$ is exact. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence. The following diagram arises:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Hom}_R(M,A) \arrow[r, "f\circ_"] \arrow[d, "_\circ q"'] & \mathrm{Hom}_R(M,B) \arrow[r, "g\circ_"] \arrow[d, "_\circ q"'] & \mathrm{Hom}_R(M,C) \arrow[d, "_\circ q"'] & \\
0 \arrow[r] & \mathrm{Hom}_R(R^I,A) \arrow[r, "f\circ_"] \arrow[d, "_\circ i"'] & \mathrm{Hom}_R(R^I,B) \arrow[r, "g\circ_"] \arrow[d, "_\circ i"'] & \mathrm{Hom}_R(R^I,C) \arrow[r] \arrow[d, "_\circ i"'] & 0. \\
0 \arrow[r] & \mathrm{Hom}_R(K,A) \arrow[r, "f\circ_"] \arrow[d] & \mathrm{Hom}_R(K,B) \arrow[r, "g\circ_"] \arrow[d] & \mathrm{Hom}_R(K,C) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

A diagram chase similar to diagram (7) shows surjectivity of $\mathrm{Hom}_R(M,B) \xrightarrow{g\circ\_} \mathrm{Hom}_R(M,C)$. $\square$

Definition 4.68. Let R be a commutative ring. Let M be an R-module. M is *projective* if the statements of Lemma 4.67 hold.

Lemma 4.69. Every free module is projective.

Proof. The direct sum of a free module with 0 is free. $\square$

Definition 4.70 (Dual to Definition 4.28)**.** Let R be a ring and let M be an R-module. A *projective resolution* of M is an exact sequence $... \to P_2 \xrightarrow{i} P_1 \xrightarrow{d_0} P_0 \xrightarrow{p} M \to 0$ with projective R-modules P_i for $i \geq 0$. If all the P_i are free, then it is called a *free resolution*.

Remark 4.71. Notice the order reversal compared to injective resolution defined in Definition 4.28. The order reversal will disappear by application of the Ext-functors. Compare Theorem 4.52 with Theorem 4.82. This is because the projective resolution is in the left-hand contravariant term and the injective resolution is in the right-hand covariant term.

Proposition 4.72 (Dual to Proposition 4.29)**.** Let R be a ring. Every R-module allows a free resolution.

Proof. We will give a sketch of the proof. Let M be an R-module. Let $\{m_i\}_{i\in I} \subset M$ be a set of generators, where I is the index set. Define $R^I \ni e_i := \underbrace{(\dots, 0, 1, 0, \dots)}_{\text{i-th position}}$. The R-module morphism $R^I \xrightarrow{\phi} M$ for which $\phi(e_i) = m_i$ $(i \in I)$ is surjective. The sequence $0 \to \ker\phi \to R^I \xrightarrow{\phi} M \to 0$ is exact. Do the same for $\ker\phi$. Let $\{n_j\}_{j\in J} \subset \ker\phi$ be a set of generators, where J is the index set. The

R-module morphism $R^J \xrightarrow{\psi} \ker\phi$ for which $\psi(e_j) = n_j$ $(j \in J)$, is surjective. Compose ψ with the inclusion of $\ker\phi$ in R^I. Image elements of the composed mapping $R^J \to R^I$ are mapped to 0 by ϕ. Conversely, if $\phi(x) = 0$ for $x \in R^I$, then $x \in \ker\phi$ and by surjectivity of ψ, there is a $y \in R^J$ in the pre-image of x under the composed mapping $R^J \to R^I$. Exactness at R^I is established. Etc. ☐

Corollary 4.73. Let R be a ring. Every R-module allows a projective resolution.

Proof. A free resolution is automatically a projective resolution by Corollary 4.72. ☐

Theorem 4.74 (Dual to Theorem 4.36)**.** Let R be a ring, let M, N be R-modules and let $f\colon M \to N$ be a morphism of R-modules. Let $(P_\bullet, d)$ be a projective resolution of M and let $(Q_\bullet, e)$ be a projective resolution of N. There exists a chain map $g\colon (P_\bullet, d) \to (Q_\bullet, e)$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r, "d_0"] \arrow[d, "g_1"'] & P_0 \arrow[r, "p"] \arrow[d, "g_0"'] & M \arrow[r] \arrow[d, "f"'] & 0 \\
\cdots \arrow[r] & Q_1 \arrow[r, "e_0"] & Q_0 \arrow[r, "q"] & N \arrow[r] & 0
\end{tikzcd}$$

This chain map g is unique up to chain homotopy.

Proof. Use property 3 of Lemma 4.67 in the same way property 3 of Lemma 4.19 was used in Theorem 4.36. ☐

Corollary 4.75 (Dual to Corollary 4.37)**.** If there exists multiple projective resolutions of M, then these resolutions are chain homotopy equivalent.

Remark 4.76. Let R be a ring. Let N and M be R-modules. Dual to Definition 4.38 the R-module $\operatorname{Ext}^i_R(M,N)$ can be constructed as follows. Take a projective resolution $(P_\bullet, d)$ of M. Apply $\operatorname{Hom}_R(\_, N)$ to $(P_\bullet, d)$ to get a cochain $\operatorname{Hom}_R((P_\bullet, d), N)$ and take the i-th cohomology of it.

Remark 4.77. For PIDs injectivity and divisibility were equivalent (Lemma 4.23 and Lemma 4.24). In a PID free and projective are equivalent as well. Let R be a PID. Take a projective R-module P. There exists an R-module Q such that $P \oplus Q$ is free. Say $P \oplus Q \cong R^I$ where I is an index set. Now P is a submodule of R^I. Let $\pi_i\colon R^I \to R$ be the projection for $i \in I$. Now $\pi_i(P)$ is a submodule of R. Submodules of R are ideals. As R is a PID, there exists a $x \in R$ such that $\pi_i(P) = (x)$. If $x = 0$, then $\pi_i(P)$ is free of rank 0. If $x \neq 0$, then $\pi_i(P)$ is free of rank 1. Hence P is free itself. This argument goes for any submodule of a free module over a PID. Therefore, we state the following lemma.

Lemma 4.78. Let R be a PID. A submodule of a free module is free.

Remark 4.79. It is necessary to let the ring be a PID. For example $\mathbb{Z}/3\mathbb{Z}$ is projective in $\mathbb{Z}/6\mathbb{Z}$ (Chinese remainder theorem), but not free. So in general, projective non-free modules exist.

Example 4.80 (Dual to Example 4.45)**.** Let R be a ring. Let M be a projective R-module. Now $0 \to M \to M \to 0$ is a projective resolution of M. So $\operatorname{Ext}^i_R(M,N) = 0$ for every R-module N and every $i \geq 1$.

Example 4.81 (Dual to Example 4.46)**.** Let M be a $\mathbb{Z}$-module. There exists a surjective $\mathbb{Z}$-module morphism $P \xrightarrow{f} M$ with P free. By Lemma 4.78, $\ker f$ is free itself. In particular, P and $\ker f$ are free. So $0 \to \ker f \to P \to M \to 0$ is a projective resolution of M. Therefore $\operatorname{Ext}^i_R(M,N) = 0$ for every $\mathbb{Z}$-module N and every $i \geq 2$.

Theorem 4.82 (Dual to Theorem 4.52)**.** Let R be a ring. Let A, B, C, M be R-modules and let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. There exists a long exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(M,A) \arrow[r, "f\circ_"] & \mathrm{Hom}_R(M,B) \arrow[r, "g\circ_"] & \mathrm{Hom}_R(M,C) \arrow[dll, "\delta_0"] \\
& \mathrm{Ext}^1_R(M,A) \arrow[r, "f\circ_"] & \mathrm{Ext}^1_R(M,B) \arrow[r, "g\circ_"] & \mathrm{Ext}^1_R(M,C) \arrow[dll, "\delta_1"] \\
& \mathrm{Ext}^2_R(M,A) \arrow[r, "f\circ_"] & \mathrm{Ext}^2_R(M,B) \arrow[r, "g\circ_"] & \mathrm{Ext}^2_R(M,C) \arrow[dll, "\delta_2"] \\
& \cdots & &
\end{tikzcd} \tag{14}$$

Remark 4.83. The converse of Lemma 4.80 holds as well. In §4.4 the Baer sum was set up to show that a module over a ring for which the Ext^1-groups vanish for any module in the left-hand term, must be injective. A dual statement holds for projective modules. In the proof of Theorem 4.64 for the injectivity a section will be found instead. For the surjectivity, using a projective resolution will lead to the same result. The proof can be dualised.

Lemma 4.84 (Dual to Lemma 4.66)**.** Let R be a ring. Let M be an R-module. The following statements are equivalent:

1. M is projective, i.e. the statements of Lemma 4.67 hold.
2. $\mathrm{Ext}^1_R(M,N) = 0$ for all R-modules N.

4.6 A projective resolution induces an injective resolution

At one point in the proof of the UCT, we will start out with a projective resolution and turn it into an injective resolution. In this subsection we will show this is done. The application of this technique is found in §16.1.

Let G be an abelian group. It allows a free resolution (see Lemma 4.72) of the form $0 \to F_1 \to F_0 \xrightarrow{f} G \to 0$ (see Example 4.81). Obviously $F_1 = \ker f$. By surjectivity, there is an isomorphism $G \cong F_0/F_1$. Now $\mathbb{Q}$ is divisible (and therefore injective over $\mathbb{Z}$) and so $F_0 \otimes \mathbb{Q}$ is divisible as well. Hence $(F_0 \otimes \mathbb{Q})/F_1$ is divisible. In this case view F_1 as the subgroup $F_1 \otimes \mathbb{Z} \subset F_0 \otimes \mathbb{Q}$. There is an embedding of F_0/F_1 into $(F_0 \otimes \mathbb{Q})/F_1$. The cokernel of the embedding is again divisible. Define $I_0 := (F_0 \otimes \mathbb{Q})/F_1$ and $I_1 := F_0/G$. It follows $0 \to G \to I_0 \to I_1 \to 0$ is an injective resolution.

Remark 4.85. In the proofs of Lemma 4.19 and Proposition 4.29, it was assumed that any module over a commutative ring is included in an injective module. Using this method of constructing an injective resolution, it can be shown for abelian groups that every module is indeed included in an injective module. This method can be generalised to modules over a PID. For PIDs, injectivity of a module is the same as divisibility (see Lemma 4.23 and Lemma 4.24). The field of fractions, when viewed as a module, is divisible and therefore injective. There is an obvious inclusion of a domain into its field of fractions. So this method works for all PIDs when $\mathbb{Q}$ is replaced by the field of fractions. ($\mathbb{Q}$ is the field of fractions of $\mathbb{Z}$). It cannot be further generalised to any domain or any ring. An injective module over a domain must be divisible, but a divisible module need not be injective. For a non-domain, there is no such thing as the field of fractions.

Part II
C*-algebras and K-theory

5 C*-algebras

This section is an introduction to some important definitions, facts about and constructions on C*-algebras. It is necessary for understanding the later sections; Busby invariants in §6, K-theory in §7 and KK-theory in §11. In order to keep this as brief as possible, proofs will be limited to a few sentences or else there will be references to other sources.

5.1 The definition of C*-algebras and a few examples

Definition 5.1. A normed algebra (over a field) that is complete with respect to the metric induced by its norm is called a *Banach algebra*.

Definition 5.2. A $*$-algebra is an algebra A over $\mathbb{C}$ combined with a function $A \xrightarrow{*} A$ (denoted by superscript) such that

$$(a+b)^* = a^* + b^*, \quad (\lambda a)^* = \overline{\lambda} a^*, \qquad (ab)^* = b^* a^*, \quad (a^*)^* = a, \qquad \forall a, b \in A, \quad \lambda \in \mathbb{C}\,.$$

Definition 5.3. A Banach $*$-algebra A is a C*-algebra if $||a^* a|| = ||a||^2$ for all $a \in A$.

Definition 5.4. A C*-algebra is *unital* if it possesses a multiplicative unit.

Example 5.5. Let $\mathcal{H}$ be a Hilbert space. The bounded operators form a C*-algebra $B(\mathcal{H})$ if the adjoint is its $*$-operation. It is always unital. The compact operators form an ideal $K(\mathcal{H}) \subset B(\mathcal{H})$. The compact operators possess a unit if and only if $\mathcal{H}$ is finite dimensional. All infinite dimensional separable Hilbert spaces are isomorphic (just connect orthonormal bases). The C*-algebra of bounded operators on an infinite dimensional separable Hilbert space are denoted by $\mathbb{B}$ and the compact operators by $\mathbb{K}$.

Example 5.6. Let X be a locally compact Hausdorff topological space. A continuous function $X \xrightarrow{f} \mathbb{C}$ *vanishes at infinity* if for every $\epsilon > 0$ the set $\{x \in X : |f(x)| \geq \epsilon\}$ is compact. The set of functions $X \xrightarrow{f} \mathbb{C}$ vanishing at infinity is denoted by $C_0(X)$. The function $C_0(X) \ni f \to \sup_{x \in X} |f(x)|$ is a norm (this is called the *supremum norm*). The operation $C_0(X) \ni f \to (x \to \overline{f(x)}) \in C_0(X)$ is a $*$-operation on $C_0(X)$. With this norm and $*$-operation, $C_0(X)$ is a C*-algebra. It is unital if and only if X is compact [21, Examples 1.1.2, 1.1.3, 2.1.2].

Definition 5.7. A $*$-homomorphism is a linear map between C*-algebras that preserves multiplication and $*$-operation.

Remark 5.8. We can speak about the category of C*-algebras with objects C*-algebras (Definition 5.3) and morphisms $*$-homomorphisms (Definition 5.7).

Lemma 5.9. $*$-homomorphisms between C*-algebras are norm-decreasing and hence continuous.

Proof. See [21, Theorem 2.1.7]. □

Definition 5.10. A $*$-homomorphism is *unital* if its domain and codomain contain a multiplicative unit and the $*$-homomorphism preserves the multiplicative unit.

Remark 5.11. Unital $*$-homomorphisms are ring homomorphisms; they preserve invertible elements. In the category of unital C*-algebras, morphisms will be required to be unital. The category of unital C*-algebras is not a full subcategory of the category of C*-algebras. The zero map is a $*$-homomorphism. For instance, if preservation of the multiplicative unit is required, then there is only one morphism from $\mathbb{C}$ to $\mathbb{C}$.

Remark 5.12. From now on, in a unital C*-algebra, the multiplicative unit is called the unit and the additive unit will be called the zero, as we do in rings.

Definition 5.13. Let A be a C*-algebra. An element $a \in A$ is called *positive* if there exists a $b \in A$ such that $a = b^* b$.

Remark 5.14. This definition is in fact the result of a theorem. See [21, Theorem 2.2.5 (1)].

Lemma 5.15. A $*$-homomorphism between C*-algebras preserves positive elements.

Proof. $*$-homomorphisms preserve the $*$-operation and multiplication. Apply this to Definition 5.13. □

5.2 Multiplier C*-algebras

Definition 5.16. Let A be a C*-algebra. A *double centraliser* on A is a pair (L, R) of bounded linear maps L, R from A to A such that $L(ab) = L(a)b$, $R(ab) = aR(b)$ and $R(a)b = aL(b)$ for all $a, b \in A$. The set of double centralisers of A is called the *multiplier algebra* and denoted by $\mathcal{M}(A)$.

Example 5.17. Let A be a C*-algebra. Let $c \in A$. Left- and right-multiplication by c forms a double centraliser. The operator norm of both the left- and right-multiplication is $||c||$ as a result of $||cc^*|| = ||c^*||^2 = ||c^*|| \cdot ||c||$. Likewise $||c^* c|| = ||c||^2 = ||c^*|| \cdot ||c||$. The left- resp. right-multiplication is denoted by L_c resp. R_c.

Lemma 5.18. Let A be a C*-algebra. $\mathcal{M}(A)$ is a vector space.

We will now construct a multiplication on the multiplier algebra, so that $\mathcal{M}(A)$ indeed becomes an algebra. See the following proposition.

Proposition 5.19. Let A be a C*-algebra. $\mathcal{M}(A)$ is an algebra under this multiplication:

$$(L_1, R_1)(L_2, R_2) := (L_1 L_2, R_2 R_1).$$

Proof. The following equations hold for all $a, b \in A$. So $(L_1 L_2, R_2 R_1) \in \mathcal{M}(A)$.

$$\begin{aligned} L_1(L_2(ab)) &= L_1(L_2(a)b) = L_1 L_2(a)b \\ R_2(R_1(ab)) &= R_2(aR_1(b)) = aR_2 R_1(b) \\ R_2(R_1(a))b &= R_1(a)L_2(b) = aL_1(L_2(b)). \end{aligned}$$

Associativity and distributivity hold for all linear operators. □

Lemma 5.20. Let (L, R) be a double centraliser on a C*-algebra A, then $||L|| = ||R||$.

Proof. By boundedness we get $\|aL(b)\| = \|R(a)b\| \leq \|R\| \cdot \|a\| \cdot \|b\|$ for $a, b \in A$. By the C*-property $\|L(b)^* L(b)\| = \|L(b)\|^2 = \|L(b)^*\| \cdot \|L(b)\|$.
Together $\|L(b)\| = \sup_{\|a\|\leq 1} \|aL(b)\| = \sup_{\|a\|\leq 1} \|R(a)b\| \leq \|R\| \cdot \|b\|$. Hence $\|L\| \leq \|R\|$.
Likewise $\|R(a)\| = \sup_{\|b\|\leq 1} \|R(a)b\| = \sup_{\|b\|\leq 1} \|aL(b)\| = \|a\| \cdot \|L\|$. So $\|R\| \leq \|L\|$. □

Proposition 5.21. Let A be a C*-algebra. Define $L^*(a) := (L(a^*))^*$ and $R^*(a) := (R(a^*))^*$ for $a \in A$. Then $(L, R)^* := (R^*, L^*)$ defines a $*$-operation on $\mathcal{M}(A)$.

Proof. L^* and R^* have the following properties:

$$L^*(\lambda a) = (L(\bar{\lambda} a^*))^* = (\bar{\lambda} L(a^*))^* = \lambda (L(a^*))^* = \lambda L^*(a) \text{ for all } a \in A, \lambda \in \mathbb{C} \tag{15}$$
$$R^*(ab) = (R((ab)^*))^* = (R(b^* a^*))^* = (b^* R(a^*))^* = (R(a^*))^* b = R^*(a)b \text{ for all } a, b \in A \tag{16}$$
$$L^*(ab) = (L((ab)^*))^* = (L(b^* a^*))^* = (L(b^*) a^*)^* = a(L(b^*))^* = aL^*(b) \text{ for all } a, b \in A \tag{17}$$
$$L^*(a)b = (L(a^*))^* b = (b^* L(a^*))^* = (R(b^*) a^*)^* = a(R(b^*))^* = aR^*(b) \text{ for all } a, b \in A \tag{18}$$
$$(L_1 L_2)^*(a) = (L_1 L_2(a^*))^* = (L_1(L_2^*(a))^*)^* = L_1^*(L_2^*(a)) \text{ for all } a \in A \tag{19}$$
$$((L_1, R_1)(L_2, R_2))^* = (L_1 L_2, R_2 R_1)^* = (R_2^* R_1^*, L_1^* L_2^*) = (L_2, R_2)^* (L_1, R_1)^* \tag{20}$$

L^* is linear by (15). The same holds for R^*. From (16), (17) and (18), it follows that $(R^*, L^*) \in \mathcal{M}(A)$. The $*$-operation is antilinear as $(\lambda L)^* = \bar{\lambda} L^*$ and $(\lambda R)^* = \bar{\lambda} R^*$. The order reversal of the multiplication in (20) follows (19) □

Remark 5.22. Equation (19) holds for R_1 and R_2 as well. There is no order reversal of linear operators L_1 and L_2 in equation (19), because the $*$-operation switches the left- and right-hand side of the double centraliser

Corollary 5.23. Let A be a C*-algebra. Then $\mathcal{M}(A)$ is a $*$-closed subalgebra of $B(A) \oplus B(A)$ (where $B(A)$ is the Banach algebra of bounded linear operators on A) and hence a C*-algebra. Moreover, $\mathcal{M}(A)$ is unital.

Proof. $\mathcal{M}(A)$ is a closed subspace of $B(A) \oplus B(A)$. A closed subset of a complete space is complete. The operator norm on the Banach algebra of bounded operators is a C*-norm. So the norm of Lemma 5.20 is a C*-norm. The element $(\mathrm{id}_A, \mathrm{id}_A) \in \mathcal{M}(A)$ forms a unit, so $\mathcal{M}(A)$ is unital. □

Remark 5.24. To see that the operator norm of the Banach algebra of bounded operators is a C*-norm, notice that $\|R^* L\| \leq \|R^*\| \cdot \|L\| = \|R\| \cdot \|L\| = \|L\|^2$ by Lemma 5.20. Conversely, if $\|a\| \leq 1$, then $\|L(a)\|^2 = \|(L(a))^* L(a)\| = \|L^*(a^*) L(a)\| = \|a^* R^* L(a)\| \leq \|R^* L(a)\| \leq \|R^* L\|$. As $\|L\|^2 = \sup_{\|a\|\leq 1} \|L(a)\|^2$, we get $\|L\|^2 \leq \|R^* L\|$. So $\|(L, R)^*(L, R)\| = \|(L, R)\|^2$. The C*-property is satisfied.

Lemma 5.25. Let A be a C*-algebra. A can be viewed as a C*-subalgebra of $\mathcal{M}(A)$ by left- and right-multiplication (see Example 5.17).

Proof. The map $c \to (L_c, R_c)$ is a $*$-homomorphism, as $(L_c, R_c)^* = ((R_c)^*, (L_c)^*) = (L_{c^*}, R_{c^*})$. Via Example 5.17, the $*$-homomorphism is an isometry. □

Lemma 5.26. A is an ideal of $\mathcal{M}(A)$ when viewed as a subalgebra of $\mathcal{M}(A)$ via Lemma 5.25.

Proof. A is a left-ideal by the left-hand column and a right-ideal by the right-hand column.

$$\begin{aligned}
(L_c \circ L)(a) &= cL(a) = R(c)a & (L \circ L_c)(a) &= L(ca) = L(c)a \\
(R \circ R_c)(a) &= R(ac) = aR(c) & (R_c \circ R)(a) &= R(a)c = aL(c) \\
(L_c, R_c)(L, R) &= (L_c \circ L, R \circ R_c) & (L, R)(L_c, R_c) &= (L \circ L_c, R_c \circ R) \\
&= (L_{R(c)}, R_{R(c)}) & &= (L_{L(c)}, R_{L(c)})
\end{aligned}$$
$\square$

Corollary 5.27. If A is unital, then $A \cong \mathcal{M}(A)$.

Proof. If A is unital, then $(L_1, R_1) \in A$ is the unit of $\mathcal{M}(A)$. An ideal that contains an invertible element cannot be proper. $\square$

Remark 5.28. We owe a lot of credits to Wegge-Olsen [34] later on. He identifies $\mathcal{M}(A)$ by using a faithful representation of A, which he requires to be non-degenerate. We have not defined all of that; it would take unnecessarily much space. His characterisation depends on the choice of representation. Our characterisation is canonical!

5.3 Unitisation and Gelfand duality

Lemma 5.29. Let A and B be C*-algebras. Then the Cartesian product $A \times B$ equipped with the following pointwise operations and norm forms a C*-algebra:

- $(a_1, b_1) + (a_2, b_2) := (a_1 + a_2, b_1 + b_2)$
- $(a_1, b_1) \cdot (a_2, b_2) := (a_1 a_2, b_1 b_2)$
- $(a, b)^* := (a^*, b^*)$
- $\|(a, b)\| := \max\{\|a\|, \|b\|\}$

Proof. This norm is subadditive, submultiplicative. It is non-negative and zero if and only if both entries are zero, i.e. the element is zero. The norm is a C*-norm as

$$\|(a \oplus b)^* \cdot (a \oplus b)\| = \max\{\|a^* a\|, \|b^* b\|\} = \max\{\|a\|^2, \|b\|^2\} = (\max\{\|a\|, \|b\|\})^2 = \|a \oplus b\|^2. \qquad \square$$

Definition 5.30. The C*-algebra of Lemma 5.29 is called the *orthogonal direct sum* of A and B. It is denoted by $A \oplus B$ and pairs $(a, b) \in A \oplus B$ are denoted by $a \oplus b$.

Definition 5.31. Let A be a C*-algebra. If A is non unital, then the C*-algebra generated by A and the unit $1_{\mathcal{M}(A)}$ is called the *unitisation* of A. Norms are taken from $\mathcal{M}(A)$.
If A is unital, then the orthogonal direct sum $A \oplus \mathbb{C}$ is the unitisation.
The unitisation is denoted by $\tilde{A}$.

Remark 5.32. A is embedded in $\tilde{A}$ as an ideal. See Lemma 5.26 for the non-unital case. For the unital case, it is obvious that $A \oplus 0 \subset A \oplus \mathbb{C}$ form an ideal. The quotient $\tilde{A}/A \cong \mathbb{C}$ in both the unital and non-unital case. In the unital case, the unitisation creates a new unit.

Lemma 5.33. A morphism between arbitrary C*-algebras induces a unique unital $*$-homomorphism between the unitisations. Unitisation is functorial.

Proof. Let $f\colon A \to B$ be a $*$-homomorphism, then $\tilde{A} \xrightarrow{(f,\mathrm{id}_{\mathbb{C}})} \tilde{B}$ is a $*$-homomorphism. It is unital since $(f,\mathrm{id}_{\mathbb{C}})(0,1) = (0,1)$. This construction clearly preserves compositions and identities. □

Remark 5.34. From now on the unitisation of a $*$-homomorphism will be denoted by $\tilde{f}$ also.

Theorem 5.35. Let X be a locally compact Hausdorff space and X_∞ denote its one-point compactification. The map $\widetilde{C_0(X)} \ni (f,\lambda) \to f + \lambda \in C(X_\infty)$ is a $*$-isomorphism.

Proof. The unit in $\widetilde{C_0(X)}$ is the added unit, denoted by $(0,1)$. The unit in $C(X_\infty)$ is the constant function 1. The map preserves the unit. It preserves addition and the $*$-operation. For $(f,\lambda),(g,\mu) \in \widetilde{C_0(X)}$, the product is $(f,\lambda)\cdot(g,\mu) = (fg, \mu f + \lambda g, \lambda\mu)$. The map clearly preserves the product. So the map is a $*$-homomorphism. The kernel is $(0,0)$, so it is injective. The image of $(C_0(X),0)$ is $C_0(X) = \{f \in C(X_\infty)\colon f(\infty) = 0)\}$. Now $C(X_\infty)/C_0(X) \cong \mathbb{C}$. Similarly the image of $(0,\mathbb{C})$ is the set of constant maps in $C(X_\infty)$. By counting dimensions, the map is surjective. □

Theorem 5.36 (Gelfand duality)**.** There is an anti-equivalence between the categories of locally compact Hausdorff topological spaces with proper (the pre-image of a compact subset is compact) continuous maps and commutative C*-algebras. Under this anti-equivalence, the category of compact Hausdorff topological spaces is anti-equivalent to the commutative unital C*-algebras.

Proof. We will show that the functor $C_0(\_)$ from Example 5.6 is an equivalence of categories. We will first restrict $C_0(\_)$ to the compact Hausdorff spaces. This restriction will be denoted by $C(\_)$. The essential surjectivity of $C(\_)$ follows from [23, Theorem 1]. For the fully faithfulness, see the remarks below [23, Theorem 1]. For the non-unital case, use Theorem 5.35. The unitisation of a commutative non-unital C*-algebra is linked to a compact Hausdorff space. A C*-algebra is a maximal ideal in its unitisation. The ideal space of a commutative unital C*-algebra forms a bijection with the points of the compact Hausdorff space it is linked to. See [7, Theorem VII.8.7] for this fact. This way, a commutative non-unital C*-algebra is uniquely (up to homeomorphism) linked to a locally compact Hausdorff space. The fully faithfulness follows immediately from the fully faithfulness of the unital case. □

Remark 5.37. Between compact Hausdorff spaces, every continuous map is proper.

5.4 Tensor products on C*-algebras

Lemma 5.38. Let A and B be $*$-algebras. Then $A \otimes_{\mathbb{C}} B$, can be endowed with the following product and $*$-operation:

- $(a \otimes b)^* = a^* \otimes b^*$ for $a \in A$, $b \in B$
- $(a_1 \otimes b_1)\cdot(a_2 \otimes b_2) = (a_1a_2 \otimes b_1b_2)$ for $a_1,a_2 \in A$, $b_1,b_2 \in B$.

Proof. The anti-linearity follows from this equation:

$$(\lambda(a\otimes b))^* = (\lambda a \otimes b)^* = (\lambda a)^* \otimes b^* = \overline{\lambda} a^* \otimes b^* \text{ for all } a \in A, b \in B, \lambda \in \mathbb{C}\,.$$

The order reversal of the $*$-operation follows from this equation:

$$\begin{aligned}((a_1\otimes b_1)(a_2\otimes b_2))^* = (a_1a_2 \otimes b_1b_2)^* = a_2^*a_1^* \otimes b_2^*b_1^* = \\ (a_2^*\otimes b_2^*)(a_1^*\otimes b_1^*) = (a_2\otimes b_2)^*(a_1\otimes b_1)^* \text{ for all } a \in A, b \in B.\end{aligned}$$

The associativity follows from the associativity of A and B. The distributivity is immediate. □

Remark 5.39. In most cases the ring of the modules is clear and omitted from the tensor product notation. For the construction of the tensor product of modules over a ring, see Definition 3.5. A $*$-algebra is always a complex vector space. Because $\mathbb{C}$ is a field, a pure tensor of two non-zero elements can never be zero.

So now one wonders, if A and B are C*-algebras, then what is the C*-norm on $A \otimes B$? No unique answer exists. Firstly, the tensor product is most of the time not complete with respect to the norm. It needs to be completed. Secondly, there is no canonical C*-norm. A different norm makes a different completion. There may be multiple ways in which a tensor product can be completed with respect to a C*-norm. At least two explicit constructions exist; the *spatial* norm ([21, p. 190])(a.k.a. the *minimal* norm, see [21, §6.4]) and the *maximal* norm. The spatial norm is built on the Gelfand-Naimark-Segal representation([21, §3.4]) of both C*-algebras (this construction implies that every C*-algebra is a $*$-subalgebra of $\mathbb{B}$). The maximal norm is the set-theoretic pointwise maximum (indeed an upper bound can be found, see [21, p.193]) of each possible C*-norm on the tensor product. This maximum of C*-norms satisfies the C*-property. This thesis will not go into the details and properties of norm constructions on tensor products. The values of the minimal and maximal norm give pointwise lower and upper bounds for all other C*-norms.

Remark 5.40. On the pure tensors, all possible norms agree, see [21, Remark 6.4.4].

Remark 5.41. Some authors use '$\odot$' instead of '$\otimes$' to denote the algebraic tensor product in order to emphasise the omission of a norm. In this thesis, tensor products of C*-algebras are always normed, so there is no need for more notation.

Definition 5.42. Let A be a C*-algebra. A is called *nuclear* if there exists only one C*-norm on $A \otimes B$ for every C*-algebra B.

Remark 5.43. Equivalently, A is nuclear when the minimal norm and maximal norm on $A \otimes B$ agree (on every element of $A \otimes B$) for every C*-algebra B.

Example 5.44. $\mathbb{K}$ is nuclear. But $\mathbb{B}$ is not nuclear. See [21, p. 216].

Remark 5.45. A tensor product with a finite dimensional C*-algebra is automatically complete. Therefore a finite dimensional C*-algebra is nuclear. An elaborate proof can be found in [21, Theorem 6.3.9]. It builds on the fact that every finite dimensional C*-algebra is the direct sum of matrix algebras ([21, Theorem 6.3.8]).

Theorem 5.46. All commutative C*-algebras are nuclear.

Proof. See [21, Theorem 6.4.15]. □

Lemma 5.47. A tensor product of nuclear C*-algebras is nuclear.

Proof. Use the nuclearity property twice. □

Remark 5.48. A tensor product with not all of the C*-algebras nuclear need not be nuclear. Tensoring with $\mathbb{C}$ leaves a C*-algebra invariant. If true, a non-nuclear C*-algebra would become nuclear.

Definition 5.49. Let A be a C*-algebras. $\mathbb{K} \otimes A$ is called the *stabilisation* of A.
A C*-algebra A is called *stable* if $A \cong \mathbb{K} \otimes A$

Remark 5.50. Some others use A^s to denote the stabilisation of A. This is mostly done in combination with multiplier and corona algebras.

Example 5.51. $\mathbb{K}$ is stable. $\mathbb{B}$ is not stable, as $\mathbb{K}\otimes\mathbb{B} \not\cong \mathbb{B}$. In general, no unital C*-algebra is stable. In particular, no finite dimensional C*-algebra is stable.

Example 5.52. Any stabilisation of a C*-algebra is stable. This follows from the stability of $\mathbb{K}$ and from the associativity of tensor products.

Definition 5.53. Let A and B be C*-algebras. A and B are *stably isomorphic* if $\mathbb{K}\otimes A \cong \mathbb{K}\otimes B$.

Remark 5.54. Stable isomorphisms do not imply isomorphisms. For instance, $\mathbb{K} \cong \mathbb{K}\otimes\mathbb{C} \cong \mathbb{K}\otimes\mathbb{K}$. But $\mathbb{C}$ and $\mathbb{K}$ are not isomorphic.

5.5 Direct limits

In the category of C*-algebras all (co)limits exist. The statement is found in [8, Proposition 19, page 8]. In particular, direct limits exist. In this subsection we will go over a few examples and show their particularities. We will return to these examples in §7, where we will calculate their K-groups.

Example 5.55. An $i\times i$-matrix can be embedded into $(i+1)\times(i+1)$-matrix by adding a column and a row of zeros. For $\mathbb{C}$-valued matrices, this embedding is a $*$-homomorphism. The map $M_i(\mathbb{C}) \ni X \to \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in M_{i+1}(\mathbb{C})$, for $i \geq 1$, defines a cotower of C*-algebras. In $\varinjlim M_n(\mathbb{C})$, the matrices $X \in M_i(\mathbb{C})$, $\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in M_{i+1}(\mathbb{C})$, $\begin{pmatrix} X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M_{i+2}(\mathbb{C})$, et cetera represent the same element. It turns out $\varinjlim M_n(\mathbb{C}) = \mathbb{K}$.

Example 5.56. Let A be a C*-algebra. The previous example can be generalised. The $*$-homomorphism $M_i(A) \ni X \to \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \in M_{i+1}(A)$, for $i \geq 1$, embeds $M_i(A)$ into $M_{i+1}(A)$ and this defines a cotower of C*-algebras. Now $\varinjlim M_n(A) = \varinjlim(M_n(\mathbb{C})\otimes A) = (\varinjlim M_n(\mathbb{C}))\otimes A = \mathbb{K}\otimes A$.

Example 5.57. $M_2(\mathbb{C})\otimes ... \otimes M_{n!}(\mathbb{C})$ can be embedded into $M_2(\mathbb{C})\otimes ... \otimes M_{(n+1)!}(\mathbb{C})$ this way:

$$M_2(\mathbb{C})\otimes ... \otimes M_{n!}(\mathbb{C}) \ni x \to x \otimes \operatorname{diag}(x,x,...,x) \in M_2(\mathbb{C})\otimes ... \otimes M_{(n+1)!}(\mathbb{C}).$$

This embedding is not obtained by adding zeros, but by repetition. It is a $*$-homomorphism. These embeddings form a cotower. The direct limit omits the structure morphisms from the notation, but they definitely matter! We will see that in Example 12.12. This C*-algebra will be used in the proof of the UCT as given in Theorem 17.3.

Remark 5.58. It is easy to determine the norm on a direct limit. In a cotower of C*-algebras

$$X_1 \xrightarrow{f_{1,2}} X_2 \longrightarrow \cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1,i}} X_i \xrightarrow{f_{i,i+1}} X_{i+1} \longrightarrow \cdots ,$$

each of the structure morphisms $f_{i,i+1}$ is a $*$-homomorphism and therefore norm decreasing (Lemma 5.9). So for $x \in X_i$, the sequence $\{\|f_{i,i+j}(x)\|_{X_j}\}_{j=1}^{\infty}$ is decreasing, but bounded from below by 0. By the monotone convergence theorem$\{\|f_{i,i+j}(x)\|_{X_j}\}_{j=1}^{\infty}$ has an infimum. For every $x \in \varinjlim X_i$, there exists an $x_i \in X_i$ such that x_i is a pre-image of $x \in \varinjlim X_i$ under the $*$-homomorphism $X_i \xrightarrow{\mu_i} \varinjlim X_i$. This infimum defines a C*-norm, see the following equations:

$$\inf\{\|f_{j,j+k}(f_{i,j}(x)y)\|_{X_k}\}_{k=1}^{\infty} \le \inf\{\|f_{i,j+k}(x)\|_{X_k}\|f_{j,j+k}(y)\|_{X_k}\}_{k=1}^{\infty} =$$
$$(\inf\{\|f_{i,j+k}(x)\|_{X_k}\}_{k=1}^{\infty})(\inf\{\|f_{j,j+k}(y)\|_{X_k}\}_{k=1}^{\infty}) \quad i \le j,\ x \in X_i,\ y \in X_j.$$

$$\inf\{\|f_{i,i+j}(\lambda x)\|_{X_j}\}_{j=1}^{\infty} = \inf\{|\lambda|\|f_{i,i+j}(x)\|_{X_j}\}_{j=1}^{\infty} = |\lambda|\inf\{\|f_{i,i+j}(x)\|_{X_j}\}_{j=1}^{\infty} \quad x \in X_i, \lambda \in \mathbb{C}\,.$$
$$\text{If } \inf\{\|f_{i,i+j}(x)\|_{X_j}\}_{j=1}^{\infty} = 0 \text{ for } x \in X_i, \text{ then } \mu_i(x) = 0 \text{ (by contradiction).}$$
$$\inf\{\|f_{i,i+j}(x^*x)\|_{X_j}\}_{j=1}^{\infty} = \inf\{\|f_{i,i+j}(x^*)f_{i,i+j}(x)\|_{X_j}\}_{j=1}^{\infty} = \inf\{\|f_{i,i+j}(x)\|_{X_j}^2\}_{j=1}^{\infty} = (\inf\{\|f_{i,i+j}(x)\|_{X_j}\}_{j=1}^{\infty})^2.$$

Definition 5.59. An *AF-algebra* is a C*-algebra A that allows a sequence $\{A_i\}_{i=1}^{\infty}$ of finite dimensional C*-subalgebras, ordered by inclusion, such that $\bigcup_{i\ge 1} A_i$ is dense in A.

Remark 5.60. Let A be an AF-algebra. Let be $\{A_i\}_{i=1}^{\infty}$ be a sequence of finite dimensional C*-subalgebras, ordered by inclusion, such that $\bigcup_{i\ge 1} A_i$ is dense in A. The inclusions $A_i \to A_{i+1}$ define a cotower. Now $A = \varinjlim A_i$. See [21, Example 6.2.4].

Example 5.61. $\mathbb{K}$ is an AF-algebra via Example 5.55. The direct limit from Example 5.57 is an AF-algebra.

Theorem 5.62. AF-algebras are nuclear.

Proof. See [21, Theorem 6.3.11]. □

5.6 Representations and postliminal C*-algebras

Definition 5.63. Let A be a C*-algebra. A *representation* of A is a $*$-homomorphisms $A \xrightarrow{\pi} B(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space.

Definition 5.64. Let A be a C*-algebra and let $\mathcal{H}$ be a Hilbert space. Let $A \xrightarrow{\pi} B(\mathcal{H})$ be a representation. A linear subspace $\mathcal{K} \subset \mathcal{H}$ is *invariant* under π if $\pi(a)h \in \mathcal{K}$ for all $a \in A$ and all $h \in \mathcal{K}$.

Definition 5.65. Let A be a C*-algebra and let $\mathcal{H} \neq 0$ be a Hilbert space. Let $A \xrightarrow{\pi} B(\mathcal{H})$ be a representation. If the only linear subspaces of $\mathcal{H}$ invariant under π are 0 and $\mathcal{H}$, then π is called *irreducible*.

Lemma 5.66. The Hilbert space of an irreducible representation of a separable C*-algebra is separable.

Proof. See [9, 2.3.3]. □

Remark 5.67. There is up to isomorphism only one non-finite separable Hilbert space by Example 5.5.

Definition 5.68. Let A be a C*-algebra. If $\pi(a)$ is compact for every $a \in A$ for every irreducible representation $A \xrightarrow{\pi} B(\mathcal{H}_\pi)$, then A is called *liminal*.

Definition 5.69. Let A be a C*-algebra. If every non-zero quotient of A possesses a non-zero liminal closed two-sided ideal, then A is called *postliminal*.

Lemma 5.70. Every postliminal C*-algebra is liminal.

Proof. See [21, p.169]. □

Remark 5.71. The converse is false, see [21, Example 5.6.4].

Lemma 5.72. Every ideal and every quotient of a (post)liminal C*-algebra is (post)liminal.

Proof. See [21, Theorem 5.6.2]. □

Definition 5.73. Let A be a C*-algebra. A *composition series* for A is a family $(I_\beta)_{\beta\leq\alpha}$ of ideals indexed by the ordinal numbers β less or equal to the fixed ordinal number α, such that:

- $I_\alpha = A$ and $I_0 = 0$.
- If $\gamma < \beta \leq \alpha$, then $I_\beta \subset I_\gamma$
- $\overline{\bigcup_{\gamma<\beta} I_\gamma} = I_\beta$.

Theorem 5.74. Let A be postliminal C*-algebra. Then there exists a composition series $(I_\beta)_{\beta\leq\alpha}$ for A such that $I_{\beta+1}/I_\beta$ is postliminal for all $\beta < \alpha$.

Proof. See [21, p. 172]. □

Theorem 5.75. Let A be separable postliminal C*-algebra. Let $(I_\beta)_{\beta\leq\alpha}$ be a composition series for A satisfying Theorem 5.74. There exist locally compact Hausdorff spaces $\{Y_\beta\}_{\beta\leq\alpha-1}$ and separable Hilbert spaces $\{\mathcal{H}_\beta\}_{\beta\leq\alpha-1}$ such that $I_{\beta+1}/I_\beta \cong C_0(Y_\beta)\otimes K(\mathcal{H}_\beta)$ for $\beta \leq \alpha - 1$.

Proof. This result is also stated in [30, p.449-450]. The proof follows from the relation between the topological space of primitive ideals of A ([21, p.156]) and the quasi-spectrum ([9, §7.2]). □

Theorem 5.76. Postliminal C*-algebras are nuclear.

Proof. See [21, p. 226]. □

5.7 Separability and σ-unitality

Definition 5.77. A C*-algebra is *separable* if it contains a countable dense subset.

Proposition 5.78. A C*-algebra is separable if and only if it is countably generated as a $*$-algebra.

Proof. Let A be a separable C*-algebra. There exists a countable and dense subset $X \subset A$. The C*-algebra generated by X must be A; it is the smallest C*-algebra that includes X. Conversely, let X be a countable set. Then $X \cup X^*$ is countable. Now the set of finite sums of $\mathbb{Q}[i]$-scalared finite products of elements of $X \cup X^*$ is countable and dense in the C*-algebra generated by X. □

Example 5.79. All finite dimensional C*-algebras are separable.

Example 5.80. $\mathbb{K}$ is separable as it is generated by rank one operators (see Definition 9.11).

Example 5.81. $\mathbb{B}$ is not separable.

Lemma 5.82. The tensor product of separable C*-algebras is separable.

Proof. Follows from Proposition 5.78; the union of the set of pairs of the generators from the respective C*-algebras plus the set of pairs of generators of one with generators of the other under the $*$-operation is countable. □

Definition 5.83. Let A be a C*-algebra. An *approximate unit* of a C*-algebra is an increasing net $\{u_\lambda\}_{\lambda\in\Lambda}$ of positive elements in the closed unit ball such $a = \lim_\lambda u_\lambda a$ for all $a \in A$.

Definition 5.84. A C*-algebra is *σ-unital* if it allows a countable approximate unit.

Example 5.85. For unital C*-algebras the constant net $\{1\}_{\lambda\in\Lambda}$ is an approximate unit. Hence all unital C*-algebras are σ-unital.

Lemma 5.86. All separable C*-algebras are σ-unital.

Proof. See [21, Remark 3.1.1]. □

Lemma 5.87. The tensor product of σ-unital C*-algebras is σ-unital.

Proof. The tensor products of approximate unit elements of the respective C*-algebras forms an approximate unit. □

Lemma 5.88. Every C*-algebra admits an approximate unit.

Proof. See [21, Theorem 3.1.1]. □

Remark 5.89. The definition and construction of approximate units build on the set of positive elements with norm up to 1.

Theorem 5.90. Let A be a separable and unital C*-algebra and B be a unital C*-algebra. Let J be a closed ideal of B. Let at least A, B/J or B be nuclear. For a unital norm-decreasing linear map $A \xrightarrow{\phi} B/J$, there exists a unital norm decreasing linear map $A \xrightarrow{\psi} B$ such that $\pi \circ \psi = \phi$, where $B \xrightarrow{\pi} B/J$ is the quotient.

Proof. The proof of a more general statement can be found in [4, Theorem 3.10]. □

Remark 5.91. Theorem 5.90 applies to $*$-homomorphisms in particular. But ψ need not be a $*$-homomorphism. In Remark 13.50 the non-multiplicativity of ψ is revisited.

Remark 5.92. A positive map is one that preserves positive elements. Let $k \in \mathbb{N}$, a k-positive map is one for which the induced map (coordinate wise) on the $k \times k$-matrices is positive. Positive maps are 1-positive. A completely positive map is a map that is k-positive for all $k \in \mathbb{N}$. Any $*$-homomorphism between C*-algebras is completely positive.

Theorem 5.93 (Generalized Theorem of Stinespring)**.** Let A be a separable unital C*-algebra and let B be a σ-unital C*-algebra. Let $A \xrightarrow{\phi} \mathcal{M}(\mathbb{K}\otimes B)$ be a unital, completely positive, linear map. Then there exists a $*$-homomorphism $A \xrightarrow{\rho} M_2(\mathcal{M}(\mathbb{K}\otimes B))$ such that $\begin{pmatrix}\phi(a) & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\rho(a)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ for all $a \in A$.

Proof. The proof of a more general statement can be found in [15, Theorem 3.2]. $\square$

Remark 5.94. Other sources that reference results similar to the results of Theorem 5.90 and Theorem 5.93 may use different definitions. For example, in [2], [16] and [4] the term *contractive map* is used, which means a norm-decreasing map. A $*$-homomorphism between C*-algebras is automatically norm-decreasing, see Lemma 5.9.

Remark 5.95. In [4, Theorem 3.10] the term *nuclear map* is used. The definition of nuclear maps can be found in [4, p. 592]. For C*-algebras, being nuclear is equivalent to the identity being nuclear under the given definition. For C*-algebras, if either the domain or codomain is nuclear, then a $*$-homomorphism is nuclear (just pre- or post-compose with the identity on the nuclear C*-algebras).

5.8 A special C*-algebra: $\mathbb{C}_1$

Definition 5.96. Define $\mathbb{C}_1$ to be the C*-algebra isomorphic to $\mathbb{C}^2$ as a vector space, with generators 1 and ϵ_1 for which $1^2 = 1$, $\epsilon_1^2 = 1$, $1 \cdot \epsilon_1 = \epsilon_1$, $\epsilon_1 \cdot 1 = \epsilon_1$, $1^* = 1$ and $\epsilon_1^* = \epsilon_1$.

Remark 5.97. $\mathbb{C}_1$ is isomorphic to $\mathbb{C}^2$ as a $*$-algebra too. The non-trivial ideals are $\{\lambda\frac{1+\epsilon_1}{2} : \lambda \in \mathbb{C}\}$ and $\{\lambda\frac{1-\epsilon_1}{2} : \lambda \in \mathbb{C}\}$. And so the multiplicative structure is the same. Denote by $\overline{\mathbb{C}^2}$ the endowment of $\mathbb{C}^2$ with a pointwise addition and $*$-operation, the multiplication $(\lambda_1, \mu_1) \cdot (\lambda_2, \mu_2) = (\lambda_1\lambda_2 + \lambda_1\mu_2 + \mu_1\lambda_2, \mu_1\mu_2)$ and the norm $\|(\lambda, \mu)\| = |\lambda + \mu|$. Now $\overline{\mathbb{C}^2}$ is a C*-algebra, see [21, p.40]. We establish a $*$-isomorphism between $\mathbb{C}_1$ and $\overline{\mathbb{C}^2}$ by linearly sending $\mathbb{C}_1 \ni 1 \to (0,1) \in \overline{\mathbb{C}^2}$ and $\mathbb{C}_1 \ni \epsilon_1 \to (-2,1) \in \overline{\mathbb{C}^2}$. It preserves the $*$-operation and multiplication, since $(-2,1)^2 = (0,1)$. It is of rank 2 and therefore surjective. The injectivity follows from the finite dimension. We endow $\mathbb{C}_1$ with the C*-norm of $\overline{\mathbb{C}^2}$. The ideal in $\overline{\mathbb{C}^2}$ corresponding to $\{\lambda\frac{1+\epsilon_1}{2} : \lambda \in \mathbb{C}\}$ is $\{(-\lambda, \lambda) : \lambda \in \mathbb{C}\}$ and the ideal in $\overline{\mathbb{C}^2}$ corresponding to $\{\lambda\frac{1-\epsilon_1}{2} : \lambda \in \mathbb{C}\}$ is $\{(\lambda, 0) : \lambda \in \mathbb{C}\}$.

Remark 5.98. $\mathbb{C}_1$ is finite dimensional and unital. Therefore $\mathbb{C}_1$ is separable, σ-unital and nuclear.

Remark 5.99. $\mathbb{C}_1$ is a Clifford algebra. More on Clifford algebras can be found in [16, §2.13] and [32, §4.1]. $\mathbb{C}_1$ is abelian. In general Clifford algebras are not abelian.

6 Extensions of C*-algebras

In this section extensions of C*-algebras are introduced. The differences and similarities with extensions of R-modules will be pointed out in the first section. In the second subsection, we will show that extensions of C*-algebras can be classified by $*$-homomorphisms, the so-called Busby invariants. In the third subsection, the unitary equivalence on extensions of C*-algebras will be defined. In the fourth subsection we will introduce several constructions related to extensions. In the last subsection we will show that extensions of C*-algebras preserve certain properties.

6.1 Ideals and quotients of C*-algebras

Categorically speaking a (co)kernel of a morphism is the (co)equalizer of that morphism with the zero morphism. For a commutative ring R, the category $R-\mathsf{Mod}$ contains kernels and cokernels. Moreover, an R-module morphism $M \xrightarrow{f} N$ induced the exact sequence (5). This raises the question whether kernels and cokernels exist for C*-algebras, i.e. are they objects in the category of C*-algebras? If so, do quotients exist? Is the cokernel formed by a quotient? In (5) $N/f(M)$ represents $\operatorname{coker} f$. For R-modules, cokernels can be written as the quotient by the image of the morphism. The quotient of a C*-algebra by an arbitrary C*-subalgebra does not exist. In this section a few result are worked through to show that (co)kernels exist in the category of C*-algebras. (If you look in the literature, for C*-algebras, all limits exist. See [8, Proposition 19, page 8]).

Lemma 6.1. Two sided closed ideals of C*-algebras are closed under the $*$-operation.

Proof. See [21, Theorem 3.12 & 3.1.3]. □

Corollary 6.2. Two sided closed ideals of C*-algebras are C*-algebras in their own right.

Proof. Ideals are algebras, since they are closed under addition and multiplication. By Lemma 6.1 they are closed under the $*$-operation. A closed subset of a complete set is complete. □

Corollary 6.3. Kernels exist in the category of C*-algebras.

Proof. A $*$-homomorphism is always continuous by Lemma 5.9. A norm defines a metric. Every metric space is Hausdorff. So singletons are closed. So the pre-image of $\{0\}$ is closed. □

Lemma 6.4. There exists a C*-norm for the quotient of a C*-algebra by a closed ideal.

Proof. Let A be a C*-algebra and let I be a closed ideal. I is a C*-algebra and possesses an approximate unit. Let $\{u_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit on I. Let $a \in A$, now for $\bar{a} \in A/I$ define $||\bar{a}|| := \lim_\lambda ||a - u_\lambda a||$. This defines a norm as $a \in I$ if and only if $\bar{a} = 0$. If $a \in I$, then $a = \lim_\lambda u_\lambda a$, so $\lim_\lambda ||a + u_\lambda a|| = ||a - a|| = ||0||$. Conversely I is closed, so $\lim_\lambda u_\lambda a \in I$. If $a \notin I$, then $a - u_\lambda a \notin I$ for any $\lambda \in \Lambda$. Hence $||\bar{a}|| \neq 0$. This norm satisfies the C*-property by [7, Lemma VIII.4.6]. □

Corollary 6.5. The image of a $*$-homomorphism between C*-algebras forms a C*-algebra.

Proof. The image of a $*$-isomorphism is isomorphic to the domain modulo the kernel. The kernel is a closed ideal. Hence the image is a C*-algebra under the norm defined in Lemma 6.4. □

Theorem 6.6. Injective $*$-homomorphisms between C*-algebras are isometries (preserve norms).

Proof. See [21, Theorem 3.1.5]. It suffices to show that an injective $*$-homomorphism preserves norms for positive elements. For this, use Gelfand representation. See [21, Theorem 2.1.10]. □

Remark 6.7. In a short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, we the image $\alpha(A)$ is an ideal of B as $\alpha(A) = \ker\beta$. Via Theorem 6.6, A and $\alpha(A)$ are isometrically $*$-isomorphic. From now on, when speaking of the ideal $\alpha(A) \subset B$ we will just write A.

Corollary 6.8. Every short exact sequence of C*-algebras can be viewed as a triple of a C*-algebra, a closed ideal and its quotient.

Proof. The left-hand object is the kernel of the right-hand morphism. The left-hand term forms an ideal of the middle element. By surjectivity of the right-hand morphisms, the right-hand object is the quotient of the middle object modulo the left-hand object.
Conversely every closed ideal of a C*-algebra can be extended to a short exact sequence by taking its embedding and quotient. □

6.2 Extensions of C*-algebras

Lemma 6.9. Let A be a C*-algebra and let $I \subset A$ be a closed ideal. The following are equivalent:

1. Let $a \in A$, $aI = 0$ if and only if $a = 0$.
2. Let $a \in A$, $Ia = 0$ if and only if $a = 0$.
3. For every non-zero closed ideal $J \subset A$ holds $I \cap J \neq 0$.

Proof. 1 implies 2: I is closed under the $*$-operation by Lemma 6.1. So $aI = 0$ if and only if $Ia^* = 0$ for $a \in A$.
2 implies 3: Suppose there is a non-zero closed ideal $J \subset A$ such that $I \cap J = 0$, then since $IJ \subset I \cap J = 0$, we have $Ij = 0$ for all $j \in J$. This contradicts statement 2.
3 implies 1: The elements $a \in A$ such that $aI = 0$ form a closed ideal of A. This closed ideal must have zero intersection with I, since I is closed under the $*$-operation. Since there exists no closed non-zero ideal in A with zero intersection with I, we get $aI = 0$ only if $a = 0$. □

Definition 6.10. Let A be a C*-algebra and let $I \subset A$ be a closed ideal. I is *essential in* A if it satisfies the statements of Lemma 6.9.

Example 6.11. Let A and B be C*-algebras. Then A and B are non-essential ideals in $A \oplus B$. This is easily seen from the fact that $(a,0)(0,b) = (0,0)$ for every $a \in A$ and $b \in B$.

Lemma 6.12. Let A be a C*-algebra and let $I \subset A$ be a closed ideal. There exists a unique $*$-homomorphism $A \xrightarrow{\phi} \mathcal{M}(I)$ such that $\phi(a) = (L_a, R_a)$ for $a \in I$. Moreover, ϕ is injective if and only if I is essential.

Proof. Take $\phi(b)(c) = (bc, cb)$ for $b \in A$ and $c \in I$. As I is an ideal, left- and right- multiplication by an element of A are linear maps on I. They are bounded by submultiplicativity of the norm. This forms a double centraliser on I. Let $A \xrightarrow{\psi} \mathcal{M}(I)$ be another $*$-homomorphism such that $\psi(a) = (L_a, R_a)$ for $a \in I$. Then $\psi(b) \circ (L_c, R_c) = \psi(b)\psi(c) = \psi(bc) = (L_{bc}, R_{bc})$ for $b \in A$ and $c \in I$, as I is an ideal. The same holds for ϕ, as $\phi(b) \circ (L_c, R_c) = (L_{bc}, R_{bc})$. Hence $\phi = \psi$.

If ϕ is not injective, then there is a non-zero element of A such that left- and right-multiplication by this element are the zero maps on I. This means there is a $b \in A$ such that $ba = 0$ and $ab = 0$ for all $a \in I$. Certainly statements 1 and 2 of Lemma 6.9 are not satisfied. So I is not essential in A.
Conversely, if I is not essential in A, then there is a non-zero $b \in A$ such that $bI = 0$, i.e. left-multiplication by b is the zero map on I. Now certainly $b^*bI = 0$. But $0 = (b^*bI)^* = Ib^*b$ (in terms of sets) by Lemma 6.1. So right-multiplication by b^*b is the zero map on I as well. So $\phi(bb^*) = 0$. As $b \neq 0$, we get $b^*b \neq 0$ by Definition 5.3. So ϕ is not injective. □

Corollary 6.13. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of C*-algebras. There exists a unique $*$-homomorphism $B \xrightarrow{\sigma} \mathcal{M}(A)$ such that $(\sigma \circ \alpha)(a) = (L_a, R_a)$.

Proof. By Remark 6.7, we get that $\alpha(A)$ and A are isometrically $*$-isomorphic. Hence $\mathcal{M}(\alpha(A))$ and $\mathcal{M}(A)$ are isomorphic. Since $\alpha(A)$ is an ideal of B, the corollary follows immediately from Lemma 6.12. □

Corollary 6.14. Every short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of C*-algebras induces a unique $*$-homomorphism $C \xrightarrow{\tau} \mathcal{M}(A)/A$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\beta"] \arrow[d, "\sigma"'] & C \arrow[r] \arrow[d, "\tau"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "{(L_\_,R_\_)}"] & \mathcal{M}(A) \arrow[r, "\pi"] & \mathcal{M}(A)/A \arrow[r] & 0. \end{tikzcd}$$

Proof. β is surjective. So for every $c \in C$ there exists a $b \in B$ such that $c = \beta(b)$. Now define $\tau(c) := \pi(\sigma(b))$. This is well-defined. If for both b_1, b_2 holds $\beta(b_1) = \beta(b_2) = c$, then $\beta(b_1 - b_2) = 0$, to $b_1 - b_2 \in \operatorname{im}\alpha$. Hence there exists an $a \in A$ such that $\alpha(a) = b_1 - b_2$. Now $\sigma(b_1 - b_2) = \sigma(\alpha(a)) = (L_a, R_a)$ and $\pi(\sigma(b_1 - b_2)) = \pi((L_a, R_a)) = 0$.
The uniqueness of τ is immediate: if there exist two such $*$-homomorphism, then they have to differ at a certain element $c \in C$. By the surjectivity of β we can pull back to an element of B and then we see that the images of τ are predetermined. Hence they are equal. □

Definition 6.15. A is a closed ideal of $\mathcal{M}(A)$ via Lemma 5.26. Now $\mathcal{M}(A)/A$ is a C*-algebra via Lemma 6.4. $\mathcal{M}(A)/A$ is called the *corona-algebra* of A and denoted by $\mathcal{Q}(A)$.

Example 6.16. For unital C*-algebras, the corona-algebra is trivial. See Corollary 5.27.

Definition 6.17. For a short exact sequence of C*-algebras $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, the $*$-homomorphism $C \xrightarrow{\tau} \mathcal{Q}(A)$ found in Corollary 6.14 is called the *Busby invariant.*

Lemma 6.18. Let A, B and C be C*-algebras and let $A \xrightarrow{\alpha} C$ and $B \xrightarrow{\beta} C$ be $*$-homomorphisms. Then $\{(a,b) \in A \oplus B : \alpha(a) = \beta(b)\}$ is the pullback of α and β.

Proof. Clearly $\{(a,b) \in A \oplus B : \alpha(a) = \beta(b)\}$ is a $*$-algebra. Take the norm $\|(a,b)\| = \|\alpha(a)\|_C = \|\beta(b)\|_C$. This norm satisfies Definition 5.3. By the continuity of α and β, the $*$-algebra is complete with respect to the norm. For the limit property, see the proof of Proposition 3.3. □

Lemma 6.19. Let A and C be C*-algebras. Let $\tau \in \mathrm{Hom}_{\mathsf{C^*-alg}}(C, \mathcal{Q}(A))$. There exists an extension $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ of C*-algebras for which τ is the Busby invariant.

Proof. Let PB be the pullback of τ and $\mathcal{M}(A) \xrightarrow{\pi} \mathcal{M}(A)/A$. We will fill this diagram:

$$\begin{tikzcd} & & PB \arrow[r] \arrow[d] & C \arrow[d, "\tau"'] & \\ 0 \arrow[r] & A \arrow[r, "{(L_\_,R_\_)}"] & \mathcal{M}(A) \arrow[r, "\pi"] & \mathcal{M}(A)/A \arrow[r] & 0. \end{tikzcd}$$

Let $c \in C$. As π is surjective, there exists a $x \in \mathcal{M}(A)$ such that $\pi(x) = \tau(c)$. Hence $(c,x) \in PB$ and so $PB \to C$ is surjective. Now $\ker(PB \to C) = \{(0,x) \in C \oplus \mathcal{M}(A) : \tau(0) = \pi(x)\}$. As $\tau(0) = 0$, we get $\pi(x) = 0$. From $\ker\pi = A$, it must follow $x \in A$. So $\ker(PB \to C) = A$. So there is an embedding $A \xrightarrow{(\mathrm{id}_A, 0)} PB$ such that following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "{(\mathrm{id}_A,0)}"] \arrow[d, "\mathrm{id}_A"'] & PB \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "\tau"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "{(L_\_,R_\_)}"] & \mathcal{M}(A) \arrow[r, "\pi"] & \mathcal{M}(A)/A \arrow[r] & 0. \end{tikzcd}$$

By Corollary 6.14, τ is the only $*$-homomorphism to make the diagram commute. Hence τ is the Busby invariant of $0 \to A \xrightarrow{(\mathrm{id}_A, 0)} PB \to C \to 0$. ☐

Definition 6.20. Let A and C be C*-algebras. Extensions of C*-algebras $0 \to A \xrightarrow{\alpha_1} B_1 \xrightarrow{\beta_1} C \to 0$ and $0 \to A \xrightarrow{\alpha_2} B_2 \xrightarrow{\beta_2} C \to 0$ of C by A are *isomorphic* if there exists a $*$-homomorphism $B_1 \xrightarrow{\phi} B_2$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha_1"] \arrow[d, "\mathrm{id}_A"'] & B_1 \arrow[r, "\beta"] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d, "\mathrm{id}_C"'] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha_2"] & B_2 \arrow[r, "\beta_2"] & C \arrow[r] & 0.
\end{tikzcd}$$

Remark 6.21. An extension of C*-algebras is an extension of $\mathbb{C}$-vector spaces, i.e. $\mathbb{C}$-modules. Hence Lemma 4.10 applies. The middle $*$-homomorphism is an $*$-isomorphism and therefore being extension isomorphic is an equivalence relation.

Remark 6.22. In Definition 4.54, we used the word *equivalent* for R-modules. In Definition 6.20 we used the word *isomorphic* for C*-algebras. Although confusing right now, it will pay off later. A unitary equivalence (Definition 6.32) is weaker than an extension isomorphism. Unitary equivalence allows us to add extensions, similar to what is done in Definition 4.59.

Corollary 6.23. The Busby invariant defines a bijection between the set of isomorphism classes of extensions of C*-algebras of C by A and $\mathrm{Hom}_{\mathsf{C^*-alg}}(C, \mathcal{Q}(A))$.

Proof. For surjectivity, see Lemma 6.19.
For injectivity, if two extensions of C by A have the same Busby invariant τ, then they are isomorphic to the extension created in Lemma 6.19 by the pullback PB of τ and π. The isomorphism is given by the map $B \xrightarrow{(\beta,\sigma)} PB$, where $B \xrightarrow{\sigma} \mathcal{M}(A)$ is the $*$-homomorphism given in Lemma 6.12. The two extensions are isomorphic to the same extension. Hence they must be isomorphic. ☐

Example 6.24. Let A and C be C*-algebras. The orthogonal extension $0 \to A \xrightarrow{i_A} A \oplus C \xrightarrow{\pi_C} C \to 0$ corresponds to $0 \in \mathrm{Hom}_{\mathsf{C^*-alg}}(C, \mathcal{Q}(A))$. Notice that $A \oplus \mathbb{C}$ is the pullback of the zero map and the quotient $\mathcal{M}(A) \xrightarrow{\pi} \mathcal{Q}(A)$.

Lemma 6.25. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an extension of C*-algebras. $\alpha(A)$ is essential in B if and only if the Busby invariant τ is injective.

Proof. Note that by Lemma 6.12, we get $\alpha(A)$ is essential in B if and only if the corresponding $*$-homomorphism $B \xrightarrow{\sigma} \mathcal{M}(A)$ is injective. By construction $\pi \circ \sigma = \tau \circ \beta$. Now if σ is injective, then $\ker(\pi \circ \sigma) = A$. Since $\ker \beta = A$, we get $\ker \tau = 0$. Conversely, if $\ker \tau = 0$, then $\pi(\sigma(b)) \neq 0$ for $b \in B \setminus A$. Hence $\sigma(b) \notin A$. In particular, $\sigma(b) \neq 0$. So σ is injective then. ☐

Definition 6.26. A short exact sequence of C*-algebras $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is *split* if there exists a $*$-homomorphism $C \xrightarrow{\gamma} B$ such that $\beta \circ \gamma = \mathrm{id}_C$. Then γ is called a *section* of β.

Remark 6.27. Besides being a linear map a section must preserve multiplication and $*$-operation. This is a stronger condition than being a section of a short exact sequence of complex vector spaces. In the following remark, we will see the equivalences of Lemma 4.8 do not hold for C*-algebras.

Remark 6.28. Every orthogonal sum is split. But not all split sequences are orthogonal. One example is the unitisation if for a non-unital C*-algebra A. There is a natural exact sequence $0 \to A \xrightarrow{(\mathrm{id}_A, 0)} \tilde{A} \xrightarrow{\pi_{\mathbb{C}}} \mathbb{C} \to 0$. Now the $*$-homomorphism $\mathbb{C} \xrightarrow{\gamma} \tilde{A}$, with $\gamma(\lambda) = (0, \lambda) \in \tilde{A}$ is a section. However, if $\tilde{A}$ were to be the orthogonal sum of A and $\mathbb{C}$, then $\mathbb{C} \cong \mathrm{Span}\{(\mathrm{id}_A, \mathrm{id}_A)\} \subset \mathcal{M}(A)$ has to be an ideal by Example 6.11. This is not the case.

Remark 6.29. There are many other words used for split extensions of C*-algebras. In [34] and [2] the word *trivial* is used. But not all split extensions are *the same*, so there is no such thing as *the* trivial extension. In [16] the word *decomposable* is used for split extensions. But what do those extensions decompose into? The word decomposable is suitable for the understanding of the later sections where a correspondence is to be found between **decomposable** extensions and **degenerate** pairs and Hilbert modules. Both words start with the same letter.

Corollary 6.30. Let A be a unital C*-algebra and let C be a C*-algebra. Every extension of C by A is orthogonal.

Proof. The isomorphism classes of extensions of C by A are in bijection with $\mathrm{Hom}_{\mathsf{C^*\text{-}alg}}(C, \mathcal{Q}(A))$ by Corollary 6.23. As A is unital, $\mathcal{Q}(A) = 0$ (Example 6.16) and therefore $\mathrm{Hom}_{\mathsf{C^*\text{-}alg}}(C, \mathcal{Q}(A)) = \{0\}$. By Example 6.24 the $*$-homomorphism 0 corresponds to the orthogonal extension. □

Lemma 6.31. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an extension of C*-algebras. The extension is split if and only if there exists a $*$-homomorphism $C \xrightarrow{\eta} \mathcal{M}(A)$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[r, "\beta"] \arrow[d, "\sigma"'] & C \arrow[r] \arrow[d, "\tau"'] \arrow[dl, "\eta"] & 0 \\
0 \arrow[r] & A \arrow[r, "{(L_-, R_-)}"] & \mathcal{M}(A) \arrow[r, "\pi"] & \mathcal{M}(A)/A \arrow[r] & 0.
\end{tikzcd}$$

Proof. If the extension is split, then there exists a section $C \xrightarrow{\gamma} B$ such that $\beta \circ \gamma = \mathrm{id}_C$. In particular, $\sigma \circ \gamma$ makes the diagram commute; $\pi \circ \sigma \circ \gamma = \tau \circ \beta \circ \gamma = \tau$.
Conversely, assume there is a $*$-homomorphism $C \xrightarrow{\eta} \mathcal{M}(A)$ such that $\eta \circ \beta = \sigma$. Let PB be the pullback of π and τ. Then $C \xrightarrow{(\mathrm{id}_C, \eta)} PB$ offers a section of $0 \to A \xrightarrow{(\mathrm{id}_A, 0)} PB \to C \to 0$. But this extension is isomorphic to $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$. So the section extends to a section on $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$. An explicit isomorphism is given by $B \xrightarrow{(\sigma, \beta)} PB$. □

6.3 Unitary equivalence of extensions of C*-algebras

In this section Busby invariants will be used to represent isomorphism classes of extensions.

Definition 6.32. Let A and C be C*-algebras. Let $\tau_1, \tau_2 \in \mathrm{Hom}_{\mathsf{C^*\text{-}alg}}(C, \mathcal{Q}(A))$ be Busby-invariants. τ_1 and τ_2 are *unitarily equivalent* if there exists a unitary $u \in \mathcal{M}(A)$ such that $\tau_2(c) = \pi(u)\tau_1(c)\pi(u^*)$ for all $c \in C$, where $\mathcal{M}(A) \xrightarrow{\pi} \mathcal{Q}(A)$ is the quotient.

Lemma 6.33. Unitary equivalence of extensions of C*-algebras is an equivalence relation.

Proof. Reflexivity: $\tau(c) = \pi(1_{\mathcal{M}(A)})\tau(c)\pi(1_{\mathcal{M}(A)})$ for $\tau \in \mathrm{Hom}_{\mathsf{C^*\text{-}alg}}(C, \mathcal{Q}(A))$ and $c \in C$.
Symmetry: If there exists a unitary $u \in \mathcal{M}(A)$ such that $\tau_2(c) = \pi(u)\tau_1(c)\pi(u^*)$ for all $c \in C$, then

$$\pi(u^*)\tau_2(c)\pi(u) = \pi(u^*)\pi(u)\tau_1(c)\pi(u^*)\pi(u) = \pi(1_{\mathcal{M}(A)})\tau_1(c)\pi(1_{\mathcal{M}(A)}) = \tau_1(c) \text{ for all } c \in C.$$

Transitivity: If there exists a unitary $u \in \mathcal{M}(A)$ such that $\tau_2(c) = \pi(u)\tau_1(c)\pi(u^*)$ for all $c \in C$ and a unitary $v \in \mathcal{M}(A)$ such that $\tau_3(c) = \pi(v)\tau_2(c)\pi(v^*)$ for all $c \in C$, then $\tau_3(c) = \pi(vu)\tau_1(c)\pi(u^*v^*)$. □

Remark 6.34. Two extensions can be unitarily equivalent while not being isomorphic. As a result of Corollary 6.23 isomorphic extensions are also unitarily equivalent. Unitary equivalence is a weaker equivalence relation than isomorphism.

Lemma 6.35. The orthogonal extension is only unitarily equivalent to itself.

Proof. The orthogonal extension corresponds to the zero Busby invariant (Corollary 6.30).
For every unitary $u \in \mathcal{Q}(A)$ holds that $u0u^* = 0$. So it can never be unitarily equivalent to any non-zero Busby invariant. □

Lemma 6.36. Let C be a unital C*-algebra. A unital extension, i.e. an extension whose Busby invariant is unital, can only be unitarily equivalent to a unital extension.

Proof. Let $\tau \in \mathrm{Hom}_{\mathsf{C^*\text{-}alg}}(C, \mathcal{Q}(A))$ be a unital $*$-homomorphism. Let $u \in \mathcal{M}(A)$ be a unitary. Now $\pi(u)\tau(1_C)\pi(u^*) = \pi(u)1_{\mathcal{Q}(A)}\pi(u^*) = 1_{\mathcal{Q}(A)}$, where $\mathcal{M}(A) \xrightarrow{\pi} \mathcal{Q}(A)$ is the quotient. □

Lemma 6.37. Let A and C be C*-algebras. Let $\tau_1, \tau_2 \in \mathrm{Hom}_{\mathsf{C^*\text{-}alg}}(C, \mathcal{Q}(A))$ be Busby invariants representing unitarily equivalent extensions. If τ_1 is split, then so is τ_2. In other words, split extensions are only unitarily equivalent to split extensions.

Proof. Lemma 6.31 says that there exists a $*$-homomorphism $C \xrightarrow{\eta} \mathcal{M}(A)$ such that $\pi \circ \eta = \tau_1$. The unitary equivalence says there exists a unitary $u \in \mathcal{M}(A)$ such that $\tau_2(c) = \pi(u)\tau_1(c)\pi(u^*)$ for all $c \in C$. As $\tau_1(c) = \pi(\eta(c))$, we get $\tau_2(c) = \pi(u)\pi(\eta(c))\pi(u^*) = \pi(u\eta(c)u^*)$ for all $c \in C$. The mapping $C \ni c \to u\eta(c)u^* \in \mathcal{M}(A)$ is additive. It is multiplicative: let $c_1, c_2 \in C$, then $u\eta(c_1)u^*u\eta(c_2)u^* = u\eta(c_1)\eta(c_2)u^* = u\eta(c_1c_2)u^*$. And it preserves the $*$-operation. Let $c \in C$, then $(u\eta(c)u^*)^* = u^{**}\eta(c)^*u^* = u\eta(c^*)u^*$. So it is a $*$-homomorphism. Lemma 6.31 says τ_2 is split. □

In §13 extensions of C*-algebras will be revisited. An addition of extensions of C*-algebras will be defined in Definition 13.1. This addition is only defined when the left-hand term of the extension is stable. Via the Busby invariant, $\mathrm{Hom}_{\mathsf{C^*\text{-}alg}}(C, \mathcal{Q}(A))$ served as the set of isomorphism classes (see Corollary 6.23) in this section. In §13, the following notation, which is exclusive to extensions with a stable left-hand term, will be used.

Definition 6.38. Let A and B be C*-algebras.
The set of isomorphism classes of extensions of A by $\mathbb{K} \otimes B$ is denoted by $\mathfrak{Ext}(A, B)$.
The set of unitary equivalence classes of extensions of A by $\mathbb{K} \otimes B$ is denoted by $\overline{\mathfrak{Ext}}(A, B)$.
The set of isomorphism classes of split extensions of A by $\mathbb{K} \otimes B$ is denoted by $\mathfrak{Dxt}(A, B)$.
The set of unitary equivalence classes of split extensions of A by $\mathbb{K} \otimes B$ is denoted by $\overline{\mathfrak{Dxt}}(A, B)$.

6.4 Useful exact sequences and constructions of C*-algebras

We will now give a few examples of extensions of C*-algebras and construct some C*-algebras that appear in certain extensions.

Example 6.39. Let A be a C*-algebra. Recall Remark 6.28. $0 \to A \to \tilde{A} \to \mathbb{C} \to 0$ is always split exact. It is orthogonal precisely if A is unital.

Example 6.40. Let A be a C*-algebra, then by construction $0 \to A \xrightarrow{(L_-, R_-)} \mathcal{M}(A) \xrightarrow{\pi} \mathcal{Q}(A) \to 0$ is exact. This will be the most recurring exact sequence throughout this thesis, in particular when A is stable. It originates from Lemma 5.26 and recall $\mathcal{Q}(A) := \mathcal{M}(A)/A$.

Example 6.41. In Example 9.18, we will see that $\mathcal{M}(\mathbb{K}) = \mathbb{B}$. Therefore $0 \to \mathbb{K} \to \mathbb{B} \to \mathbb{B}/\mathbb{K} \to 0$.

Definition 6.42. Let A be a C*-algebra. The *suspension* of A is the C*-algebra $C_0(\mathbb{R}) \otimes A$. The suspension of A is denoted by SA.

Proposition 6.43. Suspension commutes with stabilisation.

Proof. Follows from the associativity of tensor products. □

Proposition 6.44. Let A be a C*-algebra. There are the following isomorphisms $A \otimes C_0(\mathbb{R}) \cong C_0(\mathbb{R}, A) \cong C_0((0,1), A) \cong \{f \in C(\mathbb{T}, A) | f(1) = 0\}$, where $\mathbb{T}$ is the unit circle $\{x \in \mathbb{R}^2 : \|x\| = 1\}$.

Proof. The first isomorphism is immediate. The second follows from the fact that $\mathbb{R}$ and $(0,1)$ are homeomorphic. The last isomorphism follows the fact that f induces a unique continuous function from the one-point compactification to A, in which the element $\infty \in \widetilde{(0,1)}$ is sent to $0 \in A$. Now $\widetilde{(0,1)} = \mathbb{T}$ and $\mathbb{T}$ is isomorphic to $[0,1]/\sim$ where $0 \sim 1$. □

Lemma 6.45. Suspension is exact as a functor, i.e. for an exact sequence of C*-algebras $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$, the sequence $0 \to SA \xrightarrow{\alpha \otimes \mathrm{id}_{C_0(\mathbb{R})}} SB \xrightarrow{\beta \otimes \mathrm{id}_{C_0(\mathbb{R})}} SC \to 0$ is exact as well.

Proof. Write SA, SB and SC in the form $\{f \in C(\mathbb{T}, A) | f(1) = 0\}$ (see Proposition 6.44). The $*$-homomorphisms $SA \xrightarrow{\alpha \otimes \mathrm{id}_{C_0(\mathbb{R})}} SB$ and $SB \xrightarrow{\beta \otimes \mathrm{id}_{C_0(\mathbb{R})}} SC$ become postcomposition by α and β in that form. Now the exactness is immediate. □

Remark 6.46. Stabilisation is functorial. For the induced $*$-homomorphisms between suspensions, the same notation will be adopted.

Definition 6.47. Let A be a C*-algebra. Then $CA := \{f \in C([0,1], A) | f(0) = 0\}$ is the *cone* of A.

Lemma 6.48. Let A be a C*-algebra. Then CA is contractible.

Proof. $[0,1]$ is contractible. For an explicit homotopy, see [34, Proposition 6.4.7]. □

Remark 6.49. In general SA is not contractible. This is because $\mathbb{T}$ is not contractible.

Lemma 6.50. Let A be a C*-algebra. The sequence $0 \to SA \to CA \xrightarrow{\mathrm{ev}_1} A \to 0$ is exact.

Proof. Every element of SA is included in CA. Let $f \in CA$, then $f \in SA$ if and only $f(1) = 0$. □

Definition 6.51. Let A and B be C*-algebras. Let $A \xrightarrow{\alpha} B$ be a $*$-homomorphism. The *mapping cone of* α is defined as $C_\alpha := \{(a,f) \in A \oplus CB \mid \mathrm{ev}_1(f) = \alpha(a)\}$.

Proposition 6.52. Let A and B be C*-algebras. Let $A \xrightarrow{\alpha} B$ be a $*$-homomorphism. C_α is the pullback of α and $CB \xrightarrow{\mathrm{ev}_1} B$.

Proof. Pullbacks of C*-algebras can always be written in the form given in Lemma 6.18. □

Lemma 6.53. Let A and B be C*-algebras. Let $A \xrightarrow{\alpha} B$ be a $*$-homomorphism. Let $SB \xrightarrow{(0,\_)} C_\alpha$ be the inclusion of SB into CB from Lemma 6.50. The sequence $0 \to SB \xrightarrow{(0,\_)} C_\alpha \to A \to 0$ is exact. Moreover, the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & SB \arrow[r, "(0,_)"] \arrow[d, "\mathrm{id}_{SB}"'] & C_\alpha \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\alpha"'] & 0 \\
0 \arrow[r] & SB \arrow[r] & CB \arrow[r, "\mathrm{ev}_1"] & B \arrow[r] & 0.
\end{tikzcd} \tag{21}$$

Proof. It is clear that for $f \in SB$, $(0,f) \in C_\alpha$. Obviously, $SB \xrightarrow{(0,\_)} C_\alpha$ is injective. The projection $C_\alpha \to A$ is surjective. Let $a \in A$, then the map $[0,1] \ni \lambda \to \lambda\alpha(a)$ is included in CB. Now $(a, \lambda\alpha(a)) \in C_\alpha$. Let $(a,f) \in C_\alpha$, if $a = 0$, then $\mathrm{ev}_1(f) = 0$, so $f \in SB$. Clearly $\mathrm{im}(SB \xrightarrow{(0,\_)} C_\alpha) \subset \ker(C_\alpha \to A)$. In diagram (21) the right-hand square is the pullback diagram corresponding to C_α. The left-hand square consists of embeddings. □

Lemma 6.54. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of C*-algebras. The sequence $0 \to A \xrightarrow{(\alpha(\_),0)} C_\beta \to CC \to 0$ is exact.

Proof. $\beta \circ \alpha = 0$, so $(\alpha(a), 0) \in C_\beta$ for any $a \in A$. Obviously $A \xrightarrow{(\alpha(\_),0)} C_\beta$ is injective. For any $g \in CC$, there exists a $b \in B$ such that $\beta(b) = \mathrm{ev}_1(g)$ as β is surjective. So $C_\beta \to CC$ is surjective. Let $g \in CC$, if $(b,0) \in C_\beta$, then $\beta(b) = 0$, so $b \in A$. Clearly $\mathrm{im}(A \xrightarrow{(\alpha(\_),0)} C_\beta) \subset \ker(C_\beta \to CC)$. □

Definition 6.55. Let A and B be C*-algebras. Let $A \xrightarrow{\alpha} B$ be a $*$-homomorphism. The *mapping cylinder of* α is defined as $Z_\alpha := \{(a,f) \in A \oplus C([0,1],B) \mid \mathrm{ev}_1(f) = \alpha(a)\}$.

Lemma 6.56. Let A and B be C*-algebras. Let $A \xrightarrow{\alpha} B$ be a $*$-homomorphism. The sequence $0 \to C_\alpha \to Z_\alpha \xrightarrow{\mathrm{ev}_0} B \to 0$ is exact.

Proof. C_α is the ideal in Z_α of pairs (a,f) for which $f(0) = 0$. □

Lemma 6.57. Let A and B be C*-algebras. Let $A \xrightarrow{\alpha} B$ be a $*$-homomorphism. Then A is a deformation retract of Z_α.

Proof. The deformation retract follows from inclusion of A via the constant functions under the image of α and the evaluation at 1. □

Definition 6.58. Let A be a C*-algebra. Let $A \xrightarrow{\alpha} A$ be a $*$-isomorphism. The *mapping torus* of α is defined as $T_\alpha := \{f \in C([0,1],A) \mid f(1) = \alpha(f(0))\}$.

Lemma 6.59. Let A be a C*-algebra. Let $A \xrightarrow{\alpha} A$ be a $*$-isomorphism. The sequence $0 \to SA \to T_\alpha \xrightarrow{\mathrm{ev}_0} A \to 0$ is exact.

Proof. Clearly SA is a $*$-subalgebra of T_α. Take for $a \in A$ the line segment between a and $\alpha(a)$, then this defines an element of T_α. So the map $T_\alpha \xrightarrow{\mathrm{ev}_0} A$ is surjective. If $\mathrm{ev}_0(f) = 0$ for $f \in T_\alpha$, then $f(1) = \alpha(0) = 0$, so $f \in SA$. So the sequence is exact. $\square$

6.5 Extension closedness

In this subsection we will state a few results about properties extensions of C*-algebras preserve. To be precise, this means whenever C*-algebras A and B have a property, any extension of A by B has that property as well. It is important to have this information. For example, in §8 we will construct a connecting morphism. This construction builds on the mapping cone. When we restrict to a subcategory, we still want to be able to use this construction. If the mapping cone is not included in the subcategory, then the construction fails in the subcategory.

Lemma 6.60. An extension of a separable C*-algebra by a separable C*-algebra is separable.

Proof. The proof builds on Proposition 5.78. The right-hand term is separable and therefore countably generated as a $*$-algebra. Take the pre-image of the generator elements. Every element in the middle term differs from this pre-image by an element of the kernel of the right-hand morphism. Hence it differs by an element of the left-hand term, which is again separable and therefore countably generated as a $*$-algebra. So the middle term is countably generated as a $*$-algebra and therefore separable. $\square$

Lemma 6.61. An extension of a σ-unital C*-algebra by a σ-unital C*-algebra is σ-unital.

Proof. See [16, §1, Corollary 7]. $\square$

Lemma 6.62. An extension of a nuclear C*-algebra by a nuclear C*-algebra is nuclear.

Proof. See [21, Theorem 6.5.3]. $\square$

Lemma 6.63. An extension of a postliminal C*-algebra by a postliminal C*-algebra is postliminal.

Proof. See [21, Theorem 5.6.2]. $\square$

Corollary 6.64. Let A and B be separable, resp. σ-unital, resp. nuclear, resp. postliminal C*-algebras. Let $A \xrightarrow{\alpha} B$ be a $*$-homomorphism. Then C_α, Z_α, T_α are separable, resp. σ-unital, resp. nuclear, resp. postliminal.

Proof. C_α appears in the exact sequence of Lemma 6.53. Now Z_α appears in the exact sequence of Lemma 6.56. And T_α appears in the exact sequence of Lemma 6.59. Now Lemmas 6.60, 6.61, 6.62 and 6.63 finalise the proof. $\square$

7 K-theory and Bott periodicity

This section goes through the definitions and properties of the K-functors. K-theory has some valuable results. We will see that the K-functors are not unique in their behaviour. They are actually examples of a much more general concept we will encounter in §8.

7.1 Equivalence relations on projections and the K_0-functor

Definition 7.1. Let A be a C*-algebra. An element $p \in A$ is called a *projection* if $p^2 = p^* = p$.

Example 7.2. 0 is a projection. If A is unital, 1_A is a projection. These are the *trivial* projections.

Example 7.3. Let $\mathcal{H}$ be a Hilbert space, the projection on a closed linear subspace is a projection in the sense of Definition 7.1.

Definition 7.4. Let A be a C*-algebra. Projections $p, q \in A$ are *Murray-von Neumann equivalent* if there exists a $v \in A$ such that $p = v^*v$ and $q = vv^*$. The set of Murray-von Neumann equivalence classes of A is denoted by $V(A)$.

Lemma 7.5. Murray-von Neumann equivalence is an equivalence relation.

Proof. Reflexivity: p is a projection, so $p = p^2 = p^*p = pp^*$. Symmetry: Immediate.
Transitivity: If $p = v^*v$ and $q = vv^*$ and $q = w^*w$ and $r = ww^*$, then $(wv)^*(wv) = v^*w^*wv^* = v^*qv = v^*vv^*v = p^2 = p$ and $(wv)(wv)^* = wvv^*w^* = wqw^* = ww^*ww^* = r^2 = r$. □

Definition 7.6. Let A be a C*-algebra. Projections p and q are *unitarily equivalent* if there exists a unitary $u \in \tilde{A}$ such that $q = upu^*$ in $\tilde{A}$.

Lemma 7.7. Unitary equivalence of projections is an equivalence relation.

Proof. Reflexivity and symmetry are obvious. We will show transitivity, let p and q be unitarily equivalent and let q and r be unitarily equivalent. There exists a unitary $u \in \tilde{A}$ such that $q = upu^*$ and there exists a unitary $v \in \tilde{A}$ such that $r = vqv^*$. It follows that $r = uvpv^*u^*$. So p and r are unitarily equivalent. □

Definition 7.8. Let A be a C*-algebra. Projections p and q are *homotopic* if there exists a (norm)-continuous path of **projections** $\xi : [0,1] \to A$ such that $\xi(0) = p$ and $\xi(1) = q$.

Lemma 7.9. Homotopy of projections is an equivalence relation.

Proof. Identical to the proof that path connectedness in a topological space is an equivalence relation. See [28, Theorem 3.4.17 iii]. □

Theorem 7.10. Let A be a C*-algebra. Homotopy of projections implies unitary equivalence of projections, which implies Murray-von Neumann equivalence.

Proof. See [34, Proposition 5.2.10]. □

Definition 7.11. Let A be a C*-algebra. For projections $p \in M_m(A)$ and $q \in M_n(A)$, *addition* of p and q is defined as $\begin{pmatrix} p & 0_{n\times m} \\ 0_{m\times n} & q \end{pmatrix} \in M_{m+n}(A)$ and denoted by $p \oplus q$.

Lemma 7.12. Let A be a C*-algebra. Let p and p' be Murray-von Neumann equivalent projections and let q and q' be Murray-von Neumann equivalent projections, then $p \oplus q$ and $p' \oplus q'$ are Murray-von Neumann equivalent.

Proof. See [21, Theorem 7.1.1.1]. □

Corollary 7.13. The addition of Definition 7.11 defines an addition on $V(A)$.

Remark 7.14. Murray-von Neumann equivalent projections in $\mathbb{K}\otimes A$ (the direct limit in Example 5.55) need not be included in the same $M_n(A)$ for some $n\in\mathbb{N}$. See [21, Theorem 7.1.1.3].

Lemma 7.15. Let A be a C*-algebra. The addition on $V(A)$ is unital and commutative.

Proof. The class of 0 (which only contains 0) is the unit, see [21, Theorem 7.1.1.3]. For commutativity, see [21, Theorem 7.1.1.1]. □

Proposition 7.16. Let A be a C*-algebra. Then $V(A) = V(M_n(A))$ for all $n\in\mathbb{N}$.

Proof. See [34, Lemma 6.2.10]. □

Example 7.17. $V(\mathbb{C}) = V(M_n(\mathbb{C})) = V(\mathbb{K}) = \mathbb{N}_0$, $V(\mathbb{B}) = \mathbb{N}_0\cup\{\infty\}$ and $V(\mathbb{B}/\mathbb{K}) = \{0,\infty\}$ (For an explanation, see [34, Example 6.1.4]).

Proposition 7.18. Let A and B be C*-algebras and let $A \xrightarrow{f} B$ be a $*$-homomorphism. Then $V(f)([p]) = [f(p)]\in V(B)$ defines an unital subgroup morphism.

Proof. The $*$-homomorphism is applied pointwise to a matrix. $*$-homomorphisms preserve projections. Addition and 0 are preserved. Murray-von Neumann equivalence is preserved, so the mapping is independent of representative. □

Corollary 7.19. $V\colon \mathsf{C}^* - \mathsf{alg} \to \mathsf{CommMon}$ defines a functor.

Proof. It follows immediately that identities and compositions are preserved. □

Definition 7.20. Let A be a C*-algebra. Let $\mathrm{K}_{00}(A)$ be the *Grothendieck group of* $V(A)$. $[p]\in V(A)$ is viewed as $[p]-[0]\in \mathrm{K}_{00}(A)$.

Remark 7.21. A detailed explanation of the construction of Grothendieck groups from semigroups can be found in [34, Appendix G] or [21].

Remark 7.22. For those familiar with localisations of commutative rings (see [1, §3]. A Grothendieck group has striking resemblances with the field of fractions of an integral domain. In fact, every commutative ring is a commutative semigroup under multiplication. Compare the equivalence relations of [1, p.36] to [34, p. 295], the definitions of field of fractions [1, p.37] and Grothendieck group [34, p.296] and the results of [1, Proposition 3.1] and [34, Theorem p. 296].

Lemma 7.23. $\mathrm{K}_{00}\colon \mathsf{C}^* - \mathsf{alg} \to \mathsf{Ab}$ defines a functor.

Proof. The same as Corollary 7.19. □

Example 7.24. From Example 7.17 follows $\mathrm{K}_{00}(\mathbb{C}) = \mathrm{K}_{00}(\mathbb{K}) = \mathbb{Z}$ and $\mathrm{K}_{00}(\mathbb{B}) = \mathrm{K}_{00}(\mathbb{B}/\mathbb{K}) = 0$. In the last two cases, this is because the class of infinite dimensional projections satisfies [34, Prop. p. 297]. No inverse can be constructed for the class of infinite dimensional operators.

Proposition 7.25. Let A_1 and A_2 be C*-algebras. Then $V(A_1\oplus A_2) = V(A_1)\oplus V(A_2)$ and $\mathrm{K}_{00}(A_1\oplus A_2) = \mathrm{K}_{00}(A_1)\oplus \mathrm{K}_{00}(A_2)$.

Proof. For a full proof, see [34, Proposition 6.2.1]. It builds on the fact that a projection in an orthogonal sum is a pair of projections in each summand . □

Definition 7.26. Let A be a C*-algebra. Define $\mathrm{K}_0(A) := \ker(\mathrm{K}_{00}(\tilde{A}) \xrightarrow{\mathrm{K}_{00}(\pi)} \mathrm{K}_{00}(\mathbb{C}))$, where $\tilde{A} \xrightarrow{\pi} \mathbb{C}$ is the projection to the unit.

Lemma 7.27. Let A_1 and A_2 be C*-algebras. Then $\mathrm{K}_0(A_1 \oplus A_2) = \mathrm{K}_0(A_1) \oplus \mathrm{K}_0(A_2)$.

Lemma 7.28. $\mathrm{K}_0 : \mathsf{C}^* - \mathsf{alg} \to \mathsf{Ab}$ defines a functor.

Proof. Let A and B be C*-algebras. Let $A \xrightarrow{f} B$ be a $*$-homomorphism. Then $\tilde{A} \xrightarrow{\tilde{f}} \tilde{B}$ is a unital $*$-homomorphism ($\tilde{f}(a + \lambda) = f(a) + \lambda$ for $a \in A$ and $\lambda \in \mathbb{C}$). So $\mathrm{K}_{00}(\pi_B \circ \tilde{f})([p] - [q]) = \mathrm{K}_{00}(\pi_B)([\tilde{f}(p)] - [\tilde{f}(q)]) = ([\pi_B \circ \tilde{f}(p)] - [\pi_B \circ \tilde{f}(q)]) = [\pi_A(p)] - [\pi_A(q)] = \mathrm{K}_{00}(\pi_A)([p] - [q])$ for some projections $p \in M_m(\tilde{A})$ and $q \in M_n(\tilde{A})$, where $\tilde{A} \xrightarrow{\pi_A} \mathbb{C}$ and $\tilde{B} \xrightarrow{\pi_B} \mathbb{C}$ are the projections to the unit. So $\mathrm{K}_0(A)$ is mapped to $\mathrm{K}_0(B)$ by $\mathrm{K}_{00}(\tilde{f})$. Identity and composition are preserved by unitisation (Lemma 5.33) and by K_{00}(Lemma 7.23). □

Proposition 7.29. Let A be a unital C*-algebra. Then $\mathrm{K}_0(A) = \mathrm{K}_{00}(A)$.

Proof. It follows immediately from the fact that $\tilde{A} = A \oplus \mathbb{C}$ and Proposition 7.25. □

The harder C*-algebras to determine the K_0-groups for, are the non-unital ones. For instance, what is $\mathrm{K}_0(\mathbb{K})$? It turns out $\mathrm{K}_0(\mathbb{K}) = \mathbb{Z}$ and that is due to the *stability* of K_0, which is the result of the following property of K_0.

Theorem 7.30. K_0 preserves direct limits.

Proof. The proof is very technical and can be found in [34, p.116-117]. □

Corollary 7.31. K_0 is invariant under stabilisation. In other words, let A be a C*-algebra, then $\mathrm{K}_0(A) \cong \mathrm{K}_0(\mathbb{K} \otimes A)$. Moreover, K_0 is invariant under stable isomorphism.

Proof. Let A be a C*-algebra. By Example 5.56 $\mathbb{K} \otimes A = \varinjlim M_n(A)$. By Proposition 7.16 $V(A) = V(M_n(A))$ and as result of that $\mathrm{K}_{00}(A) = \mathrm{K}_{00}(M_n(A))$ and $\mathrm{K}_0(A) = \mathrm{K}_0(M_n(A))$ for all $n \in \mathbb{N}$. The group homomorphisms induced by application of K_0 to the structure morphisms of Example 5.56 are isomorphisms (as the Murray-von Neumann class is fixed by the structure morphisms). Hence $\mathrm{K}_0(\mathbb{K} \otimes A) = \mathrm{K}_0(A)$. Let A and B be stably isomorphic. Then $\mathbb{K} \otimes A \cong \mathbb{K} \otimes B$ by definition of stable isomorphism. So $\mathrm{K}_0(A) \cong \mathrm{K}_0(\mathbb{K} \otimes A) \cong \mathrm{K}_0(\mathbb{K} \otimes B) \cong \mathrm{K}_0(B)$. □

Example 7.32. $\mathrm{K}_0(\mathbb{K}) = \mathrm{K}_0(\mathbb{C}) = \mathbb{Z}$. This follows from $\mathbb{K} = \mathbb{K} \otimes \mathbb{C}$.

Lemma 7.33. K_0 is homotopy invariant, i.e. let A and B be C*-algebras and let $f, g : A \to B$ be homotopic $*$-homomorphisms, then $\mathrm{K}_0(f) = \mathrm{K}_0(g)$.

Proof. There is a path $(\xi_t)_{t \in [0,1]}$ of $*$-homomorphisms $A \xrightarrow{\xi_t} B$ such that $\xi_0 = f$ and $\xi_1 = g$. Let $p \in A$ be a projection. $f(p)$ and $g(p)$ are homotopic (by $\xi_t(p)$). Via Theorem 7.10, a homotopy of projections induces a Murray-von Neumann equivalence. Hence $\mathrm{K}_0(f) = \mathrm{K}_0(g)$. □

Corollary 7.34. Let A be a contractible C*-algebra. Then $\mathrm{K}_0(A) = 0$.

Example 7.35. Let A and B be C*-algebras. Let $A \xrightarrow{\alpha} B$ be a $*$-homomorphism. Then $\mathrm{K}_0(Z_\alpha) \cong \mathrm{K}_0(A)$. This follows from the fact that A is a deformation retract of Z_α, see Lemma 6.57.

Theorem 7.36. Let A be a C*-algebra. Then $\mathrm{K}_0(\mathcal{M}(\mathbb{K} \otimes A)) = 0$.

Proof. See [34, Theorem 10.2]. □

7.2 Equivalence relations on invertibles and unitaries and the K_1-functor

In this subsection another functor will be introduced: K_1. It has many of the properties of K_0. We will go through them, but be brief in the explanation. More important to this thesis is the way K_0 and K_1 are interlinked. That will be saved for the next subsection. First we need to introduce some definitions.

It is only possible to speak about invertibles and unitaries whenever a C*-algebras possesses a unit. In this section always a unitisation will be used, even when the C*-algebra is already unital. Let A be a C*-algebra. Denote the invertible $\tilde{A}$-valued $n \times n$-matrices by $\mathrm{GL}_n(\tilde{A})$ and denote the unitary $\tilde{A}$-valued $n \times n$-matrices by $\mathcal{U}_n(\tilde{A})$. Let $M_n(\tilde{A}) \xrightarrow{\pi_A} M_n(\mathbb{C})$ be the projection. Define the following sets of invertibles and unitaries:

$$\mathrm{GL}_n^+(A) := \{X \in \mathrm{GL}_n(\tilde{A}) : \pi_A(X) = \mathrm{I}_n\} \qquad \mathcal{U}_n^+(A) := \{X \in \mathcal{U}_n(\tilde{A}) : \pi_A(X) = \mathrm{I}_n\}.$$

Each is endowed with the subset topology from $M_n(\tilde{A})$.

The embeddings $\mathrm{GL}_n^+(A) \ni X \to \begin{pmatrix} X & 0 \\ 0 & 1_{\tilde{A}} \end{pmatrix} \in \mathrm{GL}_{n+1}^+(A)$ with $n \geq 1$ form a cotower of topological groups. The same can be done for the unitaries. We will use the following notation for the direct limits:

$$\mathrm{GL}_\infty^+(A) := \varinjlim \mathrm{GL}_n^+(A) \qquad \mathcal{U}_\infty^+(A) := \varinjlim \mathcal{U}_n^+(A).$$

This means that $X \in \mathrm{GL}_i^+(A)$ and $\begin{pmatrix} X & 0 \\ 0 & I_{j-i} \end{pmatrix} \in \mathrm{GL}_j^+(A)$ with $i \leq j$ represent the same in $\mathrm{GL}_\infty^+(A)$.

Definition 7.37. Let A be a C*-algebra. Elements $X, Y \in \mathrm{GL}_n^+(A)$ are *homotopic* if there exists a path $\xi : [0,1] \to \mathrm{GL}_n^+(A)$ such that $\xi(0) = X$ and $\xi(1) = Y$.

Remark 7.38. The same definition will be used for $\mathcal{U}_n^+(A)$.

Theorem 7.39. In $\mathrm{GL}_n^+(A)$ and $\mathcal{U}_n^+(A)$ path connected components are connected components.

Proof. See [34, Proposition 4.2.4]. There exists path connected neighbourhoods at every point. Apply [28, Proposition 3.4.25] to the connected components to conclude each of them is path connected. □

Definition 7.40. Let A be a C*-algebra. Let $n, m \in \mathbb{N} \cup \{\infty\}$. For invertibles $X \in \mathrm{GL}_m^+(A)$ and $Y \in \mathrm{GL}_n^+(A)$, the *composition* of X and Y is defined as $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \in \mathrm{GL}_{m+n}^+(A)$ and denoted by XY.

Lemma 7.41. Let A be a C*-algebra. Let $n, m \in \mathbb{N} \cup \{\infty\}$. Let $X, X' \in \mathrm{GL}_m^+(A)$ be homotopic and let $Y, Y' \in \mathrm{GL}_n^+(A)$ be homotopic, then XY and $X'Y'$ are homotopic.

Proof. A composition in the same way as in Definition 7.40 of the homotopies between X and X' and between Y and Y' gives a homotopy between XY and $X'Y'$. □

Definition 7.42. Let A be a C*-algebra. Let $n \in \mathbb{N} \cup \{\infty\}$. $\mathrm{GL}_n^+(A)_0$ is the connected component of $1_{M_n(\tilde{A})}$ in $\mathrm{GL}_n^+(A)$. Likewise $\mathcal{U}_n^+(A)_0$ is the connected component of $1_{M_n(\tilde{A})}$ in $\mathcal{U}_n^+(A)$.

Theorem 7.43. Let A be a C*-algebra. Then $\mathrm{GL}_n^+(A)/\mathrm{GL}_n^+(A)_0$ and $\mathcal{U}_n^+(A)/\mathcal{U}_n^+(A)_0$ are isomorphic for $n \in \mathbb{N} \cup \{\infty\}$.

Proof. See [34, Proposition 4.2.6]. □

Corollary 7.44. The composition of Definition 7.40 defines a composition on $\mathrm{GL}_\infty^+(A)/\mathrm{GL}_\infty^+(A)_0$

Remark 7.45. In Definition 7.40, Lemma 7.41 and Corollary 7.44 $\mathrm{GL}_*^+(A)$ can be replaced by $\mathcal{U}_*^+(A)$.

Definition 7.46. Let A be a C*-algebra. $\mathrm{K}_1(A)$ is the semigroup $\mathrm{GL}_\infty^+(A)/\mathrm{GL}_\infty^+(A)_0$

Lemma 7.47. Let A be a C*-algebra. Let $X, Y \in \mathrm{GL}_n + (A)$. The following four are homotopic: $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \begin{pmatrix} Y & 0 \\ 0 & X \end{pmatrix}, \begin{pmatrix} XY & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} YX & 0 \\ 0 & 1 \end{pmatrix}$.

Proof. See [34, Theorem 4.2.9]. □

Corollary 7.48. Let A be a C*-algebra. $\mathrm{K}_1(A)$ is an abelian group.

Proof. Lemma 7.47 gives the existence of inverses in $\mathrm{K}_1(A)$. From this lemma follows $\begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} = \begin{pmatrix} XX^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{id}_{\mathrm{K}_1(A)}$.
The commutativity follows from Lemma 7.47 as well. □

Lemma 7.49. $\mathrm{K}_1\colon \mathsf{C}^* - \mathsf{alg} \to \mathsf{Ab}$ defines a functor.

Proof. Let A and B be C*-algebras. Let $A \xrightarrow{f} B$ be a $*$-homomorphism. Then $\tilde{A} \xrightarrow{\tilde{f}} \tilde{B}$ induces a group homomorphism $\mathrm{GL}_\infty^+(A) \to \mathrm{GL}_\infty^+(B)$. Unital $*$-homomorphisms preserve invertibles and unitaries. $*$-homomorphisms are continuous by Lemma 5.9. So connected components are preserved. It is obvious that identities and compositions are preserved. □

Proposition 7.50. Let A be a C*-algebra. Then $\mathrm{K}_1(A) = \mathrm{K}_1(\tilde{A})$.

Proof. See [34, Example 7.1.11 (5)]. □

Lemma 7.51. Let A_1 and A_2 be C*-algebras. $\mathrm{K}_1(A_1 \oplus A_2) = \mathrm{K}_1(A_1) \oplus \mathrm{K}_0(A_2)$.

Proof. Every invertible element in $A_1 \oplus A_2$ is a pair of invertibles of A_1 and A_2. □

Theorem 7.52. K_1 preserves direct limits.

Proof. See [34, Proposition 7.1.7]. □

Corollary 7.53. K_1 is invariant under stabilisation. In other words, let A be a C*-algebra, then $\mathrm{K}_1(A) \cong \mathrm{K}_1(\mathbb{K} \otimes A)$. Moreover, K_1 is invariant under stable isomorphism.

Example 7.54. $\mathrm{K}_1(\mathbb{C}) = \mathrm{K}_1(\mathbb{K}) = 0$ as $\mathbb{C}$ is the unitisation of 0 and $\mathbb{K} = \varinjlim M_n(\mathbb{C})$ (see Theorem 7.52).

Example 7.55. $\mathrm{K}_1(\mathbb{B}) = 0$ and $\mathrm{K}_1(\mathbb{B}/\mathbb{K}) = \mathbb{Z}$. See [34, Example 7.1.11 (3) + (7)].

Lemma 7.56. K_1 is homotopy invariant, i.e. let A and B be C*-algebras and let $f, g : A \to B$ be homotopic $*$-homomorphisms, then $\mathrm{K}_1(f) = \mathrm{K}_1(g)$.

Proof. Same as Lemma 7.33. ☐

Corollary 7.57. Let A be a contractible C*-algebra. Then $\mathrm{K}_1(A) = 0$.

Example 7.58. Let A and B be C*-algebras. Let $A \xrightarrow{\alpha} B$ be a $*$-homomorphism. Then $\mathrm{K}_1(Z_\alpha) \cong \mathrm{K}_1(A)$. This follows from the fact that A is a deformation retract of Z_α, see Lemma 6.57.

Theorem 7.59. Let A be a C*-algebra. Then $\mathrm{K}_1(\mathcal{M}(\mathbb{K} \otimes A)) = 0$.

Proof. See [34, Theorem 10.2]. ☐

Remark 7.60. For an overview of some K_0- and K_1-groups, see [34, §6.5].

7.3 Suspensions and Bott periodicity

The K_0- and K_1-functors are related. The K_0-group of a C*-algebra is the K_1-group of the suspension and vice versa. More generally, we will construct a six term sequence K_0- and K_1-groups.

Theorem 7.61. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of C*-algebras. The following sequences are exact

$$\mathrm{K}_0(A) \xrightarrow{\mathrm{K}_0(\alpha)} \mathrm{K}_0(B) \xrightarrow{\mathrm{K}_0(\beta)} \mathrm{K}_0(C)$$
$$\mathrm{K}_1(A) \xrightarrow{\mathrm{K}_1(\alpha)} \mathrm{K}_1(B) \xrightarrow{\mathrm{K}_1(\beta)} \mathrm{K}_1(C).$$

Proof. See [34, Theorem 6.3.2] and [34, Theorem 7.1.12]. ☐

Remark 7.62. In other words, K_0 and K_1 are half-exact.

Theorem 7.63. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of C*-algebra. There exists a connecting map $\mathrm{K}_1(C) \xrightarrow{\delta} \mathrm{K}_0(A)$ such the following sequence is exact:

$$\mathrm{K}_1(A) \xrightarrow{\mathrm{K}_1(\alpha)} \mathrm{K}_1(B) \xrightarrow{\mathrm{K}_1(\beta)} \mathrm{K}_1(C) \xrightarrow{\delta} \mathrm{K}_0(A) \xrightarrow{\mathrm{K}_0(\alpha)} \mathrm{K}_0(B) \xrightarrow{\mathrm{K}_0(\beta)} \mathrm{K}_0(C). \tag{22}$$

The connecting map is natural in the sense that for a commutative diagram with short exact sequences on the horizontal rows as in diagram (23), the induced long exact sequences of the form of sequence (22) form a commutative diagram as in diagram (24).

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha_1"] \arrow[d, "\phi"'] & B \arrow[r, "\beta_1"] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d, "\chi"'] & 0 \\
0 \arrow[r] & D \arrow[r, "\alpha_2"] & E \arrow[r, "\beta_2"] & F \arrow[r] & 0.
\end{tikzcd} \tag{23}$$

$$\begin{tikzcd}
\mathrm{K}_1(A) \arrow[r, "\mathrm{K}_1(\alpha_1)"] \arrow[d, "\mathrm{K}_1(\phi)"'] & \mathrm{K}_1(B) \arrow[r, "\mathrm{K}_1(\beta_1)"] \arrow[d, "\mathrm{K}_1(\psi)"'] & \mathrm{K}_1(C) \arrow[r, "\delta"] \arrow[d, "\mathrm{K}_1(\chi)"'] & \mathrm{K}_0(A) \arrow[r, "\mathrm{K}_0(\alpha_1)"] \arrow[d, "\mathrm{K}_0(\phi)"'] & \mathrm{K}_0(B) \arrow[r, "\mathrm{K}_0(\beta_1)"] \arrow[d, "\mathrm{K}_0(\psi)"'] & \mathrm{K}_0(C) \arrow[d, "\mathrm{K}_0(\chi)"'] \\
\mathrm{K}_1(D) \arrow[r, "\mathrm{K}_1(\alpha_2)"] & \mathrm{K}_1(E) \arrow[r, "\mathrm{K}_1(\beta_2)"] & \mathrm{K}_1(F) \arrow[r, "\delta'"] & \mathrm{K}_0(D) \arrow[r, "\mathrm{K}_0(\alpha_2)"] & \mathrm{K}_0(E) \arrow[r, "\mathrm{K}_0(\beta_2)"] & \mathrm{K}_0(F).
\end{tikzcd} \tag{24}$$

Proof. For a sketch of the construction, see [34, §8.1]. ☐

Definition 7.64. The map $\mathrm{K}_1(C) \xrightarrow{\delta} \mathrm{K}_0(A)$ in sequence (22) is called the *index* map.

Lemma 7.65. Let A be a C*-algebra. Then $\mathrm{K}_1(A)$ and $\mathrm{K}_0(SA)$ are naturally isomorphic (in the sense of Definition 2.21).

Proof. Apply Theorem 7.63 to the exact sequence $0 \to SA \to CA \xrightarrow{\mathrm{ev}_1} A \to 0$ from Lemma 6.50. By Lemma 6.48, CA is contractible. Now $\mathrm{K}_0(CA) = \mathrm{K}_1(CA) = 0$. So $\mathrm{K}_1(A) \xrightarrow{\delta} \mathrm{K}_0(SA)$ is an isomorphism by the exactness of sequence (22). ☐

Theorem 7.66. Let A be a C*-algebra. Then $\mathrm{K}_0(A)$ and $\mathrm{K}_1(SA)$ are naturally isomorphic.

Proof. The construction is explained in [34, §9]. ☐

Definition 7.67. The next corollary is known as *Bott periodicity.*

Corollary 7.68. Let A be a C*-algebra. Then $\mathrm{K}_0(A)$ and $\mathrm{K}_0(S^2A)$ are naturally isomorphic and $\mathrm{K}_1(A)$ and $\mathrm{K}_1(S^2A)$ are naturally isomorphic.

Proof. Combine Lemma 7.65 and Theorem 7.66. ☐

Lemma 7.69. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of C*-algebra. There exists a natural connecting map $\mathrm{K}_1(C) \xrightarrow{\delta} \mathrm{K}_0(A)$ such that the following sequence is exact:

$$\mathrm{K}_0(A) \xrightarrow{\mathrm{K}_0(\alpha)} \mathrm{K}_0(B) \xrightarrow{\mathrm{K}_0(\beta)} \mathrm{K}_0(C) \xrightarrow{\delta} \mathrm{K}_1(A) \xrightarrow{\mathrm{K}_1(\alpha)} \mathrm{K}_1(B) \xrightarrow{\mathrm{K}_1(\beta)} \mathrm{K}_1(C). \tag{25}$$

Proof. $0 \to SA \xrightarrow{S(\alpha)} SB \xrightarrow{S(\beta)} SC \to 0$ is exact by Lemma 6.45. Now apply Theorem 7.63 to it to obtain an index map $\mathrm{K}_1(A) \xrightarrow{\delta} \mathrm{K}_0(SA)$ and the following exact sequence:

$$\mathrm{K}_1(SA) \xrightarrow{\mathrm{K}_1S(\alpha)} \mathrm{K}_1(SB) \xrightarrow{\mathrm{K}_1S(\beta)} \mathrm{K}_1(SC) \xrightarrow{\delta} \mathrm{K}_0(SA) \xrightarrow{\mathrm{K}_0S(\alpha)} \mathrm{K}_0(SB) \xrightarrow{\mathrm{K}_0S(\beta)} \mathrm{K}_0(SC). \tag{26}$$

Lemma 7.65 and Theorem 7.66 turn sequence (26) into sequence (25). ☐

Definition 7.70. The index map $\mathrm{K}_0(C) = \mathrm{K}_1(SC) \xrightarrow{\delta} \mathrm{K}_0(SA) = \mathrm{K}_1(A)$ in sequence (26) is called the *exponential* map.

Corollary 7.71 (Six term sequence)**.** Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of C*-algebra. Then the following sequence is exact:

$$\begin{tikzcd}
\mathrm{K}_0(A) \arrow[r, "\mathrm{K}_0(\alpha)"] & \mathrm{K}_0(B) \arrow[r, "\mathrm{K}_0(\beta)"] & \mathrm{K}_0(C) \arrow[d] \\
\mathrm{K}_1(C) \arrow[u] & \mathrm{K}_1(B) \arrow[l, "\mathrm{K}_1(\beta)"] & \mathrm{K}_1(A). \arrow[l, "\mathrm{K}_1(\alpha)"]
\end{tikzcd}$$

The vertical maps are the index and exponential map.

Proof. Combine the exact sequences (22) and (25). ☐

7.4 Examples and applications of the six term sequence

Example 7.72. It is not true that K_0 or K_1 is exact. The exact sequence $0 \to \mathbb{K} \to \mathbb{B} \to \mathbb{B}/\mathbb{K} \to 0$ gives a counterexample as $\mathrm{K}_0(\mathbb{K}) = \mathbb{Z}$, while $\mathrm{K}_0(\mathbb{B}) = \mathrm{K}_0(\mathbb{B}/\mathbb{K}) = 0$ and similarly, $\mathrm{K}_0(\mathbb{K}) = \mathrm{K}_0(\mathbb{B}) = 0$, while $\mathrm{K}_0(\mathbb{B}/\mathbb{K}) = \mathbb{Z}$.

Example 7.73. Let A and B be C*-algebras. Let $A \xrightarrow{\alpha} B$ be a $*$-homomorphism. Both the exact sequences $0 \to SB \xrightarrow{(0,\_)} C_\alpha \to A \to 0$ (Lemma 6.53) and $0 \to C_\alpha \to Z_\alpha \xrightarrow{\mathrm{ev}_0} B \to 0$ (Lemma 6.56) induce the following exact six term sequence:

$$\begin{tikzcd}
\mathrm{K}_1(B) \arrow[r] & \mathrm{K}_0(C_\alpha) \arrow[r] & \mathrm{K}_0(A) \arrow[d] \\
\mathrm{K}_1(A) \arrow[u] & \mathrm{K}_1(C_\alpha) \arrow[l] & \mathrm{K}_0(B). \arrow[l]
\end{tikzcd}$$

Superscripts have been left out from the arrows. This is because the arrows differ, depending on the choice of exact sequence. Recall that A is a deformation retract from Z_α (Lemma 6.57) and recall the relation between K-groups and suspensions.

Lemma 7.74. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of C*-algebra. Then $\mathrm{K}_0(C_\beta) \cong \mathrm{K}_0(A)$ and $\mathrm{K}_1(C_\beta) \cong \mathrm{K}_1(A)$.

Proof. The sequence $0 \to A \xrightarrow{(\alpha(\_),0)} C_\beta \to CC \to 0$ is exact (Lemma 6.54) and induces the following six term sequence:

$$\begin{tikzcd}
\mathrm{K}_0(A) \arrow[r, "\mathrm{K}_0((\alpha(_),0))"] & \mathrm{K}_0(C_\beta) \arrow[r] & \mathrm{K}_0(CC) \arrow[d] \\
\mathrm{K}_1(CC) \arrow[u] & \mathrm{K}_1(C_\beta) \arrow[l] & \mathrm{K}_1(A\mathbb{K}\otimes A). \arrow[l, "\mathrm{K}_1((\alpha(_),0))"']
\end{tikzcd}$$

As CC is contractible (Lemma 6.48), $\mathrm{K}_0(CC) = \mathrm{K}_1(CC) = 0$. This proves the lemma. ☐

Lemma 7.75. Let A be a C*-algebra. Then $\mathrm{K}_0(\mathcal{Q}(\mathbb{K}\otimes A)) \cong \mathrm{K}_1(A)$ and $\mathrm{K}_1(\mathcal{Q}(\mathbb{K}\otimes A)) \cong \mathrm{K}_0(A)$.

Proof. The exact sequence $0 \to \mathbb{K}\otimes A \xrightarrow{(L_\_,R_\_)} \mathcal{M}(\mathbb{K}\otimes A) \xrightarrow{\pi} \mathcal{Q}(\mathbb{K}\otimes A) \to 0$ induces the following six term sequence:

$$\begin{tikzcd}
\mathrm{K}_0(\mathbb{K}\otimes A) \arrow[r, "\mathrm{K}_0((L__,R__))"] & \mathrm{K}_0(\mathcal{M}(\mathbb{K}\otimes A)) \arrow[r, "\mathrm{K}_0(\pi)"] & \mathrm{K}_0(\mathcal{Q}(\mathbb{K}\otimes A)) \arrow[d] \\
\mathrm{K}_1(\mathcal{Q}(\mathbb{K}\otimes A)) \arrow[u] & \mathrm{K}_1(\mathcal{M}(\mathbb{K}\otimes A)) \arrow[l, "\mathrm{K}_1(\pi)"'] & \mathrm{K}_1(\mathbb{K}\otimes A). \arrow[l, "\mathrm{K}_1((L__,R__))"']
\end{tikzcd}$$

By Theorem 7.36 and Theorem 7.59, $\mathrm{K}_0(\mathcal{M}(\mathbb{K}\otimes A)) = \mathrm{K}_1(\mathcal{M}(\mathbb{K}\otimes A)) = 0$, so $\mathrm{K}_0(\mathcal{Q}(\mathbb{K}\otimes A)) = \mathrm{K}_1(\mathbb{K}\otimes A) = \mathrm{K}_1(A)$ and $\mathrm{K}_1(\mathcal{Q}(\mathbb{K}\otimes A)) = \mathrm{K}_0(\mathbb{K}\otimes A) = \mathrm{K}_0(A)$ by stability. ☐

Example 7.76. Following up on Example 7.75, let $0 \to \mathbb{K}\otimes A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an extension of C*-algebras. Then the following diagram containing the Busby invariant $C \xrightarrow{\tau} \mathcal{Q}(\mathbb{K}\otimes A)$, commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{K}\otimes A \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}_{\mathbb{K}\otimes A}"'] & B \arrow[r, "\beta"] \arrow[d, "\sigma"'] & C \arrow[r] \arrow[d, "\tau"'] & 0 \\
0 \arrow[r] & \mathbb{K}\otimes A \arrow[r, "(L_{_}, R_{_})"] & \mathcal{M}(\mathbb{K}\otimes A) \arrow[r, "\pi"] & \mathcal{Q}(\mathbb{K}\otimes A) \arrow[r] & 0.
\end{tikzcd}$$

Applying the naturality of K_0 and K_1, we get the following commutative diagram:

$$\begin{tikzcd}
\mathrm{K}_1(\mathbb{K}\otimes A) \arrow[r, "\mathrm{K}_1(\alpha)"] \arrow[d, "\mathrm{id}_{\mathrm{K}_1(\mathbb{K}\otimes A)}"'] & \mathrm{K}_1(B) \arrow[r, "\mathrm{K}_1(\beta)"] \arrow[d] & \mathrm{K}_1(C) \arrow[r, "\delta"] \arrow[d, "\mathrm{K}_1(\tau)"'] & \mathrm{K}_0(\mathbb{K}\otimes A) \arrow[r, "\mathrm{K}_0(\alpha)"] \arrow[d, "\mathrm{id}_{\mathrm{K}_0(\mathbb{K}\otimes A)}"'] & \mathrm{K}_0(B) \arrow[r, "\mathrm{K}_0(\beta)"] \arrow[d] & \mathrm{K}_0(C) \arrow[d, "\mathrm{K}_0(\tau)"'] \\
\mathrm{K}_1(\mathbb{K}\otimes A) \arrow[r] & 0 \arrow[r] & \mathrm{K}_1(\mathcal{Q}(\mathbb{K}\otimes A)) \arrow[r, "\cong"] & \mathrm{K}_0(\mathbb{K}\otimes A) \arrow[r] & 0 \arrow[r] & \mathrm{K}_0(\mathcal{Q}(\mathbb{K}\otimes A).
\end{tikzcd}$$

It follows that $\mathrm{K}_1(C) \xrightarrow{\delta} \mathrm{K}_0(\mathbb{K}\otimes A)$ coincides with $\mathrm{K}_1(C) \xrightarrow{\mathrm{K}_1(\tau)} \mathrm{K}_1(\mathcal{Q}(\mathbb{K}\otimes A)) \cong \mathrm{K}_0(\mathbb{K}\otimes A)$. The same holds when K_0 and K_1 are switched.

Lemma 7.77. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a split exact sequence of C*-algebras. The following exact sequences are split exact:

$$0 \to \mathrm{K}_0(A) \xrightarrow{\mathrm{K}_0(\alpha)} \mathrm{K}_0(B) \xrightarrow{\mathrm{K}_0(\beta)} \mathrm{K}_0(C) \to 0, \tag{27}$$

$$0 \to \mathrm{K}_1(A) \xrightarrow{\mathrm{K}_1(\alpha)} \mathrm{K}_1(B) \xrightarrow{\mathrm{K}_1(\beta)} \mathrm{K}_1(C) \to 0. \tag{28}$$

Proof. $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is split, so there exists a section $C \xrightarrow{\gamma} B$ of β. Recall sequence (22). By functoriality $\mathrm{K}_0(\beta)\circ\mathrm{K}_0(\gamma) = \mathrm{K}_0(\mathrm{id}_C) = \mathrm{id}_{\mathrm{K}_0(C)}$. So $\mathrm{K}_0(\beta)$ is surjective. Similarly, $\mathrm{K}_1(\beta)\circ\mathrm{K}_1(\gamma) = \mathrm{K}_1(\mathrm{id}_C) = \mathrm{id}_{\mathrm{K}_1(C)}$. So $\mathrm{K}_1(\beta)$ is surjective. By exactness $\ker\delta = \mathrm{K}_1(C)$. Hence $\delta = 0$. Therefore $\ker\mathrm{K}_0(\alpha) = 0$ by exactness. So $\mathrm{K}_0(\alpha)$ is injective. Via Examples 4.2 and 4.3 sequence (27) is exact. $\mathrm{K}_0(\gamma)$ offers a section of sequence (27). Statement 1 of Theorem 4.8 is satisfied. For sequence (28), do the same, but use the suspended exact sequence instead. ☐

Corollary 7.78. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a split exact sequence of C*-algebras. Then the index map and the exponential map are trivial.

Corollary 7.79. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a split exact sequence of C*-algebras. Then $\mathrm{K}_0(B) = \mathrm{K}_0(A)\oplus\mathrm{K}_0(C)$ and $\mathrm{K}_1(B) = \mathrm{K}_1(A)\oplus\mathrm{K}_1(C)$.

Remark 7.80. Fun fact; K-groups can be used to determine whether an exact sequence of C*-algebras is split. In particular, for a split exact sequence, when the middle term has trivial K-groups, the left-hand and right-hand terms must have trivial K-groups as well! For instance, the exact sequence $0 \to \mathbb{K} \to \mathbb{B} \to \mathbb{B}/\mathbb{K} \to 0$ cannot be split. Similarly, the exact sequence $0 \to SA \to CA \xrightarrow{\mathrm{ev}_1} A \to 0$ cannot be split whenever $\mathrm{K}_0(A) \neq 0$ and/or $\mathrm{K}_1(A) \neq 0$!

7.5 Thom isomorphism and Pimsner-Voiculescu

Theorem 7.81 (Thom isomorphism)**.** Let A be a C*-algebra. Let $\mathbb{R} \xrightarrow{\alpha} \mathrm{Aut}(A)$ be a continuous group homomorphism. Then $\mathrm{K}_0(A \rtimes_\alpha \mathbb{R}) \cong \mathrm{K}_1(A)$ and $\mathrm{K}_1(A \rtimes_\alpha \mathbb{R}) \cong \mathrm{K}_0(A)$.

Proof. See [6, §II]. ☐

Remark 7.82. This Thom isomorphism might be viewed as a generalisation of Theorem 7.66. The suspension can be viewed as a trivial action of $\mathbb{R}$.

Theorem 7.83. Let A be a C*-algebra. Let $A \xrightarrow{\alpha} A$ be a $*$-isomorphism. Then $\mathrm{K}_0(A \rtimes_\alpha \mathbb{Z}) \cong \mathrm{K}_1(T_\alpha)$ and $\mathrm{K}_1(A \rtimes_\alpha \mathbb{Z}) \cong \mathrm{K}_0(T_\alpha)$.

Proof. We will work through the strategy for solving [34, Exercise 9K]. Define on T_α the $\mathbb{R}$-action $\mathbb{R} \ni \lambda \xrightarrow{\tau} (T_\alpha \ni f \to \tau(\lambda)(f)(t))$ where $\tau(\lambda)(f)(t) = \alpha^n(f(t+\lambda-n))$, whenever $t+\lambda \in [n, n+1]$ for $n \in \mathbb{Z}$ with $t \in [0,1]$. Now by Takai Duality ([2, Theorem 10.1.2]) there is a $*$-isomorphism $T_\alpha \rtimes_\tau \mathbb{R} \cong (A \rtimes_\alpha \mathbb{Z}) \otimes \mathbb{K}(\mathcal{L}^2(\mathbb{T}))$. As $\mathcal{L}^2(\mathbb{T})$ is separable, $\mathbb{K}(\mathcal{L}^2(\mathbb{T})) \cong \mathbb{K}$. Applying the Thom isomorphism, the isomorphism induced by Takai duality and stability, we find that $\mathrm{K}_0(T_\alpha) \cong \mathrm{K}_1(T_\alpha \rtimes_\tau \mathbb{R}) \cong \mathrm{K}_1((A \rtimes_\alpha \mathbb{Z}) \otimes \mathbb{K}(\mathcal{L}^2(\mathbb{T}))) \cong \mathrm{K}_1(A \rtimes_\alpha \mathbb{Z})$. Similarly, $\mathrm{K}_1(T_\alpha) \cong \mathrm{K}_0(T_\alpha \rtimes_\tau \mathbb{R}) \cong \mathrm{K}_0((A \rtimes_\alpha \mathbb{Z}) \otimes \mathbb{K}(\mathcal{L}^2(\mathbb{T}))) \cong \mathrm{K}_0(A \rtimes_\alpha \mathbb{Z})$. ☐

Remark 7.84. The K-groups do not depend on the $\mathbb{Z}$-action. But the morphisms of the six term sequence do depend on the group action. See the following example.

Example 7.85. Let A be a C*-algebra. Let $A \xrightarrow{\alpha} A$ be a $*$-isomorphism. Then the exact sequence of Lemma 6.59 induces the following six sequence:

$$\begin{tikzcd}
\mathrm{K}_0(SA) \arrow[r] & \mathrm{K}_0(T_\alpha) \arrow[r, "\mathrm{K}_0(\mathrm{ev}_0)"] & \mathrm{K}_0(A) \arrow[d] \\
\mathrm{K}_1(A) \arrow[u] & \mathrm{K}_1(T_\alpha) \arrow[l, "\mathrm{K}_1(\mathrm{ev}_0)"'] & \mathrm{K}_1(SA). \arrow[l]
\end{tikzcd}$$

As $\mathrm{K}_0(SA) = \mathrm{K}_1(A)$ and $\mathrm{K}_1(SA) = \mathrm{K}_0(A)$, the following six term sequence is obtained:

$$\begin{tikzcd}
\mathrm{K}_1(A) \arrow[r] & \mathrm{K}_0(T_\alpha) \arrow[r, "\mathrm{K}_0(\mathrm{ev}_0)"] & \mathrm{K}_0(A) \arrow[d] \\
\mathrm{K}_1(A) \arrow[u] & \mathrm{K}_1(T_\alpha) \arrow[l, "\mathrm{K}_1(\mathrm{ev}_0)"'] & \mathrm{K}_0(A). \arrow[l]
\end{tikzcd}$$

Under the isomorphism $T_\alpha \rtimes_\tau \mathbb{R} \cong (A \rtimes_\alpha \mathbb{Z}) \otimes \mathbb{K}(\mathcal{L}^2(\mathbb{T}))$, the six term sequence becomes:

$$\begin{tikzcd}
\mathrm{K}_1(A) \arrow[r, "i_*"] & \mathrm{K}_1(A \rtimes_\alpha \mathbb{Z}) \arrow[r, "\mathrm{K}_0(\mathrm{ev}_0)"] & \mathrm{K}_0(A) \arrow[d] \\
\mathrm{K}_1(A) \arrow[u] & \mathrm{K}_0(A \rtimes_\alpha \mathbb{Z}) \arrow[l, "\mathrm{K}_1(\mathrm{ev}_0)"'] & \mathrm{K}_0(A). \arrow[l, "i_*"']
\end{tikzcd}$$

Here $i_* : \mathrm{K}_*(A) \to \mathrm{K}_*(A \rtimes_\alpha \mathbb{Z})$ is the inclusion $A \ni a \to (a, 0) \in A \rtimes_\alpha \mathbb{Z}$. The connecting morphisms are $\mathrm{id}_{\mathrm{K}_*(A)} - \mathrm{K}_*(\alpha)$. For an explanation of how this follows from the construction of the connecting morphism as found in [34, §8.1] (Theorem 7.63), see [2, Proposition 10.4.1].

Remark 7.86. The connecting morphisms are not induced by a $*$-homomorphism as $\mathrm{id}_A - \alpha$ is not a $*$-homomorphism.

7.6 Semigroup structures on homotopy classes of $*$-homomorphisms

Definition 7.87. Let A and B be C*-algebras. Let $[A, B]$ denote the set of homotopy classes of $*$-homomorphisms from A to B. Let $[A, B]_*$ denote the pointed set, where the special point is the class of 0. Representatives of this class are called *null-homotopic*.

Theorem 7.88. Let A be a C*-algebra and let B be a stable C*-algebras. Fix a bijection between B and $M_2(B)$ (for inspiration, see Definition 13.1). Then $[A, B]_*$ is a commutative monoid when endowed with the operation $(\alpha_1 \oplus \alpha_2)(a) = \begin{pmatrix} \alpha_1(a) & 0 \\ 0 & \alpha_2(a) \end{pmatrix}$ with $\alpha_1, \alpha_2 \in [A, B]$.

Proof. A proof can be found in [26, Theorem 3.1.a]. It only shows that the null-homotopic $*$-homomorphisms form the unit element and the commutativity. For completion, we will show independence of representatives. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{Hom}_{\mathsf{C^*-alg}}(A, B)$ be such that α_1 and β_1 are homotopic (with homotopy $(\xi_t)_{t\in[0,1]}$) α_2 and β_2 are homotopic (with homotopy $(\omega_t)_{t\in[0,1]}$). Then $\alpha_1 \oplus \beta_1$ is homotopic to $\alpha_2 \oplus \beta_2$ via $\begin{pmatrix} \chi_t & 0 \\ 0 & \omega_t \end{pmatrix}$. □

Theorem 7.89. Let A and B be C*-algebras. Then $[A, SB]_*$ is a group under the loop composition: given $\alpha_1, \alpha_2 \colon A \to SB$, we define $a_1 \cdot a_2 \colon A \to C_0((0,1), B)$ as follows:

$$(\alpha_1 \cdot \alpha_2)(a)(t) = \begin{cases} \alpha_1(a)(2t) & 0 \le t \le \frac{1}{2} \\ \alpha_2(a)(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

Proof. The proof for independence of representatives of homotopy classes, the associativity and the fact that the null-homotopic $*$-homomorphisms form a unit is similar to that of fundamental classes (see [28, Lemma 5.1.10, 5.1.11 & 5.1.12]).

The inverse of $A \xrightarrow{\alpha} C_0((0,1), B)$ is given by $A \xrightarrow{\tilde{\alpha}} C_0((0,1), B)$ where $\tilde{\alpha}(a)(t) = \alpha(a)(1-t)$. An explicit homotopy from $\alpha \cdot \tilde{\alpha}$ to 0 can be found in [26, Theorem 3.1b]. The homotopy from $\tilde{\alpha} \cdot \alpha$ to 0 is constructed in a similar way. □

Theorem 7.90. Let A be a C*-algebra and let B be a stable C*-algebra. Then the commutative monoid operation on $[A, SB]_*$ from Theorem 7.88 and the loop composition on $[A, SB]_*$ from Theorem 7.89 coincide.

Proof. In both Theorem 7.88 and Theorem 7.89 the null-homotopic $*$-homomorphisms form the unit. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [A, SB]$. Notice the following:

$$(\alpha_1 \cdot \alpha_2) \oplus (\beta_1 \cdot \beta_2) = \begin{pmatrix} \alpha_1 \cdot \alpha_2 & 0 \\ 0 & \beta_1 \cdot \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix} = (\alpha_1 \oplus \beta_1) \cdot (\alpha_2 \oplus \beta_2).$$

Now for $\alpha, \beta \in [A, SB]$ follows:

$$\beta \oplus \alpha = \alpha \oplus \beta = (0 \cdot \alpha) \oplus (\beta \cdot 0) = (0 \oplus \beta) \cdot (\alpha \oplus 0) = \beta \cdot \alpha.$$ □

Remark 7.91. Theorem 7.90 is a weaker version of the *Eckmann-Hilton argument*.

Now that we know a few properties of $[A,B]_*$ for special choices of C*-algebras, we would like to calculate what the group $[A,B]_*$ is under the commutative monoid operation of Theorem 7.88 and/or the loop composition of Theorem 7.89. We no longer need to emphasise the special element of $[A,B]$. In a monoid, the unit is automatically the special element.
In the next pages we will show a connection between homotopy classes of $*$-homomorphisms with the K-groups!

Definition 7.92. Let A and B be unital C*-algebras. Then $[A,B]_+$ defines the set of homotopy classes of unital $*$-homomorphisms (via unital homotopies).

Lemma 7.93. Let A and B be C*-algebras. Then $[A,B]$ is isomorphic to $[\tilde{A},\tilde{B}]_+$ as sets.

Proof. Follows immediately from the construction of the unitisation of C*-algebras. □

Theorem 7.94. Let A be a C*-algebra. Then $[C_0(\mathbb{R}),\mathbb{K}\otimes A] \cong \mathrm{K}_1(A)$ under the commutative monoid operation of Theorem 7.88.

Proof. We will give a proof for when A is unital. How the proof must be generalised for when A is non-unital, can be found in [26, Theorem 4.1]. It involves topological properties of the topological groups $\mathcal{U}^+_\infty(\mathbb{K}\otimes\tilde{A})$ and its subgroups and quotients.
By Lemma 7.93, $[C_0(\mathbb{R}),\mathbb{K}\otimes A] = [\widetilde{C_0(\mathbb{R})},\widetilde{\mathbb{K}\otimes A}]_+$. By Theorem 5.35, $\widetilde{C_0(\mathbb{R})} = C(\mathbb{T})$ as $\mathbb{T}$ is the one-point compactification of $\mathbb{R}$. Under Gelfand duality $C(\mathbb{T})$ corresponds to the C*-algebra generated by one unitary. See [21, Theorem 2.1.13]. A unital $*$-homomorphism from $C(\mathbb{T})$ to $\widetilde{\mathbb{K}\otimes A}$ is a choice of a unitary in $\widetilde{\mathbb{K}\otimes A}$. Two $*$-homomorphisms from $C(\mathbb{T})$ to $\widetilde{\mathbb{K}\otimes A}$ are homotopic precisely if the unitaries in $\widetilde{\mathbb{K}\otimes A}$ that define the $*$-homomorphisms are homotopic. So there is a bijection between classes in $[C_0(\mathbb{R}),\mathbb{K}\otimes A]$ and elements in $\mathcal{U}^+_\infty(\mathbb{K}\otimes A)/\mathcal{U}^+_\infty(\mathbb{K}\otimes A)_0$. By Theorem 7.43 $\mathcal{U}^+_\infty(\mathbb{K}\otimes A)/\mathcal{U}^+_\infty(\mathbb{K}\otimes A)_0$ is isomorphic to $\mathrm{GL}^+_\infty(\mathbb{K}\otimes A)/\mathrm{GL}^+_\infty(\mathbb{K}\otimes A)_0$ (Recall the definition of K_1 in Definition 7.46). The bijection between $[C_0(\mathbb{R}),\mathbb{K}\otimes A]$ and $\mathrm{K}_1(\mathbb{K}\otimes A)$ is a group isomorphism. The commutative monoid operation of Theorem 7.88 agrees with the group action on $\mathrm{GL}^+_\infty(\mathbb{K}\otimes A)/\mathrm{GL}^+_\infty(\mathbb{K}\otimes A)_0$ (see Definition 7.40). □

Theorem 7.95. Let A be a C*-algebra. There exist a C*-algebra F and a $*$-homomorphism $F \xrightarrow{\phi} S(\mathbb{K}\otimes A)$ such that $\mathrm{K}_0(F)$ and $\mathrm{K}_1(F)$ are free and the group homomorphisms $\mathrm{K}_0(F) \xrightarrow{\mathrm{K}_0(\phi)} \mathrm{K}_0(S(\mathbb{K}\otimes A))$ and $\mathrm{K}_1(F) \xrightarrow{\mathrm{K}_1(\phi)} \mathrm{K}_1(S(\mathbb{K}\otimes A))$ are surjective.
Whenever A is separable, a separable F can be chosen.

Proof. For $\mathbb{K}\otimes A$, there exists a group isomorphism $[C_0(\mathbb{R}),\mathbb{K}\otimes A] \cong \mathrm{K}_1(A)$ (Theorem 7.94) under the commutative monoid operator of Theorem 7.88. Now $\mathrm{K}_1(A)$ is an abelian group; in other words a $\mathbb{Z}$-module (see Remark 4.46). So there exists a basis $\mathbb{Z}^I \xrightarrow{g} \mathrm{K}_1(A)$, where I is some index set. Let $C_0(\mathbb{R}) \xrightarrow{\psi_i} \mathbb{K}\otimes A$ be a representative of the class in $[C_0(\mathbb{R}),\mathbb{K}\otimes A]$ that corresponds to the element $g(e_i) \in \mathrm{K}_1(A)$ under that group isomorphism, for $i \in I$.

Under the Gelfand duality $C_0(\mathbb{T})$ corresponds to the C*-algebra generated by one unitary. A unital $*$-homomorphism from $C(\mathbb{T})$ to $\widetilde{\mathbb{K}\otimes A}$ is a choice of a homotopy class of unitaries in $\widetilde{\mathbb{K}\otimes A}$. By Lemma 7.93, $[C(\mathbb{T}),\widetilde{\mathbb{K}\otimes A}]_+ = [C_0(R),\mathbb{K}\otimes A]$. The coproduct $\bigsqcup_I \mathbb{R}$ becomes the product $C_0(\bigsqcup_I \mathbb{R})$

under the Gelfand duality. Collectively, the $*$-homomorphisms $C_0(\mathbb{R}) \xrightarrow{\psi_i} \mathbb{K} \otimes A$ (with $i \in I$) induce the $*$-homomorphism $C_0(\bigsqcup_I \mathbb{R}) \xrightarrow{\psi} \mathbb{K} \otimes A$. Now ψ makes this diagram commute for $i \in I$:

$$\begin{tikzcd} C_0(\bigsqcup_I \mathbb{R}) \arrow[r, "C_0(\mathrm{incl}_i)"] \arrow[dr, "\psi"'] & C_0(\mathbb{R}) \arrow[d, "\psi_i"] \\ & \mathbb{K} \otimes A \end{tikzcd}$$

Notice that $\mathrm{K}_1(C_0(\bigsqcup_i \mathbb{R})) = \mathbb{Z}^I$ (because $\mathrm{K}_1(C_0(\mathbb{R})) = \mathbb{Z}$). As $\mathrm{im}\,\mathrm{K}_1(\psi_i) = (e_i)$ (as a subgroup of $\mathrm{K}_1(A)$), it follows $\mathrm{K}_1(\psi) = g$, which means that $\mathrm{K}_1(\psi)$ is surjective. Furthermore $\mathrm{K}_0(C_0(\bigsqcup_i \mathbb{R})) = 0$ (because $\mathrm{K}_0(C_0(\mathbb{R})) = 0$).

We repeat the steps for $S(\mathbb{K} \otimes A)$. There exists a basis $\mathbb{Z}^J \xrightarrow{h} \mathrm{K}_1(SA)$, where J is some index set. Let $C_0(\mathbb{R}) \xrightarrow{\chi_j} S(\mathbb{K} \otimes A)$ be a representative of the class in $[C_0(\mathbb{R}), S(\mathbb{K} \otimes A)]$ that corresponds to the element $h(e_j) \in \mathrm{K}_1(SA)$ under that group isomorphism, for $j \in J$. This induces a $*$-homomorphism $C_0(\bigsqcup_j \mathbb{R}) \xrightarrow{\chi} S(\mathbb{K} \otimes A)$ such that $\chi_i \circ C_0(\mathrm{incl}_j) = \chi$ for $j \in J$. Now $\mathrm{K}_1(C_0(\bigsqcup_j \mathbb{R})) = \mathbb{Z}^j$ and $\mathrm{K}_1(\chi) = h$, which means that $\mathrm{K}_1(\chi)$ is surjective. Similarly, $\mathrm{K}_0(C_0(\bigsqcup_j \mathbb{R})) = 0$.

Now take $F = SC_0(\bigsqcup_I \mathbb{R}) \oplus C_0(\bigsqcup_J \mathbb{R})$ and define ϕ to be this $*$-homomorphism

$$SC_0(\bigsqcup_I \mathbb{R}) \oplus C_0(\bigsqcup_J \mathbb{R}) \xrightarrow{S(\psi)\oplus\chi} S(\mathbb{K} \otimes A) \oplus S(\mathbb{K} \otimes A) \subset M_2(S(\mathbb{K} \otimes A)) \cong S(\mathbb{K} \otimes A). \qquad \square$$

8 (Co)homology theories

In §7, we introduced the functors K_0 and K_1 and showed a few properties they have. Two of the most important were half-exactness and homotopy invariance. In this section we will introduce a more general notion; the (co)homology functor. In §7 we constructed connecting maps between K_0 and K_1; the index and exponential map. In this section the more general (co)homology theory will be introduced. (Co)homology theories consist of multiple (co)homology functors with connection maps between them. We will take an axiomatic approach and show that the results in §7 were only an application of the constructions for (co)homology theories. In the last section we will dive into the direct limit behaviour of (co)homology theories.

8.1 Admissible subcategories

The goal of this subsection is to define what an admissible subcategory is. This concept is frequently used in the later subsections. A closed model category is a category with an extra structure on the morphisms. A subcategory of closed model category is admissible if it is a complete and cocomplete and preserves the closed model structure of the morphisms. A complete overview of all the properties of closed model categories would take us too far. We will refer to the definition in [24] and sketch a closed model structure for C*-algebras. We will show a few properties of closed model categories that will be important in the proofs of §8.3. For this, we will mostly follow [31]. Two examples of admissible subcategories of C*-algebras are the separable and separable nuclear C*-algebras.

Definition 8.1. A *closed model category* is a category with three classes of morphisms; fibrations, cofibrations and weak equivalences satisfying axioms M0-M6 in [24, §1, Def. 1 &§5, Def. 1].

Remark 8.2. Not every morphism in a closed model category needs to be a fibration, cofibration or weak equivalence, but any morphism can be factorised in two ways; both as a composition of a morphism that is a cofibration and a weak equivalence with a fibration and as a composition of a cofibration and a morphism that is a fibration and a weak equivalence. This is M2 in [24]. The titles fibration, cofibration and weak equivalence are not mutually exclusive. For example, identities are fibrations, cofibrations and weak equivalences.

Example 8.3. For readers with a background in (advanced) algebraic topology; the topological spaces admit a closed model structure with Serre fibrations ([12, p. 375-376]) as fibrations and weak homotopy equivalences ([12, p. 352]) as weak equivalences. See [24, p. 1.2].

Theorem 8.4. The category of C^*-algebras admits a closed model structure.

We will not fully prove this theorem. We will define cofibrations for the category of C^*-algebras. We will construct a factorisation for $*$-homomorphisms between C^*-algebras consisting of a cofibration and another $*$-homomorphism to convince the reader of the admission of a closed model structure without going too much into the details.

Definition 8.5. Let A and B be C^*-algebras. A $*$-homomorphism $A \xrightarrow{f} B$ has the *homotopy extension property* if for every C^*-algebra D and $*$-homomorphism $D \xrightarrow{g} A$ and any homotopy $D \xrightarrow{h} C([0,1],B)$ between $f \circ g$ and another $*$-homomorphism from D to B, there exists a homotopy $D \xrightarrow{H} C([0,1],A)$ between g and some other $*$-homomorphism from D tot A such that the following diagram commutes:

$$\begin{tikzcd}
& D \arrow[dl, "h"'] \arrow[d, "H"] \arrow[dr, "g"] & \\
C([0,1],B) & C([0,1],A) \arrow[l, "f\circ_"] \arrow[r, "\mathrm{ev}_0"'] & A.
\end{tikzcd}$$

Definition 8.6. A morphism in $\mathsf{C}^* - \mathsf{alg}$ is a *cofibration* if it satisfies the homotopy extension property.

Remark 8.7. A Serre fibration satisfies the homotopy extension property version of topological spaces for spheres, see [12, p. 375-376]. In Example 8.3 the Serre fibrations were fibrations. The Gelfand duality (Theorem 5.36) switches fibrations and cofibrations.

Lemma 8.8. Let A be a C^*-algebra and let $t \in [0,1]$. Then $C([0,1],A) \xrightarrow{\mathrm{ev}_t} A$ is a cofibration.

Proof. See [31, Lemma 1.3]. □

Lemma 8.9. Let A, B and C be C^*-algebras. Let $A \xrightarrow{f} C$ and $B \xrightarrow{g} C$ be $*$-homomorphisms. Let X be the pullback of f and g and let the $X \xrightarrow{\tilde{g}} A$ and $X \xrightarrow{\tilde{f}} B$ be the structure maps (of the pullback diagram). If f is cofibration, then so is $\tilde{f}$.

Proof. See [31, Proposition 1.5]. □

Theorem 8.10. Let A and B be C*-algebras. Let $A \xrightarrow{f} B$ be a $*$-homomorphism. There exists a $*$-homomorphism $A \xrightarrow{r} Z_f$ (see Definition 6.55) and a cofibration $Z_f \xrightarrow{j} B$ such that $j \circ r = f$.

Proof. The mapping cylinder Z_f is a pullback by construction (see Lemma 6.18). As ev_1 is a cofibration by Lemma 8.8, it follows from Lemma 8.9 that the structure map $Z_f \to A$ is a cofibration. Now define $A \xrightarrow{c} C([0,1], B)$ to be the $*$-homomorphism that sends $a \in A$ to the constant path $f(a) \in B$. Now $f \circ \mathrm{id}_A = \mathrm{ev}_1 \circ c$. By the limit property (see Example 2.36), there exists a unique $*$-homomorphism $A \xrightarrow{r} Z_f$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[dr, "r"] \arrow[drr, bend left, "c"] \arrow[ddr, bend right, "\mathrm{id}_A"'] & & \\
& Z_f \arrow[r] \arrow[d] & C([0,1], B) \arrow[d, "\mathrm{ev}_1"] \\
& A \arrow[r, "f"'] & B.
\end{tikzcd}$$

Define j to be the composition $Z_f \to C([0,1], B) \xrightarrow{\mathrm{ev}_0} B$. Then j is a cofibration by [31, Proposition 1.8]. Now f coincides with $\mathrm{ev}_1 \circ c$, which coincides with $\mathrm{ev}_0 \circ c$ as c was chosen to map to the constant paths. Hence f coincides with the composition $A \xrightarrow{r} Z_f \to C([0,1], B) \xrightarrow{\mathrm{ev}_0} B$. $\square$

Remark 8.11. Recall that r in Theorem 8.9 is the deformation retract from Lemma 6.57. So r is the inverse equivalence (of C*-algebras) of a cofibration. Note id_A is an isomorphism and therefore a weak equivalence.

Proof Theorem 8.4 (sketch). If the cofibrations defined in Definition 8.6 are the cofibrations of C*-algebras in a closed model structure as defined in Definition 8.1, then Theorem 8.10 proves that in the C*-algebras every $*$-homomorphisms can be factorised into a cofibration and an arbitrary $*$-homomorphism. If the $*$-homomorphism $A \xrightarrow{r} Z_f \to C([0,1], B)$ in the proof of Theorem 8.10 is both a fibration and weak equivalence (both have not been given for C*-algebras), then this would prove the factorisation in M2 in [24, §1, Def. 1]. $\square$

Definition 8.12. An *admissible* subcategory of the category of C*-algebras is a category that is complete and cocomplete in the sense of Definition 2.53 and inherits the closed model structure, i.e. any morphism in the subcategory that is a fibration, resp. cofibration, resp. weak equivalence in $\mathsf{C}^* - \mathsf{alg}$ (with its closed model structure) is a fibration, resp. cofibration, resp. weak equivalence.

Example 8.13. The category of separable C*-algebras is an admissible subcategory of $\mathsf{C}^* - \mathsf{alg}$. This is claimed in [31, p. 402].

Example 8.14. The category of separable nuclear C*-algebras is an admissible subcategory of $\mathsf{C}^* - \mathsf{alg}$. This is claimed in [31, p. 402].

Remark 8.15. We have not proven that the categories of separable and separable nuclear C*-algebras are (co)complete. From Lemmas 6.60 and 6.62, we can check that the mapping cone is an object via the exact sequence of Lemma 6.53 and the mapping cylinder via the exact sequence of Lemma 6.56. We can check in a similar way that the factorisation from Theorem 8.10 is preserved.

As $[0,1]$ is second countable, it follows that $C([0,1])$ is separable from [34, §1.11] (and nuclear by Theorem 5.46). So tensor products with $C([0,1])$ are objects in the categories of separable and separable nuclear C*-algebras. Hence the pullback diagram from the mapping cylinder can be used to get the $*$-homomorphism r from Theorem 8.10.

In §8.3 we will show properties of homotopy invariant half-exact functors from an admissible subcategory of the C*-algebras to the abelian groups. The proofs in §8.3 are inspired by the proofs in [34, §11]. In [34], only functors from the C*-algebras to the abelian groups §11 are covered, no subcategories. Nevertheless, the statements in [34, §11] also hold for admissible subcategories of the C*-algebras. This is, in short, because admissible subcategories are closed under mapping cones (see Definition 6.51) and the factorisation of Theorem 8.10 is preserved in an admissible subcategory. We will see this in Theorem 8.30. The proof relies on the mapping cone. We will introduce a bifunctor $\mathrm{KK}\colon \mathsf{C}^* - \mathsf{alg}^{\mathrm{opp}} \times \mathsf{C}^* - \mathsf{alg} \to \mathsf{Ab}$ (see Lemma 11.22). In general, the functors $\mathrm{KK}(A, \_)$ and $\mathrm{KK}(\_, A)$ are not half-exact. We will see in §14 that there is a subcategory for which the restriction is half-exact.

8.2 (Co)homology functors and theories

In this subsection we will only give definitions and save the results for the next subsection. The reader may replace the words *admissible subcategory of $\mathsf{C}^*-\mathsf{alg}$* by *the category of separable (nuclear) C^*-algebras* throughout this section. See Examples 8.13 and 8.14.

Definition 8.16. A *homology functor* is a half-exact and homotopy invariant functor from an admissible subcategory of $\mathsf{C}^* - \mathsf{alg}$ to Ab.

Definition 8.17. A *cohomology functor* is a half-exact and homotopy invariant contravariant functor from an admissible subcategory of $\mathsf{C}^* - \mathsf{alg}$ to Ab.

Definition 8.18. Let X be an admissible subcategory of $\mathsf{C}^*-\mathsf{alg}$. A *homology theory* is a collection of homology functors $\{H_n\colon \mathsf{X} \to \mathsf{Ab}\}_{n\in I}$, with $I = \mathbb{N}_0$ or $I = \mathbb{Z}$, such that for every exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ in X there exist connecting maps $\delta_n\colon H_n(C) \to H_{n-1}(A)$ (if $I = \mathbb{N}_0$, then $n \neq 0$) such that the following sequence is naturally exact:

$$\begin{aligned} \cdots \longrightarrow H_{n+1}(A) &\xrightarrow{H_{n+1}(\alpha)} H_{n+1}(B) \xrightarrow{H_{n+1}(\beta)} H_{n+1}(C) \xrightarrow{\delta_{n+1}} \\ \longrightarrow H_n(A) &\xrightarrow{H_n(\alpha)} H_n(B) \xrightarrow{H_n(\beta)} H_n(C) \xrightarrow{\delta_n} \\ \longrightarrow H_{n-1}(A) &\xrightarrow{H_{n-1}(\alpha)} H_{n-1}(B) \xrightarrow{H_{n-1}(\beta)} H_{n-1}(C) \xrightarrow{\delta_{n-1}} \\ \longrightarrow \cdots & \end{aligned},$$

in the sense that for every commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha_1"] \arrow[d, "\phi"'] & B \arrow[r, "\beta_1"] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d, "\chi"'] & 0 \\ 0 \arrow[r] & D \arrow[r, "\alpha_2"] & E \arrow[r, "\beta_2"] & F \arrow[r] & 0, \end{tikzcd}$$

the following diagram commutes:

$$
\begin{tikzcd}
H_{n+1}(A) \arrow[r, "H_{n+1}(\alpha_1)"] \arrow[d, "H_{n+1}(\phi)"'] & H_{n+1}(B) \arrow[r, "H_{n+1}(\beta_1)"] \arrow[d, "H_{n+1}(\psi)"'] & H_{n+1}(C) \arrow[r, "\delta"] \arrow[d, "H_{n+1}(\chi)"'] & H_n(A) \arrow[r, "H_n(\alpha_1)"] \arrow[d, "H_n(\phi)"'] & H_n(B) \arrow[r, "H_n(\beta_1)"] \arrow[d, "H_n(\psi)"'] & H_n(C) \arrow[d, "H_n(\chi)"'] \\
H_{n+1}(D) \arrow[r, "H_{n+1}(\alpha_2)"] & H_{n+1}(E) \arrow[r, "H_{n+1}(\beta_2)"] & H_{n+1}(F) \arrow[r, "\delta'"] & H_n(D) \arrow[r, "H_n(\alpha_2)"] & H_n(E) \arrow[r, "H_n(\beta_2)"] & H_n(F).
\end{tikzcd}
$$

Definition 8.19. Let X be an admissible subcategory of $\mathsf{C}^* - \mathsf{alg}$. A *cohomology theory* is a collection of cohomology functors $\{H_n: \mathsf{X}^{\text{opp}} \to \mathsf{Ab}\}_{n \in I}$, with $I = \mathbb{N}_0$ or $I = \mathbb{Z}$, such that for every exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ in X there exist connecting maps $\delta_n: H_n(C) \to H_{n+1}(A)$ such that the following sequence is naturally exact:

$$
\begin{tikzcd}
\cdots \arrow[r] & H_{n-1}(C) \arrow[r, "H_{n-1}(\beta)"] & H_{n-1}(B) \arrow[r, "H_{n-1}(\alpha)"] & H_{n+1}(A) \arrow[dll, "\delta_{n-1}"] \\
& H_n(C) \arrow[r, "H_n(\beta)"] & H_n(B) \arrow[r, "H_n(\alpha)"] & H_n(A) \arrow[dll, "\delta_n"] \\
& H_{n+1}(C) \arrow[r, "H_{n+1}(\beta)"] & H_{n+1}(B) \arrow[r, "H_{n+1}(\alpha)"] & H_{n+1}(A) \arrow[dll, "\delta_{n+1}"] \\
& \cdots & &
\end{tikzcd},
$$

so that for every commutative diagram with exact rows:

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha_1"] \arrow[d, "\phi"'] & B \arrow[r, "\beta_1"] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d, "\chi"'] & 0 \\
0 \arrow[r] & D \arrow[r, "\alpha_2"] & E \arrow[r, "\beta_2"] & F \arrow[r] & 0,
\end{tikzcd}
$$

the following diagram commutes:

$$
\begin{tikzcd}
H_n(F) \arrow[r, "H_n(\beta_2)"] \arrow[d, "H_n(\chi)"'] & H_n(E) \arrow[r, "H_n(\alpha_2)"] \arrow[d, "H_n(\psi)"'] & H_n(D) \arrow[r, "\delta"] \arrow[d, "H_n(\phi)"'] & H_{n+1}(F) \arrow[r, "H_{n+1}(\beta_2)"] \arrow[d, "H_{n+1}(\chi)"'] & H_{n+1}(E) \arrow[r, "H_{n+1}(\alpha_2)"] \arrow[d, "H_{n+1}(\psi)"'] & H_{n+1}(D) \arrow[d, "H_{n+1}(\phi)"'] \\
H_n(C) \arrow[r, "H_n(\beta_1)"] & H_n(B) \arrow[r, "H_n(\alpha_1)"] & H_n(A) \arrow[r, "\delta'"] & H_{n+1}(C) \arrow[r, "H_{n+1}(\beta_1)"] & H_{n+1}(B) \arrow[r, "H_{n+1}(\alpha_1)"] & H_{n+1}(A).
\end{tikzcd}
$$

Remark 8.20. A homology theory has degree decreasing connecting maps and a cohomology theory has a degree increasing connecting maps. This is in contrast to [34, Definition 11.1.4].

To move one (co)homology theory indexed by I to another, we obviously need morphisms of functors indexed by I. The connecting morphisms are not induced by $*$-homomorphisms and they do not automatically commute with any choice of morphism of functors. We have to add extra structure to them. We will coin the term *morphism* of (co)homologies for morphisms of functors with this extra structure. The following two definitions will state this concept precisely.

Definition 8.21. Let X be an admissible subcategory of $\mathsf{C}^* - \mathsf{alg}$. Let $\{H_n: \mathsf{X} \to \mathsf{Ab}\}_{n \in I}$ and $\{J_n: \mathsf{X} \to \mathsf{Ab}\}_{n \in I}$ be homology theories. A *morphism of homology theories* from $\{H_n: \mathsf{X} \to \mathsf{Ab}\}_{n \in I}$ to

$\{J_n{:}\mathsf{X} \to \mathsf{Ab}\}_{n\in I}$ is a collection of morphisms of functors $\{H_n \overset{\eta^n}{\Longrightarrow} J_n\}_{n\in I}$ such that for every exact sequence $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ in X the following diagram commutes:

$$\begin{array}{ccccccccccc}
H_{n+1}(A) & \xrightarrow{H_{n+1}(\alpha)} & H_{n+1}(B) & \xrightarrow{H_{n+1}(\beta)} & H_{n+1}(C) & \xrightarrow{\delta} & H_n(A) & \xrightarrow{H_n(\alpha)} & H_n(B) & \xrightarrow{H_n(\beta)} & H_n(C) \\
\eta_A^{n+1}\big\downarrow & & \eta_B^{n+1}\big\downarrow & & \eta_C^{n+1}\big\downarrow & & \eta_A^{n}\big\downarrow & & \eta_B^{n}\big\downarrow & & \eta_C^{n}\big\downarrow \\
J_{n+1}(A) & \xrightarrow{J_{n+1}(\alpha)} & J_{n+1}(B) & \xrightarrow{J_{n+1}(\beta)} & J_{n+1}(C) & \xrightarrow{\delta'} & J_n(A) & \xrightarrow{J_n(\alpha)} & J_n(B) & \xrightarrow{J_n(\beta)} & J_n(C).
\end{array}$$

Definition 8.22. Let X be an admissible subcategory of $\mathsf{C}^* - \mathsf{alg}$. Let $\{H_n{:}\mathsf{X}^{\text{opp}} \to \mathsf{Ab}\}_{n\in I}$ and $\{J_n{:}\mathsf{X}^{\text{opp}} \to \mathsf{Ab}\}_{n\in I}$ be cohomology theories. A *morphism of cohomology theories* from $\{H_n{:}\mathsf{X}^{\text{opp}} \to \mathsf{Ab}\}_{n\in I}$ to $\{J_n{:}\mathsf{X}^{\text{opp}} \to \mathsf{Ab}\}_{n\in I}$ is a collection of morphisms of functors $\{H_n \overset{\eta^n}{\Longrightarrow} J_n\}_{n\in I}$ such that for every exact sequence $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ in X the following diagram commutes:

$$\begin{array}{ccccccccccc}
H_n(C) & \xrightarrow{H_n(\beta)} & H_n(B) & \xrightarrow{H_n(\alpha)} & H_n(A) & \xrightarrow{\delta} & H_{n+1}(C) & \xrightarrow{H_{n+1}(\beta)} & H_{n+1}(B) & \xrightarrow{H_{n+1}(\alpha)} & H_{n+1}(A) \\
\eta_C^{n}\big\downarrow & & \eta_B^{n}\big\downarrow & & \eta_A^{n}\big\downarrow & & \eta_C^{n+1}\big\downarrow & & \eta_B^{n+1}\big\downarrow & & \eta_A^{n+1}\big\downarrow \\
J_n(C) & \xrightarrow{J_n(\beta)} & J_n(B) & \xrightarrow{J_n(\alpha)} & J_n(A) & \xrightarrow{\delta'} & J_{n+1}(C) & \xrightarrow{J_{n+1}(\beta)} & J_{n+1}(B) & \xrightarrow{J_{n+1}(\alpha)} & J_n(A).
\end{array}$$

8.3 Properties: generalising K_0 and K_1

Proposition 8.23. Let X be an admissible subcategory of $\mathsf{C}^* - \mathsf{alg}$. Let $\{H_n{:}\mathsf{X} \to \mathsf{Ab}\}_{n\in I}$ be a homology theory. Then H_n is split exact for every $n \in I$, unless it is the bottom element (if $I = \mathbb{N}_0$).

Proof. See Lemma 7.77. The proof fails for the bottom functor, because there is no connecting map to a preceding functor. The left-hand induced morphism need not be injective. □

Remark 8.24. Bottom becomes top in cohomology. So the proof fails for the top functor. Due to the non-existence of connecting map to a succeeding functor, the right-hand induced morphism need not be surjective.

Corollary 8.25. Let X be an admissible subcategory of $\mathsf{C}^* - \mathsf{alg}$. Let $\{H_n{:}\mathsf{X} \to \mathsf{Ab}\}_{n\in I}$ be a homology theory. Let $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ be a split exact sequence in X. The connecting maps $\delta_n{:}\ H_n(C) \to H_{n-1}(A)$ are trivial.

Corollary 8.26. Let X be an admissible subcategory of $\mathsf{C}^* - \mathsf{alg}$. Let $\{H_n{:}\mathsf{X} \to \mathsf{Ab}\}_{n\in I}$ be a homology theory. Let $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ be a split exact sequence in X. Then $H_n(B) = H_n(A) \oplus H_n(C)$ for every $n \in I$, unless for the bottom element in I.

Remark 8.27. Corollary 7.78, resp. Corollary 7.79 is a generalisation of Corollary 8.25 resp. Corollary 8.26. Because of Bott periodicity, K_0 and K_1 in alternating order form a homology theory with index $\mathbb{Z}$. There is no complicating bottom element.

Proposition 8.28. Let X be an admissible subcategory of $\mathsf{C}^* - \mathsf{alg}$. Let $\{H_n{:}\mathsf{X} \to \mathsf{Ab}\}_{n\in I}$ be a homology theory. Let $A \in \mathrm{Ob}(X)$. Then $H_n(A)$ is naturally isomorphic to $H_{n-1}(SA)$ for $n \in I$.

Proof. Similar to the proof of Lemma 7.65. □

Definition 8.29. The isomorphism established in Proposition 8.28 will be referred to as the *suspension isomorphism*.

Theorem 8.30. Let $H\colon \mathsf{C}^* - \mathsf{alg} \to \mathsf{Ab}$ be a homology functor. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence in $\mathsf{C}^* - \mathsf{alg}$. There exists a connecting map $H(SC) \xrightarrow{\delta} H(A)$ such that the following sequence is exact:

$$H(SA) \xrightarrow{HS(\alpha)} H(SB) \xrightarrow{HS(\beta)} H(SC) \xrightarrow{\delta} H(A) \xrightarrow{H(\alpha)} H(B) \xrightarrow{H(\beta)} H(C).$$

Proof. The exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ induces the exact sequences from Lemma 6.53 and Lemma 6.54:

$$0 \to SC \xrightarrow{(0,\_)} C_\beta \to B \to 0$$
$$0 \to A \xrightarrow{(\alpha(\_),0)} C_\beta \to CC \to 0.$$

Notice α is the composition $A \xrightarrow{(\alpha(\_),0)} C_\beta \to B$. Define δ so that the following diagram commutes:

$$\begin{tikzcd}
H(SB) \arrow[r, "HS(\beta)"] & H(SC) \arrow[rr, "\delta"] \arrow[dr, "H((0,_))"'] & & H(A) \arrow[r, "H(\alpha)"] & H(B). \\
& & H(C_\beta) \arrow[ur, "H((\alpha(_),0))^{-1}"] \arrow[urr] & &
\end{tikzcd}$$

As CC is contractible, $H(A) \xrightarrow{H((\alpha(\_),0))} C_\beta$ is an isomorphism. See [34, Proposition 11.1.10]. So $H((\alpha(\_),0))^{-1}$ is well-defined. $H(\alpha)$ is the composition $H(A) \xrightarrow{H((\alpha(\_),0))} H(C_\beta) \to H(B)$. So $\ker H(\alpha) = H((\alpha(\_),0))^{-1}(\ker(H(C_\beta) \to H(B)) = H((\alpha(\_),0))^{-1}(\operatorname{im} H((0,\_))) = \operatorname{im}\delta$ by the half-exactness of H. It remains to prove $\ker\delta = \operatorname{im} HS(\beta)$. Define

$$E := \{(f,g) \in C([0,1],B) \oplus C([0,1],C) | f(0) = 0, \beta(f(1)) = g(0), g(1) = 0\}.$$

Define the $*$-homomorphisms $SC \xrightarrow{\phi} E$ to be $\phi(g) = (0, t \to g(t-1))$ and $E \xrightarrow{\psi} C_\beta$ to be $\psi(f,g) = (\mathrm{ev}_1(f), t \to g(2-t))$. E appears in the following exact sequences:

$$0 \to SC \xrightarrow{\phi} E \to CB \to 0$$
$$0 \to SB \xrightarrow{(\_,0)} E \xrightarrow{\psi} C_\beta \to 0.$$

As CB is contractible, $H(SC) \xrightarrow{H(\phi)} H(E)$ is an isomorphism. Notice $(\phi \circ S\beta)(f) = \phi(\beta \circ f)$ for $f \in SB$. Now $SB \xrightarrow{(\_,0)} E$ is homotopic to $(\phi \circ S\beta)$ and $\psi \circ \phi$ is homotopic to $SC \xrightarrow{(0,\_)} C_\beta$ (not equal, notice the reversal in ψ). So $H(\psi) \circ H(\phi)$ coincides with $H(SC) \xrightarrow{H((0,\_))} H(C_\beta)$ by homotopy invariance. All the claims about exactness and homotopy are proven in [34, Proposition 11.1.12]. Now $\ker\delta = \ker(H(SC) \xrightarrow{H((0,\_))} H(C_\beta))$ as $H((\alpha(\_),0))^{-1}$ is an isomorphism. This fact and the homotopy invariance of H give the following equalities:

$$\ker\delta = \ker(H(SC) \xrightarrow{H((0,\text{-}))} H(C_\beta)) = \ker(H(\psi)\circ H(\phi)) = H(\phi)^{-1}(\ker H(\psi)) =$$
$$H(\phi)^{-1}(\operatorname{im} H(SB) \xrightarrow{H((\text{-},0))} H(E)) = H(\phi)^{-1}(\operatorname{im}(H(\phi)\circ HS(\beta))) = \operatorname{im} HS(\beta).$$

□

Theorem 8.31. The proof of Theorem 8.30 is valid for any admissible subcategory.

Proof. For any C*-algebra A in an admissible category, $C([0,1],A)$ has to be in the admissible category as well, see [31, Remark 2.12]. Recall that an admissible category is complete. The kernel of the $*$-homomorphism $C([0,1],A) \xrightarrow{\mathrm{ev}_0} A$ is CA. The kernel of the $*$-homomorphism $CA \xrightarrow{\mathrm{ev}_1} A$ is SA, see Lemma 6.50. So CA and SA are included in an admissible category, whenever A is included. To see that mapping cones are included, recall that it is a pullback, see Proposition 6.52. □

Lemma 8.32. Let X be an admissible subcategory of $\mathsf{C^*-alg}$. The connecting map constructed in Theorem 8.30 is natural in the sense that for a commutative diagram in X with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\alpha_1"] \arrow[d, "\phi"'] & B \arrow[r, "\beta_1"] \arrow[d, "\psi"'] & C \arrow[r] \arrow[d, "\chi"'] & 0 \\
0 \arrow[r] & D \arrow[r, "\alpha_2"] & E \arrow[r, "\beta_2"] & F \arrow[r] & 0,
\end{tikzcd}$$

the following diagram commutes:

$$\begin{tikzcd}
H(SA) \arrow[r, "HS(\alpha_1)"] \arrow[d, "HS(\phi)"'] & H(SB) \arrow[r, "HS(\beta_1)"] \arrow[d, "HS(\psi)"'] & H(SC) \arrow[r, "\delta"] \arrow[d, "HS(\chi)"'] & H(A) \arrow[r, "H(\alpha_1)"] \arrow[d, "H(\phi)"'] & H(B) \arrow[r, "H(\beta_1)"] \arrow[d, "H(\psi)"'] & H(C) \arrow[d, "H(\chi)"'] \\
H(SD) \arrow[r, "HS(\alpha_2)"] & H(SE) \arrow[r, "HS(\beta_2)"] & H(SF) \arrow[r, "\delta'"] & H(D) \arrow[r, "H(\alpha_2)"] & H(E) \arrow[r, "H(\beta_2)"] & H(F).
\end{tikzcd}$$

Proof. The exact sequences from Lemma 6.53 and Lemma 6.54 are natural. The following diagrams are well-defined and commute:

$$\begin{tikzcd}
0 \arrow[r] & SC \arrow[r, "(0,\text{-})"] \arrow[d, "S(\chi)"'] & C_{\beta_1} \arrow[r] \arrow[d, "(\psi,\chi\circ\text{-})"'] & B \arrow[r] \arrow[d, "\psi"'] & 0 \\
0 \arrow[r] & SF \arrow[r, "(0,\text{-})"] & C_{\beta_2} \arrow[r] & E \arrow[r] & 0,
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "(\alpha_1(\text{-}),0)"] \arrow[d, "\phi"'] & C_{\beta_1} \arrow[r] \arrow[d, "(\psi,\chi\circ\text{-})"'] & CC \arrow[r] \arrow[d, "\chi\circ\text{-}"'] & 0 \\
0 \arrow[r] & D \arrow[r, "(\alpha_2(\text{-}),0)"] & C_{\beta_2} \arrow[r] & CF \arrow[r] & 0.
\end{tikzcd}$$

We conclude that the following diagram commutes:

$$\begin{tikzcd}
 & & H(C_{\beta_1}) \arrow[rd, "H((\alpha_1(_),0))^{-1}"] \arrow[rrd] \arrow[ddd, bend right=80, "H((\psi,\chi\circ_))"'] & & \\
H(SB) \arrow[r, "HS(\beta_1)"'] \arrow[d, "HS(\psi)"] & H(SC) \arrow[ru, "H((0,_))"] \arrow[rr, "\delta"'] \arrow[d, "HS(\chi)"] & & H(A) \arrow[r, "H(\alpha_1)"'] \arrow[d, "H(\phi)"] & H(B) \arrow[d, "H(\psi)"] \\
H(SE) \arrow[r, "HS(\beta_2)"] & H(SF) \arrow[rr, "\delta'"] \arrow[rd, "H((0,_))"'] & & H(D) \arrow[r, "H(\alpha_2)"] & H(E). \\
 & & H(C_{\beta_2}) \arrow[ru, "H((\alpha_2(_),0))^{-1}"] \arrow[rru] & &
\end{tikzcd}$$

□

Corollary 8.33. Let H be a (co)homology functor, then $\{HS^n\}_{n=0}^{\infty}$ is a (co)homology theory.

Example 8.34. The connecting morphism between K_1 and K_0 is constructed in the same way as in Theorem 8.30. See [34, §8.1] and Theorem 7.63.

Theorem 8.35. Let X be an admissible subcategory of $\mathsf{C}^*-\mathsf{alg}$. Let $H{:}\,\mathsf{X}^{\mathrm{opp}} \to \mathsf{Ab}$ be a cohomology functor. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence in X. There exists a connecting map $H(SC) \xrightarrow{\delta} H(A)$ such that the following sequence is exact:

$$H(C) \xrightarrow{H(\beta)} H(B) \xrightarrow{H(\alpha)} H(A) \xrightarrow{\delta} H(SC) \xrightarrow{HS(\beta)} H(SB) \xrightarrow{HS(\alpha)} H(SA).$$

Proof. Identical to the proof of Theorem 8.30, but with arrows reversed. □

Lemma 8.36. Let X be an admissible subcategory of $\mathsf{C}^*-\mathsf{alg}$. Let $\{H_n{:}\,\mathsf{X} \to \mathsf{Ab}\}_{n\in I}$ be a homology theory. Let $A \xrightarrow{\alpha} B$ be a morphism in X. Then the connecting morphism $H_n(A) \xrightarrow{\delta} H_{n-1}(SB)$ induced by the exact sequence $0 \to SB \xrightarrow{(0,\_)} C_\alpha \to A \to 0$ coincides with α under the suspension isomorphism (see Proposition 8.28).

Proof. By the admissibility $C_\alpha \in \mathrm{Ob}(\mathsf{X})$ (it is a pullback, see Proposition 6.52).
Apply $\{H_n{:}\,\mathsf{X} \to \mathsf{Ab}\}_{n\in I}$ to diagram (21). As CB is contractible, $H_n(CB) = 0 = H_{n-1}(CB)$. This gives the following commutative diagram:

$$\begin{tikzcd}
H_n(C_\alpha) \arrow[r] \arrow[d] & H_n(A) \arrow[r, "\delta"] \arrow[d, "H_n(\alpha)"'] & H_{n-1}(SB) \arrow[r, "H_{n-1}((0,_))"] \arrow[d, "\mathrm{id}_{H_{n-1}(SB)}"'] & H_n(C_\alpha) \arrow[d] \\
0 \arrow[r] & H_n(B) \arrow[r, "\cong"] & H_{n-1}(SB) \arrow[r] & 0.
\end{tikzcd}$$

□

Lemma 8.36 will be very useful in the eventual goal of working out the proof of the UCT. However, it can also be applied to the homology theories that are already known: K_0 and K_1.

Theorem 8.37 (Essential surjectivity of K_*)**.** Let M and N be abelian groups. There exists a C*-algebra C such that $\mathrm{K}_0(C) = M$ and $\mathrm{K}_1(C) = N$.

Proof. The steps are lined out in [34, Exercise 9H]. Let $0 \to \mathbb{Z}^J \xrightarrow{\psi} \mathbb{Z}^I \xrightarrow{\pi} M \to 0$ be a free resolution, with I and J index sets, such that there exist sets of generators $\{f_i\}_{i\in I} \subset \mathbb{Z}^I$ and $\{g_j\}_{j\in J} \subset \mathbb{Z}^J$ for which the matrix expression $\psi(g_j) = \sum_{i\in I} n_{ij} f_i$ consists only of $n_{ij} \geq 0$. Only finitely many n_{ij} can be non-zero for any $i \in I$. Define $A := \bigoplus_I \mathbb{K}$ and $B := \bigoplus_J \mathbb{K}$. Now $\mathrm{K}_1(A) = \mathrm{K}_1(B) = 0$ (as $\mathrm{K}_1(\mathbb{K}) = 0$) and $\mathrm{K}_0(A) = \bigoplus_I \mathbb{Z}$ and $\mathrm{K}_0(B) = \bigoplus_J \mathbb{Z}$. Since $n_{ij} \geq 0$, there is a $*$-homomorphism $B \xrightarrow{\alpha} A \otimes \mathbb{K} = \bigoplus_{i\in I}(\mathbb{K}\otimes\mathbb{K})$ such that α acts coordinatewise by letting $\mathbb{K} \xrightarrow{\alpha_{ij}} \mathbb{K}\otimes\mathbb{K}$ be $\alpha_{ij}(x) = \mathrm{diag}(x, ..., x)$ (n_{ij}-times). Now $\mathrm{K}_0(\alpha) = \psi$. By Lemma 6.53, α induces a short exact sequence:

$$0 \to SA \xrightarrow{(0,\_)} C_\alpha \to B \to 0.$$

We will now determine the six term sequence induced by this short exact sequence. First $\mathrm{K}_1(B) = 0$ and $\mathrm{K}_0(SA) = \mathrm{K}_1(A) = 0$. By Lemma 8.36, the exponential map $\mathrm{K}_0(B) \to \mathrm{K}_1(SA) = \mathrm{K}_0(A)$ coindices with ψ. As ψ is injective, it follows that $\mathrm{K}_0(C_\alpha) \to \mathrm{K}_0(B)$ must be the zero map. But as $\mathrm{K}_0(SA) = \mathrm{K}_1(A) = 0$, it must also be injective. Hence $\mathrm{K}_0(C_\alpha) = 0$. This is what remains from the six term sequence:

$$0 \to \mathrm{K}_0(B) \xrightarrow{\psi} \mathrm{K}_0(A) \xrightarrow{\mathrm{K}_1((0,\_))} \mathrm{K}_1(C_\alpha) \to 0.$$

By exactness $M = \mathbb{Z}^I / \ker\psi = \mathrm{K}_0(A)/\ker\psi = \mathrm{K}_1(C_\alpha)$. Suspend the obtained C*-algebra to flip K_0 and K_1. Repeat this algorithm for N (but don't suspend this time). Now the orthogonal sum of the C*-algebras has K_0-group M and K_1-group N. ☐

Theorem 8.38 (Mayer-Vietoris)**.** Let X be an admissible subcategory of $\mathsf{C^*-alg}$. Let $\{H_n : \mathsf{X} \to \mathsf{Ab}\}_{n\in I}$ be a homology theory. Let $A \xrightarrow{\alpha} C$ and $B \xrightarrow{\beta} C$ be $*$-homomorphisms in X. Define $P := \{(a,b) \in A \oplus B : \alpha(a) = \beta(b)\}$. By Lemma 6.18 P is the pullback of α and β. It forms this pullback diagram:

$$\begin{tikzcd} P \arrow[r, "\pi_A"] \arrow[d, "\pi_B"] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & C. \end{tikzcd}$$

If α or β is surjective, then the following sequence is naturally exact:

$$... \to H_n(C) \xrightarrow{\delta_{n-1}} H_n(P) \xrightarrow{(\pi_A,\pi_B)} H_n(A) \oplus H_n(B) \xrightarrow{\beta-\alpha} H_n(C) \xrightarrow{\delta_n} H_{n+1}(P) \to ...$$

Proof. See [34, Exercise 11D] for the statement and [31, Theorem 4.1] for the proof. ☐

8.4 Bott functors

There is even a stronger version of a homology functor. A Bott functor is not only half-exact and homotopy invariant, but also stable. This extra property guarantees the Bott functors a six term sequence, just like K_0 and K_1 have.

Definition 8.39. A *Bott functor* is a stable homology functor.

Example 8.40. K_0 and K_1 are Bott functors for $\mathsf{C^*-alg}$.

Theorem 8.41. Let $F\colon \mathsf{C^*-alg} \to \mathsf{Ab}$ be a Bott functor. Then F and FS^2 are isomorphic functors.

Proof. For the full proof, see [34, Theorem 11.2.1]. □

Remark 8.42. The proof relies on the Toeplitz algebra $\mathcal{T}$, which is the $*$-subalgebra of $B(\ell^2(\mathbb{N}))$ generated by the unilateral shift on $\mathbb{N}$ (see [34, Exercise 3F]). As $\mathcal{T}$ appears in an exact sequence $0 \to \mathbb{K} \to \mathcal{T} \to C(\mathbb{T})$, it follows $\mathcal{T}$ is separable, σ-unital, nuclear and postliminal, see §6.5.

Remark 8.43. Theorem 8.41 gives a result about Bott functors whose domain is $\mathsf{C^*-alg}$. This raises the question; does the proof fail for subcategories? For the subcategories in this thesis; the category of separable C*-algebras, the category of separable nuclear C*-algebras and the category of σ-unital C*-algebras the answer is no! The proof is completely valid!

Corollary 8.44. Let $F\colon \mathsf{C^*-alg} \to \mathsf{Ab}$ be a Bott functor. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of C*-algebras. Then the following diagram, in which the vertical maps are the connecting morphisms, is exact:

$$\begin{tikzcd} F(A) \arrow[r, "F(\alpha)"] & F(B) \arrow[r, "F(\beta)"] & F(C) \arrow[d] \\ F(SC) \arrow[u] & F(SB) \arrow[l, "FS(\beta)"] & F(SA). \arrow[l, "FS(\alpha)"] \end{tikzcd}$$

Proof. Apply Theorem 8.30 to F and FS. Then use Theorem 8.41 to conclude that $FS^2 = F$ and close the six term sequence. □

8.5 Limit behaviour of (co)homology theories

Homology theories preserve direct limits, just like K_0 and K_1 do (Theorems 7.30 and 7.52). For cohomology theories this is not the case. The morphism of functors as established in Example 2.66 turns out to be surjective in every component in $\mathrm{Fun}(\mathbb{N}, \mathsf{C^*-alg})$ for functors in a cohomology theory. Not every component in $\mathrm{Fun}(\mathbb{N}, \mathsf{C^*-alg})$ is injective. Recall that when a component in $\mathrm{Fun}(\mathbb{N}, \mathsf{C^*-alg})$ is isomorphic, the direct/inverse limit and the cohomology functor commute.

Theorem 8.45. Let $\{H_n\colon \mathsf{C^*-alg} \to \mathsf{Ab}\}_{n\in I}$ be a homology theory. Then $H_n(\varinjlim_i A_i) = \varinjlim_i H_n(A_i)$.

Proof. The proof can be found in [31, §5]. The proof involves mapping telescopes. The eventual isomorphism is obtained via application of the homotopy invariance and half-exactness to a pullback diagram and a few short exact sequences. □

Definition 8.46. Let $(\{M_i\}_{i=1}^\infty, \{f_{i,i+1}\}_{i=1}^\infty)$ be a tower in Ab. Define the following map:

$$\partial\colon (\{M_i\}_{i=1}^\infty, \{f_{i,i+1}\}_{i=1}^\infty) \to (\{M_i\}_{i=1}^\infty, \{f_{i,i+1}\}_{i=1}^\infty) \qquad \prod_{i=1}^\infty M_i \ni (m_i)_{i=1}^\infty \xrightarrow{\partial} (m_i - f_{i,i+1}(m_{i+1}))_{i=1}^\infty \in \prod_{i=1}^\infty M_i.$$

Define $\varprojlim^1 M_i := \operatorname{coker} \partial$.

Remark 8.47. Although omitted from the notation, ∂ and $\varprojlim^1$ depend on the morphisms $f_{i,i+1} \in \mathrm{Hom}_{\mathsf{Ab}}(M_{i+1}, M_i)$. For that reason it is necessary to work with towers of abelian groups and not just products.

Proposition 8.48. Let $(\{M_i\}_{i=1}^\infty, \{f_{i,i+1}\}_{i=1}^\infty)$ be a tower in Ab. The following sequence is exact:

$$0 \to \varprojlim M_i \to \prod_{i=1}^\infty M_i \xrightarrow{\partial} \prod_{i=1}^\infty M_i \to \varprojlim{}^1 M_i \to 0.$$

Proof. There is a bijection between $\varprojlim M_i$ and sequence $(m_i)_{i=1}^\infty \in \prod_{i=1}^\infty M_i$ such that $f_{i,i+1}(m_{i+1}) = m_i$ for all $i \in \mathbb{N}$ (see Example 2.39). Clearly $\ker \partial = \varprojlim M_i$. By definition $\varprojlim{}^1 M_i = \operatorname{coker} \partial$. ☐

Lemma 8.49. $\varprojlim{}^1$ defines a functor $\varprojlim{}^1 \colon \mathrm{Fun}(\mathbb{N}^{\mathrm{opp}}, \mathsf{Ab}) \to \mathsf{Ab}$.

Proof. Let $(\{M_i\}_{i=1}^\infty, \{f_{i,i+1}\}_{i=1}^\infty)$ and $(\{N_i\}_{i=1}^\infty, \{g_{i,i+1}\}_{i=1}^\infty)$ be towers in Ab. Let $\{d_i \colon X_i \to Y_i\}_{i=1}^\infty$ be a morphism of towers. By definition $d_i(f_{i,i+1}(m_{i+1})) = g_{i,i+1}(d_{i+1}(m_{i+1}))$ for every $m_i \in M_i$. It follows $d_i(\partial(m_i)) = d_i(m_i - f_{i,i+1}(m_{i+1})) = d_i(m_i) - g_{i,i+1}(d_{i+1}(m_{i+1})) = \partial(d_i(m_i))$. So the image of ∂ is preserved by a morphism of towers. Therefore morphisms of towers define a well-define group homomorphisms between the cokernels of ∂. It is clear that this construction preserves identities and compositions. ☐

Lemma 8.50 (Equivalent to Lemma 8.49)**.** The exact sequence of Proposition 8.48 is naturally exact. It means that for towers $(\{M_i\}_{i=1}^\infty, \{f_{i,i+1}\}_{i=1}^\infty)$ and $(\{N_i\}_{i=1}^\infty, \{g_{i,i+1}\}_{i=1}^\infty)$ in Ab and a morphisms of towers $\{d_i \colon X_i \to Y_i\}_{i=1}^\infty$ between them, the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & \varprojlim M_i \arrow[r] \arrow[d, "d"] & \prod_{i=1}^\infty M_i \arrow[r, "\partial"] \arrow[d, "\{d_i\}_{i=1}^\infty"] & \prod_{i=1}^\infty M_i \arrow[r] \arrow[d, "\{d_i\}_{i=1}^\infty"] & \varprojlim{}^1 M_i \arrow[r] \arrow[d, "\{d_i\}_{i=1}^\infty"] & 0 \\ 0 \arrow[r] & \varprojlim N_i \arrow[r] & \prod_{i=1}^\infty N_i \arrow[r, "\partial"] & \prod_{i=1}^\infty N_i \arrow[r] & \varprojlim{}^1 N_i \arrow[r] & 0. \end{tikzcd} \tag{29}$$

Proof. Here d is the morphism in Ab induced by the morphism of towers $\{d_i \colon X_i \to Y_i\}_{i=1}^\infty$. See Proposition 2.64. It is by construction/definition of the vertical arrows that the left-hand and middle square of the above diagram commute. ☐

Theorem 8.51 (Milnor)**.** Let $\{H_n \colon \mathsf{C}^* - \mathsf{alg}^{\mathrm{opp}} \to \mathsf{Ab}\}_{n \in I}$ be a cohomology theory. Then the following sequence is naturally exact for every $n \in I$:

$$0 \to \varprojlim_i{}^1 H_{n-1}(A_i) \to H_n(\varinjlim A_i) \to \varprojlim_i H_n(A_i) \to 0,$$

so that for every morphism of cohomology theories $\{H_n \overset{\eta^n}{\Longrightarrow} J_n\}_{n \in I}$ from $\{H_n \colon \mathsf{C}^* - \mathsf{alg}^{\mathrm{opp}} \to \mathsf{Ab}\}_{n \in I}$ to $\{J_n \colon \mathsf{C}^* - \mathsf{alg}^{\mathrm{opp}} \to \mathsf{Ab}\}_{n \in I}$ the following sequence commutes:

$$\begin{tikzcd} 0 \arrow[r] & \varprojlim_i{}^1 H_{n-1}(A_i) \arrow[r] \arrow[d, "\varprojlim_i{}^1 \eta_i^{n-1}"] & H_n(\varinjlim A_i) \arrow[r] \arrow[d, "\eta^n_{\varinjlim A_i}"] & \varprojlim_i H_n(A_i) \arrow[r] \arrow[d, "\varprojlim_i \eta_i^n"] & 0 \\ 0 \arrow[r] & \varprojlim_i{}^1 J_{n-1}(A_i) \arrow[r] & J_n(\varinjlim A_i) \arrow[r] & \varprojlim_i J_n(A_i) \arrow[r] & 0. \end{tikzcd} \tag{30}$$

Proof. The full proof can be found in [31, §7]. Via the Mayer-Vietoris sequence (Theorem 8.38), a long exact sequence is obtained from a pullback diagram of C*-algebras. Up to an equivalence, this long exact sequence is:

$$\cdots \to H_n(\varinjlim A_i) \to \prod_{i=1}^{\infty} H_n(A_i) \xrightarrow{\partial_n} \prod_{i=1}^{\infty} H_n(A_i) \to H_{n+1}(\varinjlim A_i) \to \cdots$$

By exactness $\mathrm{im}(H_n(\varinjlim A_i) \to \prod_{i=1}^{\infty} H_n(A_i)) = \ker \partial_n$ and $\ker(H_n(\varinjlim A_i) \to \prod_{i=1}^{\infty} H_n(A_i)) = \operatorname{coker} \partial_{n-1}$. From diagram (29), it is clear that $\operatorname{coker} \partial_{n-1} = \varprojlim^1 H_{n-1}(A_i)$ and $\ker \partial_n = \varprojlim_i H_n(A_i)$. This proves the exactness. The Mayer-Vietoris sequence is natural, so diagram (30) is commutative. The right-hand square in diagram (30) is diagram (4) from Theorem 2.69 (but the functor is contravariant in this case (it is the situation, described in Remark 2.68)). The middle vertical arrow is the component of η^n at $\varinjlim A_i$. As η^n is a morphism of functors, $\eta^n_{\varinjlim A_i}$ is the morphism that makes diagram (3) commute. The right-hand vertical $\varprojlim_i \eta^n_i$-arrow is the $\varprojlim$-functor (Proposition 2.64) applied to the morphism of towers induced by a morphism of functors (Proposition 2.60). The left-hand vertical morphism is the $\varprojlim^1$-functor applied to the morphism of towers induced by a morphism of functors. □

9 Hilbert modules

The goal of this section is to get acquainted with Hilbert modules. We will show an isomorphism between the adjointable operators and multiplier algebra of the compact operators. It turns out for the standard Hilbert module of a C*-algebra the bounded operators are isomorphic to the multiplier algebra of its stabilisation.

9.1 Hilbert modules

Definition 9.1. Let A be a C*-algebra. A Hilbert A-module is a A-module E with an A-valued inner product $\langle .,. \rangle_E : E \times E \to A$ such that:

1. $\langle e_1, e_2 \cdot a \rangle_E = \langle e_1, e_2 \rangle_E \cdot a \qquad \forall e_1, e_2 \in E,\ a \in A$
2. $\langle e_1, e_2 \rangle_E = \langle e_2, e_1 \rangle_E^* \qquad \forall e_1, e_2 \in E$
3. $\langle e, e \rangle_E \geq 0 \qquad \forall e \in E$ and $\langle e, e \rangle_E = 0 \iff e = 0$
4. E is complete w.r.t. the norm induced by the inner product: $\|e\|_E := \|\langle e, e \rangle_E\|_A^{\frac{1}{2}}$ for $e \in E$.

Example 9.2. A Hilbert $\mathbb{C}$-module is just a Hilbert space.

Example 9.3. Let A be a C*-algebra, then A is an Hilbert A-module via the inner product $\langle a, b \rangle = a^* b$. In case no inner product is explicitly stated, a C*-algebra is assumed to have this inner product when viewed as a Hilbert module over itself.

Example 9.4. Let A be a C*-algebra. The *standard Hilbert A-module* is the inner product defined in equation (31) under pointwise addition and A-action:

$$\mathbb{H}_A := \{(x_k) \in \prod_1^{\infty} A \mid \sum x_k^* x_k \text{ converges in norm in } A\} \qquad \langle x, y \rangle_{\mathbb{H}_A} := \sum_{k \in \mathbb{N}} x_k^* y_k. \tag{31}$$

Its norm is induced by the inner product. Clearly $\mathbb{H}_{\mathbb{C}} = \ell_2$.

Remark 9.5. The set $\mathbb{H}_A$ is often wrongly characterised. The sets $\{(x_k) \in \prod_{k=1}^{\infty} A | \sum_{k\in\mathbb{N}} \|x_k^* x_k\| < \infty\}$ and $\{(x_k) \in \prod_{k=1}^{\infty} A | \{\|\sum_{k=1}^{n} x_k^* x_k\|\}_{n=1}^{\infty} \text{bounded}\}$ are **not** equal to $\mathbb{H}_A$. For the former, take $A = \mathbb{B}$. Each rank one projection on the span of an orthogonal basis element has operator norm 1. The sum converges to the unit in $\mathbb{B}$, however $\sum_{\mathbb{N}} 1 = \infty$. For the latter, take $A = \mathbb{K}$. The sum of the rank one projections on the span of orthogonal basis elements does not converge, since $\mathbb{K}$ is non-unital. However the operator norm of the n-th sum is 1. [34, p.239].

Definition 9.6. Let A be a C*-algebra and let E be a Hilbert A-module. The *adjoint* of F is a linear operator F^* on E such that $\langle F(a), b\rangle_E = \langle a, F^*(b)\rangle_E$ for all $a, b \in A$. The algebra of adjointable operators on E is denoted by $\mathbb{B}(E)$.

Remark 9.7. The A-action is automatically preserved for operators in $\mathbb{B}(E)$, see [2, Definition 13.2.1]. For a Hilbert space, every bounded operator allows an adjoint [29, Theorem 6.1]. This does in general not hold for Hilbert modules of other C*-algebras.

Lemma 9.8. Let A be a C*-algebra and let E be a Hilbert A-module. Then $\mathbb{B}(E)$ is a C*-algebra with the adjoint as $*$-operation and the operator norm as norm.

Proof. The sums and products of two adjointable operators are adjointable. The adjoint satisfies Definition 5.2. It follows from the definition of the operator norm and adjoint that $\mathbb{B}(E)$ is a C*-algebra. □

Remark 9.9. In this section, we assume that the domain and codomain of operators between Hilbert modules over a C*-algebra coincide. This is because $\mathbb{B}(E)$ is a C*-algebra. The other results hold for adjointable operators with a different domain and codomain. This also applies to §10.2.

Lemma 9.10. Let A be a C*-algebra. View A as a Hilbert module over itself (as in Example 9.3). Then $\mathbb{B}(A) \cong \mathcal{M}(A)$.

Proof. Let $(L, R) \in \mathcal{M}(A)$, then $L(ab) = L(a)b$, so L preserves the A-action. Proposition 5.21 shows the $*$-operation on $(L, R) \in \mathcal{M}(A)$. Notice that $\langle L(a), b\rangle_A = (L(a))^* b = L^*(a^*)b = a^* R^*(b) = \langle a, R^*(b)\rangle_A$. So R^* is the adjoint of L. So $L \in \mathbb{B}(A)$ and the mapping $\mathcal{M}(A) \ni (L, R) \to L \in \mathbb{B}(A)$ must be injective; it follows from the uniqueness of adjoints.
Let $F \in \mathbb{B}(A)$. Define $F^\dagger \in B(A)$ as $F^\dagger(a) = (F^*(a^*))^*$. Now $aF(b) = \langle a^*, F(b)\rangle_A = \langle F^*(a^*), b\rangle_A = (F^*(a^*))^* b = F^\dagger(a)b$. So $(F, F^\dagger) \in \mathcal{M}(A)$. The mapping is surjective.
This bijection preserves addition, composition and the $*$-operation. It is a $*$-isomorphism. □

9.2 Compactness and Properties of $\mathbb{H}_A$

Definition 9.11. Let A be a C*-algebra. Let E be a Hilbert A-module. A *rank one operator* is an operator in $\mathbb{B}(E)$ of the form $\theta_{x,y}(z) := x\langle y, z\rangle_E$ for $x, y, z \in E$. An operator $T \in \mathbb{B}(E)$ is *compact* if it is in the closure of the linear span of the rank one operators. The set of compact operators is denoted by $\mathbb{K}(E)$. In other words, $\mathbb{K}(E) := \overline{\operatorname{Span}}(\{\theta_{x,y} : x, y \in E\})$.

Remark 9.12. A rank one operator has an adjoint, namely $\theta_{x,y}^* = \theta_{y,x}$. See the following equation:

$$\langle \theta_{x,y}(z), w\rangle_E = \langle x\langle y, z\rangle_E, w\rangle_E = \langle w, x\langle y, z\rangle_E\rangle_E^* = (\langle w, x\rangle_E \cdot \langle y, z\rangle_E)^* =$$
$$\langle y, z\rangle_E^* \cdot \langle w, x\rangle_E^* = \langle z, y\rangle_E \cdot \langle x, w\rangle_E = \langle z, y\langle x, w\rangle_E\rangle_E = \langle z, \theta_{y,x}(w)\rangle_E.$$

So the rank one operators are contained in $\mathbb{B}(E)$ indeed.

Lemma 9.13. $\mathbb{K}(E)$ is an ideal of $\mathbb{B}(E)$.

Proof. Let $T \in \mathbb{B}(E)$, then $(T \circ \theta_{x,y})(z) = T(x\langle y,z\rangle_E) = T(x)\langle y,z\rangle_E = \theta_{T(x),y}(z)$.
Likewise $(\theta_{x,y} \circ T)(z) = x\langle y, T(z)\rangle_E = x\langle T^*(y), z\rangle_E = \theta_{x,T^*(y)}(z)$. □

Lemma 9.14. Let A be a C*-algebra. Then $\mathbb{K}(\mathbb{H}_A) \cong \mathbb{K} \otimes A$

Proof. See [34, p. 245]. □

Theorem 9.15. Let A be a C*-algebra and let E be a Hilbert A-module. Then $\mathbb{B}(E) \cong \mathcal{M}(\mathbb{K}(E))$.

Proof. The entire proof can be found in [34, Theorem 15.2.12]. The $*$-isomorphism is given by the mapping $\mathbb{B}(E) \ni T \to (L_T, R_T) \in \mathcal{M}(\mathbb{K}(E))$, where L_T and R_T denote the post- and precomposition by T. It is similar to Example 5.17. So it is a well-defined $*$-homomorphism, because $\mathbb{K}(E)$ is an ideal. If $(L_T, R_T) = 0$, then $T \circ S = 0$ and $S \circ T = 0$ for all $S \in \mathbb{K}(E)$. In particular, for the rank one operator $\theta_{x,Tx}$. Now $(L_T \circ \theta_{x,Tx})(Tx) = T(x\langle Tx, Tx\rangle_E) = T(x)\|T(x)\|^2$. Now $T(x)\|T(x)\|^2 = 0$ if and only if $T(x) = 0$. Hence $T = 0$. So its kernel is trivial.
For the surjectivity we refer to [34, Theorem 15.2.12]. □

Remark 9.16. The injectivity of the $*$-homomorphism in the proof of Theorem 9.15 shows that $\mathbb{K}(E)$ is an essential ideal of $\mathbb{B}(E)$, see Lemma 6.12.

Corollary 9.17. Let A be a C*-algebra. Then $\mathbb{B}(\mathbb{H}_A) \cong \mathcal{M}(\mathbb{K} \otimes A)$.

Example 9.18. $\mathcal{M}(\mathbb{K}) = \mathbb{B}$, as $\mathbb{K} = \mathbb{K}(\ell_2) = \mathbb{K}(\mathbb{H}_\mathbb{C}) = \mathbb{K} \otimes \mathbb{C}$ (Example 9.4), because ℓ_2 is separable ([29, Example 3.53]; it allows an orthonormal basis). Now $\mathbb{B} = \mathbb{B}(\ell_2) = \mathbb{B}(\mathbb{H}_\mathbb{C}) = \mathcal{M}(\mathbb{K} \otimes \mathbb{C}) = \mathcal{M}(\mathbb{K})$.

10 Grading

A grading indexed by an abelian group adds more structure to the already existing additive and possibly multiplicative structure of a ring, a module, an algebra. When a grading indexed by an abelian group is applied to an object, the object will be decomposed into components indexed by the abelian group. Each component may or may not be an object in the categorical sense. That depends on the category. The best way for a more precise formulation is just going through several applications of gradings. In this thesis objects (C*-algebras, Hilbert modules, abelian groups, modules over a ring, etc.) will only be endowed with a $\mathbb{Z}/2\mathbb{Z}$-grading. For clarity, $\mathbb{Z}/2\mathbb{Z}$ will be abbreviated to $\mathbb{Z}_2$ throughout this thesis.

10.1 Graded C*-algebras

Definition 10.1. A C*-algebra A is $\mathbb{Z}_2$*-graded* if it decomposes into self-adjoint closed linear subspaces $A^{(0)}$ and $A^{(1)}$ such that $A = A^{(0)} \oplus A^{(1)}$ as vector space and $xy \in A^{(i+j)}$ for $x \in A^{(i)}$ and $y \in A^{(j)}$ with $i, j \mod 2$.

Remark 10.2. $A^{(0)}$ is a C*-subalgebra (closed and self-adjoint subalgebra). $A^{(1)}$ in general is not a C*-subalgebra. It is not multiplicatively closed.

Example 10.3 (Trivial grading)**.** Every C*-algebra A can be endowed with the following grading: $A = A \oplus 0$ ($A^{(0)} = A$ and $A^{(1)} = 0$).

Example 10.4 (Even grading)**.** Let A be an arbitrary C*-algebra. $M_2(A)$ can be given the following $\mathbb{Z}_2$-grading; let $M_2(A)^{(0)}$ be the diagonal and let $M_2(A)^{(1)}$ be the off-diagonal. This is called the *even* grading on $M_2(A)$.

Example 10.5 (Odd grading)**.** $\mathbb{C}_1$ is defined in Definition 5.96. Define $\mathbb{C}_1^{(0)} = \{\lambda 1 : \lambda \in \mathbb{C}\}$ and $\mathbb{C}_1^{(1)} = \{\lambda\epsilon_1 : \lambda \in \mathbb{C}\}$, then this forms a $\mathbb{Z}_2$-grading on $\mathbb{C}_1$. The motivation for this grading is the isomorphism $\mathbb{C}_1 \cong \mathbb{C}^2$. The corresponding grading on $\mathbb{C}^2$ is $(\mathbb{C}^2)^{(0)} = \{(\lambda,\lambda) : \lambda \in \mathbb{C}\}$ and $(\mathbb{C}^2)^{(1)} = \{(\lambda,-\lambda) : \lambda \in \mathbb{C}\}$. The non-trivial ideals of $\mathbb{C}^2$ correspond to $\{\lambda\frac{1+\epsilon_1}{2} : \lambda \in \mathbb{C}\}$ and $\{\lambda\frac{1-\epsilon_1}{2} : \lambda \in \mathbb{C}\}$ in $\mathbb{C}_1$. This can be generalised to any C*-algebra A (see [2, Example 14.1.2b]). The orthogonal sum $A \oplus A$ will be $\mathbb{Z}_2$-graded in this way: $(A \oplus A)^{(0)} = \{(a,a) : a \in A\}$ and $(A \oplus A)^{(1)} = \{(a,-a) : a \in A\}$. This is called the *odd* grading.

Definition 10.6. Let A be a $\mathbb{Z}_2$-graded C*-algebra. Let $B \subset A$ be a C*-subalgebra. B is a $\mathbb{Z}_2$-*graded C*-subalgebra* of A if $B^{(0)} := A^{(0)} \cap B$ and $B^{(1)} := A^{(1)} \cap B$ define a $\mathbb{Z}_2$-grading on B. It means that if $a \in B$ and a decomposes as $a = a^{(0)} + a^{(1)}$ in A, then $a^{(0)}, a^{(1)} \in B$.

Remark 10.7. In line with Definition 10.6; a $\mathbb{Z}_2$-*graded ideal* is a $\mathbb{Z}_2$-graded subalgebra that happens to be an ideal. A $\mathbb{Z}_2$-graded ideal can be decomposed into closed self-adjoint linear subspaces, but the summands need not be ideals in their own right.

Definition 10.8. Let A be a $\mathbb{Z}_2$-graded C*-algebra. Elements of $A^{(0)}$ and $A^{(1)}$ are *homogeneous.*

Remark 10.9. Later on, (linear) maps where the outcome depends on whether an element is in $A^{(0)}$ or $A^{(1)}$ will be introduced. The degree map, $\partial : A^{(0)} \cup A^{(1)} \to \{0,1\}$, is only well-defined on homogeneous elements. In the expressions of functions, it is assumed that all elements are homogeneous and extended linearly.

10.2 Graded Hilbert modules and results

Definition 10.10. Let A be a $\mathbb{Z}_2$-graded C*-algebra. Let E be a Hilbert A-module. E is $\mathbb{Z}_2$-*graded* if there is a decomposition into closed linear subspaces $E = E^{(0)} \oplus E^{(1)}$ such that $E^{(i)}A^{(j)} \subset E^{(i+j)}$ and $\langle E^{(i)}, E^{(j)}\rangle_E \subset A^{(i+j)}$ for $i,j = 0,1 \mod 2$.

Definition 10.11. Let A be a $\mathbb{Z}_2$-graded C*-algebra and let E be a $\mathbb{Z}_2$-graded Hilbert A-module. Let $F \in \mathbb{B}(E)$. Then F is *even* if $F(E^{(0)}) \subset E^{(0)}$ and $F(E^{(1)}) \subset E^{(1)}$ and F is *odd* if $F(E^{(0)}) \subset E^{(1)}$ and $F(E^{(1)}) \subset E^{(0)}$.

Example 10.12 (Trivial grading)**.** Similar to Example 10.3 every Hilbert A-module E can be graded with the trivial grading; $E = E \oplus 0$. The Hilbert modules of trivially graded C*-algebras need not be trivially graded. For trivially graded Hilbert A-modules the only odd operator is 0.

Lemma 10.13. Let A be a $\mathbb{Z}_2$-graded C*-algebra and let E be a $\mathbb{Z}_2$-graded Hilbert A-module. Every operator $F \in \mathbb{B}(E)$ can be decomposed into $F = F^{(0)} + F^{(1)}$ for which $F^{(0)}, F^{(1)} \in \mathbb{B}(E)$ such that $F^{(0)}$ is even and $F^{(1)}$ is odd.

Proof. Decompose $z \in E$ into $z = x + y$ with $x \in E^{(0)}$, $y \in E^{(1)}$. For $F \in \mathbb{B}(E)$, $F(z) = F(x) + F(y)$. There are unique decompositions $F(x) = t + u$ and $F(y) = v + w$ with $t, v \in E^{(0)}$, $u, w \in E^{(1)}$. Define $F^{(0)}(x) := t$, $F^{(0)}(y) = w$ and $F^{(1)}(x) = u$ and $F^{(1)}(y) = v$. $F^{(0)}$ is even and $F^{(1)}$ is odd. $\square$

Corollary 10.14. Let A be a $\mathbb{Z}_2$-graded C*-algebra and let E be a $\mathbb{Z}_2$-graded Hilbert A-module. The even and odd operators in $\mathbb{B}(E)$ define a $\mathbb{Z}_2$-grading. Under this grading $\mathbb{B}(E)$ is a $\mathbb{Z}_2$-graded C*-algebra. Moreover, if E is trivially graded, then so is $\mathbb{B}(E)$.

Remark 10.15. Also adjointable operators with a different domain and codomain inherit a $\mathbb{Z}_2$-grading; even is grading preserving and odd is grading reversing.

Lemma 10.16. Let A be $\mathbb{Z}_2$-graded C*-algebra and let E be a $\mathbb{Z}_2$-graded Hilbert A-module. Then under the grading of Corollary 10.14 on $\mathbb{B}(E)$, the ideal $\mathbb{K}(E) \subset \mathbb{B}(E)$ is a $\mathbb{Z}_2$-graded ideal.

Proof. Let $x, y, z \in E$. Decompose $x = x^{(0)} + x^{(1)}$, $y = y^{(0)} + y^{(1)}$ and $z = z^{(0)} + z^{(1)}$.

$$A \ni \langle y, z\rangle_E = \langle y^{(0)} + y^{(1)}, z^{(0)} + z^{(1)}\rangle_E = \langle y^{(0)}, z^{(0)}\rangle_E + \langle y^{(1)}, z^{(0)}\rangle_E + \langle y^{(0)}, z^{(1)}\rangle_E + \langle y^{(1)}, z^{(1)}\rangle_E.$$

$\langle y^{(0)}, z^{(0)}\rangle_E, \langle y^{(1)}, z^{(1)}\rangle_E \in A^{(0)}$ and $\langle y^{(1)}, z^{(0)}\rangle_E, \langle y^{(0)}, z^{(1)}\rangle_E \in A^{(0)} \in A^{(1)}$ by Definition 10.10.
$\theta_{x,y}(z) = x\langle y, z\rangle_E \in E$ decomposes as follows:

$$(x\langle y, z\rangle_E)^{(0)} = x^{(0)}\langle y^{(0)}, z^{(0)}\rangle_E + x^{(0)}\langle y^{(1)}, z^{(1)}\rangle_E + x^{(1)}\langle y^{(1)}, z^{(0)}\rangle_E + x^{(1)}\langle y^{(0)}, z^{(1)}\rangle_E$$
$$(x\langle y, z\rangle_E)^{(1)} = x^{(1)}\langle y^{(0)}, z^{(0)}\rangle_E + x^{(1)}\langle y^{(1)}, z^{(1)}\rangle_E + x^{(0)}\langle y^{(1)}, z^{(0)}\rangle_E + x^{(0)}\langle y^{(0)}, z^{(1)}\rangle_E.$$

Clearly $\theta^{(0)}_{x,y} = \theta_{x^{(0)},y^{(0)}} + \theta_{x^{(1)},y^{(1)}}$ and $\theta^{(1)}_{x,y} = \theta_{x^{(1)},y^{(0)}} + \theta_{x^{(0)},y^{(1)}}$. So $\theta^{(0)}_{x,y}, \theta^{(1)}_{x,y} \in \mathbb{K}(E)$.
So $\mathbb{K}(E)$ is a $\mathbb{Z}_2$-graded ideal of $\mathbb{B}(E)$. □

Example 10.17. Let A be a $\mathbb{Z}_2$-graded C*-algebra. A is a $\mathbb{Z}_2$-graded Hilbert module over itself, see Example 9.3. Now $\mathbb{B}(A)$ inherits the $\mathbb{Z}_2$-grading of A by Corollary 10.14. Lemma 9.10 offers an isomorphism between $\mathbb{B}(A)$ and $\mathcal{M}(A)$, so that $\mathcal{M}(A)$ can copy the $\mathbb{Z}_2$-grading of $\mathbb{B}(A)$. This will from now on be the $\mathbb{Z}_2$-grading on multiplier algebras. By Example 10.12 the multiplier algebra of a trivially graded on C*-algebra is trivially graded.

Remark 10.18. Theorem 9.15 is a $\mathbb{Z}_2$-graded isomorphism. In the proof of [34, Theorem 15.2.12] no grading is taken into account. However $\mathbb{B}(E)$ has its grading from E via Corollary 10.14. As $\mathbb{K}(E)$ is $\mathbb{Z}_2$-graded ideal of $\mathbb{B}(E)$, it has the same grading. The isomorphism of Lemma 9.10 for the C*-algebra $\mathbb{K}(E)$ is $\mathbb{B}(\mathbb{K}(E)) \cong \mathcal{M}(\mathbb{K}(E))$. Example 10.17 let $\mathcal{M}(\mathbb{K}(E))$ copy the $\mathbb{Z}_2$-grading from $\mathbb{B}(\mathbb{K}(E))$.

Definition 10.19. Let A be a $\mathbb{Z}_2$-graded C*-algebra and let E be a $\mathbb{Z}_2$-graded Hilbert A-module. Then the *opposite* of E is the Hilbert A-module E^{opp} for which $E^{\mathrm{opp}(0)} = E^{(1)}$ and $E^{\mathrm{opp}(1)} = E^{(0)}$.

Example 10.20. A $\mathbb{Z}_2$-graded C*-algebra A induces a natural grading on $\mathbb{H}_A$; $\mathbb{H}^{(0)}_A$ consists of all sequences of $\mathbb{H}_A$ with values in $A^{(0)}$ and $\mathbb{H}^{(1)}_A$ consists of all sequences of $\mathbb{H}_A$ with values in $A^{(1)}$. This grading will be assumed on $\mathbb{H}_A$ from now on. Define $\widehat{\mathbb{H}_A} := \mathbb{H}_A \oplus \mathbb{H}^{\mathrm{opp}}_A$. If A is trivially graded, then so is $\mathbb{H}_A$ and $\widehat{\mathbb{H}_A}$ is in fact the direct sum of two copies of $\mathbb{H}_A$, so that $\widehat{\mathbb{H}_A}^{(0)} = \mathbb{H}_A$ $\widehat{\mathbb{H}_A}^{(1)} = \mathbb{H}_A$.

Lemma 10.21. Let A be a trivially graded C*-algebra. Then $\mathbb{B}(\widehat{\mathbb{H}_A}) \cong M_2(\mathcal{M}(\mathbb{K} \otimes A))$.

Proof. Since A is trivially graded, $\widehat{\mathbb{H}_A} = \mathbb{H}_A \oplus \mathbb{H}_A$ and every element of $\mathbb{B}(\widehat{\mathbb{H}_A})$ can be written as a 2×2-matrix of elements of $\mathbb{B}(\mathbb{H}_A)$. Using Corollary 9.17, every element of $\mathbb{B}(\widehat{\mathbb{H}_A})$ can be written as a 2×2-matrix of elements of $\mathcal{M}(\mathbb{K} \otimes A)$.
It is easy to see that the gradings of $M_2(\mathcal{M}(\mathbb{K} \otimes A))$ and $\mathbb{B}(\widehat{\mathbb{H}_A})$ agree. An even morphism preserves the grading and the image of each summand will be included in itself. So an even morphism is zero on the off-diagonal. An odd morphism switches the grading and the image of each summand will be included in the other summand. So an odd morphism is zero on the diagonal. □

Theorem 10.22 (Kasparov's stabilisation theorem)**.** Let B be a $\mathbb{Z}_2$-graded C*-algebra and let E be a countably generated $\mathbb{Z}_2$-graded Hilbert B-module. There is a $\mathbb{Z}_2$-graded isomorphism $\widehat{\mathbb{H}_B} \oplus E \cong \widehat{\mathbb{H}_B}$.

Proof. The proof can be found in [2, Theorem 13.6.2] for the ungraded case described in the remark below. The modifications for the $\mathbb{Z}_2$-graded case are explained in [2, Theorem 14.6.1]. □

Remark 10.23. There is an ungraded version of this theorem. It can be obtained by endowing B with the trivial grading. Then $\widehat{\mathbb{H}_B} = \mathbb{H}_B \oplus \mathbb{H}_B$ and $\mathbb{H}_B \oplus \mathbb{H}_B \oplus E = \widehat{\mathbb{H}_B} \oplus E \cong \widehat{\mathbb{H}_B} = \mathbb{H}_B \oplus \mathbb{H}_B$. There is an isomorphism between $\mathbb{H}_B$ and $\mathbb{H}_B \oplus \mathbb{H}_B$. It can be established by creating a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ and mapping the indices accordingly. Hence $\mathbb{H}_B \oplus E \cong \mathbb{H}_B$.

10.3 Grading on tensor products of C*-algebras

Definition 10.24. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. Denote by $A\hat{\otimes}B$ the endowment of the algebraic tensor product $A \otimes B$ with the following structures:

- $(A\hat{\otimes}B)^{(0)} = \overline{\mathrm{Span}}(\{a\hat{\otimes}b : a \in A^{(0)}, b \in B^{(0)}\} \cup \{a\hat{\otimes}b : a \in A^{(1)}, b \in B^{(1)}\})$
- $(A\hat{\otimes}B)^{(1)} = \overline{\mathrm{Span}}(\{a\hat{\otimes}b : a \in A^{(0)}, b \in B^{(1)}\} \cup \{a\hat{\otimes}b : a \in A^{(1)}, b \in B^{(0)}\})$
- $(a_1\hat{\otimes}b_1)(a_2\hat{\otimes}b_2) = (-1)^{\partial(b_1)\partial(a_2)}(a_1a_2\hat{\otimes}b_1b_2)$
- $(a\hat{\otimes}b)^* = (-1)^{\partial(a)\partial(b)}(a^*\hat{\otimes}b^*)$.

Proposition 10.25. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. $A\hat{\otimes}B$ with the structure of Definition 10.24 is a $\mathbb{Z}_2$-graded $*$-algebra.

Proof. $(A\hat{\otimes}B)^{(0)}$ and $(A\hat{\otimes}B)^{(1)}$ are closed by construction. $A^{(0)}, A^{(1)}, B^{(0)}$ and $B^{(1)}$ are closed under the $*$-operation. Hence $(A\hat{\otimes}B)^{(0)}$ and $(A\hat{\otimes}B)^{(1)}$ are closed under adjunction. Only the signs differ from Lemma 5.38 and so $A\hat{\otimes}B$ is $\mathbb{Z}_2$-graded.

The $*$-operation is anti-linear, a minus sign does not change that. To see that the $*$-operation and product commute in reversed order, check that (32) and (33) are both expressions for $((a_1\hat{\otimes}b_1)(a_2\hat{\otimes}b_2))^*$.

$$((-1)^{\partial(b_1)\partial(a_2)}(a_1a_2\hat{\otimes}b_1b_2))^* = (-1)^{\partial(b_1)\partial(a_2)}(-1)^{\partial(a_1a_2)\partial(b_1b_2)}(a_2^*a_1^*\hat{\otimes}b_2^*b_1^*) \tag{32}$$

$$(a_2\hat{\otimes}b_2)^*(a_1\hat{\otimes}b_1)^* = (-1)^{\partial(a_2)\partial(b_2)}(a_2^*\hat{\otimes}b_2^*)(-1)^{\partial(a_1)\partial(b_1)}(a_1^*\hat{\otimes}b_1^*) = \\ (-1)^{\partial(a_2)\partial(b_2)}(-1)^{\partial(a_1)\partial(b_1)}(-1)^{\partial(a_1)\partial(b_2)}(a_2^*a_1^*\hat{\otimes}b_2^*b_1^*). \tag{33}$$

It needs to be verified that the signs of (32) and (33) agree, the following equation holds:

$$(-1)^{\partial(b_1)\partial(a_2)}(-1)^{\partial(a_1a_2)\partial(b_1b_2)} = (-1)^{\partial(a_2)\partial(b_2)}(-1)^{\partial(a_1)\partial(b_1)}(-1)^{\partial(a_1)\partial(b_2)}. \tag{34}$$

The degree of a product is the sum of degrees, so:

$$\partial(b_1)\partial(a_2) + \partial(a_1a_2)\partial(b_1b_2) = \partial(b_1)\partial(a_2) + (\partial(a_1) + \partial(a_2))(\partial(b_1) + \partial(b_2)) = \\ \partial(a_1)\partial(b_1) + \partial(a_1)\partial(b_2) + \partial(a_2)\partial(b_2).$$

From this follows that equation (34) holds and so the signs of (32) and (33) agree.
It needs to be proven the product is associative. Equations (35) and (36) take products in different orders:

$$((a_1\hat{\otimes}b_1)(a_2\hat{\otimes}b_2))(a_3\hat{\otimes}b_3) = (-1)^{\partial(b_1)\partial(a_2)}(a_1a_2\hat{\otimes}b_1b_2))(a_3\hat{\otimes}b_3) = \\ (-1)^{\partial(b_1)\partial(a_2)}(-1)^{\partial(b_1b_2)\partial(a_3)}(a_1a_2a_3\hat{\otimes}b_1b_2b_3) \quad (35)$$

$$(a_1\hat{\otimes}b_1)((a_2\hat{\otimes}b_2)(a_3\hat{\otimes}b_3)) = (a_1\hat{\otimes}b_1)(-1)^{\partial(b_2)\partial(a_3)}(a_2a_3 \otimes b_2b_3) = \\ (-1)^{\partial(b_1)\partial(a_2a_3)}(-1)^{\partial(b_2)\partial(a_3)}(a_1a_2a_3 \otimes b_1b_2b_3). \quad (36)$$

It needs to be verified that the signs of (35) and (36) agree, i.e. the following equation holds:

$$(-1)^{\partial(b_1)\partial(a_2)}(-1)^{\partial(b_1b_2)\partial(a_3)} = (-1)^{\partial(b_1)\partial(a_2a_3)}(-1)^{\partial(b_2)\partial(a_3)}. \quad (37)$$

The degree of a product is the sum of degrees, so:

$$\partial(b_1)\partial(a_2) + \partial(b_1b_2)\partial(a_3) = \partial(b_1)\partial(a_2) + (\partial(b_1) + \partial(b_2))\partial(a_3) = \partial(b_1)\partial(a_2a_3) + \partial(b_2)\partial(a_3).$$

From this follows equation (37) holds and (35) and (36) agree. The distributivity follows from the distributivity of the ungraded tensor product (from Lemma 5.38). Addition of a minus sign will not change that. □

Definition 10.26. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. The $\mathbb{Z}_2$-grading on $A\hat{\otimes}B$ from Definition 10.24 is called the *tensor product $\mathbb{Z}_2$-grading.*

Remark 10.27. The norm on the graded tensor product does not need to be altered; there are only changes of signs. Any norm is invariant under sign changes.

Example 10.28. The $\mathbb{Z}_2$-graded tensor product $\mathbb{C}_1 \hat{\otimes} \mathbb{C}_1$ is gradedly isomorphic to $M_2(\mathbb{C})$ with even grading (see Example 10.4). $\mathbb{C}_1 \hat{\otimes} \mathbb{C}_1$ has dimension 4 and cannot be trivially graded as $\mathbb{C}_1$ is not trivially graded. The set $\{1\hat{\otimes}1, 1\hat{\otimes}\epsilon_1, \epsilon_1\hat{\otimes}1, \epsilon_1\hat{\otimes}\epsilon_1\}$ forms a basis of $\mathbb{C}_1 \hat{\otimes} \mathbb{C}_1$. Now $1\hat{\otimes}1 \in (\mathbb{C}_1 \hat{\otimes} \mathbb{C}_1)^{(0)}$ is the unit. The odd basis elements $1\hat{\otimes}\epsilon_1, \epsilon_1\hat{\otimes}1 \in (\mathbb{C}_1 \hat{\otimes} \mathbb{C}_1)^{(1)}$ are self-adjoint. The self-adjoint off-diagonal 2×2-matrices are $\pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\pm\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Now $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ commute up to a minus sign. We check that $(1\hat{\otimes}\epsilon_1)(\epsilon\hat{\otimes}1) = -(\epsilon\hat{\otimes}1)(1\hat{\otimes}\epsilon_1)$. If we send $(1\hat{\otimes}\epsilon_1)$ to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $(\epsilon_1\hat{\otimes}1)$ to $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ and $(\epsilon_1\hat{\otimes}\epsilon_1)$ to $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, then this forms an isomorphism. We check that $(\epsilon_1\hat{\otimes}\epsilon_1)^* = -\epsilon_1\hat{\otimes}\epsilon_1$.

Example 10.29. Let A and B be trivially graded C*-algebras. Then $A \otimes B$ endowed with the trivial grading is isomorphic to $A\hat{\otimes}B$.

Example 10.30. Let A be a trivially graded C*-algebra. Let B be any $\mathbb{Z}_2$-graded C*-algebra. Since A has a zero odd part, the $*$-operation and product of $A\hat{\otimes}B$ will never contract a minus sign. This holds, in particular, for $A\hat{\otimes}\mathbb{C}_1$.

Remark 10.31. Lemma 9.14 and Corollary 9.17 hold for $\mathbb{Z}_2$-graded C*-algebras as well. View $\mathbb{K}$ as trivially graded. Lemma 10.21 only holds for trivially graded C*-algebras.

Lemma 10.32. Let A be a $\mathbb{Z}_2$-graded C*-algebra. Then $\mathbb{H}_{A\hat{\otimes}\mathbb{C}_1} \cong \mathbb{H}_A \hat{\otimes} \mathbb{C}_1$ as $\mathbb{Z}_2$-graded Hilbert $A\hat{\otimes}\mathbb{C}_1$- modules.

Proof. $\mathbb{H}^{(0)}_{A\hat{\otimes}\mathbb{C}_1}$ consists of all the sequences of $\mathbb{H}_{A\hat{\otimes}\mathbb{C}_1}$ with values in $(A^{(0)}\hat{\otimes}\mathbb{C}_1^{(0)}) \oplus (A^{(1)}\hat{\otimes}\mathbb{C}_1^{(1)})$ and $\mathbb{H}^{(1)}_{A\hat{\otimes}\mathbb{C}_1}$ consists of all the sequences of $\mathbb{H}_{A\hat{\otimes}\mathbb{C}_1}$ with values in $(A^{(0)}\hat{\otimes}\mathbb{C}_1^{(1)}) \oplus (A^{(1)}\hat{\otimes}\mathbb{C}_1^{(0)})$. □

Corollary 10.33. Let A be a $\mathbb{Z}_2$-graded C*-algebra. Then $\widehat{\mathbb{H}_{A\hat{\otimes}\mathbb{C}_1}} \cong \widehat{\mathbb{H}_A} \hat{\otimes} \mathbb{C}_1$ as $\mathbb{Z}_2$-graded Hilbert $A\hat{\otimes}\mathbb{C}_1$-modules.

Proof. By Lemma 10.32:

$$\widehat{\mathbb{H}_{A\hat{\otimes}\mathbb{C}_1}} := \mathbb{H}_{A\hat{\otimes}\mathbb{C}_1} \oplus \mathbb{H}^{\mathrm{opp}}_{A\hat{\otimes}\mathbb{C}_1} \cong (\mathbb{H}_A\hat{\otimes}\mathbb{C}_1) \oplus (\mathbb{H}_A\hat{\otimes}\mathbb{C}_1)^{\mathrm{opp}} = (\mathbb{H}_A\hat{\otimes}\mathbb{C}_1) \oplus (\mathbb{H}_A^{\mathrm{opp}}\hat{\otimes}\mathbb{C}_1) \cong (\mathbb{H}_A \oplus \mathbb{H}_A^{\mathrm{opp}})\hat{\otimes}\mathbb{C}_1 = \widehat{\mathbb{H}_A}\hat{\otimes}\mathbb{C}_1 \,.$$

□

Lemma 10.34. Let A be a C*-algebra. $\mathcal{M}(\mathbb{K}\otimes A\hat{\otimes}\mathbb{C}_1) \cong \mathcal{M}(\mathbb{K}\otimes A)\hat{\otimes}\mathbb{C}_1$ as $\mathbb{Z}_2$-graded C*-algebras.

Proof. According to Lemma 9.10 and Remark 10.18, $\mathcal{M}(\mathbb{K}\otimes A\hat{\otimes}\mathbb{C}_1) \cong \mathbb{B}(\mathbb{K}\otimes A\hat{\otimes}\mathbb{C}_1)$ as $\mathbb{Z}_2$-graded C*-algebras. Let $F \in \mathbb{B}(\mathbb{K}\otimes A\hat{\otimes}\mathbb{C}_1)$, then $F((x_1\hat{\otimes}y_1)(x_2\hat{\otimes}y_2)) = (F(x_1\hat{\otimes}y_1))(x_2\hat{\otimes}y_2)$ for $x_1, x_2 \in \mathbb{K}\otimes A$ and $y_1, y_2 \in \mathbb{C}_1$ as F preserves the $\mathbb{K}\otimes A\hat{\otimes}\mathbb{C}_1$-action. Now F can be decomposed as $F = F^{(0)} + F^{(1)}$ with $F^{(0)} \in \mathbb{B}(\mathbb{K}\otimes A\hat{\otimes}\mathbb{C}_1)^{(0)}$ and $F^{(1)} \in \mathbb{B}(\mathbb{K}\otimes A\hat{\otimes}\mathbb{C}_1)^{(1)}$. It is known what elements of $(\mathbb{K}\otimes A\hat{\otimes}\mathbb{C}_1)^{(0)}$ look like, they are of the form $x\hat{\otimes}1$ for $x \in \mathbb{K}\otimes A$. Similarly, elements of $(\mathbb{K}\otimes A\hat{\otimes}\mathbb{C}_1)^{(1)}$ are of the form $x\hat{\otimes}\epsilon_1$, for $x \in \mathbb{K}\otimes A$. $\mathbb{C}_1$ is unital, so $\mathcal{M}(\mathbb{C}_1) \cong \mathbb{C}_1$. Both $F^{(0)}$ and $F^{(1)}$ preserve the right-action, so $F^{(0)}$ is of the form $G\hat{\otimes}1$ and $F^{(1)}$ is of the form $G\hat{\otimes}\epsilon_1$ for $G \in \mathcal{M}(\mathbb{K}\otimes A)$. □

Remark 10.35. An alternative proof can be given, so that Lemma 10.34 actually holds for $\mathbb{Z}_2$-graded C*-algebras A. Use Lemma 9.14, Lemma 10.32 and Remark 10.18 and the fact that $\mathbb{C}_1$ is finite dimensional and unital to get the follows $\mathbb{Z}_2$-graded $*$-isomorphisms:

$$\mathcal{M}(\mathbb{K}\otimes A\hat{\otimes}\mathbb{C}_1) \cong \mathcal{M}(\mathbb{K}(\mathbb{H}_{A\hat{\otimes}\mathbb{C}_1})) \cong \mathcal{M}(\mathbb{K}(\mathbb{H}_A\hat{\otimes}\mathbb{C}_1)) \cong \mathcal{M}(\mathbb{K}(\mathbb{H}_A)\hat{\otimes}\mathbb{C}_1) \cong \mathcal{M}(\mathbb{K}(\mathbb{H}_A))\hat{\otimes}\mathbb{C}_1 \cong \mathcal{M}(\mathbb{K}\otimes A)\hat{\otimes}\mathbb{C}_1 \,.$$

As we saw in the proof of Lemma 10.34, if A is ungraded, then $\mathcal{M}(\mathbb{K}\otimes A)$ is ungraded and homogeneous elements of $\mathcal{M}(\mathbb{K}\otimes A)\hat{\otimes}\mathbb{C}_1$ have a nice form.

Remark 10.36. It is not true that $\mathcal{M}(A\hat{\otimes}B) \cong \mathcal{M}(A)\hat{\otimes}B$ for $\mathbb{Z}_2$-graded C*-algebras A and B in general; for example $\mathbb{B} \cong \mathcal{M}(\mathbb{K}) \cong \mathcal{M}(\mathbb{K}\otimes\mathbb{K}) \ncong \mathcal{M}(\mathbb{K})\otimes\mathbb{K} \cong \mathbb{B}\otimes\mathbb{K}$, see Examples 5.51 and 9.18.

10.4 Different $\mathbb{Z}_2$-gradings on $M_2(\mathbb{K}\otimes A)$

Throughout this thesis C*-algebras are considered to be trivially graded. Let A be a C*-algebra. Now $\mathbb{K}\otimes A$ can be trivially graded. The following (non-graded) isomorphisms have been established; $\mathbb{K}\otimes A \cong \mathbb{K}(\mathbb{H}_A) \cong \mathbb{K}(\mathbb{H}_A \oplus \mathbb{H}_A) = M_2(\mathbb{K}(\mathbb{H}_A)) = M_2(\mathbb{K}\otimes A)$ (for inspiration, see Definition 13.1). There is also a graded isomorphism $\mathbb{K}(\widehat{\mathbb{H}_A}) \cong M_2(\mathbb{K}\otimes A)$ (see Lemma 10.21). In the latter case, $M_2(\mathbb{K}\otimes A)$ is endowed with the even grading on 2×2-matrices (see Example 10.4). The even and the trivial gradings on $M_2(\mathbb{K}\otimes A)$ are different. However tensored with $\mathbb{C}_1$, they will get the same grading. We will establish an isomorphism between them.

Definition 10.37. Let A be a C*-algebra. Denote by $M_2(\mathbb{K}\otimes A)^{\text{even}}$ the endowment of $M_2(\mathbb{K}\otimes A)$ with the even grading and denote by $M_2(\mathbb{K}\otimes A)^{\text{triv}}$ the endowment with the trivial grading.

Lemma 10.38. The following function preserves the $\mathbb{Z}_2$-grading:

$$M_2(\mathbb{K}\otimes A)^{\text{even}}\hat{\otimes}\,\mathbb{C}_1 \ni x\hat{\otimes}y \to x\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{\partial(y)}\hat{\otimes}\, y\epsilon_1^{\partial(x)} \in M_2(\mathbb{K}\otimes A)^{\text{triv}}\hat{\otimes}\,\mathbb{C}_1\,. \tag{38}$$

Proof. The easiest thing to do is to write down all possible homogeneous pure tensors. We will distinguish four cases. Let $a,b,c,d\in\mathbb{K}\otimes A$ and $\lambda,\mu\in\mathbb{C}$. In $M_2(\mathbb{K}\otimes A)^{\text{triv}}\hat{\otimes}\,\mathbb{C}_1$ a homogeneous pure tensor is even precisely when the $\mathbb{C}_1$-side is even and odd precisely when the $\mathbb{C}_1$-side is odd.

- An even element of $M_2(\mathbb{K}\otimes A)^{\text{even}}$ tensored with an even element of $\mathbb{C}_1$ gives an even element in $M_2(\mathbb{K}\otimes A)^{\text{even}}\hat{\otimes}\,\mathbb{C}_1$. The image under this function is even in $M_2(\mathbb{K}\otimes A)^{\text{triv}}\hat{\otimes}\,\mathbb{C}_1$ as

$$M_2(\mathbb{K}\otimes A)^{\text{even}}\hat{\otimes}\,\mathbb{C}_1 \ni \begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}\hat{\otimes}\lambda 1 \to \begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}\hat{\otimes}\lambda 1.$$

- An even element of $M_2(\mathbb{K}\otimes A)^{\text{even}}$ tensored with an odd element of $\mathbb{C}_1$ gives an odd element in $M_2(\mathbb{K}\otimes A)^{\text{even}}\hat{\otimes}\,\mathbb{C}_1$. The image under this function is odd in $M_2(\mathbb{K}\otimes A)^{\text{triv}}\hat{\otimes}\,\mathbb{C}_1$ as

$$M_2(\mathbb{K}\otimes A)^{\text{even}}\hat{\otimes}\,\mathbb{C}_1 \ni \begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}\hat{\otimes}\mu\epsilon_1 \to \begin{pmatrix}a & 0\\ 0 & -d\end{pmatrix}\hat{\otimes}\mu\epsilon_1.$$

- An odd element of $M_2(\mathbb{K}\otimes A)^{\text{even}}$ tensored with an even element of $\mathbb{C}_1$ gives an odd element in $M_2(\mathbb{K}\otimes A)^{\text{even}}\hat{\otimes}\,\mathbb{C}_1$. The image under this function is odd in $M_2(\mathbb{K}\otimes A)^{\text{triv}}\hat{\otimes}\,\mathbb{C}_1$ as

$$M_2(\mathbb{K}\otimes A)^{\text{even}}\hat{\otimes}\,\mathbb{C}_1 \ni \begin{pmatrix}0 & b\\ c & 0\end{pmatrix}\hat{\otimes}\lambda 1 \to \begin{pmatrix}0 & b\\ c & 0\end{pmatrix}\hat{\otimes}\lambda\epsilon_1.$$

- An odd element of $M_2(\mathbb{K}\otimes A)^{\text{even}}$ tensored with an odd element of $\mathbb{C}_1$ gives an even element in $M_2(\mathbb{K}\otimes A)^{\text{even}}\hat{\otimes}\,\mathbb{C}_1$. The image under this function is even in $M_2(\mathbb{K}\otimes A)^{\text{triv}}\hat{\otimes}\,\mathbb{C}_1$ as

$$M_2(\mathbb{K}\otimes A)^{\text{even}}\hat{\otimes}\,\mathbb{C}_1 \ni \begin{pmatrix}0 & b\\ c & 0\end{pmatrix}\hat{\otimes}\mu\epsilon_1 \to \begin{pmatrix}0 & b\\ -c & 0\end{pmatrix}\hat{\otimes}\mu\epsilon_1^2 = \begin{pmatrix}0 & b\\ -c & 0\end{pmatrix}\hat{\otimes}\mu 1.$$

By inspection of the above cases; the mapping preserves grading. □

Lemma 10.39. The function of Lemma 10.38 defines a $*$-homomorphism.

Proof. We will first show the function of Lemma 10.38 preserves the $*$-operation. For homogeneous elements in $M_2(\mathbb{K}\otimes A)^{\text{triv}}\hat{\otimes}\,\mathbb{C}_1$ the $*$-operation differs only from the $*$-operation in $M_2(\mathbb{K}\otimes A)^{\text{even}}\hat{\otimes}\,\mathbb{C}_1$ in the case of an off-diagonal 2×2-matrix tensored with an odd element of $\mathbb{C}_1$. In that case the $*$-operations differ by a minus sign. Let $x\in M_2(\mathbb{K}\otimes A)^{\text{even}}$ and $y\in\mathbb{C}_1$ be homogeneous, then by equation (38):

$$M_2(\mathbb{K}\otimes A)^{\text{even}}\hat{\otimes}\,\mathbb{C}_1 \ni (x\hat{\otimes}y)^* \to (-1)^{\partial(x)\partial(y)}x^*\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{\partial(y)}\hat{\otimes}\, y^*\epsilon_1^{\partial(x)} \in M_2(\mathbb{K}\otimes A)^{\text{triv}}\hat{\otimes}\,\mathbb{C}_1,\ \text{while}$$

$$M_2(\mathbb{K}\otimes A)^{\mathrm{triv}}\hat{\otimes}\mathbb{C}_1 \ni \left(x\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{\partial(y)}\right)^* \hat{\otimes} \left(y\epsilon_1^{\partial(x)}\right)^* = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{\partial(y)} x^*\hat{\otimes}\epsilon_1^{\partial(x)}y^*.$$

As $\mathbb{C}_1$ is commutative and $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ commutes with the diagonal matrices of $M_2(\mathbb{K}\otimes A)$ and anti-commutes with the off-diagonal matrices of $M_2(\mathbb{K}\otimes A)$, it follows that the function of Lemma 10.38 preserves the $*$-operation.

We will show that the map of Lemma 10.38 preserves the multiplication. Let $x_1, x_2 \in M_2(\mathbb{K}\otimes A)^{\mathrm{even}}$ and $y_1, y_2 \in \mathbb{C}_1$ be homogeneous. The multiplication $(x_1\hat{\otimes}y_1)(x_2\hat{\otimes}y_2)$ in $M_2(\mathbb{K}\otimes A)^{\mathrm{even}}$ differs from the multiplication in $M_2(\mathbb{K}\otimes A)^{\mathrm{triv}}$ only if y_1 and x_2 are odd. In that case the multiplications differ by a minus sign. By equation (38):

$$M_2(\mathbb{K}\otimes A)^{\mathrm{even}}\hat{\otimes}\mathbb{C}_1 \ni x_1x_2\hat{\otimes}y_1y_2 \to x_1x_2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{\partial(y_1y_2)} \hat{\otimes}\, y_1y_2\epsilon_1^{\partial(x_1x_2)} =$$

$$x_1x_2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{\partial(y_1)+\partial(y_2)} \hat{\otimes}\, y_1y_2\epsilon_1^{\partial(x_1)+\partial(x_2)} \in M_2(\mathbb{K}\otimes A)^{\mathrm{triv}}\hat{\otimes}\mathbb{C}_1, \text{ while}$$

$$M_2(\mathbb{K}\otimes A)^{\mathrm{triv}}\hat{\otimes}\mathbb{C}_1 \ni \left(x_1\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{\partial(y_1)} \hat{\otimes}\, y_1\epsilon_1^{\partial(x_1)}\right)\left(x_2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{\partial(y_2)} \hat{\otimes}\, y_2\epsilon_1^{\partial(x_2)}\right) =$$

$$(-1)^{\partial(y_1)\partial(x_2)}\left(x_1x_2\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{\partial(y_1)+\partial(y_2)} \hat{\otimes} y_1y_2\epsilon^{\partial(x_1)+\partial(x_2)}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{\partial(y)} x^*\hat{\otimes}\epsilon_1^{\partial(x)}y^*.$$

As $\mathbb{C}_1$ is commutative and $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ commutes with the diagonal matrices of $M_2(\mathbb{K}\otimes A)$ and anti-commutes with the off-diagonal matrices of $M_2(\mathbb{K}\otimes A)$, it follows that the function of Lemma 10.38 preserves the multiplication. ☐

Lemma 10.40. The function of Lemma 10.38 is bijective.

Proof. Any tensor in $M_2(\mathbb{K}\otimes A)^{\mathrm{even}}\hat{\otimes}\mathbb{C}_1$ is a linear sum of the four forms of homogeneous pure tensors described in the proof of Lemma 10.38. The four forms are linearly independent. The proof of Lemma 10.38 shows clearly that the mapping gives a bijection on those four forms. Therefore the kernel of the mapping is 0. The mapping is injective.
Any of the four forms of homogeneous pure tensors is included in the image. Therefore there exists a basis of $M_2(\mathbb{K}\otimes A)^{\mathrm{triv}}\hat{\otimes}\mathbb{C}_1$ that is included in the image. Hence the mapping is surjective. ☐

Proposition 10.41. $M_2(\mathbb{K}\otimes A)^{\mathrm{triv}}\hat{\otimes}\mathbb{C}_1$ and $M_2(\mathbb{K}\otimes A)^{\mathrm{even}}\hat{\otimes}\mathbb{C}_1$ are isomorphic as $\mathbb{Z}_2$-graded C*-algebras.

Proof. The function in equation (38) in Lemma 10.38 is a $\mathbb{Z}_2$-graded bijective $*$-homomorphism by Lemma 10.38, Lemma 10.40 and Lemma 10.39. ☐

Corollary 10.42. $M_2(\mathcal{M}(\mathbb{K}\otimes A))^{\mathrm{triv}}\hat{\otimes}\mathbb{C}_1$ and $M_2(\mathcal{M}(\mathbb{K}\otimes A))^{\mathrm{even}}\hat{\otimes}\mathbb{C}_1$ are isomorphic as $\mathbb{Z}_2$-graded C*-algebras.

Proof. As $M_2(\mathbb{K}\otimes A)^{\text{triv}}\hat{\otimes}\mathbb{C}_1$ and $M_2(\mathbb{K}\otimes A)^{\text{even}}\hat{\otimes}\mathbb{C}_1$ are isomorphic as $\mathbb{Z}_2$-graded C*-algebras, so are their multiplier algebras. Remark 10.35 implies the following isomorphisms:

$$\mathcal{M}(M_2(\mathbb{K}\otimes A)^{\text{triv}}\hat{\otimes}\mathbb{C}_1) \cong \mathcal{M}(M_2(\mathbb{K}\otimes A)^{\text{triv}})\hat{\otimes}\mathbb{C}_1 \cong M_2(\mathcal{M}(\mathbb{K}\otimes A))^{\text{triv}}\hat{\otimes}\mathbb{C}_1$$
$$\mathcal{M}(M_2(\mathbb{K}\otimes A)^{\text{even}}\hat{\otimes}\mathbb{C}_1) \cong \mathcal{M}(M_2(\mathbb{K}\otimes A)^{\text{even}})\hat{\otimes}\mathbb{C}_1 \cong M_2(\mathcal{M}(\mathbb{K}\otimes A))^{\text{even}}\hat{\otimes}\mathbb{C}_1\,.$$

□

Remark 10.43. Alternatively, the function in equation (38) in Lemma 10.38 gives a well-defined $\mathbb{Z}_2$-graded function for $a,b,c,d\in\mathcal{M}(\mathbb{K}\otimes A)$. By the same arguments, the analogous mapping between multiplier algebras is a bijective $*$-homomorphism. It was chosen to give the proof for $\mathbb{K}\otimes A$ instead of $\mathcal{M}(\mathbb{K}\otimes A)$ for notational reasons.

Corollary 10.44. There is a bijection between even elements of $\mathbb{B}(\widehat{\mathbb{H}_{A\hat{\otimes}\mathbb{C}_1}})$ and $\mathcal{M}(\mathbb{K}\otimes A)\hat{\otimes}\mathbb{C}_1$. In particular, every even element of $\mathbb{B}(\widehat{\mathbb{H}_{A\hat{\otimes}\mathbb{C}_1}})$ can be written as $\psi\hat{\otimes}1$ with $\psi\in\mathcal{M}(\mathbb{K}\otimes A)$.
Every odd element of $\mathbb{B}(\widehat{\mathbb{H}_{A\hat{\otimes}\mathbb{C}_1}})$ can be written in the form $\psi\hat{\otimes}\epsilon_1$ with $\psi\in\mathcal{M}(\mathbb{K}\otimes A)$.

Proof. $\mathbb{B}(\widehat{\mathbb{H}_A})\cong M_2(\mathcal{M}(\mathbb{K}\otimes A))^{\text{even}}$ follows from Lemma 10.21 and $\mathbb{B}(\widehat{\mathbb{H}_{A\hat{\otimes}\mathbb{C}_1}})\cong$
$M_2(\mathcal{M}(\mathbb{K}\otimes A\hat{\otimes}\mathbb{C}_1))^{\text{even}}\cong M_2(\mathcal{M}(\mathbb{K}\otimes A))^{\text{even}}\hat{\otimes}\mathbb{C}_1$ follows from Lemma 10.34.
$M_2(\mathcal{M}(\mathbb{K}\otimes A))^{\text{even}}\hat{\otimes}\mathbb{C}_1\cong M_2(\mathcal{M}(\mathbb{K}\otimes A))^{\text{triv}}\hat{\otimes}\mathbb{C}_1$ follows from Corollary 10.42. Under the isomorphism $\mathbb{H}_A\oplus\mathbb{H}_A\cong\mathbb{H}_A$, the graded isomorphism $M_2(\mathcal{M}(\mathbb{K}\otimes A))^{\text{triv}}\hat{\otimes}\mathbb{C}_1\cong\mathcal{M}(\mathbb{K}\otimes A)\hat{\otimes}\mathbb{C}_1$ in which $\mathcal{M}(\mathbb{K}\otimes A)$ has the trivial grading, is established. This proves the statement. □

Remark 10.45. Let $A\xrightarrow{\phi'}\mathbb{B}(\widehat{\mathbb{H}_{B\hat{\otimes}\mathbb{C}_1}})$ be an even $*$-homomorphism. Now $\phi'(a)\in\mathbb{B}(\widehat{\mathbb{H}_{B\hat{\otimes}\mathbb{C}_1}})$ can be written as $\phi_a\hat{\otimes}1$ with $\phi_a\in\mathcal{M}(\mathbb{K}\otimes B)$. This assignment $A\ni a\to\phi_a\in\mathcal{M}(\mathbb{K}\otimes B)$ defines a $*$-homomorphisms. The linearity is obvious. Let $a\in A$, then $\phi'(a^*)=\phi'(a)^*=\phi_a^*\otimes 1$. So $\phi_{a^*}=\phi_a^*$. Let $a,b\in A$, then $\phi'(a)$ and $\phi'(b)$ can be written as $\phi_a\hat{\otimes}1$ and $\phi_b\hat{\otimes}1$, with $\phi_a,\phi_b\in\mathcal{M}(\mathbb{K}\otimes B)$. Now $\phi'(ab)=\phi'(a)\phi'(b)=(\phi_a\hat{\otimes}1)(\phi_b\hat{\otimes}1)=\phi_a\phi_b\hat{\otimes}1$. So $\phi_{ab}=\phi_a\phi_b$. We conclude ϕ' can be written as $\phi\hat{\otimes}1$, where $A\xrightarrow{\phi}\mathcal{M}(\mathbb{K}\otimes B)$ is a $*$-homomorphism.

10.5 Functoriality of Hilbert modules

Definition 10.46. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. Let $A\xrightarrow{f}B$ be an even $*$-homomorphism. Let E be a $\mathbb{Z}_2$-graded Hilbert A-module. Define $E\otimes_f B$ to be the B-module in which $e\otimes f(a)b=ea\otimes b$ for $e\in E$, $a\in A$ and $b\in B$. Define $E\hat{\otimes}_f B$ to be the $\mathbb{Z}_2$-graded completion with respect to the the inner product $\langle e_1\otimes b_1,e_2\otimes b_2\rangle_{E\hat{\otimes}_f B}=\langle b_1,f(\langle e_1,e_2\rangle_E)b_2\rangle_B=b_1^*f(\langle e_1,e_2\rangle_E)b_2$ for $e_1,e_2\in E$ and $b_1,b_2\in B$.

Remark 10.47. In the tensor product $E\hat{\otimes}_f B$ we treat B as a Hilbert module over itself. Recall B is endowed the inner product given in Example 9.3.

10.6 Graded K-groups

Rings can be $\mathbb{Z}_2$-graded in the same way C*-algebras are $\mathbb{Z}_2$-graded. Modules of a ring can be $\mathbb{Z}_2$-graded in the same way Hilbert modules are $\mathbb{Z}_2$-graded. The purpose of this subsection is to underscore that K_0- and K_1-groups together form a $\mathbb{Z}_2$-graded abelian group. Assume any ring is trivially graded. For more information on grading of rings and their modules, see [17, p. 172 & p. 427].

Definition 10.48. Let A be a C*-algebra. Define $\mathrm{K}_*(A) := \mathrm{K}_0(A) \oplus \mathrm{K}_1(A)$. $\mathrm{K}_0(A)$ will be the even part and $\mathrm{K}_1(A)$ will be the odd part of $\mathrm{K}_*(A)$.

Lemma 10.49. Let A be a C*-algebra. $\mathrm{K}_*(A)^{\mathrm{opp}} = \mathrm{K}_*(SA)$.

Proof. This follows from Lemma 7.65 and Theorem 7.66; $\mathrm{K}_0(SA) = \mathrm{K}_1(A)$ and $\mathrm{K}_1(SA) = \mathrm{K}_0(A)$. □

Example 10.50. Let A and B be C*-algebras. Just as adjointable operators between Hilbert modules inherit a $\mathbb{Z}_2$-grading, so do homomorphisms (of abelian groups). From additivity follows:

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) &= \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A)\oplus\mathrm{K}_1(A),\mathrm{K}_0(B)\oplus\mathrm{K}_1(B)) \\ &= \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)) \\ &\oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B)).\end{aligned} \tag{39}$$

In terms of Corollary 10.14, $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ has the following grading:

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))^{(0)} = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B))$$

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))^{(1)} = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)).$$

Example 10.51. Example 10.50 can be generalised; Lemma 4.48 shows that $\mathrm{Hom}_{\mathbb{Z}}$ and $\mathrm{Ext}^0_{\mathbb{Z}}$ are isomorphic functors. But similar to Example 10.50, from additivity follows

$$\begin{aligned}\mathrm{Ext}^i_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) &= \mathrm{Ext}^i_{\mathbb{Z}}(\mathrm{K}_0(A)\oplus\mathrm{K}_1(A),\mathrm{K}_0(B)\oplus\mathrm{K}_1(B)) \\ &= \mathrm{Ext}^i_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Ext}^i_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)) \\ &\oplus \mathrm{Ext}^i_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)) \oplus \mathrm{Ext}^i_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B)) \text{ for } i \geq 0.\end{aligned} \tag{40}$$

In terms of Corollary 10.14, $\mathrm{Ext}^i_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ has the following grading:

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))^{(0)} = \mathrm{Ext}^i_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Ext}^i_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B)) \text{ for } i \geq 0$$

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))^{(1)} = \mathrm{Ext}^i_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)) \oplus \mathrm{Ext}^i_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)) \text{ for } i \geq 0.$$

$\mathbb{Z}$ is a PID, $\mathrm{Hom}_{\mathbb{Z}}$ and $\mathrm{Ext}^1_{\mathbb{Z}}$ are the only non-constant Ext-functors (see Example 4.46).

Part III
KK-theory

11 First glance at KK-theory

In this section we will introduce KK-theory. We will define Kasparov modules and define a few equivalence relations on Kasparov modules of $\mathbb{Z}_2$-graded C*-algebras in the first subsection. The most important of them is homotopy. We will show how $*$-homomorphisms form a map between Kasparov modules. A $*$-homomorphism induces a map between homotopy classes. This way the homotopy classes form a functor; KK. We will state the most important properties of KK. We will finish this section by adding a biadditive structure to KK.

11.1 Kasparov modules and equivalence relations

Definition 11.1. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. A *Kasparov module* for the pair (A, B) is a triple (E, ϕ, F) consisting of a countably generated Hilbert B-module E, an even $*$-homomorphism $\phi: A \to \mathbb{B}(E)$ and an odd operator $F \in \mathbb{B}(E)$ such that $[F, \phi(a)], (F^2 - 1)\phi(a)$ and $(F - F^*)\phi(a)$ are in $\mathbb{K}(E)$ for all $a \in A$. The set of Kasparov A-B-modules is denoted by $\mathbb{E}(A, B)$. If $[F, \phi(a)] = 0$, $(F^2 - 1)\phi(a) = 0$ and $(F - F^*)\phi(a) = 0$ for all $a \in A$, then (E, ϕ, F) is called *degenerate*. The set of degenerate Kasparov A-B-modules is denoted by $\mathbb{D}(A, B)$.

Remark 11.2. Some authors do not require the Hilbert B-module of a Kasparov A-B-module to be countably generated. Later on, this will make a difference when we restrict to special subcategories of C*-algebras. We stick to the definition in [2].

Remark 11.3. The brackets $[.,.]$ denote the graded brackets on $\mathbb{B}(E)$, not commutator brackets. For its properties, see [2, Proposition 14.1.3]. The behaviour of the brackets depends on the grading. Graded brackets are bilinear. Via decomposition into homogeneous elements, its complete behaviour can be determined.

Remark 11.4. Kasparov modules are defined over $\mathbb{Z}_2$-graded C*-algebras and Hilbert modules. Using Example 10.12, Kasparov modules can be constructed for any (non-graded) C*-algebra. A Kasparov A-B-module (E, ϕ, F) with E trivially graded satisfies $F = 0$ by Corollary 10.14.

Example 11.5. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. Let $A \xrightarrow{f} B$ be an even $*$-homomorphism. Then $(B, f, 0) \in \mathbb{E}(A, B)$. Here B is the Hilbert B-module defined in Example 9.3. The even $*$-homomorphisms f sends elements of A to adjointable operators on B, as $B \subset \mathcal{M}(B) \cong \mathbb{B}(B)$ by Lemma 9.10.

Definition 11.6. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. Kasparov A-B-modules (E, ϕ, F) and (E', ϕ', F) are *unitarily equivalent* if there exists an even unitary $u \in \mathbb{B}(E, E')$ such that $u\phi(a)u^* = \phi'(a)$ for all $a \in A$ and $uFu^* = F'$. Notation: $(E, \phi, F) \approx_u (E', \phi', F)$.

Definition 11.7. Let A and B be $\mathbb{Z}_2$-graded C*-algebras and let (E_0, ϕ_0, F_0) and (E_1, ϕ_1, F_1) be Kasparov A-B-modules. A *homotopy* of the Kasparov A-B-modules (E_0, ϕ_0, F_0) and (E_1, ϕ_1, F_1) is a Kasparov A-$C([0,1], B)$-module (E, ϕ, F) for which $(E \hat{\otimes}_{\mathrm{ev}_0} B, \phi \hat{\otimes}_{\mathrm{ev}_0} 1_B, F \hat{\otimes}_{\mathrm{ev}_0} 1_B) \approx_u (E_0, \phi_0, F_0)$ and $(E \hat{\otimes}_{\mathrm{ev}_1} B, \phi \hat{\otimes}_{\mathrm{ev}_1} 1_B, F \hat{\otimes}_{\mathrm{ev}_1} 1_B) \approx_u (E_1, \phi_1, F_1)$. Notation: $(E_0, \phi_0, F_0) \approx_h (E_1, \phi_1, F_1)$. The set

of homotopy classes of Kasparov A-B-modules is denoted by $\mathrm{KK}(A,B)$.
An *operator homotopy* is a homotopy (E,ϕ,F) in which E and ϕ are constant over $[0,1]$. Operator homotopy will be denoted by $\approx_{oh}$. The set of operator homotopy classes of Kasparov A-B-modules is denoted by $\mathrm{KK}_{oh}(A,B)$.

Lemma 11.8. Unitary equivalence is an equivalence relation.

Proof. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. Reflexivity: Let $(E,\phi,F)\in\mathbb{E}(A,B)$, then $\mathrm{id}_E\,\phi(a)\,\mathrm{id}_E=\phi(a)$ for all $a\in A$ and $\mathrm{id}_E\,F\,\mathrm{id}_E=F$.
Symmetry: Let $(E,\phi,F),(E',\phi',F')\in\mathbb{E}(A,B)$. If there exists an even unitary $u\in\mathbb{B}(E,E')$ such that $u\phi(a)u^*=\phi'(a)$ for all $a\in A$ and $uFu^*=F'$, then

$$u^*\phi'(a)u=u^*u\phi(a)u^*u=\mathrm{id}_E\,\phi(a)\,\mathrm{id}_E=\phi(a)\text{ for all }a\in A$$
$$u^*F'u=u^*uFu^*u=\mathrm{id}_E\,F\,\mathrm{id}_E=F.$$

Transitivity: Let $(E,\phi,F),(E',\phi',F'),(E'',\phi'',F'')\in\mathbb{E}(A,B)$. If there exists an even unitary $u\in\mathbb{B}(E,E')$ such that $u\phi(a)u^*=\phi'(a)$ for all $a\in A$ and $uFu^*=F'$ and there exists an even unitary $v\in\mathbb{B}(E',E'')$ such that $v\phi'(a)v^*=\phi''(a)$ for all $a\in A$ and $vFv^*=F''$, then

$$\phi''(a)=v\phi'(a)v^*=vu\phi(a)u^*v^*\text{ for all }a\in A\text{ and }F''=vF'v^*=vuFu^*v^*.$$ □

Lemma 11.9. Homotopy and operator homotopy are equivalence relations.

Proof. The proof for an operator homotopy is the same as for a homotopy. We will show that homotopy is an equivalence relation. Let A and B be C*-algebras. Reflexivity: Let $(E,\phi,F)\in\mathbb{E}(A,B)$. Then $(E,\phi,F)\approx_h(E,\phi,F)$ via $(C([0,1],E),\phi,F)$, where ϕ and F are constant over $[0,1]$.
Symmetry: Let $(E_0,\phi_0,F_0),(E_1,\phi_1,F_1)\in\mathbb{E}(A,B)$. Let (E_0,ϕ_0,F_0) and (E_1,ϕ_1,F_1) be homotopic via $(E,\phi,F)\in\mathbb{E}(A,C([0,1],B))$. Define E' to be the Hilbert $C([0,1],B)$-module linearly isomorphic to E, but endowed with the following $C([0,1],B)$-action: $e\cdot_{E'}f(t):=e\cdot_E f(1-t)$ for all $e\in E$, $f\in C([0,1],B)$ and $t\in[0,1]$. Define ϕ' to be the following $*$-homomorphism $\phi'(a)(t):=\phi(a)(1-t)$ for all $a\in A$ and $t\in[0,1]$. Define F' to be the odd operator $F'(t):=F(1-t)$ for all $t\in[0,1]$. Now $(E_1,\phi_1,F_1)\approx_h(E_0,\phi_0,F_0)$ via (E',ϕ',F').
Transitivity: Let $(E_0,\phi_0,F_0),(E_1,\phi_1,F_1),(E_2,\phi_2,F_2)\in\mathbb{E}(A,B)$. If $(E_0,\phi_0,F_0)\approx_h(E_1,\phi_1,F_1)$ via $(E,\phi,F)\in\mathbb{E}(A,C([0,1],B))$ and $(E_1,\phi_1,F_1)\approx_h(E_2,\phi_2,F_2)$ via $(E',\phi',F')\in\mathbb{E}(A,C([1,2],B))$, then $(E_0,\phi_0,F_0)\approx(E_2,\phi_2,F_2)$ via the composition homotopy $(E'',\phi'',F'')\in\mathbb{E}(A,C([0,1],B)$ in which $E'':=E\oplus E'$, $\phi''(a):=\begin{pmatrix}\phi(a)&0\\0&\phi'(a)\end{pmatrix}$ and $F'':=\begin{pmatrix}F&0\\0&F'\end{pmatrix}$. Note that $(E\hat{\otimes}_{\mathrm{ev}_s}B,\phi\hat{\otimes}_{\mathrm{ev}_s}1_B,F\hat{\otimes}_{\mathrm{ev}_s}1_B)=(0,0,0)$ for $s>1$ and $(E'\hat{\otimes}_{\mathrm{ev}_t}B,\phi'\hat{\otimes}_{\mathrm{ev}_t}1_B,F'\hat{\otimes}_{\mathrm{ev}_t}1_B)=(0,0,0)$ for $t<1$. □

Remark 11.10. Operator homotopy is a stronger equivalence relation than homotopy, but not necessarily strictly stronger. In some cases they agree. In Theorem 11.21 we will see a case in which homotopy and operator homotopy agree.

Definition 11.11. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. For Kasparov A-B-modules (E_1,ϕ_1,F_1) and (E_2,ϕ_2,F_2) , we define the *addition* as $(E_1,\phi_1,F_1)+(E_2,\phi_2,F_2):=(E_1\oplus E_2,(\phi_1,\phi_2),(F_1,F_2))$.

Remark 11.12. It is obvious that the addition gives a Kasparov A-B-module. Furthermore the addition is associative and commutative (up to unitary equivalence).

Lemma 11.13. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. If $(E_1, \phi_1, F_1) \approx_h (E_1', \phi_1', F_1')$ and $(E_2, \phi_2, F_2) \approx_h (E_2', \phi_2', F_2')$, then $(E_1 \oplus E_2, (\phi_1, \phi_2), (F_1, F_2)) \approx_h (E_1' \oplus E_2', (\phi_1', \phi_2'), (F_1', F_2'))$.

Proof. Add the respective homotopies in the sense of Definition 11.11 to obtain the homotopy between the two additions. □

Lemma 11.14. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. If $(E_1, \phi_1, F_1) \approx_{oh} (E_1', \phi_1', F_1')$ and $(E_2, \phi_2, F_2) \approx_{oh} (E_2', \phi_2', F_2')$, then $(E_1 \oplus E_2, (\phi_1, \phi_2), (F_1, F_2)) \approx_{oh} (E_1' \oplus E_2', (\phi_1', \phi_2'), (F_1', F_2'))$.

Proof. Similar to the proof of Lemma 11.13. □

Corollary 11.15. The addition of Kasparov modules gives a well-defined addition of KK-classes and KK_{oh}-classes.

Proof. Lemmas 11.13 and 11.14 show that the addition defined in Definition 11.11 is independent of the choice of representatives in the (operator) homotopy class. □

Theorem 11.16. All degenerate Kasparov A-B-modules are homotopic to $(0, 0, 0)$.

Proof. Take $(E, \phi, F) \in \mathbb{D}(A, B)$. Then $C_0([0,1), E)$ is a Hilbert $C_0([0,1), B)$-module via pointwise multiplication, when $C_0([0,1), B)$ and $C_0([0,1), E)$ inherit the $\mathbb{Z}_2$-gradings of B and E. Notice that $C_0([0,1), E)$ can be written as a $C_0([0,1), B)\hat{\otimes}E$ and $(C_0([0,1), E), 1_{C_0([0,1),B)}\hat{\otimes}\phi, 1_{C_0([0,1),B)}\hat{\otimes}F)$ is a Kasparov A-$C_0([0,1), B)$-module. As (E, ϕ, F) is degenerate, $[F, \phi(a)] = 0$, $(F^2 - 1)\phi(a) = 0$ and $(F - F^*)\phi(a) = 0$ for all $a \in A$. So $(C_0([0,1), E), 1_{C_0([0,1),B)}\hat{\otimes}\phi, 1_{C_0([0,1),B)}\hat{\otimes}F)$ is degenerate. Now $(C_0([0,1), E)\hat{\otimes}_{\mathrm{ev}_0}B, 1_{C_0([0,1),B)}\hat{\otimes}\phi\hat{\otimes}_{\mathrm{ev}_0}1_B, 1_{C_0([0,1),B)}\hat{\otimes}F\hat{\otimes}_{\mathrm{ev}_0}1_B) = (E, \phi, F)$ and $(C_0([0,1), E)\hat{\otimes}_{\mathrm{ev}_1}B, 1_{C_0([0,1),B)}\hat{\otimes}\phi\hat{\otimes}_{\mathrm{ev}_1}1_B, 1_{C_0([0,1),B)}\hat{\otimes}F\hat{\otimes}_{\mathrm{ev}_1}1_B) = (0, 0, 0)$. □

Remark 11.17. The homotopy in the proof of Theorem 11.16 is not a Kasparov A-$C([0,1], B)$-module, but a A-$C([0,1), B)$-module. This construction is still valid, because $[0,1]$ is the one-point compactification of $[0,1)$. This way $C_0([0,1), B)$ forms a $*$-subalgebra of $C([0,1], B)$. It is the same idea as in Proposition 6.44.

Theorem 11.18. Let A and B be C*-algebras, then $\mathrm{KK}(A, B)$ is an abelian group.

Proof. The group action is the addition that follows from Corollary 11.15. The degenerate Kasparov modules (Theorem 11.16) form the unit. Associativity and commutativity follow immediately.
We will now construct the inverses. Let (E, ϕ, F) be a Kasparov A-B-module. Define $\phi' : A \to \mathbb{B}(E^{\mathrm{opp}})$ to be $\phi'(a) = \phi'(a^{(0)} + a^{(1)}) = \phi(a^{(0)}) - \phi(a^{(1)})$.
The Kasparov module $\left(C([0,1], E \oplus E^{\mathrm{opp}}), \begin{pmatrix} \phi & 0 \\ 0 & \phi' \end{pmatrix}, \begin{pmatrix} F\cos(\frac{1}{2}\pi t) & \sin(\frac{1}{2}\pi t) \\ \sin(\frac{1}{2}\pi t) & -F\cos(\frac{1}{2}\pi t) \end{pmatrix}\right)$ forms a homotopy of $(E, \phi, F) + (E^{\mathrm{opp}}, \phi', -F)$ and $\left(E \oplus E^{\mathrm{opp}}, \begin{pmatrix} \phi & 0 \\ 0 & \phi' \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$. Notice that the latter is degenerate. So $[(E^{\mathrm{opp}}, \phi', -F)] \in \mathrm{KK}(A, B)$ is the inverse of $[E, \phi, F)] \in \mathrm{KK}(A, B)$. □

Remark 11.19. $\mathrm{KK}_{oh}(A, B)$ is a group as well. The homotopy in the proof of Theorem 11.18 is an operator homotopy.

Lemma 11.20. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. Let $(\widehat{\mathbb{H}_B}, \phi, F(t)) \in \mathbb{E}(A, C([0,1], B))$ be an operator homotopy. Then there exists a norm continuous function $u : [0,1] \to \mathbb{B}(\widehat{\mathbb{H}_B}) = M_2(\mathcal{M}(\mathbb{K} \otimes B))^{\mathrm{even}}$ (see Lemma 10.21 for the $\mathbb{Z}_2$-graded-isomorphism) such that:

1. $u(0) = 1_{M_2(\mathcal{M}(\mathbb{K}\otimes B))}$
2. $u(t)$ is an invariant even unitary element of $M_2(\mathcal{M}(\mathbb{K}\otimes B))^{\text{even}}$ for all $0 \le t \le 1$
3. $u(t)\phi(a) - \phi(a)u(t) \in M_2(\mathbb{K}\otimes B)$ for all $0 \le t \le 1$ and all $a \in A$
4. $(F(t)u(t) - u(t)F(0))\phi(a) \in M_2(\mathbb{K}\otimes B)$ for all $0 \le t \le 1$ and all $a \in A$.

Proof. See [16, §6, Lemma 1] for the proof. □

Theorem 11.21. Let A be a separable C*-algebra and let B be a σ-unital C*-algebra. On the Kasparov A-B-modules homotopy and operator homotopy agree as equivalence relations.

Proof. See [2, Theorem 17.10.7]. □

11.2 Functoriality

In the previous subsection we defined the KK-group for ($\mathbb{Z}_2$-graded) C*-algebras. In this subsection we will show how a $*$-homomorphism induces a group homomorphism between the KK-groups.

An even $*$-homomorphism $f : A_1 \to A_2$ sends $[(E,\phi,F)] \in \mathrm{KK}(A_2,B)$ to $[(E,\phi\circ f,F)] \in \mathrm{KK}(A_1,B)$. It is an easy observation that a homotopy of two Kasparov A_2-B-modules extends to a homotopy of Kasparov A_1-B-modules by precomposition with f. So the mapping is independent of the choice of representatives and therefore well-defined.

An even $*$-homomorphism $g : B_1 \to B_2$ sends $[(E,\phi,F)] \in \mathrm{KK}(A,B_1)$ to $[(E\hat{\otimes}_g B_2, \phi\hat{\otimes}_g 1_{B_2}, F\hat{\otimes}_g 1_{B_2})] \in \mathrm{KK}(A,B_2)$ (see Definition 10.46). Tensor products are associative, so a homotopy of Kasparov A-B_1-modules extends to a homotopy of Kasparov A-B_2-modules.

Lemma 11.22. $\mathrm{KK}\colon \mathsf{C}^* - \mathsf{alg}^{\mathrm{opp}} \times \mathsf{C}^* - \mathsf{alg} \to \mathsf{Ab}$ defines a bifunctor.

Proof. In the left-hand term identities are preserved and compositions are reversed. In the right-hand identities and compositions are preserved. Tensor products and direct sums commute. The mappings induced by the $*$-homomorphisms $f : A_1 \to A_2$ and $g : B_1 \to B_2$ preserve the addition, i.e. they induce group homomorphisms. It is an easy observation that the following diagram commutes for any pair of $*$-homomorphisms $f\colon A_1 \to A_2$ and $g\colon B_1 \to B_2$:

$$\begin{array}{ccc}
\mathrm{KK}(A_2,B_1) & \xrightarrow{A_1 \xrightarrow{f} A_2} & \mathrm{KK}(A_1,B_1) \\
{\scriptstyle B_1 \xrightarrow{g} B_2}\big\downarrow & & \big\downarrow{\scriptstyle B_1 \xrightarrow{g} B_2} \\
\mathrm{KK}(A_2,B_2) & \xrightarrow{A_1 \xrightarrow{f} A_2} & \mathrm{KK}(A_1,B_2).
\end{array}$$

□

Example 11.23. Let A, B and C be $\mathbb{Z}_2$-graded C*-algebras. Let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ be even $*$-homomorphisms. A $*$-homomorphism forms a Kasparov module, see Example 11.5. Applying a $*$-homomorphism to a KK-class given by a $*$-homomorphism gives the same class as the composition of the $*$-homomorphisms, as $\mathrm{KK}(f,\mathrm{id}_C)([(C,g,0)]) = [(C,g\circ f,0)] \in \mathrm{KK}(A,C)$ and $\mathrm{KK}(\mathrm{id}_B,g)([(B,f,0)]) = [(B\hat{\otimes}_g C, f\hat{\otimes}_g 1_C, 0)] = [(\overline{g(B)C}, g\circ f, 0)]$. There is a homotopy of $(\overline{g(B)C}, g\circ f, 0)$ and $(C, g\circ f, 0)$ by [2, Example 17.8.2].

Lemma 11.24. Let B be a $\mathbb{Z}_2$-graded C*-algebra. The functor $\mathrm{KK}(\_, B)\colon \mathsf{C}^* - \mathsf{alg}^{\mathrm{opp}} \to \mathsf{Ab}$ is homotopy invariant.

Proof. Let A_1 and A_2 be C*-algebras and let $f, g\colon A_1 \to A_2$ be homotopic $*$-homomorphisms. There exists a path $(\xi_t)_{t\in[0,1]}$ of $*$-homomorphisms $A_1 \xrightarrow{\xi_t} A_2$ such that $\xi_0 = f$ and $\xi_1 = g$. Let $(E, \phi, F) \in \mathbb{E}(A_2, B)$. Now $(E, \phi \circ f, F)$ and $(E, \phi \circ g, F)$ are homotopic via the Kasparov A_1-$C([0,1], B)$-module $(E, \phi \circ \xi_t, F)$ where E is viewed as a Hilbert $C([0,1], B)$-module with $(e \cdot h)(t) = e \cdot h(t)$ for $e \in E$, $h \in C([0,1], B)$ and $t \in [0,1]$ and $A_1 \ni a \to \xi_t(a) \to \phi(\xi_t(a)) \in \mathbb{B}(E)$ for $t \in [0,1]$. F is independent of $t \in [0,1]$. □

Lemma 11.25. Let A be a $\mathbb{Z}_2$-graded C*-algebra. The functor $\mathrm{KK}(A, \_)\colon \mathsf{C}^* - \mathsf{alg} \to \mathsf{Ab}$ is homotopy invariant.

Proof. Let B_1 and B_2 be C*-algebras and let $f, g\colon B_1 \to B_2$ be homotopic $*$-homomorphisms. There exists a path $(\xi_t)_{t\in[0,1]}$ of $*$-homomorphisms $B_1 \xrightarrow{\xi_t} B_2$ such that $\xi_0 = f$ and $\xi_1 = g$. Let $(E, \phi, F) \in \mathbb{E}(A_2, B)$. Now $(E \hat{\otimes}_f B_2, \phi \hat{\otimes}_f 1_{B_2}, F \hat{\otimes}_f 1_{B_2})$ and $(E \hat{\otimes}_g B_2, \phi \hat{\otimes}_g 1_{B_2}, F \hat{\otimes}_g 1_{B_2})$ are homotopic via the Kasparov A-$C([0,1], B_2)$-module $(E \hat{\otimes}_{\xi_t} B_2, \phi \hat{\otimes}_{\xi_t} 1_{B_2}, F \hat{\otimes}_{\xi_t} 1_{B_2})$. □

11.3 Properties

Definition 11.26. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. Define $\mathrm{KK}_1(A, B) := \mathrm{KK}(A, B \hat{\otimes} \mathbb{C}_1)$ (see Definition 5.96 for the definition of C_1).

Remark 11.27. To match Example 10.50 and Example 10.51, we create a $\mathbb{Z}_2$-graded abelian group $\mathrm{KK}_*(A, B)$ with even part $\mathrm{KK}(A, B)$ and odd part $\mathrm{KK}_1(A, B)$. To highlight the evenness, we may write $\mathrm{KK}_0(A, B)$ for $\mathrm{KK}(A, B)$ at times.

Lemma 11.28. Let A and B be C*-algebras. Then there are natural isomorphisms

$$\mathrm{KK}(A, B \hat{\otimes} \mathbb{C}_1) \cong \mathrm{KK}(A \hat{\otimes} \mathbb{C}_1, B)$$
$$\mathrm{KK}(A, B) \cong \mathrm{KK}(A \hat{\otimes} \mathbb{C}_1, B \hat{\otimes} \mathbb{C}_1).$$

Proof. See [16, §5, Theorem 4]. □

Lemma 11.29. KK is stable in both terms. In other words, let A and B be $\mathbb{Z}_2$-graded C*-algebras. Then there are natural isomorphisms:

$$\mathrm{KK}(A, B) = \mathrm{KK}(\mathbb{K} \hat{\otimes} A, B) = \mathrm{KK}(A, \mathbb{K} \hat{\otimes} B) = \mathrm{KK}(\mathbb{K} \hat{\otimes} A, \mathbb{K} \hat{\otimes} B).$$

Proof. See [2, Corollary 17.8.8]. □

Lemma 11.30. KK is Bott periodic in both terms. In other words, let A and B be $\mathbb{Z}_2$-graded C*-algebras. Then there are natural isomorphisms

$$\mathrm{KK}_1(A, B) \cong \mathrm{KK}(SA, B) \cong \mathrm{KK}(A, SB)$$
$$\mathrm{KK}(A, B) \cong \mathrm{KK}(S^2 A, B) \cong \mathrm{KK}(A, S^2 B) \cong \mathrm{KK}(SA, SB).$$

Proof. See [16, §5,Theorem 7]. □

11.4 Nice representatives and the Fredholm picture

Lemma 11.31. Let B be a σ-unital C*-algebra, then $\mathbb{H}_B$ is countably generated.

Proof. See [2, Theorem 14.6.1]. □

Corollary 11.32. Let A be a $\mathbb{Z}_2$-graded C*-algebra and let B be a σ-unital $\mathbb{Z}_2$-graded C*-algebra. Every class in $\mathrm{KK}(A,B)$ can be represented by a Kasparov module of the form $(\widehat{\mathbb{H}_B},\phi,F)\in\mathbb{E}(A,B)$.

Proof. Take $(E,\phi,F)\in\mathbb{E}(A,B)$ and $(\widehat{\mathbb{H}_B},0,0)\in\mathbb{D}(A,B)$. Then $(E,\phi,F)+(\widehat{\mathbb{H}_B},0,0)=(\widehat{\mathbb{H}_B},\phi,F)$ as a result of Theorem 10.22. □

Remark 11.33. Theorem 10.22 is true for any $\mathbb{Z}_2$-graded C*-algebra. Classes in the KK-group can only be represented by $\widehat{\mathbb{H}_B}$ if B is σ-unital. This is because we explicitly require the Kasparov A-B-module to consist of a countably generated Hilbert B-module. This is where we follow [2, Definition 17.1.1] and deviate from [16, Definition 4.1].

Lemma 11.34. Let A and B be $\mathbb{Z}_2$-graded C*-algebras. Every class in $\mathrm{KK}(A,B)$ can be represented by a Kasparov module $(E,\phi,F)\in\mathbb{E}(A,B)$ such that $F^*=F$ and $\|F\|\leq 1$.

Proof. See [2, Proposition 17.4.2] and [2, Proposition 17.4.3]. □

Lemma 11.35. Let A be a unital $\mathbb{Z}_2$-graded C*-algebra and let B be $\mathbb{Z}_2$-graded C*-algebras. Every class in $\mathrm{KK}(A,B)$ can be represented by a Kasparov module $(E,\phi,F)\in\mathbb{E}(A,B)$ such that ϕ is unital.

Proof. See [2, p.153]. □

Theorem 11.36 (Fredholm picture)**.** Let B be a trivially graded σ-unital C*-algebra. Then $\mathrm{KK}(\mathbb{C},B)=\mathrm{K}_1(\mathcal{Q}(\mathbb{K}\otimes B))=\mathrm{K}_0(B)$ and $\mathrm{KK}_1(\mathbb{C},B)=\mathrm{K}_0(\mathcal{Q}(\mathbb{K}\otimes B))=\mathrm{K}_1(B)$.

Proof. We will show there is an addition/composition preserving bijection between the homotopy classes of unitary elements in $\mathcal{Q}(\mathbb{K}\otimes B)$ and homotopy classes of Kasparov $\mathbb{C}$-B-modules. The classes in $\mathrm{KK}(\mathbb{C},B)$ can be represented by Kasparov modules of the form $(\widehat{\mathbb{H}_B},\phi,F)\in\mathbb{E}(\mathbb{C},B)$ such that ϕ is unital, $F^*=F$ and $\|F\|\leq 1$ by Corollary 11.32 and Lemmas 11.34 and 11.35. In [2, §17.4], explicit algorithms are given to obtain representatives of KK-classes with the properties of Lemmas 11.34 and 11.35. Hence, there exists a Kasparov module that has all the properties. As $\mathbb{B}(\widehat{\mathbb{H}_B})=M_2(\mathcal{M}(\mathbb{K}\otimes B))^{\mathrm{even}}$ by Lemma 10.21, F can be written in the form $\begin{pmatrix}0 & T\\ T^* & 0\end{pmatrix}$ with $T\in\mathcal{M}(\mathbb{K}\otimes B)$ such that $\|T\|\leq 1$. There is only one unital $*$-homomorphism $\mathbb{C}\to M_2(\mathcal{M}(\mathbb{K}\otimes B))^{\mathrm{even}}$. We apply a compactness relation of the Kasparov module to get $\left(\begin{pmatrix}0 & T\\ T^* & 0\end{pmatrix}^2-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\phi(1)\in\mathcal{M}_2(\mathbb{K}\otimes B)$. Hence $TT^*-1_{\mathcal{M}(\mathbb{K}\otimes B)},T^*T-1_{\mathcal{M}(\mathbb{K}\otimes B)}\in\mathbb{K}\otimes B$. In other words $\pi(T)\in\mathcal{Q}(\mathbb{K}\otimes B)$ is a unitary, with $\mathcal{M}(\mathbb{K}\otimes B)\xrightarrow{\pi}\mathcal{Q}(\mathbb{K}\otimes B)$ the quotient.

Let $S,T\in\mathcal{Q}(\mathbb{K}\otimes B)$ be homotopic unitaries via the homotopy $\xi\colon[0,1]\to\mathcal{Q}(\mathbb{K}\otimes B)$, then $\left(\widehat{\mathbb{H}_B},\phi,\begin{pmatrix}0 & T\\ T^* & 0\end{pmatrix}\right)$ and $\left(\widehat{\mathbb{H}_B},\phi,\begin{pmatrix}0 & S\\ S^* & 0\end{pmatrix}\right)$ are homotopic via the homotopy $\left(\widehat{\mathbb{H}_B},\phi,\begin{pmatrix}0 & \xi\\ \xi^* & 0\end{pmatrix}\right)$. The composition of unitaries in Definition 7.40 coincides with the addition of elements in $\mathrm{KK}(\mathbb{C},B)$ (Definition 11.11), because $\widehat{\mathbb{H}_B} \oplus \widehat{\mathbb{H}_B} = \widehat{\mathbb{H}_B}$ by Kasparov's stabilisation theorem and because there is only one unital $*$-homomorphism from $\mathbb{C}$ to $M_2(\mathcal{M}(\mathbb{K} \otimes B))^{\text{even}}$. So there is a well-defined surjective group homomorphism from $\mathrm{K}_1(\mathcal{Q}(\mathbb{K} \otimes B))$ to $\mathrm{KK}(\mathbb{C}, B)$. We will now determine the kernel. We get $\left(\widehat{\mathbb{H}_B}, \phi, \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}\right) \in \mathbb{D}(\mathbb{C}, B)$ if and only if $T \in \mathcal{M}(\mathbb{K} \otimes B)$ is a unitary. So T comes from $\mathrm{K}_1(\mathcal{M}(\mathbb{K} \otimes B)) = 0$ (by Theorem 7.59) and $\pi(T) \in \mathcal{Q}(\mathbb{K} \otimes B)$ is represented by $0 \in \mathrm{K}_1(\mathcal{Q}(\mathbb{K} \otimes B))$. This shows that the group homomorphism is injective. We conclude there is a group isomorphism between $\mathrm{KK}(\mathbb{C}, B)$ and $\mathcal{Q}(\mathbb{K} \otimes B)$. By Lemma 7.75, $\mathrm{K}_1(\mathcal{Q}(\mathbb{K} \otimes B)) \cong \mathrm{K}_0(B)$. Apply Bott periodicity to conclude $\mathrm{KK}(\mathbb{C}, SB) \cong \mathrm{K}_0(SB) = \mathrm{K}_1(B)$. □

11.5 Kasparov product and connections

Kasparov has defined an extra structure on the KK-groups; the *Kasparov product*, see [16, §3-4]. He constructed the Kasparov product as a following bilinear mapping (with A_1, A_2, B_1, B_2 and D $\mathbb{Z}_2$-graded C*-algebras):

$$\mathrm{KK}(A_1, B_1 \hat{\otimes} D) \times \mathrm{KK}(D \hat{\otimes} A_2, B_2) \to \mathrm{KK}(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2).$$

A special case of the Kasparov product is when $B_1 = A_2 = \mathbb{C}$. Then the Kasparov product becomes:

$$\mathrm{KK}(A_1, D) \times \mathrm{KK}(D, B_2) \to \mathrm{KK}(A_1, B_2).$$

Even in this special case, Kasparov products are not easily calculated. The 'goal' of the Kasparov product in the special case is to map classes $[(E_1, \phi_1, F_1)] \in \mathrm{KK}(A, B)$ and $[E_2, \phi_2, F_2)] \in \mathrm{KK}(B, C)$ to the class $[(E_1 \hat{\otimes}_{\phi_2} E_2, \phi_1 \hat{\otimes}_{\phi_2} 1_{E_2}, F_1 \# F_2)] \in \mathrm{KK}(A, C)$ (with A, B and C $\mathbb{Z}_2$-graded C*-algebras), where $F_1 \# F_2$ is an odd operator in $\mathbb{B}(E_1 \hat{\otimes}_{\phi_2} E_2)$, that we will define later. Important to notice is that $F_1 \hat{\otimes}_{\phi_2} F_2$ is not a well-defined operator on $E_1 \hat{\otimes}_{\phi_2} E_2$. For $b \in B$, $e_1 \in E_1$ and $e_2 \in E_2$, the tensors $e_1 b \hat{\otimes} e_2$ and $e_1 \hat{\otimes} \phi_2(b)(e_2)$ represent the same element. But in general $F_1(e_1 b) \hat{\otimes} F_2(e_2) = F(e_1) b \hat{\otimes} F_2(e_2) = F_1(e_1) \hat{\otimes} \phi_2(b)(F_2(e_2)) \neq F_1(e_1) \hat{\otimes} F_2(\phi_2(b)(e_2))$. We will have to find another construction for $F_1 \# F_2$.

Luckily, Connes and Skandalis found a way to express the Kasparov product in a relatively easy way via *connections*. For certain C*-algebras, the definition of *a* Kasparov product as given by Connes and Skandalis coincides with the original definition of the Kasparov product as given by Kasparov. We will go through the definitions and results step by step.

Definition 11.37. Let A, B and C be $\mathbb{Z}_2$-graded C*-algebras. Let $(E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ and $(E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$. Define $T_x \in \mathbb{B}(E_2, E_1 \hat{\otimes}_{\psi_2} E_2)$ for $x \in E_1$ as $T_x(y) = x \hat{\otimes} y \in E_1 \hat{\otimes}_{\psi_2} E_2$ for $y \in E_2$.

Lemma 11.38. Let A, B and C be $\mathbb{Z}_2$-graded C*-algebras. Let $(E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ and let $(E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$. The adjoint of $T_x \in \mathbb{B}(E_2, E_1 \hat{\otimes}_{\phi_2} E_2)$ from Definition 11.37 exists and is given by $T_x^*(z \hat{\otimes} w) := \phi_2(\langle x, z \rangle_{E_1})(w)$ with $z \in E_1$ and $w \in E_2$.

Proof. We will use the tensor inner product from Definition 10.46 to check that $\langle T_x(y), z \hat{\otimes} w \rangle_{E_1 \hat{\otimes}_{\phi_2} E_2} = \langle y, \psi_2(\langle x, z \rangle_{E_1})(w) \rangle_{E_2}$. This operator preserves the C-action as ψ_2 and the right-hand side of $\langle -, - \rangle_{E_2}$ do. We conclude that $\psi_2(\langle x, z \rangle_{E_1}) \in \mathbb{B}(E_2)$. □

Definition 11.39. Let A, B and C be $\mathbb{Z}_2$-graded C*-algebras. Let $(E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ and let $(E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$. An operator $F \in \mathbb{B}(E_1 \hat{\otimes}_{\phi_2} E_2)$ is called an F_2-*connection* for E_1 if for all $x \in E_1$ the following conditions are satisfied:

- $T_x \circ F_2 - (-1)^{\partial(x)} F \circ T_x \in \mathbb{K}(E_2, E_1 \hat{\otimes}_{\phi_2} E_2)$
- $F_2 \circ T_x^* - (-1)^{\partial(x)} T_x^* \circ F \in \mathbb{K}(E_1 \hat{\otimes}_{\phi_2} E_2, E_2)$.

Theorem 11.40. Let A, B and C be $\mathbb{Z}_2$-graded C*-algebras. Let $(E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ and let $(E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$. There exists an F_2-connection for E_1.

Proof. See [2, Theorem 18.3.3]. □

Definition 11.41. Let A, B and C be $\mathbb{Z}_2$-graded C*-algebras. Let $(E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ and let $(E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$. Let $F \in \mathbb{B}(E_1 \hat{\otimes}_{\phi_2} E_2)$ be an F_2-connection for E_1. Then $(E_1 \hat{\otimes}_{\psi_2} E_2, \phi_1 \hat{\otimes}_{\phi_2} 1_{E_2}, F)$ is a *Kasparov product* for (E_1, ϕ_1, F_1) and (E_2, ϕ_2, F_2) if the following properties are satisfied:

- $(E_1 \hat{\otimes}_{\psi_2} E_2, \phi_1 \hat{\otimes}_{\phi_2} 1_{E_2}, F) \in \mathbb{E}(A, C)$
- $(\phi_1(a) \hat{\otimes}_{\phi_2} 1_{E_2})[F_1 \hat{\otimes} 1_{E_2}, F](\phi_1(a)^* \hat{\otimes}_{\phi_2} 1_{E_2})$ is positive in $\mathbb{B}(E_1 \hat{\otimes}_{\psi_2} E_2) / \mathbb{K}(E_1 \hat{\otimes}_{\psi_2} E_2)$ for all $a \in A$.

The set of all $F \in \mathbb{B}(E_1 \hat{\otimes}_{\psi_2} E_2)$ appearing in Kasparov products is denoted by $F_1 \#_B F_2$.

Theorem 11.42. Let A be a separable $\mathbb{Z}_2$-graded C*-algebra, let B be a σ-unital $\mathbb{Z}_2$-graded C*-algebra and let C be a $\mathbb{Z}_2$-graded C*-algebra. Let $(E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ and let $(E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$. Then $F_1 \#_B F_2 \neq \varnothing$. Moreover, all Kasparov products in the sense of Definition 11.41 are operator homotopic.

Proof. See [2, Theorem 18.3.3]. □

Definition 11.43. Let A, B and C be $\mathbb{Z}_2$-graded C*-algebras. Let $(E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ and let $(E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$. The Kasparov product between (E_1, ϕ_1, F_1) and (E_2, ϕ_2, F_2) is the unique class (Theorem 11.42) in $\mathrm{KK}(A, C)$ that represents the Kasparov product in Definition 11.41. The Kasparov product will be denoted by $[(E_1, \phi_1, F_1)] \otimes_B [(E_2, \phi_2, F_2)]$.

Theorem 11.44. The Kasparov product is associative. In other words, let A, B, C and D be $\mathbb{Z}_2$-graded C*-algebras. Let $X \in \mathrm{KK}_i(A, B)$, $Y \in \mathrm{KK}_k(B, C)$ and $Z \in \mathrm{KK}_k(C, D)$, then $(X \otimes_B Y) \otimes_C Z = X \otimes_B (Y \otimes_C Z)$.

Proof. See [2, Theorem 18.6.1]. □

Lemma 11.45. Let A, B and C be $\mathbb{Z}_2$-graded C*-algebras. Let $(E_1, \phi_1, F_1) \in \mathbb{E}(A, B)$ and let $(E_2, \phi_2, F_2) \in \mathbb{E}(B, C)$. Let $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ be even $*$-homomorphisms. The Kasparov modules of the form of Example 11.5 give the following Kasparov products:

$$\begin{aligned}
[(E_1, \phi_1, F_1)] \otimes_B [(C, g, 0)] &= \mathrm{KK}(\mathrm{id}_A, g)([(E_1, \phi_1, F_1)]) \\
[(B, f, 0)] \otimes_B [(E_2, \phi_2, F_2)] &= \mathrm{KK}(f, \mathrm{id}_C)([(E_2, \phi_2, F_2)]).
\end{aligned}$$

Proof. See [2, Examples 18.4.2 a+b]. □

Corollary 11.46. Let A be a $\mathbb{Z}_2$-graded C*-algebra. Then $\mathrm{KK}_*(A, A)$ is a $\mathbb{Z}_2$-graded ring when the Kasparov product is the multiplication.

Proof. $\mathrm{KK}_0(A,A)$ and $\mathrm{KK}_1(A,A)$ are abelian groups by Theorem 11.18. The Kasparov product is associative by Theorem 11.44 and distributive by the biadditivity of the Kasparov product. From Lemma 11.45 follows that the multiplicative unit is given by $[(A,\mathrm{id}_A,0)] \in \mathrm{KK}_0(A,A)$. The Kasparov product is a mapping $\mathrm{KK}_i(A,A) \times \mathrm{KK}_j(A,A) \to \mathrm{KK}_{i+j}(A,A)$. So the Kasparov product respects the $\mathbb{Z}_2$-structure. □

Definition 11.47. Let A be a $\mathbb{Z}_2$-graded C*-algebra. The KK-class $[(A,\mathrm{id}_A,0)] \in \mathrm{KK}_0(A,A)$ is denoted by id_A.

Lemma 11.48. Let A, B, C and D be C*-algebras. The Kasparov product commutes with group homomorphisms induced by $*$-homomorphisms in the following way:

1. Let $X \in \mathrm{KK}_i(A,B)$ and $Y \in \mathrm{KK}_j(B,C)$. Let $C \xrightarrow{f} D$ be a $*$-homomorphism, then $X \otimes_B \mathrm{KK}_j(\mathrm{id}_B, f)(Y) = \mathrm{KK}_{i+j}(\mathrm{id}_A, f)(X \otimes_B Y)$.

2. Let $X \in \mathrm{KK}_i(A,B)$ and $Y \in \mathrm{KK}_j(B,C)$. Let $D \xrightarrow{g} A$ be a $*$-homomorphism, then $\mathrm{KK}_i(g,\mathrm{id}_B)(X) \otimes_B Y = \mathrm{KK}_{i+j}(g,\mathrm{id}_C)(X \otimes_B Y)$.

3. Let $X \in \mathrm{KK}_i(A,B)$ and $Y \in \mathrm{KK}_j(C,D)$. Let $B \xrightarrow{h} C$ be a $*$-homomorphism, then $\mathrm{KK}_i(\mathrm{id}_A,h)(X) \otimes_C Y = X \otimes_B \mathrm{KK}_j(h,\mathrm{id}_D)(Y)$.

Proof. The $*$-homomorphisms can be written as a Kasparov product, see Lemma 11.45. Now the associativity of the Kasparov product proves the lemma:

1.
$$\begin{aligned} X \otimes_B \mathrm{KK}_j(\mathrm{id}_B,f)(Y) = X \otimes_B (Y \otimes_C [(D,f,0)]) = \\ (X \otimes_B Y) \otimes_C [(D,f,0)] = \mathrm{KK}_{i+j}(\mathrm{id}_A,f)(X \otimes_B Y) \end{aligned}$$

2.
$$\begin{aligned} \mathrm{KK}_i(g,\mathrm{id}_B)(X) \otimes_B Y = ([(A,g,0)] \otimes_A X) \otimes_B Y = \\ [(A,g,0)] \otimes_A (X \otimes_B Y) = \mathrm{KK}_{i+j}(g,\mathrm{id}_C)(X \otimes_B Y) \end{aligned}$$

3.
$$\begin{aligned} \mathrm{KK}_i(\mathrm{id}_A,h)(X) \otimes_C Y = (X \otimes_B [(D,h,0)]) \otimes_C Y = \\ X \otimes_B ([(D,h,0)] \otimes_C Y) = X \otimes_B \mathrm{KK}_j(h,\mathrm{id}_D)(Y). \end{aligned}$$

□

11.6 KK-equivalence

Definition 11.49. The $\mathbb{Z}_2$-graded C*-algebras A and B are *KK-equivalent* if there exist a class $X \in \mathrm{KK}_0(A,B)$ and a class $Y \in \mathrm{KK}_0(B,A)$ such that $X \otimes_B Y = \mathrm{id}_A \in \mathrm{KK}_0(A,A)$ and $Y \otimes_A X = \mathrm{id}_B \in \mathrm{KK}_0(B,B)$. If there exist a class $X \in \mathrm{KK}_1(A,B)$ and a class $Y \in \mathrm{KK}_1(B,A)$ such that $X \otimes_B Y = \mathrm{id}_A \in \mathrm{KK}_0(A,A)$ and $Y \otimes_A X = \mathrm{id}_B \in \mathrm{KK}_0(B,B)$, then A and B are called *KK-anti-equivalent*.

Lemma 11.50. KK-equivalence is an equivalence relation.

Proof. Reflexivity and symmetry are immediate. For the transitivity, say that $\mathbb{Z}_2$-graded C*-algebras A and B are KK-equivalent via $X \in \mathrm{KK}_0(A,B)$ and $Y \in \mathrm{KK}_0(B,A)$ and C*-algebras B and C are KK-equivalent via $Z \in \mathrm{KK}(B,C)$ and $W \in \mathrm{KK}_0(C,B)$. Now A and C become KK-equivalent via $X \otimes_B Z \in \mathrm{KK}_0(A,C)$ and $W \otimes_B Y \in \mathrm{KK}_0(C,A)$ as $(X \otimes_B Z) \otimes_C (W \otimes_B Y) = X \otimes_B \mathrm{id}_B \otimes_B Y = \mathrm{id}_A \in \mathrm{KK}_0(A,A)$ and $(W \otimes_B Y) \otimes_A (X \otimes_B Z) = W \otimes_B \mathrm{id}_B \otimes_B Z = \mathrm{id}_C \in \mathrm{KK}_0(C,C)$ by the associativity of the KK-product. ☐

Remark 11.51. KK-anti-equivalence is not an equivalence relation! It is not transitive. If $X \in \mathrm{KK}_1(A,B)$ and $Z \in \mathrm{KK}_1(B,C)$, then $X \otimes_B Z \in \mathrm{KK}_0(A,C)$.

Lemma 11.52. Let A and B be KK-equivalent $\mathbb{Z}_2$-graded C*-algebras, then $\mathrm{KK}_0(A,C) \cong \mathrm{KK}_0(B,C)$ and $\mathrm{KK}_0(C,A) \cong \mathrm{KK}_0(C,B)$ for every $\mathbb{Z}_2$-graded C*-algebra.

Proof. Let $X \in \mathrm{KK}_0(A,B)$ and $Y \in \mathrm{KK}_0(B,A)$ be such that $X \otimes_B Y = \mathrm{id}_A$ and $Y \otimes_A X = \mathrm{id}_B$. Now the group homomorphism $\mathrm{KK}_0(A,C) \xrightarrow{Y \otimes_A -} \mathrm{KK}_0(B,C)$ has an inverse $\mathrm{KK}_0(B,C) \xrightarrow{X \otimes_B -} \mathrm{KK}_0(A,C)$. Similarly, the group homomorphisms $\mathrm{KK}_0(C,A) \xrightarrow{- \otimes_A X} \mathrm{KK}_0(C,B)$ has an inverse $\mathrm{KK}_0(C,B) \xrightarrow{- \otimes_B Y} \mathrm{KK}_0(C,A)$. ☐

Remark 11.53. Because $\mathrm{KK}_1(C,D) = \mathrm{KK}_0(C, D \otimes \mathbb{C}_1)$ for C*-algebras C and D, the proof of Lemma 11.52 holds when KK_0 is replaced by KK_1.

Example 11.54. Let A be a C*-algebra. Then A and $\mathbb{K} \otimes A$ are KK-equivalent.

Lemma 11.55. An even $*$-isomorphism between $\mathbb{Z}_2$-graded C*-algebras induces a KK-equivalence.

Proof. Let A and B be C*-algebras and let $A \xrightarrow{f} B$ be an even $*$-isomorphism. Now f and f^{-1} represent the Kasparov modules $(B, f, 0) \in \mathbb{E}(A,B)$ and $(A, f^{-1}.0) \in \mathbb{E}(B,A)$ by Example 11.5. The compositions in both orders are homotopic to $(A, \mathrm{id}_A, 0)$ and $(B, \mathrm{id}_B, 0)$ by Example 11.23. This forms the KK-equivalence between A and B. ☐

Corollary 11.56. An even stable isomorphism induces a KK-equivalence.

Proof. Let A and B be evenly stably isomorphic C*-algebras. By Example 11.54 A and $\mathbb{K} \otimes A$ are KK-equivalent, just as B and $\mathbb{K} \otimes B$ are KK-equivalent. The isomorphism between $\mathbb{K} \otimes A$ and $\mathbb{K} \otimes B$ implies KK-equivalence between $\mathbb{K} \otimes A$ and $\mathbb{K} \otimes B$. The transitivity of KK-equivalence makes A and B KK-equivalent. ☐

11.7 Thom isomorphism

Theorem 11.57 (Thom isomorphism)**.** Let A be a separable C*-algebra. Let $\mathbb{R} \xrightarrow{\alpha} \mathrm{Aut}(A)$ be a continuous group homomorphism. Then $A \rtimes_\alpha \mathbb{R}$ and A are KK-anti-equivalent. In other words, there exists an $X \in \mathrm{KK}_1(A, A \rtimes_\alpha \mathbb{R})$ and a $Y \in \mathrm{KK}_1(A \rtimes_\alpha \mathbb{R}, A)$ such that $X \otimes_{A \rtimes_\alpha \mathbb{R}} Y = \mathrm{id}_A$ and $Y \otimes_A X = \mathrm{id}_{A \rtimes_\alpha \mathbb{R}}$.

Proof. See [10, Theorem 1]. ☐

Remark 11.58. The Thom isomorphisms of Theorem 7.81 is a special case of this Thom isomorphism. See [10, Remark 2, p. 8].

12 The Universal Coefficient Theorem for C*-algebras

We now have gone through sufficient definitions to understand what the Universal Coefficient Theorem (UCT) states. In the first subsection we will show how a KK-class induces a group homomorphism between K-groups. We will introduce the UCT and the subcategory for which we will prove it. We will show how $\mathbb{Z}_2$-grading allows the exact sequence of the UCT to be separated into two exact sequences. In the second subsection we will state the Künneth theorem; an important tool in calculating K-groups for tensor products. The proof of the UCT will not be given in this section. In Corollary 7.71, we showed that the K-groups allow a six term sequence. We took a more axiomatic approach in §8 to show that any Bott functor has a six term sequence. To prove the UCT, we will first need to show the six term sequence of the KK-groups. To do this, we will show that the KK-functor, when restricted to the right subcategories, is a Bott functor. This will be done in §14.

12.1 Statement of the theorem and some observations

Every K-group can be written as a KK-group under the Fredholm picture (Theorem 11.36). Now a KK-class will induce a group homomorphism between the K-groups via the Kasparov product. We will formalise this with the following definition.

Definition 12.1. Let A and B be C*-algebras. Define $\gamma(A,B)$ to be the group homomorphism:

$$\mathrm{KK}_*(A,B) \ni Y \to (\mathrm{KK}_*(\mathbb{C},A) \ni X \to X \otimes_A Y \in \mathrm{KK}_*(\mathbb{C},B)).$$

Remark 12.2. By Theorem 11.36, there is a natural isomorphism $\mathrm{KK}_*(\mathbb{C},B) \cong \mathrm{K}_*(B)$. This way $\gamma(A,B)$ defines a group homomorphism from $\mathrm{K}_*(A)$ to $\mathrm{K}_*(B)$.

Lemma 12.3. The group homomorphism $\mathrm{KK}_*(A,B) \xrightarrow{\gamma} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ in Definition 12.1 is even.

Proof. Let $Y \in \mathrm{KK}_0(A,B)$, then $X \otimes_A Y \in \mathrm{KK}_j(\mathbb{C},B)$ for $X \in \mathrm{KK}_j(\mathbb{C},A)$. Let $Y \in \mathrm{KK}_1(A,B)$, then $X \otimes_A Y \in \mathrm{KK}_{j+1}(\mathbb{C},B)$ for $X \in \mathrm{KK}_j(\mathbb{C},A)$. □

We might wonder what other properties of the γ are. If $\gamma(A,B)$ is injective, then this means that every class in $\mathrm{KK}_*(A,B)$ defines a unique group homomorphism from $\mathrm{K}_*(A)$ to $\mathrm{K}_*(B)$. If $\gamma(A,B)$ is surjective, then this means that every group homomorphism between $\mathrm{K}_*(A)$ and $\mathrm{K}_*(B)$ is induced by a class in $\mathrm{KK}_*(A,B)$. For general C*-algebra A and B, we cannot say that $\gamma(A,B)$ is injective or surjective. The UCT states that for certain choices of A and B, $\gamma(A,B)$ is surjective and the kernel of $\gamma(A,B)$ is isomorphic to $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$. To get these results, at least A needs to be in the following category.

Definition 12.4. N denotes the smallest full subcategory of the category of separable nuclear C*-algebras, which contains the separable type-I (postliminal) C*-algebras and is closed under KK-equivalence, inductive limits, extensions and crossed products by $\mathbb{R}$ and $\mathbb{Z}$.

Theorem 12.5 (Universal Coefficient Theorem)**.** Let $A \in \mathrm{Ob}(\mathsf{N})$ and let B be a σ-unital C*-algebra. Then there exists an odd group homomorphism $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) \xrightarrow{\delta(A,B)} \mathrm{KK}_*(A,B)$ such that the following sequence is exact:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) \xrightarrow{\delta} \mathrm{KK}_*(A,B) \xrightarrow{\gamma} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) \to 0. \tag{41}$$

Remark 12.6. We will give the definition of δ in Definition 17.1. From sequence (41) is clear that δ is some embedding of the kernel of γ into $\mathrm{KK}_*(A,B)$.

Remark 12.7. The groups in Theorem 12.5 have a $\mathbb{Z}_2$-grading. The $\mathbb{Z}_2$-grading is not obtained by just filling in 0 or 1 on the dots. That is because the even part of $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ is not $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B))$ and the odd part is not $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B))$. The $\mathbb{Z}_2$-grading of $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ is explained in Example 10.51. The $\mathbb{Z}_2$-grading of $\mathrm{KK}_*(A,B)$ is explained in Remark 11.27. The $\mathbb{Z}_2$-grading of $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ is explained in Example 10.50. As γ is even and δ is odd, the exact sequence (41) can be separated into two parts:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B)) \xrightarrow{\delta} \mathrm{KK}_1(A,B) \xrightarrow{\gamma} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)) \to 0 \tag{42}$$

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)) \xrightarrow{\delta} \mathrm{KK}_0(A,B) \xrightarrow{\gamma} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B)) \to 0. \tag{43}$$

Sequence (42) contains the even part of $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ and the odd parts of $\mathrm{KK}_*(A,B)$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$. Sequence (43) contains the odd part of $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ and the even parts of $\mathrm{KK}_*(A,B)$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$.

Lemma 12.8. Sequence (42) is exact if and only if sequence (43) is exact.

Proof. Fill in SB in either location instead of B. Then use the natural isomorphisms between $\mathrm{K}_*(SB)$ and $\mathrm{K}_{*+1}(B)$ and $\mathrm{KK}_*(A,SB)$ and $\mathrm{KK}_{*+1}(A,B)$. ☐

Theorem 12.9. Let $\tau \in \mathfrak{Ext}(A,B)$. View $[\tau] \in \mathrm{Ext}(A,B)$ as an element of $\mathrm{KK}_1(A,B)$ under the isomorphism of Corollary 13.63. Then $\gamma([\tau]) \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(\mathbb{K}\otimes B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(\mathbb{K}\otimes B))$ gives the index and exponential maps of the extension τ.

Proof. A far more general statement is proven in [16, §7, Theorem 2]. ☐

Remark 12.10. A somewhat similar result is found in Example 7.76, which says that the boundary and index maps are the application of K_0 and K_1 to the Busby invariant.

12.2 The K-groups for objects of N

In N there is an extra result that can be used to calculate the K_0-group and the K_1-group of tensor products. This is called the *Künneth Theorem.* We will only use a special case. We can use this result to calculate the K-groups of the direct limit in Example 5.57.

Theorem 12.11 (Künneth)**.** Let $A \in \mathrm{Ob}(\mathsf{N})$ and let B be a C*-algebra such that $\mathrm{K}_*(B)$ is torsion free. Then there is a $\mathbb{Z}_2$-graded isomorphism $\mathrm{K}_*(A)\hat{\otimes}\mathrm{K}_*(B) \cong \mathrm{K}_*(A\otimes B)$.

Proof. This is a special case of the Künneth theorem in [30, p. 443]. ☐

Example 12.12. The Künneth theorem and the results from §7 make it possible to calculate the K-groups of the direct limit in Example 5.57. It is known $\mathrm{K}_0(M_i(\mathbb{C})) = \mathbb{Z}$ and $\mathrm{K}_1(M_i(\mathbb{C})) = 0$ for all $i \geq 1$. We now apply the Künneth theorem to calculate the K-groups of $M_2(\mathbb{C}) \otimes M_{3!}(\mathbb{C})$:

$$\mathrm{K}_0(M_2(\mathbb{C}) \otimes M_{3!}(\mathbb{C})) = (\mathrm{K}_0(M_2(\mathbb{C})) \otimes_{\mathbb{Z}} \mathrm{K}_0(M_{3!}(\mathbb{C}))) \oplus (\mathrm{K}_1(M_2(\mathbb{C})) \otimes_{\mathbb{Z}} \mathrm{K}_1(M_{3!}(\mathbb{C}))) = \mathbb{Z}$$
$$\mathrm{K}_1(M_2(\mathbb{C}) \otimes M_{3!}(\mathbb{C})) = (\mathrm{K}_0(M_2(\mathbb{C})) \otimes_{\mathbb{Z}} \mathrm{K}_1(M_{3!}(\mathbb{C}))) \oplus (\mathrm{K}_1(M_2(\mathbb{C})) \otimes_{\mathbb{Z}} \mathrm{K}_0(M_{3!}(\mathbb{C}))) = 0.$$

From $\mathrm{K}_1(M_2(\mathbb{C}) \otimes M_{3!}(\mathbb{C})) = 0$ and $\mathrm{K}_1(M_{4!}(\mathbb{C})) = 0$, it follows $\mathrm{K}_1(M_2(\mathbb{C}) \otimes M_{3!}(\mathbb{C}) \otimes M_{4!}(\mathbb{C})) = 0$. So it follows from induction that $\mathrm{K}_1(M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C})) = 0$ for all $n \geq 2$. From Theorem 7.52 follows $\mathrm{K}_1(\varinjlim M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C})) = \varinjlim \mathrm{K}_1(M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C})) = 0$.
In a similar way $\mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C}))$ can be calculated from $\mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{n!}(\mathbb{C}))$ and $\mathrm{K}_0(M_{n+1}!(\mathbb{C})))$. By the Künneth Theorem we get the following calculation:

$$\begin{aligned}\mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C})) = \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{n!}(\mathbb{C})) \otimes_{\mathbb{Z}} \mathrm{K}_0(M_{(n+1)!}(\mathbb{C})) &= \\ \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{n!}(\mathbb{C})) \otimes_{\mathbb{Z}} \mathbb{Z}&.\end{aligned}$$

As $\mathrm{K}_0(M_2(\mathbb{C}) \otimes M_{3!}(\mathbb{C})) = \mathbb{Z}$, it follows $\mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C})) = \mathbb{Z}$ for all $n \geq 2$.
The structure morphism $M_2(\mathbb{C}) \otimes ... \otimes M_{n!}(\mathbb{C}) \ni x \to x \otimes \mathrm{diag}(x, x, ..., x) \in M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C})$ induces an $(n+1)$-multiplication with respect to the K_0-groups. Now $1 \in \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{n!}(\mathbb{C})) = \mathbb{Z}$ is sent to $(n+1) \in \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C})) = \mathbb{Z}$ and so the image of a generator in $\mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{n!}(\mathbb{C}))$ becomes divisible by $(n+1)$. Finally, we use the preservation of direct limits (Theorem 7.30) to get $\mathrm{K}_0(\varinjlim M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C})) = \varinjlim \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C}))$. The direct limit structure is given in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z} \arrow[rr, "-\cdot n"] \arrow[drr, "-\cdot\frac{1}{(n-1)!}"'] & & \mathbb{Z} \arrow[rr, "-\cdot(n+1)"] \arrow[d, "-\cdot\frac{1}{n!}"'] & & \mathbb{Z} \arrow[r] \arrow[dll, "-\cdot\frac{1}{(n+1)!}"] & \cdots. \\
& & & \varinjlim \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C})) & & &
\end{tikzcd}$$

We have abbreviated $\mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{n!}(\mathbb{C}))$ to $\mathbb{Z}$ for every $n \geq 2$. The element in $\varinjlim \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C}))$ corresponding to $1 \in \mathrm{K}_0(\mathbb{C}) = \mathbb{Z}$ can be divided by every element of $\mathbb{Z}$. We conclude from the diagram that $\varinjlim \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C}))$ is divisible as a $\mathbb{Z}$-group (see Definition 4.21). We will now establish an isomorphism between $\varinjlim \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C}))$ and $\mathbb{Q}$. We use the mappings $\mathbb{Z} = \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{n!}(\mathbb{C})) \ni x \to \frac{x}{n!} \in \mathbb{Q}$. This mapping is surjective, because let $\frac{k}{m} \in \mathbb{Q}$ (with $m \neq 0$), then $\frac{k}{m} = \frac{k \cdot (m-1)!}{m!}$. This mapping is injective, because if $\frac{x}{n!} = \frac{y}{m!}$ with $m, n \in \mathbb{N}$, $x \in \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{n!}(\mathbb{C}))$ and $y \in \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{m!}(\mathbb{C}))$, then $n!y = m!x$. Without loss of generality, if $m < n$, then x is divisible by $m+1, m+2, ..., n$. It means that $x \in \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{n!}(\mathbb{C}))$ comes from $\mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{m!}(\mathbb{C}))$. So x and y represent the same element in $\varinjlim \mathrm{K}_0(M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C}))$.

13 Addition of extensions and its structure

In §4.4 the Baer sum gives an addition for extensions of R-modules M by N, for a commutative ring R. It was shown in Theorem 4.62 and Theorem 4.64 that $\mathrm{Ext}_R(M, N)$ is an abelian group isomorphic to $\mathrm{Ext}^1_R(M, N)$. For extensions of C*-algebras an addition of extension can be defined as well if the left-hand term is stable. This time it will be done by characterising isomorphism classes of extensions (Definition 6.20) via their corresponding Busby invariant (Definition 6.17). From now on, an extension of C*-algebras will always be viewed as a representative of its isomorphism class. Contrary to the extensions of R-modules, the unitary equivalence classes extensions of C*-algebras do not form a group in general. However, for a special choice of C*-algebras, it will be a group; an isomorphism between the semigroup of unitary equivalence classes modulo split extensions and the KK-group will be established. We will make use of Kasparov pairs. They will be defined in §13.3. We will construct an isomorphism between classes of Kasparov pairs and classes of extensions in §13.4 and an isomorphism between classes of Kasparov pairs and KK-groups in §13.5. The two isomorphism combined will form the isomorphism between extensions of C*-algebras and KK-groups.

13.1 The construction of the addition and its properties

The addition of extensions of C*-algebras is only well-defined if the left-hand term is stable. A stable C*-algebra can be written as $\mathbb{K}\otimes B$ with B a C*-algebra. The set of isomorphism classes of extensions of A by $\mathbb{K}\otimes B$ was denoted by $\mathfrak{Ext}(A,B)$ (Definition 6.38). Recall $\mathbb{K}(\mathbb{H}_B) = \mathbb{K}\otimes B$ and $\mathbb{B}(\mathbb{H}_B) = \mathcal{M}(\mathbb{K}\otimes B)$ by Lemma 9.14 and Corollary 9.17. If B is assumed to be trivially graded, then $\mathbb{B}(\widehat{\mathbb{H}_B}) \cong M_2(\mathcal{M}(\mathbb{K}\otimes B))$ (Lemma 10.21) as $\mathbb{Z}_2$-graded C*-algebras. Recall $\widehat{\mathbb{H}_B} = \mathbb{H}_B \oplus \mathbb{H}_B$. Additionally, $\mathbb{H}_B \cong \mathbb{H}_B \oplus \mathbb{H}_B$. This isomorphism can be established by creating a bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$ and mapping the indices accordingly.

Definition 13.1. Let A and B be C*-algebras. Fix a bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$. The mapping of indices induces an isomorphism $\mathbb{H}_B \cong \mathbb{H}_B \oplus \mathbb{H}_B$. This isomorphism induces the following $*$-isomorphisms:

$$\mathbb{K}\otimes B \cong M_2(\mathbb{K}\otimes B) \qquad \mathcal{M}(\mathbb{K}\otimes B) \cong M_2(\mathcal{M}(\mathbb{K}\otimes B)) \qquad \mathcal{Q}(\mathbb{K}\otimes B) \cong M_2(\mathcal{Q}(\mathbb{K}\otimes B)).$$

For $\phi_1, \phi_2 \in \mathfrak{Ext}(A,B)$, define $(\phi_1 \oplus \phi_2)(a) := \begin{pmatrix} \phi_1(a) & 0 \\ 0 & \phi_2(a) \end{pmatrix} \in M_2(\mathcal{Q}(\mathbb{K}\otimes B))$, for $a \in A$. Use the (inverse) $*$-isomorphism between $M_2(\mathcal{Q}(\mathbb{K}\otimes B))$ and $\mathcal{Q}(\mathbb{K}\otimes B)$ to view it as an extension, i.e. an element of $\mathfrak{Ext}(A,B)$. $\phi_1 \oplus \phi_2$ is called the *addition* of ϕ_1 and ϕ_2.

Remark 13.2. Obviously $\phi_1 \oplus \phi_2$ defines a $*$-homomorphism from A to $M_2(\mathcal{Q}(\mathbb{K}\otimes B))$. Under the isomorphism between $\mathbb{H}_B$ and $\mathbb{H}_B \oplus \mathbb{H}_B$, this gives a $*$-homomorphism from A to $\mathcal{Q}(\mathbb{K}\otimes B)$ and therefore defines an extension of A by $\mathbb{K}\otimes B$.

Remark 13.3. The addition is not canonical. It depends on the choice of the bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$. However, later on we will find that the addition on $\mathfrak{Ext}(A,B)$ extends to an addition on $\overline{\mathfrak{Ext}}(A,B)$. This addition will be canonically defined. For this result, see Lemma 13.13.

Remark 13.4. Contrary to the addition of rings and modules over a ring, this addition is not commutative. However the addition on the unitary equivalence classes is commutative. For this result, see Lemma 13.10.

Remark 13.5. The associativity of the addition follows from the associativity of composition of adjointable operators between Hilbert B-modules.

Remark 13.6. The addition in Definition 13.1 turns $\mathfrak{Ext}(A,B)$ into a semigroup. After Lemma 13.21, it will be shown that $\mathfrak{Ext}(A,B)$ and $\overline{\mathfrak{Ext}}(A,B)$ need not have a unit. The Busby invariant corresponding to the orthogonal extension, $0 \in \mathfrak{Ext}(A,B)$, is not a unit under the addition.

Remark 13.7. In general, extensions cannot be added in the way of Definition 13.1. It is necessary that the left-hand term is stable. For an explanation, see [34, p. 62].

Lemma 13.8. Let A and B be C*-algebras. $\mathfrak{Drt}(A,B)$ is a subsemigroup of $\mathfrak{Ert}(A,B)$ under the action defined in Definition 13.1.

Proof. By Lemma 6.31, a split extension factors through $\mathcal{M}(\mathbb{K}\otimes B)$.
Let $\eta_1,\eta_2 \in \mathrm{Hom}_{\mathsf{C^*-alg}}(A,\mathcal{M}(\mathbb{K}\otimes B))$, then $\begin{pmatrix}\pi(\eta_1(a)) & 0\\ 0 & \pi(\eta_2(a))\end{pmatrix} = \tilde{\pi}\left(\begin{pmatrix}\eta_1(a) & 0\\ 0 & \eta_2(a)\end{pmatrix}\right)$ for $a\in A$, where $\mathcal{M}(\mathbb{K}\otimes\mathcal{M}) \xrightarrow{\pi} \mathcal{Q}(\mathbb{K}\otimes\mathcal{M})$ and $M_2(\mathcal{M}(\mathbb{K}\otimes\mathcal{M})) \xrightarrow{\tilde{\pi}} M_2(\mathcal{Q}(\mathbb{K}\otimes\mathcal{M}))$ define the quotient maps. Hence $\pi(\eta_1)\oplus\pi(\eta_2)$ defines a $*$-homomorphism from A to $M_2(\mathcal{Q}(\mathbb{K}\otimes B))$ that can be factored through $M_2(\mathcal{M}(\mathbb{K}\otimes B))$. Hence $\pi(\eta_1)\oplus\pi(\eta_2)$ can be factored through $\mathcal{M}(\mathbb{K}\otimes B)$ and therefore defines a split extension; $\pi(\eta_1)\oplus\pi(\eta_2)\in\mathfrak{Drt}(A,B)$. $\mathfrak{Drt}(A,B)$ is closed under the action of Definition 13.1. □

Lemma 13.9. Let A and B be C*-algebras. The addition of Definition 13.1 gives a semigroup structure on $\overline{\mathfrak{Ert}}(A,B)$.

Proof. The associativity is made clear by Remark 13.5. It remains to be shown the addition is independent of representative. Let $\phi_1,\phi_2,\psi_1,\psi_2\in\mathfrak{Ert}(A,B)$ such that ϕ_1 and ψ_1 are unitarily equivalent and ϕ_2 and ψ_2 are unitarily equivalent. There exist unitaries $u,v\in\mathcal{M}(\mathbb{K}\otimes B)$ such that $\psi_1(a)=\pi(u)\phi_1(a)\pi(u^*)$ and $\psi_2(a)=\pi(v)\phi_2(a)\pi(v^*)$ for all $a\in A$, where $\mathcal{M}(\mathbb{K}\otimes B)\xrightarrow{\pi}\mathcal{Q}(\mathbb{K}\otimes B)$ is the quotient. Now $\begin{pmatrix}u & 0\\ 0 & v\end{pmatrix}$ is a unitary in $M_2(\mathcal{M}(\mathbb{K}\otimes B))$ and

$$\begin{pmatrix}\psi_1(a) & 0\\ 0 & \psi_2(a)\end{pmatrix} = \begin{pmatrix}\pi(u) & 0\\ 0 & \pi(v)\end{pmatrix}\begin{pmatrix}\phi_1(a) & 0\\ 0 & \phi_2(a)\end{pmatrix}\begin{pmatrix}\pi(u^*) & 0\\ 0 & \pi(v^*)\end{pmatrix} \text{ for all } a\in A.$$

Unital $*$-isomorphisms preserve and reflect unitaries. So $\phi_1\oplus\phi_2$ is unitarily equivalent to $\psi_1\oplus\psi_2$. □

Lemma 13.10. Let A and B be C*-algebras. $\overline{\mathfrak{Ert}}(A,B)$ is an abelian semigroup.

Proof. $\begin{pmatrix}0 & 1_{\mathcal{Q}(\mathbb{K}\otimes B)}\\ 1_{\mathcal{Q}(\mathbb{K}\otimes B)} & 0\end{pmatrix}$ is a unitary in $M_2(\mathcal{Q}(\mathbb{K}\otimes B))$. And

$$\begin{pmatrix}\phi_2(a) & 0\\ 0 & \phi_1(a)\end{pmatrix} = \begin{pmatrix}0 & 1_{\mathcal{Q}(\mathbb{K}\otimes B)}\\ 1_{\mathcal{Q}(\mathbb{K}\otimes B)} & 0\end{pmatrix}\begin{pmatrix}\phi_1(a) & 0\\ 0 & \phi_2(a)\end{pmatrix}\begin{pmatrix}0 & 1_{\mathcal{Q}(\mathbb{K}\otimes B)}\\ 1_{\mathcal{Q}(\mathbb{K}\otimes B)} & 0\end{pmatrix}$$

for $\phi_1,\phi_2\in\mathfrak{Ert}(A,B)$ and all $a\in A$.

Hence $\phi_1\oplus\phi_2$ is unitarily equivalent to $\phi_2\oplus\phi_1$. □

Corollary 13.11. Let A and B be C*-algebras. $\overline{\mathfrak{Drt}}(A,B)$ is a subsemigroup of $\overline{\mathfrak{Ert}}(A,B)$.

Definition 13.12. Let A and B be C*-algebras. Define $\mathrm{Ext}(A,B) := \overline{\mathfrak{Ert}}(A,B)/\overline{\mathfrak{Drt}}(A,B)$.

Lemma 13.13. The addition on $\overline{\mathfrak{Ert}}(A,B)$ is independent of the choice of bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$.

Proof. Let $S,T:\mathbb{N}\to\mathbb{N}\times\mathbb{N}$ be two bijections. Then $S^{-1}\circ T:\mathbb{N}\to\mathbb{N}$ forms a permutation of $\mathbb{N}$. This permutation induces a permutation of indices on $\mathbb{H}_B$. Any permutation has an inverse and using the inner product on $\mathbb{H}_B$ ($\langle x,y\rangle_{\mathbb{H}_B} := \sum_{\mathbb{N}} x_k^* y_k$), it immediately becomes clear that the inverse of the permutation is the adjoint. So $S^{-1}\circ T$ induces a unitary element of $\mathbb{B}(\mathbb{H}_B) = \mathcal{M}(\mathbb{K}\otimes B)$. Hence the additions defined by S and T are unitarily equivalent and represent the same class in $\overline{\mathfrak{Ext}}(A,B)$. $\square$

Remark 13.14. Another proof of Lemma 13.13 is given in [34, Proposition 3.3.4]. It builds on [34, Proposition 1.10.2]. That approach requires quite some analysis and feels less conceptual than ours.

13.2 Absorption

Definition 13.15. Let A and B be C*-algebras. An extension $\phi\in\mathfrak{Ext}(A,B)$ is *absorbing* if $\phi\oplus\psi$ is unitarily equivalent to ϕ for all $\psi\in\mathfrak{Dxt}(A,B)$.

Lemma 13.16. Let A and B be C*-algebras. Let $\phi_1,\phi_2\in\mathfrak{Ext}(A,B)$ be unitarily equivalent extensions (so they represent the same class in $\overline{\mathfrak{Ext}}(A,B)$). If ϕ_1 is absorbing, then so is ϕ_2.

Proof. Let $\phi_1,\phi_2\in\mathfrak{Ext}(A,B)$ be unitarily equivalent. Let ϕ_1 be absorbing. By definition $\phi_1\oplus\psi$ is unitarily equivalent to ϕ_1 for all $\psi\in\mathfrak{Dxt}(A,B)$. Now ϕ_2 is unitarily equivalent to ϕ_1, which is unitarily equivalent to $\phi_1\oplus\psi$ for all $\psi\in\mathfrak{Dxt}(A,B)$. By Lemma 13.9 the addition of unitary equivalence classes is well-defined. As ϕ_1 and ϕ_2 are unitarily equivalent, it follows $\phi_1\oplus\psi$ is unitarily equivalent to $\phi_2\oplus\psi$ for all $\psi\in\mathfrak{Dxt}(A,B)$. So ϕ_2 is unitarily equivalent to $\phi_2\oplus\psi$ for all $\psi\in\mathfrak{Dxt}(A,B)$. Hence ϕ_2 is absorbing. $\square$

Definition 13.17. The set of unitary equivalence classes of absorbing extensions of A by $\mathbb{K}\otimes B$ is denoted by $\mathrm{Ext}_a(A,B)$.

Lemma 13.18. Let A and B be C*-algebras. Let $\phi\in\mathfrak{Ext}(A,B)$ be an absorbing extension. Then $\phi\oplus\omega$ is absorbing for all $\omega\in\mathfrak{Ext}(A,B)$.

Proof. ϕ is absorbing, so ϕ is unitarily equivalent to $\phi\oplus\psi$ for all $\psi\in\mathfrak{Dxt}(A,B)$. Now $\phi\oplus\omega$ is unitarily equivalent to $\phi\oplus\psi\oplus\omega$ for all $\psi\in\mathfrak{Dxt}(A,B)$. By Lemma 13.10 $\phi\oplus\psi\oplus\omega$ is unitarily equivalent to $\phi\oplus\omega\oplus\psi$ for all $\psi\in\mathfrak{Dxt}(A,B)$. So $\phi\oplus\omega$ is unitarily equivalent to $\phi\oplus\omega\oplus\psi$ for all $\psi\in\mathfrak{Dxt}(A,B)$. Hence $\phi\oplus\omega$ is absorbing. $\square$

Remark 13.19. Lemma 13.18 clarifies the name *absorbing*.

Remark 13.20. By Lemma 13.16 and Lemma 13.18, $\mathrm{Ext}_a(A,B)$ forms a subsemigroup of $\overline{\mathfrak{Ext}}(A,B)$. Unlike $\mathrm{Ext}(A,B)$, $\mathrm{Ext}_a(A,B)$ is not defined as the quotient of the absorbing classes by the absorbing split classes. This is because the addition of a split class by definition does not change an absorbing class. There is nothing to be divided out.

Lemma 13.21. Let A be a unital C*-algebra and B an arbitrary C*-algebra. A unital extension in $\mathfrak{Ext}(A,B)$ cannot be absorbing.

Proof. Let $\phi\in\mathfrak{Ext}(A,B)$ be a unital absorbing extension. Then $\phi\oplus\psi$ is unitarily equivalent to ϕ for all $\psi\in\mathfrak{Dxt}(A,B)$. But by Lemma 6.36, $\phi\oplus\psi$ must be unital for all $\psi\in\mathfrak{Dxt}(A,B)$. Now if $\phi\oplus\psi$ is unital, then $\begin{pmatrix}\phi(1_A) & 0\\ 0 & \psi(1_A)\end{pmatrix} = \begin{pmatrix}1_{\mathcal{Q}(\mathbb{K}\otimes B)} & 0\\ 0 & 1_{\mathcal{Q}(\mathbb{K}\otimes B)}\end{pmatrix}$ and so $\psi(1_A) = 1_{\mathcal{Q}(\mathbb{K}\otimes B)}$. So every $\psi\in\mathfrak{Dxt}(A,B)$ is unital. That is not true: $0\in\mathfrak{Dxt}(A,B)$ and 0 is certainly not unital. $\square$

Remark 13.22. The main takeaway of Lemma 13.21 is that $\phi \oplus 0$ does not need to be unitarily equivalent to ϕ and in particular not equal. So 0 is not a unit under addition in $\mathfrak{Ext}(A,B)$ or $\overline{\mathfrak{Ext}}(A,B)$.

Remark 13.23. The unital extensions form subsemigroups of $\mathfrak{Ext}(A,B)$ and $\overline{\mathfrak{Ext}}(A,B)$.

Lemma 13.24 (Generalised Theorem of Voiculescu)**.** Let A be a separable C*-algebra and B be a σ-unital C*-algebra. Let at least A or B be nuclear. An embedding $A \hookrightarrow \mathcal{M}(\mathbb{K}) \xrightarrow{-\otimes \mathrm{id}_B} \mathcal{M}(\mathbb{K}\otimes B)$ for which $A \cap \mathbb{K} = 0$, postcomposed by $\mathcal{M}(\mathbb{K}\otimes B) \xrightarrow{\pi} \mathcal{Q}(\mathbb{K}\otimes B)$, is absorbing.

Proof. This is a special case of [15, Theorem 6]. □

Remark 13.25. An embedding $A \hookrightarrow \mathcal{M}(\mathbb{K})$ such that $A \cap \mathbb{K} = 0$ exists when A is separable, see [15, Lemma 8.1.3].

Theorem 13.26. Let A be a separable C*-algebra and let B be a σ-unital C*-algebra. If A or B is nuclear, then $\mathrm{Ext}_a(A,B) \cong \mathrm{Ext}(A,B)$.

Proof. There is always an inclusion $\mathrm{Ext}_a(A,B) \subset \mathrm{Ext}(A,B)$ by Lemma 13.20. Conversely, via Lemma 13.24, there exists a split absorbing extension. The extension constructed in Lemma 13.24 factors through $\mathcal{M}(\mathbb{K}\otimes B)$ (see Lemma 6.31). Adding a split extension does not change the class in $\mathrm{Ext}(A,B)$. An addition containing an absorbing extension is absorbing by Lemma 13.18. So every class in $\mathrm{Ext}(A,B)$ has an absorbing representative. From this follows $\mathrm{Ext}(A,B) \cong \mathrm{Ext}_a(A,B)$. □

Remark 13.27. Theorem 13.26 does **not** imply that every extension is absorbing when A or B is nuclear. For this the orthogonal extension would need to be absorbing. It only means that every class of unitarily equivalent extensions differ by a split extension from an absorbing class of unitary equivalent extensions. This is because inverses need not exist.

13.3 Kasparov Pairs

Definition 13.28. Let A and B be C*-algebras. Let $\phi \in \mathrm{Hom}_{\mathsf{C^*\text{-}alg}}(A, \mathcal{M}(\mathbb{K}\otimes B))$ and let $P \in \mathcal{M}(\mathbb{K}\otimes B)$. The pair (ϕ, P) is a *Kasparov-A-B-pair* if

$$\phi(a)P - P\phi(a) \in \mathbb{K}\otimes B \qquad (P^2 - P)\phi(a) \in \mathbb{K}\otimes B \qquad (P - P^*)\phi(a) \in \mathbb{K}\otimes B \quad \text{for all } a \in A. \tag{44}$$

The set of Kasparov A-B-pairs is denoted by $\mathcal{E}^1(A,B)$.

Definition 13.29. Let A and B be C*-algebras. A Kasparov A-B-pair (ϕ, P) is *degenerate* if the three terms in (44) equal 0 for all $a \in A$. The set of degenerate Kasparov A-B-pairs is denoted by $\mathcal{D}^1(A,B)$.

Definition 13.30. Let A and B be C*-algebras. Let $(\phi_1, P_1), (\phi_2, P) \in \mathcal{E}^1(A,B)$.
Define $(\phi_1, P_1) \oplus (\phi_2, P) := (\phi_1 \oplus \phi_2, P_1 \oplus P_2)$, where

$$P_1 \oplus P_2 := \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \in \mathbb{B}(\mathbb{H}_B \oplus \mathbb{H}_B) \cong \mathbb{B}(\mathbb{H}_B) \cong \mathcal{M}(\mathbb{K}\otimes B).$$

$(\phi_1, P_1) \oplus (\phi_2, P)$ is called the *addition* of (ϕ_1, P_1) and (ϕ_2, P_2).

Remark 13.31. It is easy to see $(\phi_1 \oplus \phi_2, P_1 \oplus P_2)$ is again a Kasparov A-B-pair as

$$\begin{pmatrix} \phi_1(a) & 0 \\ 0 & \phi_2(a) \end{pmatrix}\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} - \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\begin{pmatrix} \phi_1(a) & 0 \\ 0 & \phi_2(a) \end{pmatrix} \in M_2(\mathbb{K} \otimes B) \cong \mathbb{K} \otimes B \text{ for all } a \in A.$$

Lemma 13.32. The addition of Definition 13.30 gives $\mathcal{E}^1(A,B)$ a semigroup structure. Under this addition $\mathcal{D}^1(A,B)$ is a subsemigroup of $\mathcal{E}^1(A,B)$.

Proof. Similar to Lemma 13.8. $\square$

Remark 13.33. It is very easy to create some Kasparov A-B-pairs. A few examples are:

- Let $P \in \mathbb{K} \otimes B$ and let $\phi \in \mathrm{Hom}_{\mathsf{C^*-alg}}(A, \mathcal{M}(\mathbb{K} \otimes B))$, then $(\phi, P) \in \mathcal{E}^1(A,B)$.
- Let $\phi \in \mathrm{Hom}_{\mathsf{C^*-alg}}(A, \mathcal{M}(\mathbb{K} \otimes B))$, then $(\phi, 1_{\mathcal{M}(\mathbb{K}\otimes B)}) \in \mathcal{D}^1(A,B)$. More generally, an element of $\mathrm{Hom}_{\mathsf{C^*-alg}}(A, \mathcal{M}(\mathbb{K} \otimes B))$ paired with a central element of $\mathcal{M}(\mathbb{K} \otimes B)$ forms a degenerate Kasparov pair.
- Let $P \in \mathcal{M}(\mathbb{K} \otimes B)$, then $(0, P) \in \mathcal{D}^1(A,B)$.

Definition 13.34. Let A and B be C*-algebras. Kasparov A-B-pairs (ϕ_1, P_1) and (ϕ_2, P_2) are *unitarily equivalent* if there exists a unitary $u \in \mathcal{M}(\mathbb{K} \otimes B)$ such that $\phi_2(a) = u\phi_1(a)u^*$ for all $a \in A$ and $P_2 = uP_1u^*$.

Lemma 13.35. Unitary equivalence of Kasparov pairs is an equivalence relation.

Proof. The proof is almost identical to the proof of Lemma 6.33. Nonetheless;
Reflexivity: $\mathcal{M}(\mathbb{K} \otimes B)$ is unital. $\phi(a) = 1_{\mathcal{M}(\mathbb{K}\otimes B)}\phi(a)1_{\mathcal{M}(\mathbb{K}\otimes B)}$ for all $a \in A$ and $P = 1_{\mathcal{M}(\mathbb{K}\otimes B)}P1_{\mathcal{M}(\mathbb{K}\otimes B)}$ for $(\phi, P) \in \mathcal{E}^1(A,B)$.
Symmetry: If there exists a unitary $u \in \mathcal{M}(\mathbb{K} \otimes B)$ such that $\phi_2(a) = u\phi_1(a)u^*$ for all $a \in A$ and $P_2 = uP_1u^*$, then $u^*\phi_2(a)u = u^*u\phi_1(a)u^*u = \phi_1(a)$ for all $a \in A$ and $u^*P_2u = u^*uP_1u^*u = P_1$.
Transitivity: If there exists a unitary $u \in \mathcal{M}(\mathbb{K} \otimes B)$ such that $\phi_2(a) = u\phi_1(a)u^*$ for all $a \in A$ and $P_2 = uP_1u^*$ and there exists a unitary $v \in \mathcal{M}(\mathbb{K} \otimes B)$ such that $\phi_3(a) = v\phi_2(a)v^*$ for all $a \in A$ and $P_3 = vP_2v^*$, then $\phi_3(a) = vu\phi_1(a)u^*v^*$ for all $a \in A$ and $P_3 = vuP_1u^*v^*$. $\square$

Lemma 13.36. Let A and B be C*-algebras. Let (ϕ_1, P_1) and (ϕ_2, P_2) be unitarily equivalent Kasparov A-B-pairs. If (ϕ_1, P_1) is degenerate, then so is (ϕ_2, P_2).

Proof. Let $u \in \mathcal{M}(\mathbb{K} \otimes B)$ be a unitary such that $\phi_2(a) = u\phi_1(a)u^*$ for all $a \in A$ and $P_2 = uP_1u^*$.

$$\begin{aligned} \phi_2(a)P_2 - P_2\phi_2(a) &= u\phi_1(a)u^*uP_1u^* - uP_1u^*u\phi_1(a)u^* = u\phi_1(a)P_1u^* - uP_1\phi_1(a)u^* = \\ &\qquad u(\phi_1(a)P_1 - P_1\phi_1(a))u^* = u0u^* = 0 \text{ for all } a \in A \\ (P_2^2 - P_2)\phi_2(a) &= ((uP_1u^*)^2 - uP_1u^*)u\phi_1(a)u^* = (u(P_1^2 - P_1)u^*)u\phi_1(a)u^* = \\ &\qquad u(P_1^2 - P_1)\phi_1(a)u^* = u0u^* = 0 \text{ for all } a \in A \\ (P_2 - P_2^*)\phi_2(a) &= (uP_1u^* - (uP_1u^*)^*)u\phi_1(a)u^* = (uP_1u^* - uP_1^*u^*)u\phi_1(a)u^* = \\ &\qquad u(P_1 - P_1^*)\phi_1(a)u^* = u0u^* = 0 \text{ for all } a \in A. \end{aligned}$$

$\square$

Lemma 13.37. The addition of Definition 13.30 gives a well-defined commutative semigroup action on the classes of unitarily equivalent Kasparov pairs.

Proof. The well-definedness is similar to Lemma 13.9 and the commutativity is similar to Lemma 13.10. ☐

Definition 13.38. Let A and B be C*-algebras. Kasparov A-B-pairs (ϕ_1, P_1) and (ϕ_2, P_2) are *homological* if $P_1\phi_1(a) - P_2\phi_2(a) \in \mathbb{K} \otimes B$ for all $a \in A$.

Lemma 13.39. Let A and B be C*-algebras. Homology of Kasparov A-B-pairs is an equivalence relation.

Proof. Reflexivity: Clearly $P\phi(a) - P\phi(a) = 0 \in \mathbb{K} \otimes B$ for all $a \in A$.
Symmetry: If $P_1\phi_1(a) - P_2\phi_2(a) \in \mathbb{K} \otimes B$ for all $a \in A$, then

$$P_2\phi_2(a) - P_1\phi_1(a) = -(P_1\phi_1(a) - P_2\phi_2(a)) \in \mathbb{K} \otimes B \text{ for all } a \in A.$$

Transitivity: If $P_1\phi_1(a) - P_2\phi_2(a) \in \mathbb{K} \otimes B$ and $P_2\phi_2(a) - P_3\phi_3(a) \in \mathbb{K} \otimes B$ for all $a \in A$, then

$$P_1\phi_1(a) - P_3\phi_3(a) = P_1\phi_1(a) - P_2\phi_2(a) + P_2\phi_2(a) - P_3\phi_3(a) \in \mathbb{K} \otimes B \text{ for all } a \in A. \qquad \square$$

Remark 13.40. Let (ϕ_1, P_1) and (ϕ_2, P_2) be homological Kasparov A-B-pairs. (ϕ_1, P_1) can be degenerate, while (ϕ_2, P_2) is not. For example $(0, P)$ and (ϕ, Q), for $\phi \in \mathrm{Hom}_{\mathsf{C^*-alg}}(A, \mathcal{M}(\mathbb{K} \otimes B))$ with $P \in \mathcal{M}(\mathbb{K} \otimes B)$ and $Q \in \mathbb{K} \otimes B$. $(0, P)$ is degenerate, while (ϕ, Q) is not (necessarily). However $0 - \phi(a)Q \in \mathbb{K} \otimes B$ for all $a \in A$. Therefore it is impossible to speak about degenerate homology classes of Kasparov pairs. This is in stark contrast to unitary equivalence, see Lemma 13.36.

Lemma 13.41. Let A and B be C*-algebras. The addition of Definition 13.30 gives a well-defined addition on homology classes.

Proof. Let (ϕ_1, P_1) and (ϕ_2, P_2) be homological Kasparov A-B-pairs and let (ψ_1, Q_1) and (ψ_2, Q_2) be homological Kasparov A-B-pairs. The following sums are homological:

$$(\phi_1, P_1) \oplus (\phi_2, P_2) = \left(\phi_1 \oplus \phi_2, \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}\right) \qquad (\psi_1, Q_1) \oplus (\psi_2, Q_2) = \left(\psi_1 \oplus \psi_2, \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}\right), \text{ as}$$

$$\begin{pmatrix} P_1\phi_1(a) & 0 \\ 0 & P_2\phi_2(a) \end{pmatrix} - \begin{pmatrix} Q_1\psi_1(a) & 0 \\ 0 & Q_2\psi_2(a) \end{pmatrix} = \begin{pmatrix} P_1\phi_1(a) - Q_1\psi_1(a) & 0 \\ 0 & P_2\phi_2(a) - Q_2\psi_2(a) \end{pmatrix} \in M_2(\mathbb{K} \otimes B) \cong \mathbb{K} \otimes B \text{ for all } a \in A.$$

So $(\phi_1, P_1) \oplus (\phi_2, P_2)$ and $(\psi_1, Q_1) \oplus (\psi_2, Q_2)$ are homological. ☐

Definition 13.42. Let A and B be C*-algebras. $\overline{\mathcal{E}}^1(A, B)$ is defined as the set of Kasparov A-B-pairs modulo unitary equivalence and homology. $\overline{\mathcal{D}}^1(A, B)$ is defined as the set of Kasparov A-B-pairs modulo unitary equivalence and homology that have a degenerate representative. The quotient is denoted by $E^1(A, B) := \overline{\mathcal{E}}^1(A, B)/\overline{\mathcal{D}}^1(A, B)$.

Remark 13.43. Unitary equivalence and homology are two different equivalence relations and neither is stronger than the other. Unitarily equivalent Kasparov pairs need not be homological.

Though conjugation by unitaries of $\mathcal{M}(\mathbb{K}\otimes B)$ fixes $\mathbb{K}\otimes B$, this does not mean that the conjugation is the identity on $\mathcal{Q}(\mathbb{K}\otimes B)$. Homological Kasparov pairs need not be unitarily equivalent, see Remark 13.40. Being unitarily equivalent **and/or** homological is not an equivalence relation; it is not transitive. However, unitary equivalence **and** homology generate an equivalence relation when reflexivity, symmetry and transitivity are applied. These are exactly the equivalence classes $\overline{\mathcal{E}}^1(A,B)$ consists of.

Corollary 13.44. $E^1(A,B)$ is a unital abelian semigroup under the addition of Definition 13.30.

Proof. The semigroup structure on $\overline{\mathcal{E}}^1(A,B)$ given by Definition 13.30, is well-defined because the addition of Kasparov pairs is independent of representative of unitary equivalence class or homology class (Lemma 13.37 and Lemma 13.41).
The addition is commutative, because any sum is unitarily equivalent to the sum in the opposite order and therefore lands in the same class (Lemma 13.37).
$\overline{\mathcal{D}}^1(A,B)$ is a subsemigroup of $\overline{\mathcal{E}}^1(A,B)$, because any class in $\overline{\mathcal{D}}^1(A,B)$ can be represented by a degenerate Kasparov pair and the sum of two degenerate Kasparov pairs is degenerate (Definition 13.30). So $E^1(A,B)$ is a semigroup. Any representative of a class in $\overline{\mathcal{D}}^1(A,B)$ represents the unit of $E^1(A,B)$. ☐

Remark 13.45. See Remark 13.33 to conclude $\mathcal{D}^1(A,B)\neq\varnothing$.

13.4 Connection between Kasparov pairs and extensions

Lemma 13.46. Let A and B be C*-algebras. The mapping $\gamma(\phi,P)=P\phi \mod \mathbb{K}\otimes B$ defines a semigroup morphism $\gamma\colon E^1(A,B)\to \operatorname{Ext}(A,B)$.

Proof. First we will prove that for a Kasparov A-B-pair $(\phi,P)\in\mathcal{E}^1(A,B)$, the map $A\ni a\to P\phi(a)$ mod $\mathbb{K}\otimes B$ defines a $*$-homomorphisms from A to $\mathcal{Q}(B)$. The map $A\ni a\to P\phi(a) \mod \mathbb{K}\otimes B$ is clearly additive. To prove that the map $A\ni a\to P\phi(a) \mod \mathbb{K}\otimes B$ is multiplicative, the compactness relations for Kasparov pairs (see Definition 13.28) are exploited. This is done in equation (45). Finally, to prove that the map $A\ni a\to P\phi(a) \mod \mathbb{K}\otimes B$ preserves the $*$-operation, the properties of the $*$-operation and compactness relations are exploited. This is done in equation (46).

$$\begin{aligned}P\phi(a_1a_2) \mod \mathbb{K}\otimes B \equiv P^2\phi(a_1a_2) \mod \mathbb{K}\otimes B \equiv \\ P^2\phi(a_1)\phi(a_2) \mod \mathbb{K}\otimes B \equiv P\phi(a_1)P\phi(a_2) \mod \mathbb{K}\otimes B \text{ for } a_1,a_2\in A\end{aligned} \tag{45}$$

$$\begin{aligned}(P\phi(a))^* \mod \mathbb{K}\otimes B \equiv (\phi(a)P)^* \mod \mathbb{K}\otimes B \equiv P^*\phi(a)^* \mod \mathbb{K}\otimes B \equiv \\ P^*\phi(a^*) \mod \mathbb{K}\otimes B \equiv P\phi(a^*) \mod \mathbb{K}\otimes B \text{ for } a\in A.\end{aligned} \tag{46}$$

It follows from the following equation that γ preserves addition:

$$\begin{aligned}\gamma((\phi_1,P_2)\oplus(\phi_2,P_2))(a) = \begin{pmatrix} P_1\phi_1(a) & 0 \\ 0 & P_2\phi_2(a)\end{pmatrix} \mod \mathbb{K}\otimes B \equiv \\ (\gamma(\phi_1,P_1)\oplus\gamma(\phi_2,P_2))(a) \text{ for all } a\in A.\end{aligned}$$

Let (ϕ_1, P_2) and (ϕ_2, P_2) be unitarily equivalent. There exists a unitary $u \in \mathcal{M}(\mathbb{K} \otimes B)$ such that $\phi_2(a) = u\phi_1(a)u^*$ for all $a \in A$ and $P_2 = uP_1u^*$. Now $P_2\phi_2(a) = uP_1u^*u\phi_1(a)u^* = uP_1\phi_1(a)u^*$. So $\gamma(\phi_2, P_2)$ is unitarily equivalent to $\gamma(\phi_1, P_1)$ and therefore represents the same element in $\mathrm{Ext}(A, B)$. It is obvious that γ is invariant under homology.

Each class in $\overline{\mathcal{D}}^1(A, B)$ can be represented by a degenerate Kasparov pair. For degenerate Kasparov pairs, everything in the first paragraph of this proof is true without the ' mod $\mathbb{K} \otimes B$'-statement. For $(\phi, P) \in \mathcal{D}^1(A, B)$, the mapping $A \ni a \to P\phi(a) \in \mathcal{M}(\mathbb{K} \otimes B)$ defines an element of $\mathrm{Hom}_{\mathsf{C^*-alg}}(A, \mathcal{M}(\mathbb{K} \otimes B))$. So $\gamma(\phi, P)$ is split by Lemma 6.31. Hence γ is independent of the representative in $E^1(A, B)$. $\square$

Remark 13.47. Notation has been abused in the proof; sometimes the images of γ were viewed as extensions rather than classes of extensions (modulo split classes). The use of brackets or overlines to denote classes or modulo relations would not have made it any clearer.

Lemma 13.48. Let A and B be C*-algebras. The morphism $\gamma: E^1(A, B) \to \mathrm{Ext}(A, B)$ from Lemma 13.46 is injective.

Proof. Let $(\phi_1, P_1), (\phi_2, P_2) \in \mathcal{E}^1(A, B)$. If $\gamma(\phi_1, P_1) = \gamma(\phi_2, P_2)$ in $\mathrm{Ext}(A, B)$, then there exist a $u \in \mathcal{M}(\mathbb{K} \otimes B)$ and $\psi_1, \psi_2 \in \mathfrak{Dxt}(A, B)$ such that $\gamma(\phi_1, P_1) \oplus \psi_1$ equals $u\gamma(\phi_2, P_2)u^* \oplus \psi_2$ as extensions. Notice that $\gamma(\psi_1, 1_{\mathcal{M}(\mathbb{K} \otimes B)}) = \psi_1 \mod \mathbb{K} \otimes B$ and $\gamma(\psi_2, 1_{\mathcal{M}(\mathbb{K} \otimes B)}) = \psi_2 \mod \mathbb{K} \otimes B$. So ψ_1 and ψ_2 can be viewed as image elements of γ. From the equality of extensions between $\gamma(\phi_1, P_1) \oplus \psi_1$ and $u\gamma(\phi_2, P_2)u^* \oplus \psi_2$ follows that $(\phi_1, P_1) \oplus (\psi_1, 1_{\mathcal{M}(\mathbb{K} \otimes B)})$ and $(u\phi_2u^*, uP_2u^*) \oplus (\psi_2, 1_{\mathcal{M}(\mathbb{K} \otimes B)})$ are homological. In particular, $(\phi_1, P_1) \oplus (\psi_1, 1_{\mathcal{M}(\mathbb{K} \otimes B)})$ and $(u\phi_2u^*, uP_2u^*) \oplus (\psi_2, 1_{\mathcal{M}(\mathbb{K} \otimes B)})$ represent the same class in $E^1(A, B)$. Both $(\psi_1, 1_{\mathcal{M}(\mathbb{K} \otimes B)})$ and $(\psi_2, 1_{\mathcal{M}(\mathbb{K} \otimes B)})$ are degenerate Kasparov pairs. They represent the unit element in $E^1(A, B)$. It follows that (ϕ_1, P_1) and $(u\phi_2u^*, uP_2u^*)$ represent the same class in $E^1(A, B)$. The Kasparov pair $(u\phi_2u^*, uP_2u^*)$ is unitarily equivalent to (ϕ_2, P_2) and therefore they represent the same class in $\mathcal{E}^1(A, B)$, in particular in $E^1(A, B)$. Hence (ϕ_1, P_1) and (ϕ_2, P_2) represent the same class in $E^1(A, B)$. $\square$

Lemma 13.49. Let A be a separable nuclear C*-algebra and B be a σ-unital C*-algebra. The morphism $\gamma: E^1(A, B) \to \mathrm{Ext}(A, B)$ from Lemma 13.46 is surjective.

Proof. A is nuclear and so is its unitisation $\tilde{A}$. This is because there is a natural exact sequence $0 \to A \to \tilde{A} \xrightarrow{\pi_{\mathbb{C}}} \mathbb{C} \to 0$. See Remark 6.28. The left-hand and the right-hand term are nuclear and so the middle term is nuclear (Lemma 6.62). $\mathcal{M}(\mathbb{K} \otimes B)$ is unital and $\mathcal{Q}(\mathbb{K} \otimes B)$ is the quotient of $\mathcal{M}(\mathbb{K} \otimes B)$ by $\mathbb{K} \otimes B$. Let $\phi \in \mathfrak{Ext}(A, B)$, by Theorem 5.90 its unitisation $\tilde{A} \xrightarrow{\tilde{\phi}} \mathcal{Q}(\mathbb{K} \otimes B)$ allows a unital, completely positive and linear map $\tilde{A} \xrightarrow{\chi} \mathcal{M}(\mathbb{K} \otimes B)$ such for which $\pi \circ \chi = \tilde{\phi}$, where $\mathcal{M}(\mathbb{K} \otimes B) \xrightarrow{\pi} \mathcal{Q}(\mathbb{K} \otimes B)$ is the quotient. We apply Theorem 5.93 (Stinespring) to χ. There exists a unital $*$-homomorphism $\tilde{A} \xrightarrow{\psi} M_2(\mathcal{M}(\mathbb{K} \otimes B))$ for which

$$\chi(a) \oplus 0 = \begin{pmatrix} 1_{\mathcal{M}(\mathbb{K} \otimes B)} & 0 \\ 0 & 0 \end{pmatrix} \psi(a) \begin{pmatrix} 1_{\mathcal{M}(\mathbb{K} \otimes B)} & 0 \\ 0 & 0 \end{pmatrix} \text{ for all } a \in \tilde{A}.$$

We will now prove that $\left(\psi, \begin{pmatrix} 1_{\mathcal{M}(\mathbb{K} \otimes B)} & 0 \\ 0 & 0 \end{pmatrix}\right)$ is a Kasparov $\tilde{A}$-B pair. The linear map χ does not need to be a $*$-homomorphism, as $\tilde{A} \ni a \to \begin{pmatrix} 1_{\mathcal{M}(\mathbb{K} \otimes B)} & 0 \\ 0 & 0 \end{pmatrix} \psi(a) \begin{pmatrix} 1_{\mathcal{M}(\mathbb{K} \otimes B)} & 0 \\ 0 & 0 \end{pmatrix}$ is in general not multiplicative. But $\pi\circ\chi=\tilde{\phi}$ is a $*$-homomorphism. Write $M_2(\mathcal{M}(\mathbb{K}\otimes B))\ni\psi(a)=\begin{pmatrix}\psi_{11}(a) & \psi_{12}(a)\\ \psi_{21}(a) & \psi_{22}(a)\end{pmatrix}$. We will set up relations between $\psi_{11},\psi_{12},\psi_{21}$ and ψ_{22} first. The multiplicativity of ψ implies

$$\begin{pmatrix}\psi_{11}(ab) & \psi_{12}(ab)\\ \psi_{21}(ab) & \psi_{22}(ab)\end{pmatrix}=\psi(ab)=\psi(a)\psi(b)=\begin{pmatrix}\psi_{11}(a) & \psi_{12}(a)\\ \psi_{21}(a) & \psi_{22}(a)\end{pmatrix}\begin{pmatrix}\psi_{11}(b) & \psi_{12}(b)\\ \psi_{21}(b) & \psi_{22}(b)\end{pmatrix}=$$
$$\begin{pmatrix}\psi_{11}(a)\psi_{11}(b)+\psi_{12}(a)\psi_{21}(b) & \psi_{11}(a)\psi_{12}(b)+\psi_{12}(a)\psi_{22}(b)\\ \psi_{21}(a)\psi_{11}(b)+\psi_{22}(a)\psi_{21}(b) & \psi_{21}(a)\psi_{12}(b)+\psi_{22}(a)\psi_{22}(b)\end{pmatrix}\text{ for } a,b\in\tilde{A}.$$

From the multiplicativity of $\pi\circ\chi$, it follows that
$\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}\psi(ab)\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}\psi_{11}(a)\psi_{11}(b)+\psi_{12}(a)\psi_{21}(b) & 0\\ 0 & 0\end{pmatrix}$ and
$\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}\psi(a)\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}\psi(b)\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}\psi_{11}(a)\psi_{11}(b) & 0\\ 0 & 0\end{pmatrix}$ are equal modulo $M_2(\mathbb{K}\otimes B)$ for $a,b\in\tilde{A}$. It follows that $\psi_{12}(a)\psi_{21}(b)\in\mathbb{K}\otimes B$ for $a,b\in\tilde{A}$.
As ψ commutes with the $*$-operation, it follows that

$$\begin{pmatrix}\psi_{11}(a^*) & \psi_{12}(a^*)\\ \psi_{21}(a^*) & \psi_{22}(a^*)\end{pmatrix}=\psi(a^*)=\psi(a)^*=\begin{pmatrix}\psi_{11}(a)^* & \psi_{21}(a)^*\\ \psi_{12}(a)^* & \psi_{22}(a)^*\end{pmatrix}\text{ for } a\in\tilde{A}.$$

So ψ_{11} and ψ_{22} commute with the $*$-operation and $\psi_{12}(a)^*=\psi_{21}(a^*)$ and $\psi_{21}(a)^*=\psi_{12}(a^*)$ for $a\in\tilde{A}$. In particular $\psi_{12}(a)\psi_{21}(a^*)=\psi_{12}(a)\psi_{12}(a)^*$and $\psi_{12}(a^*)\psi_{21}(a)=\psi_{21}(a)^*\psi_{21}(a)$ for all $a\in\tilde{A}$. As $\psi_{12}(a)\psi_{21}(b)\in\mathbb{K}\otimes B$ for $a,b\in\tilde{A}$, it follows $\psi_{12}(a)\psi_{12}(a)^*,\psi_{21}(a)^*\psi_{21}(a)\in\mathbb{K}\otimes B$. So $\psi_{12}(a),\psi_{21}(a)\in\mathbb{K}\otimes B$ for $a\in\tilde{A}$.
Now we will show that $\left(\psi,\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}\right)$ satisfies Definition 13.28. Notice:

$$\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}^2=\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}\text{ and }\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}^*=\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}.$$

It remains to show:

$$\psi(a)\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}-\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}\psi(a)=\begin{pmatrix}0 & -\psi_{12}(a)\\ \psi_{21}(a) & 0\end{pmatrix}\in M_2(\mathbb{K}\otimes B)\text{ for all } a\in\tilde{A}.$$

It follows that $\left(\psi,\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}\right)$ is Kasparov $\tilde{A}$-B pair.

We will now show that $\gamma\left(\psi,\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}\right)=\tilde{\phi}\oplus 0$. For this, notice

$$\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}\psi(a)=\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}^2\psi(a)\equiv$$
$$\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}\psi(a)\begin{pmatrix}1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0\\ 0 & 0\end{pmatrix}\quad\text{mod } M_2(\mathbb{K}\otimes B)\text{ for all } a\in\tilde{A}.$$

Let $A \xrightarrow{i_A} \tilde{A}$ be the inclusion. Precompose both $\tilde{\phi}$ and ψ with i_A to get that $\gamma\left(\psi \circ i_A, \begin{pmatrix} 1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0 \\ 0 & 0 \end{pmatrix}\right) = \phi \oplus 0$. In $\operatorname{Ext}(A,B)$, ϕ and $\phi \oplus 0$ represent the same class. We have found a pre-image for $[\phi] \in \operatorname{Ext}(A,B)$. So γ is surjective. $\square$

Remark 13.50. The χ in the proof of Lemma 13.49 does not need to be a $*$-homomorphism. In fact χ is a $*$-homomorphism precisely if the $\phi \in \mathfrak{Ext}(A,B)$ from Lemma 13.49 is split.
If χ is a $*$-homomorphism, then $\begin{pmatrix} 1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0 \\ 0 & 0 \end{pmatrix} \psi \begin{pmatrix} 1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0 \\ 0 & 0 \end{pmatrix}$ a $*$-homomorphism.
From the multiplicativity of χ follows that $\psi_{12}(a)\psi_{21}(b) = 0$ for $a, b \in \tilde{A}$. From the fact that $\psi_{12}(a^*) = \psi_{21}(a)^*$ for $a \in \tilde{A}$, it follows $\psi_{12} = 0$ and $\psi_{21} = 0$. Then

$$\psi(a)\begin{pmatrix} 1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0 \\ 0 & 0 \end{pmatrix}\psi(a) = \begin{pmatrix} 0 & -\psi_{12}(a) \\ \psi_{21}(a) & 0 \end{pmatrix} = 0 \text{ for all } a \in \tilde{A}.$$

So $\left(\psi, \begin{pmatrix} 1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0 \\ 0 & 0 \end{pmatrix}\right)$ is a degenerate Kasparov pair.

Proposition 13.51. Let A be a separable nuclear C*-algebra and B be a σ-unital C*-algebra. The morphism $\gamma: E^1(A,B) \to \operatorname{Ext}(A,B)$ from Lemma 13.46 is an isomorphism.

Proof. By Lemma 13.46, $\gamma(\phi, P) = P\phi \mod \mathbb{K}\otimes B$ defines a semigroup morphism $\gamma: E^1(A,B) \to \operatorname{Ext}(A,B)$. The semigroup morphism $\gamma: E^1(A,B) \to \operatorname{Ext}(A,B)$ is injective by Lemma 13.48 and surjective by Lemma 13.49. $\square$

13.5 Connection between Kasparov pairs and KK-groups

This subsection will use the notation of Remark 10.45. Remark 10.45 shows that every Kasparov A-$B\hat{\otimes}\mathbb{C}_1$-module with the Hilbert module $\widehat{\mathbb{H}_B}\hat{\otimes}\mathbb{C}_1$ can be written in the form $(\widehat{\mathbb{H}_B}\hat{\otimes}\mathbb{C}_1, \phi\hat{\otimes}1, F\hat{\otimes}\epsilon_1)$. This notation allows a much clearer display of the mapping from Kasparov modules to Kasparov pairs. It is elaborately explained in §10.3 how this result is obtained.

Lemma 13.52. Let A and B be C*-algebras. If $(\widehat{\mathbb{H}_B}\hat{\otimes}\mathbb{C}_1, \phi\hat{\otimes}1, F\hat{\otimes}\epsilon_1) \in \mathbb{E}(A, B\hat{\otimes}\mathbb{C}_1)$, then $\left(\phi, \frac{F+1}{2}\right) \in \mathcal{E}^1(A,B)$.

Proof. As $(\widehat{\mathbb{H}_B}\hat{\otimes}\mathbb{C}_1, \phi\hat{\otimes}1, F\hat{\otimes}\epsilon_1) \in \mathbb{E}(A, B\hat{\otimes}\mathbb{C}_1)$, by definition the following three things hold:

- $[F\hat{\otimes}\epsilon_1, \phi(a)\hat{\otimes}1] \in \mathbb{K}(\widehat{\mathbb{H}_B}\hat{\otimes}\mathbb{C}_1)$ for all $a \in A$. So $[F, \phi(a)] \in \mathbb{K}\otimes B$ for all $a \in A$.
- $((F\hat{\otimes}\epsilon_1)^2 - 1_{\mathcal{M}(\mathbb{K}\otimes B)}\hat{\otimes}1)(\phi(a)\hat{\otimes}1) \in \mathbb{K}(\widehat{\mathbb{H}_B}\hat{\otimes}\mathbb{C}_1)$ for all $a \in A$. So $(F^2 - 1_{\mathcal{M}(\mathbb{K}\otimes B)})\phi(a) \in \mathbb{K}\otimes B$ for all $a \in A$.
- $((F\hat{\otimes}\epsilon_1) - (F\hat{\otimes}\epsilon_1)^*)(\phi(a)\hat{\otimes}1) \in \mathbb{K}(\widehat{\mathbb{H}_B}\hat{\otimes}\mathbb{C}_1)$ for all $a \in A$. So $(F - F^*)\phi(a) \in \mathbb{K}\otimes B$ for all $a \in A$.

Now $\phi \in \operatorname{Hom}_{\mathsf{C^*\text{-}alg}}(A, \mathcal{M}(\mathbb{K}\otimes B))$ and $\frac{F + 1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2} \in \mathcal{M}(\mathbb{K}\otimes B)$ satisfy Definition 13.28.

1. $\mathcal{M}(\mathbb{K}\otimes B)$ is trivially graded. Therefore the graded brackets agree with the commutator brackets.
$$\phi(a)\frac{F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2} - \frac{F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\phi(a) = \phi(a)\frac{F}{2} - \frac{F}{2}\phi(a) = \frac{[\phi(a),F]}{2} \in \mathbb{K}\otimes B \text{ for all } a \in A.$$
2. $$\left(\left(\frac{F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\right)^2 - \frac{F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\right)\phi(a) = \left(\frac{F^2+2F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{4} - \frac{2F+2_{\mathcal{M}(\mathbb{K}\otimes B)}}{4}\right)\phi(a)$$
$$= \frac{F^2-1_{\mathcal{M}(\mathbb{K}\otimes B)}}{4}\phi(a) \in \mathbb{K}\otimes B \text{ for all } a \in A.$$
3. $$\left(\frac{F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2} - \frac{F^*+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\right)\phi(a) = \frac{F-F^*}{2}\phi(a) \in \mathbb{K}\otimes B \text{ for all } a \in A.$$

So $\left(\phi, \frac{F+1}{2}\right)$ forms an element of $\mathcal{E}^1(A,B)$. □

Lemma 13.53. Let A and B be C*-algebras. If $(\phi,P) \in \mathcal{E}^1(A,B)$, then $(\widehat{\mathbb{H}_B}\hat{\otimes}\mathbb{C}_1, \phi\hat{\otimes}1, (2P-1_{\mathcal{M}(\mathbb{K}\otimes B)})\hat{\otimes}\epsilon_1) \in \mathbb{E}(A, B\hat{\otimes}\mathbb{C}_1)$.

Proof. $\frac{(2P-1_{\mathcal{M}(\mathbb{K}\otimes B)})+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2} = \frac{2P}{2} = P$. The rest is inverting the proof of Lemma 13.52. □

Lemma 13.54. Let A and B be C*-algebras. If $\left(\widehat{\mathbb{H}_B}\hat{\otimes}\mathbb{C}_1, \phi\hat{\otimes}1, F\hat{\otimes}\epsilon_1\right) \in \mathbb{D}(A, B\hat{\otimes}\mathbb{C}_1)$, then $\left(\phi, \frac{F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\right) \in \mathcal{D}^1(A,B)$.

Proof. Copy the proof of Lemma 13.52.

1. $\mathcal{M}(\mathbb{K}\otimes B)$ is trivially graded. Therefore the graded brackets agree with the commutator brackets.
$$\phi(a)\frac{F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2} - \frac{F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\phi(a) = \phi(a)\frac{F}{2} - \frac{F}{2}\phi(a) = \frac{[\phi(a),F]}{2} = 0 \text{ for all } a \in A.$$
2. $$\left(\left(\frac{F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\right)^2 - \frac{F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\right)\phi(a) = \left(\frac{F^2+2F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{4} - \frac{2F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{4}\right)\phi(a)$$
$$= \frac{F^2-1_{\mathcal{M}(\mathbb{K}\otimes B)}}{4}\phi(a) = 0 \text{ for all } a \in A.$$
3. $$\left(\frac{F+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2} - \frac{F^*+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\right)\phi(a) = \frac{F-F^*}{2}\phi(a) = 0 \text{ for all } a \in A.$$ □

Lemma 13.55. Let A and B be C*-algebras. If $(\phi,P) \in \mathcal{D}^1(A,B)$, then $(\widehat{\mathbb{H}_B}\hat{\otimes}\mathbb{C}_1, \phi\hat{\otimes}1, (2P-1_{\mathcal{M}(\mathbb{K}\otimes B)})\hat{\otimes}\epsilon_1) \in \mathbb{D}(A, B\hat{\otimes}\mathbb{C}_1)$.

Proof. Similar to Lemma 13.54, but with '0 =' added at the left-hand side instead. □

Remark 13.56. Let A and B be C*-algebras. It is immediate from the definitions of addition of Kasparov A-B-modules (Definition 11.11) and addition of Kasparov A-B-pairs (Definition 13.30) that the construction of Lemma 13.52 preserves and reflects direct sums.

Lemma 13.57. Let A and B be C*-algebras. Let $(\phi, P), (\psi, Q) \in \mathcal{E}^1(A, B)$ be unitarily equivalent Kasparov pairs. Then $(\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \phi\hat{\otimes}1, (2P - 1_{\mathcal{M}(\mathbb{K}\otimes B)})\hat{\otimes}\epsilon_1)$ and $(\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \phi\hat{\otimes}1, (2Q - 1_{\mathcal{M}(\mathbb{K}\otimes B)})\hat{\otimes}\epsilon_1)$ are unitarily equivalent Kasparov A-B-modules.

Proof. There exists a unitary $u \in \mathcal{M}(\mathbb{K}\otimes B)$ such that $\psi(a) = u\phi(a)u^*$ for every $a \in A$ and $Q = uPu^*$. Now $u\hat{\otimes}1 \in \mathbb{B}(\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1)$ is the even unitary such that $\psi(a)\hat{\otimes}1 = (u\hat{\otimes}1)(\phi(a)\hat{\otimes}1)(u^*\hat{\otimes}1)$ for every $a \in A$ and $(2Q - 1_{\mathcal{M}(\mathbb{K}\otimes B)})\hat{\otimes}\epsilon_1 = (u\hat{\otimes}1)((2P - 1_{\mathcal{M}(\mathbb{K}\otimes B)})\hat{\otimes}\epsilon_1)(u^*\hat{\otimes}1)$. □

Lemma 13.58. Let A and B be C*-algebras. Let $(\phi, P), (\psi, Q) \in \mathcal{E}^1(A, B)$ be homological Kasparov pairs. Then $(\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \phi\hat{\otimes}1, (2P - 1_{\mathcal{M}(\mathbb{K}\otimes B)})\hat{\otimes}\epsilon_1) + (\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \psi\hat{\otimes}1, -1_{\mathcal{M}(\mathbb{K}\otimes B)}\hat{\otimes}\epsilon_1)$ and $(\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \phi\hat{\otimes}1, -1_{\mathcal{M}(\mathbb{K}\otimes B)}\hat{\otimes}\epsilon_1) + (\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \psi\hat{\otimes}1, (2Q - 1_{\mathcal{M}(\mathbb{K}\otimes B)})\hat{\otimes}\epsilon_1)$ are homotopic.

Proof. Notice the following additions:

$$(\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \phi\hat{\otimes}1, (2P - 1_{\mathcal{M}(\mathbb{K}\otimes B)})\hat{\otimes}\epsilon_1) + (\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \psi\hat{\otimes}1, -1_{\mathcal{M}(\mathbb{K}\otimes B)}\hat{\otimes}\epsilon_1) = \left(\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix} \hat{\otimes}1, \begin{pmatrix} 2P - 1_{\mathcal{M}(\mathbb{K}\otimes B)} & 0 \\ 0 & -1 \end{pmatrix} \hat{\otimes}\epsilon_1\right), \tag{47}$$

$$(\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \phi\hat{\otimes}1, -1_{\mathcal{M}(\mathbb{K}\otimes B)}\hat{\otimes}\epsilon_1) + (\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \psi\hat{\otimes}1, (2Q - 1_{\mathcal{M}(\mathbb{K}\otimes B)})\hat{\otimes}\epsilon_1) = \left(\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix} \hat{\otimes}1, \begin{pmatrix} -1 & 0 \\ 0 & 2Q - 1_{\mathcal{M}(\mathbb{K}\otimes B)} \end{pmatrix} \hat{\otimes}\epsilon_1\right). \tag{48}$$

We claim that an operator homotopy of (47) and (48) is given by:

$$\left(\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}, \begin{pmatrix} \frac{1}{1+t^2} 2P - 1 & \frac{1}{1+t^2} 2tPQ \\ \frac{1}{1+t^2} 2tQP & \frac{1}{1+t^2} 2t^2 Q - 1 \end{pmatrix}\right) \text{ with } 0 \le t < \infty. \tag{49}$$

Notice that for $t = 0$, the module becomes $\left(\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}, \begin{pmatrix} 2P - 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$.

Notice $\lim_{t\to\infty} \left(\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}, \begin{pmatrix} \frac{1}{1+t^2} 2P - 1 & \frac{1}{1+t^2} 2tPQ \\ \frac{1}{1+t^2} 2tQP & \frac{1}{1+t^2} 2t^2 Q - 1 \end{pmatrix}\right) = \left(\widehat{\mathbb{H}_B} \hat{\otimes} \mathbb{C}_1, \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2Q - 1 \end{pmatrix}\right)$.

It remains to prove that (49) is a Kasparov A-$C_0([0,\infty), B)$-module (see Definition 11.1).
Recall the compactness relations obtained by the Kasparov A-B-pairs (ϕ, P) and (ψ, Q) and their homology:

$$\begin{gathered} \phi(a)P - P\phi(a), \quad \psi(a)Q - Q\psi(a) \in \mathbb{K}\otimes B \text{ for all } a \in A \\ (P^2 - P)\phi(a), \quad (Q^2 - Q)\psi(a) \in \mathbb{K}\otimes B \text{ for all } a \in A \\ (P - P^*)\phi(a), \quad (Q - Q^*)\psi(a) \in \mathbb{K}\otimes B \text{ for all } a \in A \\ P\phi(a) - Q\psi(a) \in \mathbb{K}\otimes B \text{ for all } a \in A \end{gathered}$$

The operator homotopy (49) is a pre-image of $\left(\begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}, \frac{1}{1+t^2}\begin{pmatrix} P & tPQ \\ tQP & t^2 Q \end{pmatrix}\right) \in \mathcal{E}^1(A, C_0([0,\infty), B))$ under the mapping of Lemma 13.52. We verify that $\left(\begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}, \frac{1}{1+t^2}\begin{pmatrix} P & tPQ \\ tQP & t^2 Q \end{pmatrix}\right) \in \mathcal{E}^1(A, C_0([0,\infty), B))$.

1. $$\frac{1}{1+t^2}\begin{pmatrix}\phi(a) & 0\\ 0 & \psi(a)\end{pmatrix}\begin{pmatrix}P & tPQ\\ tQP & t^2Q\end{pmatrix}-\frac{1}{1+t^2}\begin{pmatrix}P & tPQ\\ tQP & t^2Q\end{pmatrix}\begin{pmatrix}\phi(a) & 0\\ 0 & \psi(a)\end{pmatrix}=$$
$$\frac{1}{1+t^2}\begin{pmatrix}\phi(a)P-P\phi(a) & t(\phi(a)PQ-PQ\psi(a))\\ t(\psi(a)QP-QP\phi(a)) & t^2(\psi(a)Q-Q\psi(a))\end{pmatrix}\equiv$$
$$\frac{1}{1+t^2}\begin{pmatrix}\phi(a)P-P\phi(a) & t(\psi(a)Q^2-P^2\phi(a))\\ t(\phi(a)P^2-Q^2\psi(a)) & t^2(\psi(a)Q-Q\psi(a))\end{pmatrix}\equiv 0 \mod M_2(\mathbb{K}\otimes B) \text{ for all } a\in A.$$

2. $$\left(\frac{1}{(1+t^2)^2}\begin{pmatrix}P & tPQ\\ tQP & t^2Q\end{pmatrix}^2-\frac{1}{1+t^2}\begin{pmatrix}P & tPQ\\ tQP & t^2Q\end{pmatrix}\right)\begin{pmatrix}\phi(a) & 0\\ 0 & \psi(a)\end{pmatrix}=$$
$$\left(\frac{1}{(1+t^2)^2}\begin{pmatrix}P^2+t^2PQ^2P & tP^2Q+t^3PQ^2\\ tQP^2+t^3Q^2P & t^2QP^2Q+t^4Q^2\end{pmatrix}-\frac{1}{1+t^2}\begin{pmatrix}P & tPQ\\ tQP & t^2Q\end{pmatrix}\right)\begin{pmatrix}\phi(a) & 0\\ 0 & \psi(a)\end{pmatrix}=$$
$$\frac{1}{(1+t^2)^2}\begin{pmatrix}P^2\phi(a)+t^2PQ^2P\phi(a) & tP^2Q\psi(a)+t^3PQ^2\psi(a)\\ tQP^2\phi(a)+t^3Q^2P\phi(a) & t^2QP^2Q\psi(a)+t^4Q^2\psi(a)\end{pmatrix}-\frac{1}{1+t^2}\begin{pmatrix}P\phi(a) & tPQ\psi(a)\\ tQP\phi(a) & t^2Q\psi(a)\end{pmatrix}\equiv$$
$$\frac{1}{(1+t^2)^2}\begin{pmatrix}P\phi(a)+t^2P\phi(a) & tP\phi(a)+t^3P\phi(a)\\ tQ\psi(a)+t^3Q\psi(a) & t^2Q\psi(a)+t^4Q\psi(a)\end{pmatrix}-\frac{1}{1+t^2}\begin{pmatrix}P\phi(a) & tP\phi(a)\\ tQ\psi(a) & t^2Q\psi(a)\end{pmatrix}\equiv$$
$$\frac{1}{1+t^2}\begin{pmatrix}P\phi(a) & tP\phi(a)\\ tQ\psi(a) & t^2Q\psi(a)\end{pmatrix}-\frac{1}{1+t^2}\begin{pmatrix}P\phi(a) & tP\phi(a)\\ tQ\psi(a) & t^2Q\psi(a)\end{pmatrix}\equiv 0 \mod M_2(\mathbb{K}\otimes B) \text{ for all } a\in A.$$

This follows from repetitively switching $P\phi(a)$ and $Q\psi(a)$ and applying compactness relations. Two examples:

$$PQ^2\psi(a)\equiv PQ\psi(a)\equiv P^2\phi(a)\equiv P\phi(a) \mod \mathbb{K}\otimes B$$
$$QP^2Q\psi(a)\equiv QP^3\phi(a)\equiv QP^2\phi(a)\equiv QP\phi(a)\equiv Q^2\psi(a)\equiv Q\psi(a) \mod \mathbb{K}\otimes B.$$

3. $$\frac{1}{1+t^2}\left(\begin{pmatrix}P & tPQ\\ tQP & t^2Q\end{pmatrix}-\begin{pmatrix}P & tPQ\\ tQP & t^2Q\end{pmatrix}^*\right)\begin{pmatrix}\phi(a) & 0\\ 0 & \psi(a)\end{pmatrix}=$$
$$\frac{1}{1+t^2}\left(\begin{pmatrix}P & tPQ\\ tQP & t^2Q\end{pmatrix}-\begin{pmatrix}P^* & tP^*Q^*\\ tQ^*P^* & t^2Q^*\end{pmatrix}\right)\begin{pmatrix}\phi(a) & 0\\ 0 & \psi(a)\end{pmatrix}=$$
$$\frac{1}{1+t^2}\left(\begin{pmatrix}P\phi(a) & tPQ\psi(a)\\ tQP\phi(a) & t^2Q\psi(a)\end{pmatrix}-\begin{pmatrix}P^*\phi(a) & tP^*Q^*\psi(a)\\ tQ^*P^*\phi(a) & t^2Q^*\psi(a)\end{pmatrix}\right)\equiv$$
$$\frac{1}{1+t^2}\left(\begin{pmatrix}P\phi(a) & tP\phi(a)\\ tQ\psi(a) & t^2Q\psi(a)\end{pmatrix}-\begin{pmatrix}P\phi(a) & tP\phi(a)\\ tQ\psi(a) & t^2Q\psi(a)\end{pmatrix}\right)\equiv 0 \mod M_2(\mathbb{K}\otimes B) \text{ for all } a\in A.$$

Indeed $\left(\begin{pmatrix}\phi & 0\\ 0 & \psi\end{pmatrix}, \frac{1}{1+t^2}\begin{pmatrix}P & tPQ\\ tQP & t^2Q\end{pmatrix}\right)\in\mathcal{E}^1(A, C_0([0,\infty),B))$. It follows from Lemma 13.53 that (49) is a Kasparov A-$C_0([0,\infty),B)$ module. □

Remark 13.59. It is assumed in Lemma 13.58 that $(\phi,P),(\psi,Q)\in\mathcal{E}^1(A,B)$ are homological. Also $(\phi,0),(\psi,0)\in\mathcal{D}^1(A,B)$ are homological. Addition of homology classes is well-defined. So $\left(\begin{pmatrix}\phi & 0\\ 0 & \psi\end{pmatrix},\begin{pmatrix}P & 0\\ 0 & 0\end{pmatrix}\right)$ and $\left(\begin{pmatrix}\psi & 0\\ 0 & \phi\end{pmatrix},\begin{pmatrix}Q & 0\\ 0 & 0\end{pmatrix}\right)$ are homological. In the proof of Theorem 13.58 the order of the addition was reversed for clarity reasons. Recall the addition is commutative up to unitary equivalence.

Remark 13.60. The operator homotopy (49)in the proof of Theorem 11.16 is not a Kasparov A-$C([0,1],B)$-module, but a A-$C([0,\infty),B)$-module. This construction is still valid, because $[0,\infty)$ is homeomorphic to $[0,1)$. The one-point compactification of $[0,1)$ is $[0,1]$. This way $C_0([0,1),B)$ forms a $*$-subalgebra of $C([0,1],B)$. It is the same idea as in Proposition 6.44.

Lemma 13.61. Let A and B be C*-algebras. Let $(\widehat{\mathbb{H}_B}\hat{\otimes}\mathbb{C}_1, \phi\hat{\otimes}1, F(t)\hat{\otimes}\epsilon_1) \in \mathbb{E}(A, C([0,1], B\hat{\otimes}\mathbb{C}_1))$ be an operator homotopy. Then $\left(\phi, \dfrac{F(0)+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\right)$ and $\left(\phi, \dfrac{F(1)+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\right)$ represent the same class in $\overline{\mathcal{E}}^1(A,B)$.

Proof. The map $u\colon[0,1] \to \mathbb{B}(\widehat{\mathbb{H}}_{B\hat{\otimes}\mathbb{C}_1}) = \mathcal{M}(\mathbb{K}\otimes B)\hat{\otimes}\mathbb{C}_1$ of Lemma 11.20 can be written in the form $u'(t)\hat{\otimes}1$, where $u'\colon[0,1] \to \mathcal{M}(\mathbb{K}\otimes B)$ satisfies the conditions of Lemma 11.20. By definition $\left(\phi, \dfrac{F(0)+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\right)$ and $\left(u'(1)\phi u'(1)^*, u'(1)\dfrac{F(0)+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}u'(1)^*\right)$ are unitarily equivalent. Now $\left(u'(1)\phi u'(1)^*, u'(1)\dfrac{F(0)+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}u'(1)^*\right)$ and $\left(\phi, \dfrac{F(1)+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\right)$ are homological as:

$$
\begin{aligned}
&u'(1)\frac{F(0)+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}u'(1)^*u'(1)\phi(a)u'(1)^* - \frac{F(1)+1_{\mathcal{M}(\mathbb{K}\otimes B)}}{2}\phi(a) = \\
&\qquad \frac{u'(1)F(0)\phi(a)u'(1)^* - F(1)\phi(a)}{2} + \frac{u'(1)\phi(a)u'(1)^* - \phi(a)}{2} \equiv \\
&\qquad \frac{u'(1)F(0)u'(1)^* - F(1)}{2}\phi(a) + \frac{u'(1)\phi(a)u'(1)^* - \phi(a)}{2} \equiv \\
&\qquad \frac{F(1)u'(1)u'(1)^* - F(1)}{2}\phi(a) + \frac{\phi(a)u'(1)u'(1)^* - \phi(a)}{2} \equiv 0 \mod \mathbb{K}\otimes B. \quad \square
\end{aligned}
$$

Theorem 13.62. The mapping of Lemma 13.52 induces a semigroup morphism between $\mathrm{KK}(A, B\hat{\otimes}\mathbb{C}_1)$ and $E^1(A,B)$.

Proof. Any class in $\mathrm{KK}(A, B\hat{\otimes}\mathbb{C}_1)$ can be represented by an element of the form $(\widehat{\mathbb{H}_B}\hat{\otimes}\mathbb{C}_1, \phi\hat{\otimes}1, F\hat{\otimes}\epsilon_1) \in \mathbb{E}(A, B\hat{\otimes}\mathbb{C}_1)$. See Corollary 11.32. The construction of Lemma 13.52 preserves addition. We will show that the map of Lemma 13.52 induces a well-defined bijection from $\mathrm{KK}(A, B\hat{\otimes}\mathbb{C}_1)$ to $E^1(A,B)$.

We will now show that the map of Lemma 13.52 is independent of representative in $\mathrm{KK}(A, B\hat{\otimes}\mathbb{C}_1)$. Two Kasparov A-$B\hat{\otimes}\mathbb{C}_1$-modules that represent the same class in $\mathrm{KK}(A, B\hat{\otimes}\mathbb{C}_1)$ are homotopic. By Lemma 13.61 the images of homotopic Kasparov A-$B\hat{\otimes}\mathbb{C}_1$-modules represent the same class in $\overline{\mathcal{E}}^1(A,B)$. Degenerate Kasparov A-$B\hat{\otimes}\mathbb{C}_1$-modules are mapped to degenerate Kasparov pairs by Lemma 13.54. We conclude that the mapping of Lemma 13.52 gives a well-defined mapping from $\mathrm{KK}(A, B\hat{\otimes}\mathbb{C}_1)$ to $E^1(A,B)$.

We will now show that the mapping of Lemma 13.52 is injective. If two Kasparov A-$B\hat{\otimes}\mathbb{C}_1$-modules are sent to the same class in $E^1(A,B)$, then there is a chain of unitary equivalences and homologies until the images differ by a degenerate Kasparov pair. If the images in $\mathcal{E}^1(A,B)$ are unitarily equivalent, then this means that the Kasparov A-$B\hat{\otimes}\mathbb{C}_1$-modules were unitarily equivalent by Lemma 13.57. If the images in $\mathcal{E}^1(A,B)$ are homological, then this means that the Kasparov

A-$B\hat{\otimes}\mathbb{C}_1$-modules were homotopic by Lemma 13.58.

We will now show that the mapping of Lemma 13.52 is surjective. It follows from Lemma 13.53 that for every Kasparov A-B-pair in $\mathcal{E}^1(A,B)$, there exists a Kasparov A-$B\hat{\otimes}\mathbb{C}_1$-module that is mapped to that Kasparov pair under the mapping of Lemma 13.52. In particular, any class in $E^1(A,B)$ has a pre-image. □

Corollary 13.63. Let A be a separable nuclear C*-algebra and let B be a σ-unital C*-algebra. Then $\mathrm{Ext}(A,B) \cong \mathrm{KK}(A, B\hat{\otimes}\mathbb{C}_1)$.

Proof. Compose the semigroup isomorphism of Proposition 13.51 and Theorem 13.62. □

Remark 13.64. $\mathrm{KK}(A,B\hat{\otimes}\mathbb{C}_1)$ is a group. The isomorphisms of Proposition 13.51 and Theorem 13.62 are semigroup isomorphisms. Under those isomorphisms $E^1(A,B)$ and $\mathrm{Ext}(A,B)$ obtain a unique group structure. The inverses of elements in $E^1(A,B)$ and $\mathrm{Ext}(A,B)$ are found by using the isomorphisms to get the corresponding element in $\mathrm{KK}(A,B\hat{\otimes}\mathbb{C}_1)$, inverting it in $\mathrm{KK}(A,B\hat{\otimes}\mathbb{C}_1)$ and sending it back.

14 Half-exactness and the six term sequence for KK-theory

In this section we will reap the rewards of Corollary 13.63. We will show that KK defines a Bott functor when restricted to the separable nuclear C*-algebras in the left-hand term and the σ-unital C*-algebras in the right-hand term. It was already shown that KK is stable in either term and that KK is homotopy invariant in either term. It remains to be proven that KK, when restricted to the right subcategories, is half-exact, which will be done in this section. Once the half-exactness is settled, the many corollaries of homology theories and Bott functors can be applied to it.

14.1 Half-exactness of KK

Theorem 14.1. Let D be a separable nuclear C*-algebra. Then $\mathrm{KK}(D,\_)\colon \sigma\mathsf{C}^* - \mathsf{alg} \to \mathsf{Ab}$ is half-exact.

Proof. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of σ-unital C*-algebra. The goal is to show that $\mathrm{KK}(D,A) \xrightarrow{\mathrm{KK}(\mathrm{id}_D,\alpha)} \mathrm{KK}(D,B) \xrightarrow{\mathrm{KK}(\mathrm{id}_D,\beta)} \mathrm{KK}(D,C)$ is exact. As $\beta\circ\alpha = 0$, it follows $\mathrm{KK}(\mathrm{id}_D,\beta)\circ\mathrm{KK}(\mathrm{id}_D,\alpha) = 0$, so $\operatorname{im}\mathrm{KK}(\mathrm{id}_D,\alpha) \subset \ker\mathrm{KK}(\mathrm{id}_D,\beta)$.

We will now prove that $\ker\mathrm{KK}(\mathrm{id}_D,\beta) \subset \operatorname{im}\mathrm{KK}(\mathrm{id}_D,\alpha)$. For this we will use the isomorphism between $\mathrm{KK}_1(D,B)$ and $\mathrm{Ext}(D,B)$ from Corollary 13.63. Let $\phi \in \mathfrak{Ext}(D,B)$ be such that $[\beta\circ\phi] = 0 \in \mathrm{Ext}(D,C)$. Then this means that there exists a $\psi \in \mathfrak{Dxt}(D,C)$ such that $(\beta\circ\phi)\oplus\psi$ is decomposable. Let χ be the composition of an embedding $D \hookrightarrow \mathcal{M}(\mathbb{K}) \xrightarrow{\_\otimes\mathrm{id}_B} \mathcal{M}(\mathbb{K}\otimes B)$ for which $D\cap\mathbb{K} = 0$ (such an embedding exists when D is separable, see Remark 13.25) with the quotient $\mathcal{M}(\mathbb{K}\otimes B) \xrightarrow{\pi} \mathcal{Q}(\mathbb{K}\otimes B)$. By construction χ is split, see Lemma 6.31. By Theorem 13.24, χ is absorbing. As β is surjective, the quotient π and β appear in the following commutative diagram (see [34, Proposition 2.2.16]) where $\mathcal{M}(\mathbb{K}\otimes C) \xrightarrow{\tilde{\pi}} \mathcal{Q}(\mathbb{K}\otimes C)$ is the quotient:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{K}\otimes B \arrow[r, "(L_{_},R_{_})"] \arrow[d, "\mathrm{id}_{\mathbb{K}}\otimes\beta"] & \mathcal{M}(\mathbb{K}\otimes B) \arrow[r, "\pi"] \arrow[d, "\mathrm{id}_{\mathbb{K}}\otimes\beta"] & \mathcal{Q}(\mathbb{K}\otimes B) \arrow[r] \arrow[d, "\mathrm{id}_{\mathbb{K}}\otimes\beta"] & 0 \\
0 \arrow[r] & \mathbb{K}\otimes C \arrow[r, "(L_{_},R_{_})"] & \mathcal{M}(\mathbb{K}\otimes C) \arrow[r, "\tilde{\pi}"] & \mathcal{Q}(\mathbb{K}\otimes C) \arrow[r] & 0.
\end{tikzcd} \tag{50}$$

From this follows $\beta\circ\chi \in \mathfrak{Drt}(D,C)$. Theorem 13.24 also applies to $\beta\circ\chi$, so $\beta\circ\chi$ is absorbing. As $\psi\in\mathfrak{Drt}(D,C)$, this means that $\psi\oplus(\beta\circ\chi)$ is unitarily equivalent to $\beta\circ\chi$. The addition on $\overline{\mathfrak{Drt}}(D,C)$ is well-defined, so $(\beta\circ\phi)\oplus\psi\oplus(\beta\circ\chi)$ is unitarily equivalent to $(\beta\circ\phi)\oplus(\beta\circ\chi) = \beta\circ(\phi\oplus\chi)$. By assumption $(\beta\circ\phi)\oplus\psi$ was split. As $\beta\circ\chi$ is split, the addition $(\beta\circ\phi)\oplus\psi\oplus(\beta\circ\chi)$ is split. Split extensions can only be unitarily equivalent to split extensions (Lemma 6.37). Hence $\beta\circ(\phi\oplus\chi)$ is split. By Lemma 6.31 there exists a lifting $D\xrightarrow{\omega}\mathcal{M}(\mathbb{K}\otimes C)$ such that $\tilde{\pi}\circ\omega = \beta(\phi\oplus\chi)$. Define $E{:=}\{(x,y)\in\mathcal{Q}(\mathbb{K}\otimes B)\oplus\mathcal{M}(\mathbb{K}\otimes C){:}\,\beta(x)=\tilde{\pi}(y)\}$. Now E is the pullback of $\mathcal{Q}(\mathbb{K}\otimes B)\xrightarrow{\beta}\mathcal{Q}(\mathbb{K}\otimes\mathbb{C})$ and $\mathcal{M}(\mathbb{K}\otimes C)\xrightarrow{\tilde{\pi}}\mathcal{Q}(\mathbb{K}\otimes C)$. There is the following pullback diagram:

$$\begin{tikzcd}
D \arrow[dr, "\exists!\,\eta"] \arrow[drr, bend left, "\phi\oplus\chi"] \arrow[ddr, bend right, "\omega"'] & & \\
& E \arrow[r] \arrow[d] & \mathcal{Q}(\mathbb{K}\otimes B) \arrow[d, "\beta"] \\
& \mathcal{M}(\mathbb{K}\otimes C) \arrow[r, "\tilde{\pi}"'] & \mathcal{Q}(\mathbb{K}\otimes C).
\end{tikzcd}$$

We define the following map $\mathcal{M}(\mathbb{K}\otimes B)\ni z\to(z \mod \mathbb{K}\otimes B,\beta(z))\in\mathcal{Q}(\mathbb{K}\otimes B)\oplus\mathcal{M}(\mathbb{K}\otimes C)$ to show E is isomorphic to $\mathcal{M}(\mathbb{K}\otimes B)/(\mathbb{K}\otimes A)$. By the commutativity of diagram (50), this defines indeed an element of E. Every element $(x,y)\in E$ is of the form $(z \mod \mathbb{K}\otimes B,\beta(z))$ for $z\in\mathcal{M}(\mathbb{K}\otimes B)$. Let $(x,y)\in E$, by the surjectivity of π, there exists a $z\in\mathcal{M}(\mathbb{K}\otimes B)$ such that $\pi(z)=x$. Now $\pi^{-1}(x)=\{z+w{:}\,w\in\mathbb{K}\otimes B\}$ and $\beta(\pi^{-1}(x))=\{\beta(z)+\beta(w){:}\,w\in\mathbb{K}\otimes B\}=\{\beta(z)+v{:}\,v\in\mathbb{K}\otimes C\}$. As $(x,y)\in E$, $\beta(x)=\tilde{\pi}(y)$, so $y\in\beta(\pi^{-1}(x))$. Write $y=\beta(z)+v$, with $z\in\mathcal{M}(\mathbb{K}\otimes B)$ and $v\in\mathbb{K}\otimes C$. Now we only need to find a pre-image for v under β (which must exist by surjective of β). Say $\beta(w)=v$ with $w\in\mathbb{K}\otimes B$. Now $z+w$ is the element such that $(z+w \mod \mathbb{K}\otimes B,\beta(z+w))=(x,y)$. We will determine the kernel of $\mathcal{M}(\mathbb{K}\otimes B)\ni z\to(z \mod \mathbb{K}\otimes B,\beta(z))\in\mathcal{Q}(\mathbb{K}\otimes B)\oplus\mathcal{M}(\mathbb{K}\otimes C)$. Obviously, an element of the kernel comes from $\mathbb{K}\otimes B$. Furthermore, an element of the kernel must satisfy $\beta(z)=0$. From the exactness follows $\ker(\mathrm{id}_{\mathbb{K}}\otimes\beta)=\mathrm{im}(\mathrm{id}_K\otimes A)=\mathbb{K}\otimes A$. We conclude that $E\cong\mathcal{M}(\mathbb{K}\otimes B)/\mathbb{K}\otimes A$. We view $\mathbb{K}\otimes A\subset\mathbb{K}\otimes B\subset\mathcal{M}(\mathbb{K}\otimes B)$ as an ideal. By Lemma 6.12, there is a unique $*$-homomorphism $M(\mathbb{K}\otimes B)\xrightarrow{\xi}\mathcal{M}(\mathbb{K}\otimes A)$ that fixes $\mathbb{K}\otimes A$. The following diagram, in which $\mathcal{M}(\mathbb{K}\otimes A)\xrightarrow{\pi'}\mathcal{Q}(\mathbb{K}\otimes A)$ is the quotient, commutes:

$$\begin{tikzcd}
& & \mathcal{M}(\mathbb{K}\otimes B) \arrow[d, "\exists!\xi"'] & & \\
0 \arrow[r] & \mathbb{K}\otimes A \arrow[ur] \arrow[r, "(L_{_},R_{_})"'] & \mathcal{M}(\mathbb{K}\otimes A) \arrow[r, "\pi'"'] & \mathcal{Q}(\mathbb{K}\otimes A) \arrow[r] & 0.
\end{tikzcd}$$

Now $\ker \pi' = \mathbb{K} \otimes A$, so $\pi' \circ \xi$ induces a $*$-homomorphism $\mathcal{M}(\mathbb{K} \otimes B)/\mathbb{K} \otimes A \xrightarrow{\xi'} \mathcal{Q}(\mathbb{K} \otimes A)$. The composition $D \xrightarrow{\eta} \mathcal{M}(\mathbb{K} \otimes B)/\mathbb{K} \otimes A \xrightarrow{\xi'} \mathcal{Q}(\mathbb{K} \otimes A)$ forms an extension in $\mathfrak{Ext}(D, A)$. The pullback diagram shows $\phi \oplus \chi$ coincides with the composition $D \xrightarrow{\eta} \mathcal{M}(\mathbb{K} \otimes B)/\mathbb{K} \otimes A \to \mathcal{Q}(\mathbb{K} \otimes B)$. It remains to show that $[\alpha \circ \xi' \circ \eta]$ and $[\pi \oplus \chi]$ represent the same class in $\mathrm{Ext}(D, B)$. To show this, we will follow the construction in Lemma 13.49, which combined Theorem 5.90 and Theorem 5.93. For $D \xrightarrow{\eta} \mathcal{M}(\mathbb{K} \otimes B)/\mathbb{K} \otimes A$, there exists a linear lifting $D \xrightarrow{\zeta} \mathcal{M}(\mathbb{K} \otimes B)$ by Theorem 5.90. Let $\tilde{D} \xrightarrow{\tilde{\zeta}} \mathcal{M}(\mathbb{K} \otimes B)$ be the unitisation. By Theorem 5.93, there exists a $*$-homomorphisms $\tilde{D} \xrightarrow{\rho} M_2(\mathcal{M}(\mathbb{K} \otimes B))$ such that $\begin{pmatrix} \tilde{\zeta}(d) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rho(d) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for all $d \in \tilde{D}$. Although $\tilde{\zeta}$ might not be a $*$-homomorphism, it is a $*$-homomorphism under the quotient $\mathcal{M}(\mathbb{K} \otimes B) \to \mathcal{M}(\mathbb{K} \otimes B)/\mathbb{K} \otimes A$ as η is $*$-homomorphism. In a similar way as in the proof of Lemma 13.49, it follows that

$$\rho(d) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rho(d) \in M_2(\mathbb{K} \otimes A) \quad \text{for all } d \in \tilde{D}. \tag{51}$$

It follows that $\left(\rho, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$ is a Kasparov $\tilde{D}$-B-pair. We precompose ρ with the inclusion of D in $\tilde{D}$ to make it a Kasparov D-B-pair. Because of equation (51), it follows that $\left(\xi \circ \rho, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$ is a Kasparov D-A pair. Now $\alpha \left(\xi \circ \rho, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \left(\rho, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$ under the isomorphisms in [16, §Theorem 2.2].

From $\eta(d) \oplus 0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rho(d) \mod M_2(\mathbb{K} \otimes A)$ for all $d \in \tilde{D}$, we conclude $(\xi \circ \eta) \oplus 0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \xi(\rho(d)) \mod M_2(\mathbb{K} \otimes A)$ for all $d \in D$. Via the isomorphism of Proposition 13.51, we conclude that the class of $[\alpha \circ \xi' \circ \eta] \in \mathrm{Ext}(D, B)$ is the class of the extensions $D \xrightarrow{\eta} \mathcal{M}(\mathbb{K} \otimes B)/\mathbb{K} \otimes A \to \mathcal{Q}(\mathbb{K} \otimes B)$. From the pullback diagram followed that this extension is $\phi \oplus \chi$. So $[\alpha \circ \xi' \circ \eta] = [\phi \oplus \chi] = [\phi]$ in $\mathrm{Ext}(B, D)$, as $\chi \in \mathfrak{Drt}(A, B)$. $\square$

Remark 14.2. We have abused notation in the proof. The $*$-homomorphism $\mathrm{id}_{\mathbb{K}} \otimes \beta$ is surjective and therefore induces the $*$-homomorphism $\mathcal{M}(\mathbb{K} \otimes B) \to \mathcal{M}(\mathbb{K} \otimes C)$ in (50). Postcomposition with this $*$-homomorphism is denoted by $\beta \circ \_$. We do something similar for α, which is not surjective. Because A, B and C are σ-unital, the Hilbert B-modules $\mathbb{H}_A \otimes_\alpha B$ is a direct summand of $\mathbb{H}_B$ by Theorem 10.22. This way α induces a $*$-homomorphism $\mathcal{M}(\mathbb{K} \otimes A) = \mathbb{B}(\mathbb{H}_A) \to \mathbb{B}(\mathbb{H}_B) = \mathcal{M}(\mathbb{K} \otimes B)$. This $*$-homomorphism was denoted by $\alpha \circ \_$.

Remark 14.3. It is not true that $\mathrm{KK}(D, \_)$ is half-exact for general C*-algebras D. Also for D separable nuclear or not, $\mathrm{KK}(D, \_) \colon \mathsf{C}^* - \mathsf{alg} \to \mathsf{Ab}$ need not be half-exact.

Theorem 14.4. Let D be a σ-unital C*-algebra. Then $\mathrm{KK}(\_, D) \colon \mathsf{SepNucC}^* - \mathsf{alg}^{\mathrm{opp}} \to \mathsf{Ab}$ is half-exact.

Proof. The proof is somewhat similar to the proof of Theorem 14.1. The full proof can be found in [16, §7, Lemma 7]. $\square$

Remark 14.5. In response to Remark 14.2, a $*$-homomorphism can be applied to an extension of C*-algebras by precomposition. Let $C \xrightarrow{\tau} \mathcal{Q}(\mathbb{K}\otimes D)$ be an extension of C*-algebras, then $B \xrightarrow{\tau\circ\beta} \mathcal{Q}(\mathbb{K}\otimes B)$ is the extension obtained from application of $B \xrightarrow{\beta} C$.

Remark 14.6. It is not true that $\mathrm{KK}(\_, D)$ is half-exact for general C*-algebras D. Also for D σ-unital or not, $\mathrm{KK}(\_, D)\colon \mathsf{C}^* - \mathsf{alg}^{\mathrm{opp}} \to \mathsf{Ab}$ need not be half-exact.

Lemma 14.7. Let D be σ-unital C*-algebra. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of separable nuclear C*-algebras. Define $\mathrm{KK}(A,D) \xrightarrow{\delta} \mathrm{KK}_1(C,D)$ to be the composition $\mathrm{KK}(A,D) \cong \mathrm{KK}(C_\beta, D) \xrightarrow{\mathrm{KK}((0,\_),\mathrm{id}_D)} \mathrm{KK}(SC,D) \cong \mathrm{KK}_1(C,D)$. Now the following sequence is exact:

$$\dots \to \mathrm{KK}(B,D) \xrightarrow{\mathrm{KK}(\alpha,\mathrm{id}_D)} \mathrm{KK}(A,D) \xrightarrow{\delta} \mathrm{KK}_1(C,D) \xrightarrow{\mathrm{KK}_1(\beta,\mathrm{id}_D)} \mathrm{KK}_1(B,D) \to \dots$$

Proof. $\mathrm{KK}(\_, D)$ is half-exact and homotopy invariant. The category of separable nuclear C*-algebras is admissible. Therefore the connecting morphism construction in the proof of Theorem 8.30 (with reversed arrows) does the job. □

The category of separable nuclear C*-algebras is admissible and therefore the construction from Theorem 8.30 can be applied to $\mathrm{KK}(\_, D)$ whenever D is σ-unital. We don't know whether the category of σ-unital C*-algebras is admissible. It will still be possible to construct an exact connecting morphism in the same way for $\mathrm{KK}(D, \_)$, when D is separable nuclear. We will see how this is done in the following lemma.

Lemma 14.8. Let D be a separable nuclear C*-algebra. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of σ-unital C*-algebras. Define $\mathrm{KK}_1(D,C) \xrightarrow{\delta} \mathrm{KK}(D,A)$ to be the composition $\mathrm{KK}_1(D,C) \cong \mathrm{KK}(D,SC) \xrightarrow{\mathrm{KK}(\mathrm{id}_D,\mathrm{KK}((0,\_))} \mathrm{KK}(D,C_\beta) \cong \mathrm{KK}(D,A)$. Now the following sequence is exact:

$$\dots \to \mathrm{KK}_1(D,B) \xrightarrow{\mathrm{KK}_1(\mathrm{id}_D,\beta)} \mathrm{KK}_1(D,C) \xrightarrow{\delta} \mathrm{KK}(D,A) \xrightarrow{\mathrm{KK}(\mathrm{id}_D,\alpha)} \mathrm{KK}_1(D,B) \to \dots$$

Proof. C_β is σ-unital by Corollary 6.64. The same holds for the C*-algebra E in the proof of Theorem 8.30. As $\mathrm{KK}(D, \_)$ is half-exact and homotopy invariant, the connecting morphism constructed in Theorem 8.30 exists and makes the sequence exact. □

Corollary 14.9. Let D be a σ-unital C*-algebra. Then $\mathrm{KK}(\_, D)\colon \mathsf{SepNucC}^* - \mathsf{alg}^{\mathrm{opp}} \to \mathsf{Ab}$ is a Bott functor. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of separable nuclear C*-algebras, then the following diagram in which the vertical maps are the connecting morphisms, is exact:

$$\begin{tikzcd}
\mathrm{KK}(C,D) \arrow[r, "\mathrm{KK}(\beta,\mathrm{id}_D)"] & \mathrm{KK}(B,D) \arrow[r, "\mathrm{KK}(\alpha,\mathrm{id}_D)"] & \mathrm{KK}(A,D) \arrow[d] \\
\mathrm{KK}_1(A,D) \arrow[u] & \mathrm{KK}_1(B,D) \arrow[l, "\mathrm{KK}_1(\alpha,\mathrm{id}_D)"] & \mathrm{KK}_1(C,D). \arrow[l, "\mathrm{KK}_1(\beta,\mathrm{id}_D)"]
\end{tikzcd}$$

Proof. It was already shown that $\mathrm{KK}(\_, D)$ is stable in Lemma 11.29 and homotopy invariant in Lemma 11.24. The half-exactness is proven in Theorem 14.4. By Lemma 11.30, $\mathrm{KK}_1(D, \_) \cong$

$\mathrm{KK}(D, S(\_))$ are isomorphic functors. All the C*-algebras in the proof of Theorem 8.41 ([34, Theorem 11.2.1]) are separable and nuclear. Hence that proof is valid for this corollary. The exactness of the six term sequence follows from Corollary 8.44. □

Corollary 14.10. Let D be a separable nuclear C*-algebra. Then $\mathrm{KK}(D, \_)\colon \sigma\mathsf{C}^* - \mathsf{alg} \to \mathsf{Ab}$ is a Bott functor. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of σ-unital C*-algebras, then the following diagram in which the vertical maps are the connecting morphisms, is exact:

$$\begin{tikzcd}
\mathrm{KK}(D,A) \arrow[r, "\mathrm{KK}(\mathrm{id}_D,\alpha)"] & \mathrm{KK}(D,B) \arrow[r, "\mathrm{KK}(\mathrm{id}_D,\beta)"] & \mathrm{KK}(D,C) \arrow[d] \\
\mathrm{KK}_1(D,C) \arrow[u] & \mathrm{KK}_1(D,B) \arrow[l, "\mathrm{KK}_1(\mathrm{id}_D,\beta)"] & \mathrm{KK}_1(D,A). \arrow[l, "\mathrm{KK}_1(\mathrm{id}_D,\alpha)"]
\end{tikzcd}$$

Proof. Similar to Corollary 14.9. □

The connecting morphisms in Lemma 14.7 and Lemma 14.8 are natural in the sense of Lemma 8.32. In the following two lemmas we will see that the connecting morphisms are also natural with respect to the Kasparov product in the sense of Lemma 11.48.

Lemma 14.11. Let D and E be σ-unital C*-algebra. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of separable nuclear C*-algebras. Define $\mathrm{KK}_j(A, D) \xrightarrow{\delta_D} \mathrm{KK}_{j+1}(C, D)$ and $\mathrm{KK}_j(A, E) \xrightarrow{\delta_E} \mathrm{KK}_{j+1}(C, E)$ to be connecting morphisms as in Lemma 14.7. Let $X \in \mathrm{KK}_j(D, E)$. The following diagram commutes:

$$\begin{tikzcd}
\mathrm{KK}_j(A,D) \arrow[r, "\delta_D"] \arrow[d, "_\otimes_D X"] & \mathrm{KK}_{j+1}(C,D) \arrow[d, "_\otimes_D X"] \\
\mathrm{KK}_j(A,E) \arrow[r, "\delta_E"] & \mathrm{KK}_{j+1}(C,E).
\end{tikzcd}$$

Proof. The connecting morphisms are constructed by composing group homomorphisms induced by $*$-homomorphisms and suspensions. Each of them commutes with the Kasparov product. Hence the following diagram commutes:

$$\begin{tikzcd}
\mathrm{KK}_j(A,D) \arrow[r] \arrow[d, "_\otimes_D X"] & \mathrm{KK}_j(C_\beta,D) \arrow[r, "\mathrm{KK}_j((0,_),\mathrm{id}_D)"] \arrow[d, "_\otimes_D X"] & \mathrm{KK}_j(SC,D) \arrow[r] \arrow[d, "_\otimes_D X"] & \mathrm{KK}_{j+1}(C,D) \arrow[d, "_\otimes_D X"] \\
\mathrm{KK}_j(A,E) \arrow[r] & \mathrm{KK}_j(C_\beta,E) \arrow[r, "\mathrm{KK}_j((0,_),\mathrm{id}_E)"'] & \mathrm{KK}_j(SC,E) \arrow[r] & \mathrm{KK}_{j+1}(C,E).
\end{tikzcd}$$

□

Lemma 14.12. Let D and E be separable nuclear C*-algebra. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of σ-unital C*-algebras. Define $\mathrm{KK}_j(D, C) \xrightarrow{\delta_D} \mathrm{KK}_{j-1}(D, A)$ and $\mathrm{KK}_j(E, C) \xrightarrow{\delta_E} \mathrm{KK}_{j-1}(E, A)$ to be connecting morphisms as in Lemma 14.8. Let $X \in \mathrm{KK}_j(D, E)$. The following diagram commutes:

$$\begin{tikzcd}
\mathrm{KK}_j(E,C) \arrow[r, "\delta_E"] \arrow[d, "X\otimes_E _"] & \mathrm{KK}_{j-1}(E,A) \arrow[d, "X\otimes_E _"] \\
\mathrm{KK}_j(D,C) \arrow[r, "\delta_E"] & \mathrm{KK}_{j-1}(D,A).
\end{tikzcd}$$

Proof. Similar to Lemma 14.11. □

14.2 Pimsner-Voiculescu

Theorem 14.13. Let A be a $\mathbb{Z}_2$-graded separable nuclear C*-algebra. Let B be a trivially graded σ-unital C*-algebra. Let $A \xrightarrow{\alpha} A$ be an even $*$-isomorphism. Then $\mathrm{KK}_*(A \rtimes_\alpha \mathbb{Z}, B) = \mathrm{KK}_{*+1}(T_\alpha, B)$. Furthermore the exact sequence from Lemma 6.59 gives the following six term sequence with respect to KK:

$$\begin{tikzcd}
\mathrm{KK}_0(A,B) \arrow[d, "\mathrm{id}_{\mathrm{KK}_0(A,B)} - \mathrm{KK}_0(\alpha,\mathrm{id}_B)"'] & \mathrm{KK}_0(A \rtimes_\alpha \mathbb{Z}) \arrow[l, "\mathrm{KK}_0(\mathrm{ev}_0,\mathrm{id}_B)"'] & \mathrm{KK}_1(A,B) \arrow[l] \\
\mathrm{KK}_0(A,B) \arrow[r] & \mathrm{KK}_1(A \rtimes_\alpha \mathbb{Z}) \arrow[r, "\mathrm{KK}_1(\mathrm{ev}_0,\mathrm{id}_B)"] & \mathrm{KK}_1(A,B). \arrow[u, "\mathrm{id}_{\mathrm{KK}_1(A,B)} - \mathrm{KK}_1(\alpha,\mathrm{id}_B)"']
\end{tikzcd}$$

Proof. With respect to $\mathrm{KK}_*(A \rtimes_\alpha \mathbb{Z}, B) = \mathrm{KK}_{*+1}(T_\alpha, B)$, the proof is similar to the proof of Theorem 7.83. For the six term sequence, see Example 7.85, note the arrow reversal. For the connecting morphisms, see [2, Theorem 19.6.1]. □

Part IV
The Universal Coefficient Theorem

We have now gone through all the preparation for understanding and proving the Universal Coefficient Theorem. We have stated the theorem in §12. In that section the subcategory for which it holds, is defined. In §15 we will prove a special case of the UCT. We will prove the UCT whenever $\mathrm{K}_*(B)$ is an injective module. In §16 we will construct a projective resolution for $\mathrm{K}_*(B)$; a free resolution induced by a $*$-homomorphism. Via the technique of §4.6, the projective resolution becomes an injective resolution. In §17 we will finalise the proof by applying the results of §15 to the injective modules in the injective resolution.

15 The UCT in case $\mathrm{K}_*(B)$ is injective

In this section we will prove Theorem 12.5 when $\mathrm{K}_*(B)$ is injective. If $\mathrm{K}_*(B)$ is injective, then proving Theorem 12.5 is reduced to showing that $\gamma(A,B)$ is an isomorphism, because $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ vanishes by Lemma 4.66.

Theorem 15.1. Let $A \in \mathrm{Ob}(\mathsf{N})$ and let B be a σ-unital C*-algebra such that $\mathrm{K}_*(B)$ is injective. Then $\gamma(A,B)\colon \mathrm{KK}_*(A,B) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ is an isomorphism.

To prove Theorem 15.1, we show that γ defines a morphism of functors from $\mathsf{N}^{\mathrm{opp}} \times \sigma\mathsf{C}^*\text{-}\mathsf{alg} \xrightarrow{\mathrm{KK}_*(\_,\_)} \mathsf{Ab}$ to $\mathsf{N}^{\mathrm{opp}} \times \sigma\mathsf{C}^*\text{-}\mathsf{alg} \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\_),\mathrm{K}_*(\_))} \mathsf{Ab}$. The precise statement is found in the next lemma.

Lemma 15.2. Let A be a separable nuclear C*-algebra and let B be a σ-unital C*-algebra, such that $\mathrm{K}_*(B)$ is injective. Then $\mathrm{KK}_*(\_,B)$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\_),\mathrm{K}_*(B))$ are cohomology theories and $\gamma(\_,B)$ is a morphism of cohomology theories between them as in Definition 8.22.

Proof. $\mathrm{KK}_*(\_,B)$ is a contravariant functor. $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\_),\mathrm{K}_*(B))$ is the composition of $\mathrm{K}_*(\_)$ and $\mathrm{Hom}_{\mathbb{Z}}(\_,\mathrm{K}_*(B))$; a composition of a covariant and a contravariant functor.

$\mathrm{KK}_*(\_,B)$ is homotopy invariant by Lemma 11.24. $\mathrm{K}_*(\_)$ is homotopy invariant by Lemma 7.33 and Lemma 7.56. So $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\_),\mathrm{K}_*(B))$ is homotopy invariant.
As B is σ-unital, $\mathrm{KK}_*(\_,B)$ has a six term sequence by Corollary 14.9. K_* has a six term sequence by Corollary 7.71. A six term sequence of K-groups can be cut into short exact sequences consisting of the cokernel of the previous arrow, an object and the kernel of the next arrow. $\mathrm{Hom}_{\mathbb{Z}}(\_,\mathrm{K}_*(B))$ is exact as $\mathrm{K}_*(B)$ is injective. So $\mathrm{Hom}_{\mathbb{Z}}(\_,\mathrm{K}_*(B))$ preserves short exact sequences. See Lemma 4.19. So $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\_),\mathrm{K}_*(B))$ has a six term sequence.

$\gamma(\_,B)$ is given by the Kasparov product. It commutes with $*$-homomorphisms by Lemma 11.48 and it commutes with connecting morphisms by Lemma 14.11. □

We will show γ is an isomorphism of functors, by showing it is an isomorphism at each component. See Lemma 2.23. This is precisely what we will do in the remainder of this section. We will go over all the objects of N and show that γ is an isomorphism. The order in which we do this is very important. We will start with the extensions in N; if γ is an isomorphism for two of the three C*-algebras in an extension, then so for the third. This result will be used later, for example in the proof that γ is an isomorphism for commutative C*-algebras.

15.1 The UCT for extensions and direct limits

Proposition 15.3. Let A be a separable nuclear C^*-algebra and let B be a σ-unital C^*-algebra, such that $\mathrm{K}_*(B)$ is injective. Let $J \subset A$ be a closed ideal. If two of the following maps are isomorphic:

$$\begin{aligned}\gamma(J,B)\colon \mathrm{KK}_*(J,B) &\to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(J),\mathrm{K}_*(B))\\ \gamma(A,B)\colon \mathrm{KK}_*(A,B) &\to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))\\ \gamma(A/J,B)\colon \mathrm{KK}_*(A/J,B) &\to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A/J),\mathrm{K}_*(B)),\end{aligned}$$

then so is the third.

Proof. As $J \subset A$ is an ideal, it induces an exact sequence: $0 \to J \xrightarrow{i} A \xrightarrow{q} A/J \to 0$, where i is the inclusion and q is the quotient. The $\mathrm{KK}_*(\_,B)$-functor has the following six term sequence:

$$\begin{tikzcd} \mathrm{KK}(A/J,B) \arrow[r, "\mathrm{KK}(q,\mathrm{id}_B)"] & \mathrm{KK}(A,B) \arrow[r, "\mathrm{KK}(i,\mathrm{id}_B)"] & \mathrm{KK}(A,B) \arrow[d, "\delta_0"] \\ \mathrm{KK}_1(J,B) \arrow[u, "\delta_1"] & \mathrm{KK}_1(A,B) \arrow[l, "\mathrm{KK}_1(i,\mathrm{id}_B)"] & \mathrm{KK}_1(A/J,B). \arrow[l, "\mathrm{KK}_1(q,\mathrm{id}_B)"] \end{tikzcd}$$

The K_*-functor has the following six term sequence:

$$\begin{tikzcd} \mathrm{K}_0(J) \arrow[r, "\mathrm{K}_0(i)"] & \mathrm{K}_0(A) \arrow[r, "\mathrm{K}_0(q)"] & \mathrm{K}_0(A/J) \arrow[d, "\tilde{\delta}_0"] \\ \mathrm{K}_1(A/J) \arrow[u, "\tilde{\delta}_1"] & \mathrm{K}_1(A) \arrow[l, "\mathrm{K}_1(q)"] & \mathrm{K}_1(J). \arrow[l, "\mathrm{K}_1(i)"] \end{tikzcd}$$

The six term sequence of the $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\_),\mathrm{K}_*(B))$-functor is obtained by applying the $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\_),\mathrm{K}_*(B))$-functor to the above six term sequence. Lemma 15.2 says that $\gamma(\_,B)$ is a morphism of cohomology theories. In other words, the following diagram commutes:

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{KK}_0(J,B) \arrow[r, "\gamma(J,B)"] \arrow[d, "\delta_0"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(J),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(J),\mathrm{K}_1(B)) \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(\tilde{\delta_1},\mathrm{id}_{\mathrm{K}_0(B)})"', "\mathrm{Hom}_{\mathbb{Z}}(\tilde{\delta_0},\mathrm{id}_{\mathrm{K}_1(B)})"] \\
\mathrm{KK}_1(A/J,B) \arrow[r, "\gamma(A/J,B)"] \arrow[d, "\mathrm{KK}_1(q,\mathrm{id}_B)"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A/J),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A/J),\mathrm{K}_1(B)) \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(q),\mathrm{id}_{\mathrm{K}_0(B)})"', "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(q),\mathrm{id}_{\mathrm{K}_1(B)})"] \\
\mathrm{KK}_1(A,B) \arrow[r, "\gamma(A,B)"] \arrow[d, "\mathrm{KK}_1(i,\mathrm{id}_B)"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)) \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(i),\mathrm{id}_{\mathrm{K}_0(B)})"', "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(i),\mathrm{id}_{\mathrm{K}_1(B)})"] \\
\mathrm{KK}_1(J,B) \arrow[r, "\gamma(J,B)"] \arrow[d, "\delta_1"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(J),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(J),\mathrm{K}_1(B)) \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(\tilde{\delta_0},\mathrm{id}_{\mathrm{K}_0(B)})"', "\mathrm{Hom}_{\mathbb{Z}}(\tilde{\delta_1},\mathrm{id}_{\mathrm{K}_1(B)})"] \\
\mathrm{KK}_0(A/J,B) \arrow[r, "\gamma(A,B)"] \arrow[d, "\mathrm{KK}_0(q,\mathrm{id}_B)"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A/J),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A/J),\mathrm{K}_1(B)) \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(q),\mathrm{id}_{\mathrm{K}_0(B)})"', "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(q),\mathrm{id}_{\mathrm{K}_1(B)})"] \\
\mathrm{KK}_0(A,B) \arrow[r, "\gamma(A/J,B)"] \arrow[d, "\mathrm{KK}_0(i,\mathrm{id}_B)"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B)) \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(i),\mathrm{id}_{\mathrm{K}_0(B)})"', "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(i),\mathrm{id}_{\mathrm{K}_1(B)})"] \\
\mathrm{KK}_0(J,B) \arrow[r, "\gamma(J,B)"] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(J),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(J),\mathrm{K}_1(B)). \arrow[d] \\
\vdots & \vdots
\end{tikzcd}
\tag{52}
$$

The vertical sequences are the six term sequences stretched out. The Five lemma can be applied.

If $\gamma(J,B)$ and $\gamma(A,B)$ are isomorphic, then the following maps are isomorphic:

$$
\mathrm{KK}_0(A/J,B) \xrightarrow{\gamma(A/J,B)} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A/J),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A/J),\mathrm{K}_1(B))
$$
$$
\mathrm{KK}_1(A/J,B) \xrightarrow{\gamma(A/J,B)} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A/J),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A/J),\mathrm{K}_1(B)).
$$

Hence $\gamma(A/J,B)$ is isomorphic as a $\mathbb{Z}_2$-graded group homomorphism.

If $\gamma(J,B)$ and $\gamma(A/J,B)$ are isomorphic, then the following maps are isomorphic:

$$
\mathrm{KK}_0(A,B) \xrightarrow{\gamma(A,B)} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B))
$$
$$
\mathrm{KK}_1(A,B) \xrightarrow{\gamma(A,B)} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)).
$$

Hence $\gamma(A,B)$ is isomorphic as a $\mathbb{Z}_2$-graded group homomorphism.

If $\gamma(A,B)$ and $\gamma(A/J,B)$ are isomorphic, then the following maps are isomorphic:

$$\mathrm{KK}_0(J,B) \xrightarrow{\gamma(J,B)} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(J),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(J),\mathrm{K}_1(B))$$
$$\mathrm{KK}_1(J,B) \xrightarrow{\gamma(J,B)} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(J),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(J),\mathrm{K}_1(B)).$$

Hence $\gamma(J,B)$ is isomorphic as a $\mathbb{Z}_2$-graded group homomorphism. □

Remark 15.4. As N is a full subcategory of $\mathsf{SepNucC^* - alg}$, the result of Lemma 14.9 is valid. Although Theorem 15.1 is only true for objects of N, Proposition 15.3 holds in greater generality.

Proposition 15.5. Let B be a σ-unital C^*-algebra, such that $\mathrm{K}_*(B)$ is injective. For $A = \varinjlim A_i$ in $\mathsf{SepNucC^* - alg}$, if $\gamma(A_i,B)$ is an isomorphism for every $i \in \mathbb{N}$, then $\gamma(A,B)$ is an isomorphism.

Proof. As $\mathrm{KK}_*(\_,B)$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\_),\mathrm{K}_*(B))$ are cohomology theories and $\gamma(\_,B)$ is a morphism of cohomology theories. Theorem 8.51 gives the following natural exact sequences:

$$0 \to \varprojlim{}^1 \mathrm{KK}_1(A_i,B) \to \mathrm{KK}_0(\varinjlim A_i,B) \to \varprojlim \mathrm{KK}_0(A_i,B) \to 0$$
$$0 \to \varprojlim{}^1 \mathrm{KK}_0(A_i,B) \to \mathrm{KK}_1(\varinjlim A_i,B) \to \varprojlim \mathrm{KK}_1(A_i,B) \to 0$$

$$\begin{aligned}0 \to \varprojlim{}^1(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A_i),\mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A_i),\mathrm{K}_0(B))) \to \\ \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(\varinjlim A_i),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(\varinjlim A_i),\mathrm{K}_1(B)) \to \\ \varprojlim(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A_i),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A_i),\mathrm{K}_1(B))) \to 0\end{aligned}$$

$$\begin{aligned}0 \to \varprojlim{}^1(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A_i),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A_i),\mathrm{K}_1(B))) \to \\ \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(\varinjlim A_i),\mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(\varinjlim A_i),\mathrm{K}_0(B)) \to \\ \varprojlim(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A_i),\mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A_i),\mathrm{K}_0(B))) \to 0.\end{aligned}$$

By Proposition 2.60, the $\mathbb{Z}_2$-graded morphisms $\{\gamma(A_i,B)\}_{i=1}^{\infty}$ make two morphisms of towers:

$$\mathrm{KK}_0(A_i,B) \xrightarrow{\gamma^0(A_i,B)} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A_i),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A_i),\mathrm{K}_1(B))$$
$$\mathrm{KK}_1(A_i,B) \xrightarrow{\gamma^1(A_i,B)} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A_i),\mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A_i),\mathrm{K}_0(B)).$$

In the following diagram, the horizontal rows are towers of $\mathbb{Z}_2$-graded modules. The vertical arrows form the morphism of towers γ^0 and γ^1.

$$\begin{tikzcd}
 & & \mathrm{KK}_*(A,B) \arrow[dl, "\mathrm{KK}_*(\mu_{i-1},\mathrm{id}_B)"'] \arrow[d, "\mathrm{KK}_*(\mu_i,\mathrm{id}_B)"] \arrow[dr, "\mathrm{KK}_*(\mu_{i+1},\mathrm{id}_B)"] \arrow[dddd, bend right=90, "\gamma(A,B)"'] & & \\
\cdots & \mathrm{KK}_*(A_{i-1},B) \arrow[l] \arrow[d, "\gamma(A_{i-1},B)"] & \mathrm{KK}_*(A_i,B) \arrow[l, "\mathrm{KK}_*(f_{i-1,i},\mathrm{id}_B)"'] \arrow[d, "\gamma(A_i,B)"] & \mathrm{KK}_*(A_{i+1},B) \arrow[l, "\mathrm{KK}_*(f_{i,i+1},\mathrm{id}_B)"'] \arrow[d, "\gamma(A_{i+1},B)"] & \cdots \arrow[l] \\
\cdots & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A_{i-1}),\mathrm{K}_*(B)) \arrow[l] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A_i),\mathrm{K}_*(B)) \arrow[l, "_\circ\mathrm{K}_*(f_{i-1,i})"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A_{i+1}),\mathrm{K}_*(B)) \arrow[l, "_\circ\mathrm{K}_*(f_{i,i+1})"'] & \cdots \arrow[l] \\
 & & & & \\
 & & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) \arrow[uul, "_\circ\mathrm{K}_*(\mu_{i-1})"] \arrow[uu, "_\circ\mathrm{K}_*(\mu_i)"] \arrow[uur, "_\circ\mathrm{K}_*(\mu_{i+1})"'] & &
\end{tikzcd}$$

As $\gamma(A_i, B)$ is isomorphic for every $i \in \mathbb{N}$, the morphisms of towers γ^0 and γ^1 are isomorphisms. Now we can apply the naturality of Theorem 8.51. This is the diagram for the even degrees:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\varprojlim^1 \mathrm{KK}_1(A_i, B) \arrow[r, "\varprojlim^1(\gamma 1)"] \arrow[d] & \varprojlim^1(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A_i), \mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A_i), \mathrm{K}_0(B))) \arrow[d] \\
\mathrm{KK}_0(A, B) \arrow[r, "\gamma(A,B)"] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A), \mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A), \mathrm{K}_1(B)) \arrow[d] \\
\varprojlim \mathrm{KK}_0(A_i, B) \arrow[r, "\varprojlim(\gamma^0)"] \arrow[d] & \varprojlim(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A_i), \mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A_i), \mathrm{K}_1(B))). \arrow[d] \\
0 & 0
\end{tikzcd}$$

Functors preserve isomorphisms and γ^0 and γ^1 are isomorphisms of towers. Hence $\varprojlim^1(\gamma^1)$ and $\varprojlim(\gamma^0)$ are isomorphisms. Now the vertical exact sequences are equivalent (see Definition 4.54) and in particular, the middle horizontal arrow $\mathrm{KK}_0(A, B) \xrightarrow{\gamma(A,B)} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A), \mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A), \mathrm{K}_1(B))$ must be an isomorphisms by the Five lemma.

This is the diagram for the odd degrees:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\varprojlim^1 \mathrm{KK}_0(A_i, B) \arrow[r, "\varprojlim^1(\gamma^0)"] \arrow[d] & \varprojlim^1(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A_i), \mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A_i), \mathrm{K}_1(B))) \arrow[d] \\
\mathrm{KK}_1(A, B) \arrow[r, "\gamma(A,B)"] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A), \mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A), \mathrm{K}_0(B)) \arrow[d] \\
\varprojlim \mathrm{KK}_1(A_i, B) \arrow[r, "\varprojlim(\gamma^1)"] \arrow[d] & \varprojlim(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A_i), \mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A_i), \mathrm{K}_0(B))). \arrow[d] \\
0 & 0
\end{tikzcd}$$

The proof here is similar. γ^0 and γ^1 are isomorphisms of towers. So $\varprojlim^1(\gamma^0)$ and $\varprojlim(\gamma^1)$ are isomorphisms. The vertical exact sequences are equivalent and the middle horizontal arrow $\mathrm{KK}_1(A, B) \xrightarrow{\gamma(A,B)} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A), \mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A), \mathrm{K}_0(B))$ is an isomorphism.
As $\gamma(A, B)$ is an isomorphism in both degrees, it is an isomorphism of $\mathbb{Z}_2$-graded modules. □

Remark 15.6. From Theorem 2.74 and Theorems 7.30 and 7.52 follows

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\varinjlim A_i), \mathrm{K}_*(B)) = \mathrm{Hom}_{\mathbb{Z}}(\varinjlim \mathrm{K}_*(A_i), \mathrm{K}_*(B)) = \varprojlim \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A_i), \mathrm{K}_*(B)).$$

So $\varprojlim^1 \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A_i), \mathrm{K}_*(B)) = 0$. As $\varprojlim^1(\gamma)$ is an isomorphism, it follows $\varprojlim^1 \mathrm{KK}_*(A_i, B) = 0$.

15.2 The UCT for commutative C*-algebras

Remark 15.7. We will work with the n-spheres $S^n := \{x \in \mathbb{R}^n : \|x\| = 1\} \subset \mathbb{R}^{n+1}$ in this subsection. The n-spheres are compact Hausdorff topological spaces. In Proposition 6.44, $\mathbb{T}$ was used to denote the unit circle S^1. This was done to prevent confusion with the suspension functor. It is not true that $S^n = \mathbb{T}^n = S^1 \times \cdots \times S^1$ (n-*times*) in general.

Definition 15.8. A topological space is *second countable* if it allows a countable basis.

Theorem 15.9. A second countable compact Hausdorff topological space is metrizable.

Proof. A compact Hausdorff space is normal ([28, Example 3.5.11b]). Urysohn's metrization theorem ([28, Theorem 4.1.10]) finalises the proof. ☐

Definition 15.10. Let X, Y, Z be topological space and let $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Z$ be continuous maps. Define $Y \cup_X Z := Y \sqcup Z / \sim$, where $\sim$ denotes the equivalence relation generated by $f(x) \sim g(x)$ with $x \in X$.

Lemma 15.11. Let X, Y, Z be topological space and let $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Z$ be continuous maps. $Y \cup_X Z$ is the pushout of f and g in Top.

Proof. This follows from the fact that for a topological space W and any two continuous maps $Y \xrightarrow{k} W$ and $Z \xrightarrow{l} W$ such that $k \circ f = l \circ g$, there exists a unique continuous map from $Y \cup_X Z$ to W, such that we get a commutative diagram as in Example 2.48:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d] \arrow[ddr, bend left, "k"] & \\
Z \arrow[r] \arrow[drr, bend right, "l"'] & Y \cup_X Z \arrow[dr, bend right, "\exists! h"] & \\
 & & W.
\end{tikzcd}$$

☐

Definition 15.12. Let X be a topological space and let $S^{n-1} \xrightarrow{f} X$ be a continuous map. A topological space Y *arises from* X whenever the following diagram is a pushout diagram:

$$\begin{tikzcd}
S^{n-1} \arrow[r, "f"] \arrow[d, "i"] & X \arrow[d] \\
D^n \arrow[r] & Y.
\end{tikzcd}$$

In this diagram $S^{n-1} \xrightarrow{i} D^n$ is the inclusion. In this construction f is called the *attaching map*. The image of f in Y is called an n-*cell* attached to X.

Remark 15.13. Although omitted from the notation, $Y \cup_X Z$ in Definition 15.10 depends on the choice of continuous maps $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Z$. The spaces arising from different attaching maps in Definition 15.12 are non homeomorphic. For example, $S^{n-1} \cup_{S^{n-1}} D^n$, obtained from the attaching map $S^{n-1} \xrightarrow{\mathrm{id}_{S^{n-1}}} S^{n-1}$, is homeomorphic to D^n. But $S^{n-1} \cup_{S^{n-1}} D^n$, obtained from a constant attaching map $S^{n-1} \xrightarrow{\text{const}} S^{n-1}$ is homeomorphic to $S^{n-1} \vee S^n$ (one-point union).

Remark 15.14. Because pushouts are unique up to unique isomorphism, we will from now on use the topological space $D^n \cup_{S^{n-1}} X$ to denote the space obtained from cell attachment.

Remark 15.15. In Definition 15.12 one n-cell is attached. Multiple n-cells can be attached to X at once. Even (un)countably many at once. This is done via index sets, see [12, p. 519].

Definition 15.16. A topological X is called a *CW-complex relative* to a topological space A if there exists a sequence:

$$A = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n \subset X_{n+1} \subset \cdots \subset X,$$

in which X_n arises from X_{n-1} by attaching n-cells.
X_n is called the *n-skeleton* of X. A CW-complex is *finite dimensional* if $X = X_n$ for some $n \in \mathbb{N}_0$. A *finite* CW-complex is finite dimensional and X_n arises from X_{n-1} by attaching only finitely many n-cells for every $n \in \mathbb{N}_0$. A CW-complex relative to $\varnothing$ is an *absolute* CW-complex.

Lemma 15.17. A CW-complex relative to a Hausdorff topological space is Hausdorff.

Proof. See [12, Proposition A.3]. □

Lemma 15.18. A finite CW-complex relative to a compact topological space is compact.

Proof. See [12, Corollary A.10]. □

Corollary 15.19. A finite absolute CW-complex is compact.

Lemma 15.20. Let X be a metrizable compact Hausdorff topological space. Then there exists a tower $(\{X_i\}_{i=1}^{\infty}, \{f_{i,i+1}\}_{i=1}^{\infty})$ in the category of locally compact Hausdorff spaces, in which X_i is a finite absolute CW-complex for every $i \in \mathbb{N}$ such that $X = \varprojlim X_i$.

Proof. See [11, Satz 1, p. 229]. □

Remark 15.21. The concept of CW-complexes was introduced by Whitehead in 1949. It generalised the concept of polyhedrons; geometric realisations of finite simplicial complexes (spaces constructed from gluing n-simplices $\Delta^n := \{(x_0, ..., x_n) \in \mathbb{R}^n \,|\, t_i \geq 0, \sum_{i=0}^n t_i = 1\})$. The proof referred to in Lemma 15.20 was published in 1936. Every polyhedron has a finite CW-structure, so it is a valid proof for Lemma 15.20.

Proposition 15.22. Let $A \in \mathrm{Ob}(\mathsf{N})$ be KK-equivalent to a commutative C*-algebra in N. Let B be a σ-unital C*-algebra, such that $\mathrm{K}_*(B)$ is injective. Then $\gamma(A, B)$ is an isomorphism.

Proof. It suffices to prove the proposition for commutative C*-algebras A only. This is because both $\mathrm{KK}(\_, B)$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\_), \mathrm{K}_*(B))$ are invariant under KK-equivalence by Lemma 11.52. Recall that $\mathrm{K}_0(\_) = \mathrm{KK}_0(\mathbb{C}, \_)$ and $\mathrm{K}_1(\_) = \mathrm{KK}_1(\mathbb{C}, \_)$ by Theorem 11.36. Furthermore $\gamma(\_, B)$ commutes with any Kasparov product by Theorem 11.44, in particular KK-equivalences. Hence, if $\gamma(A, B)$ is an isomorphism for every commutative C*-algebra A, then also for every C*-algebra that is KK-equivalent to A.

First we will prove the proposition when $A \in \mathrm{Ob}(\mathsf{N})$ is a commutative unital C*-algebra. By Gelfand duality (Theorem 5.36), there exists a compact Hausdorff topological space X such that $A = C(X)$. Every object of N is separable. A commutative separable C*-algebra corresponds under

Gelfand duality to a second countable locally compact Hausdorff topological space. See [34, §1.11]. So X is second countable. By Theorem 15.9, X is metrizable. By Lemma 15.20, there exists a tower of finite absolute CW-complex X_i such that $X = \varprojlim X_i$. For every such X_i the n-skeleton $(X_i)_n$ arises from the $(n-1)$-skeleton $(X_i)_{n-1}$ by attaching finitely many n-cells. In case there is one n-cell, then this is the attachment diagram:

$$\begin{tikzcd}
S^{n-1} \arrow[r, "f"] \arrow[d, "i"] & (X_i)_{n-1} \arrow[d] \\
D^n \arrow[r] & (X_i)_n .
\end{tikzcd}$$

The attachment diagram is a pushout square. Equivalences of categories reflect and preserve (co)limits (Theorem 2.54). So under Gelfand duality, it becomes the following pullback diagram:

$$\begin{tikzcd}
C((X_i)_n) \arrow[r] \arrow[d] & C((X_i)_{n-1}) \arrow[d, "C_0(f)"] \\
C_0(D^n) \arrow[r, "C_0(i)"] & C(S^{n-1}) .
\end{tikzcd}$$

As $C_0(i)$ is surjective (see Remark 15.23), applying Mayer-Vietoris (Theorem 8.38) gives the following commutative diagram:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{KK}_0(S^{n-1},B) \arrow[r, "\gamma(S^{n-1},B)"] \arrow[d, "\delta_0"'] & (\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(S^{n-1}),\mathrm{K}_*(B)))_0 \arrow[d, "\tilde{\delta_0}"] \\
\mathrm{KK}_1((X_i)_n,B) \arrow[r, "\gamma((X_i)_n,B)"] \arrow[d] & (\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*((X_i)_n),\mathrm{K}_*(B)))_1 \arrow[d] \\
\mathrm{KK}_1(D^n,B) \oplus \mathrm{KK}_1((X_i)_{n-1},B) \arrow[r, "(\gamma(D^n,B){,}\gamma((X_i)_{n-1},B))"] \arrow[d, "\mathrm{KK}_1(i,\mathrm{id}_B)'' - \mathrm{KK}_1(f,\mathrm{id}_B)"] & (\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(D^n),\mathrm{K}_*(B)))_1 \oplus (\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*((X_i)_{n-1}),\mathrm{K}_*(B)))_1 \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(K_*(i),\mathrm{id}_{\mathrm{K}_*(B)}) - \mathrm{Hom}_{\mathbb{Z}}(K_*(f),\mathrm{id}_{\mathrm{K}_*(B)})"] \\
\mathrm{KK}_1(S^{n-1},B) \arrow[r, "\gamma(S^{n-1},B)"] \arrow[d, "\delta_1"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(S^{n-1}),\mathrm{K}_*(B))_1 \arrow[d, "\tilde{\delta_1}"'] \\
\mathrm{KK}_0((X_i)_n,B) \arrow[r, "\gamma((X_i)_n,B)"] \arrow[d] & (\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*((X_i)_n),\mathrm{K}_*(B)))_0 \arrow[d] \\
\mathrm{KK}_0(D^n,B) \oplus \mathrm{KK}_0((X_i)_{n-1},B) \arrow[r, "(\gamma(D^n,B){,}\gamma((X_i)_{n-1},B))"] \arrow[d, "\mathrm{KK}_0(i,\mathrm{id}_B)'' - \mathrm{KK}_0(f,\mathrm{id}_B)"] & (\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(D^n),\mathrm{K}_*(B)))_0 \oplus (\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*((X_i)_{n-1}),\mathrm{K}_*(B)))_0 \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(K_*(i),\mathrm{id}_{\mathrm{K}_*(B)}) - \mathrm{Hom}_{\mathbb{Z}}(K_*(f),\mathrm{id}_{\mathrm{K}_*(B)})"] \\
\mathrm{KK}_0(S^{n-1},B) \arrow[r, "\gamma(S^{n-1},B)"] \arrow[d] & (\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(S^{n-1}),\mathrm{K}_*(B)))_0 . \arrow[d] \\
\vdots & \vdots
\end{tikzcd} \tag{53}$$

In this diagram C_0 is omitted from the objects and morphisms for brevity.
Now the n-sphere S^n is the one-point compactification of $\mathbb{R}^n$. Also S^n is the one-point compactification of D^{n-1}. Using Theorem 5.35, it induces the following two split exact sequences (Remark 6.28):

$$0 \to C_0(\mathbb{R}^n) \to C(S^n) \to \mathbb{C} \to 0$$
$$0 \to C_0(D^{n-1}) \to C(S^n) \to \mathbb{C} \to 0$$

The UCT holds for $\mathbb{C}$. By Bott periodicity the UCT also holds for $C_0(\mathbb{R}^n)$. Hence by Proposition 15.3, it follows that the UCT holds for $C(S^n)$. By the same arguments the UCT holds for $C_0(D^{n-1})$. In diagram (53), the horizontal arrows $\gamma(S^{n-1}, B)$ and $\gamma(D^n, B)$ are isomorphisms.
Suppose $\gamma(C((X_i)_{n-1}), B)$ is an isomorphism, then by the Five Lemma $\gamma(C((X_i)_n), B)$ is an isomorphism as well. This is the induction step. As X_i is a finite absolute CW-complex, $(X_i)_0$ consists of finitely many loose points. As $C(\{*\}) = \mathbb{C}$, $\gamma(C((X_i)_0), B)$ is an isomorphism. The induction is completed; $\gamma(C(X_i), B)$ is an isomorphism. By Proposition 15.5, $\gamma(C(X), B)$ is an isomorphism.

If $A \in \mathrm{Ob}(\mathsf{N})$ is a commutative non-unital C*-algebra, then A corresponds to a second countable locally compact Hausdorff space under Gelfand Duality. The one-point compactification X_∞ is a compact second countable Hausdorff space. By Theorem 5.35, $\widetilde{C_0(X)} \cong C(X_\infty)$. The unitisation of $C_0(X)$ gives the following exact sequence:

$$0 \to C_0(X) \to C(X_\infty) \to \mathbb{C} \to 0.$$

As the UCT holds for $\mathbb{C}$ and $C(X_\infty)$, it follows that the UCT holds for $C_0(X)$ by Proposition 15.3. □

Remark 15.23. D^n is homeomorphic to the southern hemisphere (excluding equator) of the n-sphere. As S^n is the one-point compactification of D^n, remove the north pole $z \in S^n$ to get D^n. There is a homeomorphism between S^{n-1} and the equator of the n-sphere. Using this characterisation, take a function in $f \in C(S^{n-1})$. Let $x \in S^{n-1}$. Let $y \in D^n$ be the south pole. The shortest path from y to x forms a line segment. Now let $\lambda \in [0,1)$ represent a point on the line segment from y to x (0 represent y and 1 represents x). This way every point in D^n is on such a unique line segment. Send λ to $\lambda f(x) \in \mathbb{C}$. This defines a continuous function from D^n to $\mathbb{C}$. The set of elements in D^n for which this function has an absolute value higher than ϵ is closed (and therefore compact as D^n is a subset of S^n). This construction proves the surjectivity of $C_0(D^n) \xrightarrow{C_0(i)} C(S^{n-1})$.

15.3 Completion of the proof of Theorem 15.1

Proposition 15.24. Let A be a separable postliminal C*-algebra. Let B be a σ-unital C*-algebra, such that $\mathrm{K}_*(B)$ is injective. Then $\gamma(A, B)$ is an isomorphism.

Proof. Theorem 5.75 says there exists a composition series $(I_\beta)_{\beta \le \alpha}$ for A satisfying Theorem 5.74 and locally compact Hausdorff spaces $\{Y_\beta\}_{\beta \le \alpha - 1}$ and separable Hilbert spaces $\{\mathcal{H}_\beta\}_{\beta \le \alpha - 1}$ such that $I_{\beta+1}/I_\beta \cong C_0(Y_\beta) \otimes K(\mathcal{H}_\beta)$ for $\beta \le \alpha - 1$. Notice $I_1 = I_1/I_0 \cong C_0(Y_1) \otimes K(\mathcal{H}_1)$. Either $\mathcal{H}_1$ is infinite dimensional, then $K(\mathcal{H}_1) \cong \mathbb{K}$ and $C_0(Y_1) \otimes K(\mathcal{H}_1)$ is a stabilisation of a commutative C*-algebra. Or $\mathcal{H}_1$ is finite dimensional, then $C_0(Y_1) \otimes K(\mathcal{H}_1)$ consists of finitely many copies of a commutative C*-algebra. In both cases I_1 is KK-equivalent to a commutative C*-algebra. By the same argument, any of the quotients $I_{\beta+1}/I_\beta$ is KK-equivalent to a commutative C*-algebra for

$1 \leq \beta \leq \alpha - 1$. So $\gamma(I_1, B)$ is an isomorphism and $\gamma(I_{\beta+1}/I_\beta, B)$ is an isomorphism for $1 \leq \beta \leq \alpha - 1$ by Proposition 15.22. Now the inclusion $I_\beta \subset I_{\beta+1}$ induces this exact sequence:

$$0 \to I_\beta \to I_{\beta+1} \to I_{\beta+1}/I_\beta \to 0.$$

If we assume that $\gamma(I_\beta, B)$ is an isomorphism, then it must follow from Proposition 15.3 that $\gamma(I_{\beta+1}, B)$ is an isomorphism. As $I_0 = 0$, it follows that $\gamma(I_1, B)$ is an isomorphism. This completes the induction. By construction A can be written as $A = \varinjlim_\beta I_\beta$. By Proposition 15.5, $\gamma(A, B)$ must be an isomorphism. $\square$

Proposition 15.25. Let A be a separable nuclear C*-algebra. Let B be a σ-unital C*-algebra, such that $\mathrm{K}_*(B)$ is injective. Let $\mathbb{R} \xrightarrow{\alpha} \mathrm{Aut}(A)$ be a continuous group homomorphism. If $\gamma(A, B)$ is an isomorphism, then so is $\gamma(A \rtimes_\alpha \mathbb{R}, B)$.

Proof. The Thom isomorphism for KK-groups is given by a KK-anti- equivalence. See Theorem 11.57. The maps $\gamma(A, B)$ and $\gamma(A \rtimes_\alpha \mathbb{R}, B)$ are given by a Kasparov product. Kasparov products are associative. Therefore this diagram commutes:

$$\begin{tikzcd}
\mathrm{KK}_*(A, B) \arrow[r, "\mathrm{Thom} \times_A -"] \arrow[d, "\gamma(A,B)"] & \mathrm{KK}_{*+1}(A \rtimes_\alpha \mathbb{R}, B) \arrow[d, "\gamma(A \rtimes_\alpha \mathbb{R}, B)"] \\
\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \arrow[r, "\mathrm{Hom}(\mathrm{Thom}, \mathrm{id}_{\mathrm{K}_*(B)})"] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_{*+1}(A \rtimes_\alpha \mathbb{R}), \mathrm{K}_*(B)).
\end{tikzcd}$$

The horizontal arrows are isomorphisms. By assumption $\gamma(A, B)$ is an isomorphism. It follows that $\gamma(A \rtimes_\alpha \mathbb{R}, B)$ is an isomorphism. $\square$

Proposition 15.26. Let A be a separable nuclear C*-algebra. Let B be a σ-unital C*-algebra, such that $\mathrm{K}_*(B)$ is injective. Let $A \xrightarrow{\alpha} A$ be a $*$-isomorphism. If $\gamma(A, B)$ is an isomorphism, then so is $\gamma(A \rtimes_\alpha \mathbb{Z}, B)$.

Proof. The exact sequence from Lemma 6.59 produces the six term sequence from Example 7.85 when the K_* is applied. When $\mathrm{KK}_*(\_, B)$ is applied, it produces the six term sequence from

Theorem 14.13. Now γ connects the two. Together this commutative diagram is formed:

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\mathrm{KK}_1(A,B) \arrow[r, "\gamma(A,B)"] \arrow[d, "\mathrm{id}_{\mathrm{KK}_1(A,B)} - \mathrm{KK}_1(\alpha,\mathrm{id}_B)"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)) \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{id}_{\mathrm{K}_0(A)} - \mathrm{K}_0(\alpha),\mathrm{id}_{\mathrm{K}_1(B)})"', "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{id}_{\mathrm{K}_1(A)} - \mathrm{K}_1(\alpha),\mathrm{id}_{\mathrm{K}_0(B)})"] \\
\mathrm{KK}_1(A,B) \arrow[r, "\gamma(A,B)"] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)) \arrow[d] \\
\mathrm{KK}_1(A \rtimes_\alpha \mathbb{Z},B) \arrow[r, "\gamma(A \rtimes_\alpha \mathbb{Z},B)"] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A \rtimes_\alpha \mathbb{Z}),\mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A \rtimes_\alpha \mathbb{Z}),\mathrm{K}_0(B)) \arrow[d] \\
\mathrm{KK}_0(A,B) \arrow[r, "\gamma(A,B)"] \arrow[d, "\mathrm{id}_{\mathrm{KK}_0(A,B)} - \mathrm{KK}_0(\alpha,\mathrm{id}_B)"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B)) \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{id}_{\mathrm{K}_0(A)} - \mathrm{K}_0(\alpha),\mathrm{id}_{\mathrm{K}_0(B)})"', "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{id}_{\mathrm{K}_1(A)} - \mathrm{K}_1(\alpha),\mathrm{id}_{\mathrm{K}_1(B)})"] \\
\mathrm{KK}_0(A,B) \arrow[r, "\gamma(A,B)"] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B)) \arrow[d] \\
\mathrm{KK}_0(A \rtimes_\alpha \mathbb{Z},B) \arrow[r, "\gamma(A \rtimes_\alpha \mathbb{Z},B)"] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A \rtimes_\alpha \mathbb{Z}),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A \rtimes_\alpha \mathbb{Z}),\mathrm{K}_1(B)). \arrow[d] \\
\vdots & \vdots
\end{tikzcd}
$$

By assumption $\gamma(A,B)$ is an isomorphism. The Five lemma finalises the proof in a similar way as it did in diagram (52). Hence $\gamma(A \rtimes_\alpha \mathbb{Z},B)$ is an isomorphism. $\square$

Proof Theorem 15.1. By Definition 12.4 N is the smallest full subcategory of the category of separable nuclear C*-algebras, which contains the separable type-I (postliminal) C*-algebras and is closed under KK-equivalence, inductive limits, extensions and crossed products by $\mathbb{R}$ and $\mathbb{Z}$. Let B be a σ-unital C*-algebra such that $\mathrm{K}_*(B)$ is injective. We will now show that the UCT holds for each of the objects of N.

Whenever A is a separable postliminal C*-algebra, then $\gamma(A,B)$ is an isomorphism by Proposition 15.24. For a KK-equivalence, recall that both $\mathrm{KK}(\_,B)$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\_),\mathrm{K}_*(B))$ are invariant under KK-equivalence by Lemma 11.52. And recall that $\gamma(\_,B)$ commutes with any Kasparov product by Theorem 11.44, in particular KK-equivalences. So if $\gamma(A,B)$ is an isomorphism, then $\gamma(C,B)$ is an isomorphism for every C*-algebra C that is KK-equivalent to A. For inductive limits, see Proposition 15.5. For extensions, see Proposition 15.3. For crossed products by $\mathbb{R}$, see Proposition 15.25. For crossed products by $\mathbb{Z}$, see Proposition 15.26. $\square$

16 Constructing a special injective resolution of $\mathrm{K}_*(B)$

If $\mathrm{K}_*(B)$ is not injective, we can still prove the UCT by creating an injective resolution in which each of the modules is a K-group induced by a C*-algebra and the group homomorphisms are induced by $*$-homomorphisms. Such an injective resolution exists! The UCT holds for all the modules in the injective resolution. That is what we proved in §15. To obtain an explicit injective resolution of that form, we must take a slight detour. We will build a projective resolution in which each of the modules is a K-group induced by a C*-algebra and the group homomorphisms are induced by $*$-homomorphisms. This projective resolution can be turned into an injective resolution. This method is described in §4.6. The definition of projectivity and the dual results of the first four subsections of §4 can be found in §4.5. At first creating injective resolutions from projective resolutions seems an unnecessarily laborious way compared to our direct proof of the existence of injective resolutions, but this approach allows the invocation of Theorem 7.95, which otherwise could not be used.

Recall that we have not defined the map δ (see Definition 17.1) yet. To highlight the necessity of the approach we will take, we have included in this thesis another approach in which we try to guess δ in a naive manner. This wrong approach can be found in §16.2. If you prefer to take the quickest route to the proof of the UCT, then you can skip that subsection entirely.

16.1 The construction of the injective resolution of $\mathrm{K}_*(B)$

In §4.6 it was shown how a projective resolution can be turned into an injective resolution. Now we will apply this construction to the result of Theorem 7.95, to turn a $*$-homomorphism that is surjective on the K-groups into a $*$-homomorphism that is injective on the K-groups. The result is the following theorem.

Theorem 16.1. Let B be a C*-algebra. There exist a C*-algebra D and a $*$-homomorphism $S^2B \xrightarrow{f} D$ such that $\mathrm{K}_0(D)$ and $\mathrm{K}_1(D)$ are injective and the group homomorphisms $\mathrm{K}_0(S^2B) \xrightarrow{\mathrm{K}_0(f)} \mathrm{K}_0(D)$ and $\mathrm{K}_1(S^2B) \xrightarrow{\mathrm{K}_1(f)} \mathrm{K}_1(D)$ are injective.

Proof. By Theorem 7.95, there exist a C*-algebra F and $*$-homomorphism $F \xrightarrow{\phi} S(\mathbb{K} \otimes B)$ such that $\mathrm{K}_0(F)$ and $\mathrm{K}_1(F)$ are free and the group homomorphisms $\mathrm{K}_0(F) \xrightarrow{\mathrm{K}_0(\phi)} \mathrm{K}_0(\mathbb{K} \otimes SB)$ and $\mathrm{K}_1(F) \xrightarrow{\mathrm{K}_1(\phi)} \mathrm{K}_1(\mathbb{K} \otimes SB)$ are surjective. Now ϕ induces the exact sequence (Lemma 6.53):

$$0 \to \mathbb{K} \otimes S^2B \xrightarrow{(0,\_)} C_\phi \xrightarrow{s} F \to 0. \tag{54}$$

The right-hand morphism is the projection (see Definition 6.51). We will need it on several occasions; it will be referred to as s. Sequence (54) induces the following six term sequence:

$$\begin{tikzcd}
\mathrm{K}_0(\mathbb{K} \otimes S^2B) \arrow[r, "\mathrm{K}_0((0,_))"] & \mathrm{K}_0(C_\phi) \arrow[r, "\mathrm{K}_0(s)"] & \mathrm{K}_0(F) \arrow[d] \\
\mathrm{K}_1(F) \arrow[u] & \mathrm{K}_1(C_\phi) \arrow[l, "\mathrm{K}_1(s)"] & \mathrm{K}_1(\mathbb{K} \otimes S^2B). \arrow[l, "\mathrm{K}_1((0,_))"]
\end{tikzcd}$$

From Lemma 8.36, it follows that under the suspension isomorphism, the index and exponential maps coincide with the group homomorphisms $\mathrm{K}_0(F) \xrightarrow{\mathrm{K}_0(\phi)} \mathrm{K}_0(\mathbb{K} \otimes S^2B)$ and $\mathrm{K}_1(F) \xrightarrow{\mathrm{K}_1(\phi)}$

$\mathrm{K}_1(\mathbb{K}\otimes S^2B)$. They give the following diagram:

$$\begin{tikzcd}
\mathrm{K}_1(\mathbb{K}\otimes SB) \arrow[r, "\mathrm{K}_0((0,_))"] & \mathrm{K}_0(C_\phi) \arrow[r, "\mathrm{K}_0(s)"] & \mathrm{K}_0(F) \arrow[d, "\mathrm{K}_0(\phi)"] \\
\mathrm{K}_1(F) \arrow[u, "\mathrm{K}_1(\phi)"] & \mathrm{K}_1(C_\phi) \arrow[l, "\mathrm{K}_1(s)"] & \mathrm{K}_0(\mathbb{K}\otimes SB). \arrow[l, "\mathrm{K}_1((0,_))"]
\end{tikzcd}$$

As $\mathrm{K}_0(\phi)$ and $\mathrm{K}_1(\phi)$ are surjective, they split the diagram into two exact sequences. Notice:

$$\ker \mathrm{K}_0((0,\_)) = \operatorname{im} \mathrm{K}_1(\phi) = \mathrm{K}_1(\mathbb{K}\otimes SB) \qquad \ker \mathrm{K}_0(s) = \operatorname{im} \mathrm{K}_0((0,\_)) = 0$$
$$\ker \mathrm{K}_1((0,\_)) = \operatorname{im} \mathrm{K}_0(\phi) = \mathrm{K}_0(\mathbb{K}\otimes SB) \qquad \ker \mathrm{K}_1(s) = \operatorname{im} \mathrm{K}_1((0,\_)) = 0.$$

So $\mathrm{K}_0(s)$ and $\mathrm{K}_1(s)$ are injective. Hence, these are the exact sequences:

$$0 \to \mathrm{K}_0(C_\phi) \xrightarrow{\mathrm{K}_0(s)} \mathrm{K}_0(F) \xrightarrow{\mathrm{K}_0(\phi)} \mathrm{K}_0(\mathbb{K}\otimes SB) \to 0$$
$$0 \to \mathrm{K}_1(C_\phi) \xrightarrow{\mathrm{K}_1(s)} \mathrm{K}_1(F) \xrightarrow{\mathrm{K}_1(\phi)} \mathrm{K}_1(\mathbb{K}\otimes SB) \to 0.$$

By Theorem 8.37 there exists a C*-algebra N such that $\mathrm{K}_0(N) = \mathbb{Q}$ and $\mathrm{K}_1(N) = 0$. The direct limit from Example 5.57 has these K-groups (see Example 12.12). In this proof, we will denote this C*-algebra $\varinjlim M_2(\mathbb{C}) \otimes ... \otimes M_{(n+1)!}(\mathbb{C})$ by N. By Theorem 12.11 (Künneth), $\mathrm{K}_0(F\otimes N) = \mathrm{K}_0(F)\otimes\mathbb{Q}$ and $\mathrm{K}_1(F\otimes N) = \mathrm{K}_1(F)\otimes\mathbb{Q}$ (see [34, p.171]). Recall $\mathrm{K}_0(F)\otimes\mathbb{Q}$ and $\mathrm{K}_1(F)\otimes\mathbb{Q}$ are divisible. By Lemma 4.24 divisible $\mathbb{Z}$-modules are injective.

Let $F \xrightarrow{t} F\otimes N$ be the $\ast$-homomorphism $F \ni x \to x\otimes 1 \in F\otimes N$ (there is a unit in N). Now $\mathrm{K}_0(t)$ and $\mathrm{K}_1(t)$ are the inclusions of $\mathrm{K}_0(F)$ into $\mathrm{K}_0(F)\otimes\mathbb{Q}$ and $\mathrm{K}_1(F)$ into $\mathrm{K}_1(F)\otimes\mathbb{Q}$. This is because $\mathrm{K}_0(F)$ and $\mathrm{K}_1(F)$ are free. As $\mathbb{Z}$ is a PID, $\mathrm{K}_*(F)$ is in particular torsion free. So there exist no $0 \neq x \in \mathrm{K}_*(F)$ and $0 \neq y \in \mathbb{Q}$ such that $x\otimes y = 0$. This means that $\mathrm{K}_0(t)$ and $\mathrm{K}_1(t)$ have a trivial kernel. They are injective.
The composition $C_\phi \xrightarrow{s} F \xrightarrow{t} F\otimes N$ induces the following exact sequence (Lemma 6.53):

$$0 \to S(F\otimes N) \xrightarrow{(0,\_)} C_{ts} \to C_\phi \to 0.$$

It induces the following six term sequence:

$$\begin{tikzcd}
\mathrm{K}_1(F\otimes N) \arrow[r, "\mathrm{K}_0((0,_))"] & \mathrm{K}_0(C_{ts}) \arrow[r] & \mathrm{K}_0(C_\phi) \arrow[d, "\mathrm{K}_0(ts)"] \\
\mathrm{K}_1(C_\phi) \arrow[u, "\mathrm{K}_1(ts)"] & \mathrm{K}_1(C_{ts}) \arrow[l] & \mathrm{K}_0(F\otimes N). \arrow[l, "\mathrm{K}_1((0,_))"]
\end{tikzcd}$$

The index and exponential maps in the diagram follow from Lemma 8.36. As $\mathrm{K}_0(ts)$ and $\mathrm{K}_1(ts)$ are injective (by functoriality, they are the composition of injective group homomorphisms), they split the diagram into two exact sequences:

$$0 \to \mathrm{K}_0(C_\phi) \xrightarrow{\mathrm{K}_0(ts)} \mathrm{K}_0(F\otimes N) \xrightarrow{\mathrm{K}_1((0,\_))} \mathrm{K}_1(C_{ts}) \to 0$$
$$0 \to \mathrm{K}_1(C_\phi) \xrightarrow{\mathrm{K}_1(ts)} \mathrm{K}_1(F\otimes N) \xrightarrow{\mathrm{K}_0((0,\_))} \mathrm{K}_0(C_{ts}) \to 0.$$

It follows that $\mathrm{K}_1(C_{ts})$ is a quotient of $\mathrm{K}_0(F\otimes N) = \mathrm{K}_0(F)\otimes\mathbb{Q}$. A quotient of a divisible $\mathbb{Z}$-module is divisible and therefore injective (Lemma 4.24). Similarly, $\mathrm{K}_0(C_{ts})$ being a quotient of $\mathrm{K}_1(F\otimes N)$ makes it injective. By [31, Theorem 2.9], there exists a $*$-homomorphism $C_s \xrightarrow{u} C_{ts}$ such that the following diagram commutes up to homotopy:

$$\begin{tikzcd}
SC_\phi \arrow[r, "S(s)"] \arrow[d, "\mathrm{id}_{SC_\phi}"] & SF \arrow[r] \arrow[d, "S(t)"] & C_s \arrow[r] \arrow[d, "u"] & C_\phi \arrow[r, "s"] \arrow[d, "\mathrm{id}_{C\phi}"] & F \arrow[d, "t"] \\
SC_\phi \arrow[r, "S(ts)"] & S(F\otimes N) \arrow[r] & C_{ts} \arrow[r] & C_\phi \arrow[r, "ts"] & F\otimes N.
\end{tikzcd} \tag{55}$$

As K_0 and K_1 are homotopy invariant, diagram (55) induces a commutative diagram of K-groups (the K-functors are homotopy invariant, see Lemma 7.33 and Lemma 7.56). As $\mathrm{K}_0(s)$ and $\mathrm{K}_0(ts)$ and $\mathrm{K}_1(s)$ and $\mathrm{K}_1(ts)$ are injective, the six term sequences of K-groups induced by the horizontal rows can be split (in both cases). It gives these diagrams:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_0(C_\phi) \arrow[r, "\mathrm{K}_0(s)"] \arrow[d, "\mathrm{id}_{\mathrm{K}_0(C_\phi)}"] & \mathrm{K}_0(F) \arrow[r] \arrow[d, "\mathrm{K}_0(t)"] & \mathrm{K}_1(C_s) \arrow[r] \arrow[d, "\mathrm{K}_1(u)"] & 0 \\
0 \arrow[r] & \mathrm{K}_0(C_\phi) \arrow[r, "\mathrm{K}_0(ts)"] & \mathrm{K}_0(F\otimes N) \arrow[r] & \mathrm{K}_1(C_{ts}) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{K}_1(C_\phi) \arrow[r, "\mathrm{K}_1(s)"] \arrow[d, "\mathrm{id}_{\mathrm{K}_1(C_\phi)}"] & \mathrm{K}_1(F) \arrow[r] \arrow[d, "\mathrm{K}_1(t)"] & \mathrm{K}_0(C_s) \arrow[r] \arrow[d, "\mathrm{K}_0(u)"] & 0 \\
0 \arrow[r] & \mathrm{K}_1(C_\phi) \arrow[r, "\mathrm{K}_1(ts)"] & \mathrm{K}_1(F\otimes N) \arrow[r] & \mathrm{K}_0(C_{ts}) \arrow[r] & 0.
\end{tikzcd}$$

As $\ker\mathrm{K}_0(t) = 0 = \mathrm{K}_1(t)$ and $\operatorname{coker}\mathrm{id}_{\mathrm{K}_0(C_\phi)} = 0 = \mathrm{id}_{\mathrm{K}_1(C_\phi)}$, it follows from the Snake lemma $\ker\mathrm{K}_0(u) = 0 = \ker\mathrm{K}_1(u)$ (see [17, Lemma 9.1]). Sequence (54) induces the following exact sequence (see Lemma 6.54):

$$0 \to \mathbb{K}\otimes S^2B \xrightarrow{v} C_s \to CF \to 0$$

Here v is the map $((0,\_),0)$. Now CF is contractible (see Lemma 6.48), so there are isomorphisms $\mathrm{K}_0(\mathbb{K}\otimes S^2B) \cong \mathrm{K}_0(C_s)$ and $\mathrm{K}_1(\mathbb{K}\otimes S^2B) \cong \mathrm{K}_1(C_s)$. Define $S^2B \xrightarrow{w} \mathbb{K}\otimes S^2B$ to be an inclusion by tensoring with a rank one projection in $\mathbb{K}$. The composition $S^2B \xrightarrow{w} \mathbb{K}\otimes S^2B \xrightarrow{v} C_s \xrightarrow{u} C_{ts} \equiv D$ induces injective group homomorphisms $\mathrm{K}_0(uvw)$ and $\mathrm{K}_1(uvw)$. This finalises the proof. ☐

16.2 A wrong approach

Going back to the exact sequence (43), it is clear that δ is supposed to be some embedding of the kernel of γ into $\mathrm{KK}_1(A,B)$. In Corollary 13.63, we found that $\mathrm{KK}_1(A,B)$ is isomorphic to $\mathrm{Ext}(A,B)$ whenever A is separable nuclear and B is σ-unital. Under this isomorphisms γ coincides with the index and exponential maps of the induced six term sequence of the extensions (Theorem 12.9). In Theorem 4.64, it was shown that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B))$ (resp. $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B))$) is isomorphic to the group of isomorphism classes of extensions of $\mathrm{K}_0(A)$ by $\mathrm{K}_0(B)$ (resp. $\mathrm{K}_1(A)$ by $\mathrm{K}_1(B)$) under the Baer sum. Recall that extensions of C*-algebras can only be added in the way of Definition 13.1 if the left-hand term is stable. As K_* is stable, under the Baer sum elements of $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(A), \mathrm{K}_0(B))$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(A), \mathrm{K}_1(B))$ can be viewed as extensions

$$0 \to \mathrm{K}_0(\mathbb{K} \otimes B) \xrightarrow{f_0} M \xrightarrow{g_0} \mathrm{K}_0(A) \to 0 \tag{56}$$

$$0 \to \mathrm{K}_1(\mathbb{K} \otimes B) \xrightarrow{f_1} N \xrightarrow{g_1} \mathrm{K}_1(A) \to 0. \tag{57}$$

Via the essential surjectivity of K_* (Theorem 8.37), it is possible to find C*-algebra C such that $\mathrm{K}_0(C) = M$ and $\mathrm{K}_1(C) = N$. Using this result, the extensions (56) and (57) can be written as

$$0 \to \mathrm{K}_0(\mathbb{K} \otimes B) \xrightarrow{f_0} \mathrm{K}_0(C) \xrightarrow{g_0} \mathrm{K}_0(A) \to 0 \tag{58}$$

$$0 \to \mathrm{K}_1(\mathbb{K} \otimes B) \xrightarrow{f_1} \mathrm{K}_1(C) \xrightarrow{g_1} \mathrm{K}_1(A) \to 0. \tag{59}$$

Now this fails to induce an extensions of C*-algebras of the form $0 \to \mathbb{K} \otimes B \to C \to A \to 0$. It need not be true that f_0, f_1, g_0 or g_1 come from a $*$-homomorphism, i.e. are in the images of the following maps:

$$\mathrm{Hom}_{\mathsf{C^*-alg}}(\mathbb{K} \otimes B, C) \xrightarrow{\mathrm{K}_0} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(\mathbb{K} \otimes B), \mathrm{K}_0(C))$$
$$\mathrm{Hom}_{\mathsf{C^*-alg}}(\mathbb{K} \otimes B, C) \xrightarrow{\mathrm{K}_1} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(\mathbb{K} \otimes B), \mathrm{K}_1(C))$$
$$\mathrm{Hom}_{\mathsf{C^*-alg}}(C, A) \xrightarrow{\mathrm{K}_0} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(C), \mathrm{K}_0(A))$$
$$\mathrm{Hom}_{\mathsf{C^*-alg}}(C, A) \xrightarrow{\mathrm{K}_1} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(C), \mathrm{K}_1(A)).$$

Remark 16.2. In other words; K_0 and K_1 need not be full.

Remark 16.3. In Theorem 7.61, it is stated that K_0 and K_1 are *half*-exact. So an extension of C*-algebras need not induce an extension under K_0 and K_1. In our quest to make an educated guess of what δ is, we did the converse. We tried to bring an extension of abelian groups that happens to be the image of K_0 and K_1 at the object level back to an extension of C*-algebras.

Remark 16.4. Even if f_0, f_1, g_0 and g_1 are all induced by $*$-homomorphisms, then it need not be true that an extension $0 \to \mathbb{K} \otimes B \to C \to A \to 0$ is induced by (58) and (59). Although $g_0 \circ f_0 = 0$ and $g_1 \circ f_1 = 0$, it need not be true the group homomorphisms $g_0 \circ f_0$ and $g_1 \circ f_1$ come from the $*$-homomorphism 0. This is because K_0 and K_1 are not faithful (not even up to homotopy).

Another issue is the non-uniqueness of the failing induction. In Lemma 7.77 it was proven that K_0 and K_1 are split exact. This meant a split extension of C*-algebras will give a split extension under K_0 and K_1. However, it might be true that a non-split extension of C*-algebras becomes a split extension under K_0 and K_1. This would lead to different classes in $\mathrm{Ext}(A, B)$ giving the same equivalence classes of extensions under application K_0 by K_1. If this happens, δ has to make a choice in its assignment of an extension of K-groups to an extension of C*-algebras. There is no obvious choice for this and besides it need not be true that there exists a choice such that δ preserves the additive structure (the Baer sum for modules (see Definition 4.59) and the addition of Definition 13.1 for C*-algebras).

17 The proof of the UCT for general $\mathrm{K}_*(B)$

Now the big moment has come to reveal the definition of δ! The definition in this section looks different from the δ in Theorem 12.5, but we will see in Theorem 17.3 that the δ from Definition 17.1

gives the δ from Theorem 12.5.

Let A be a separable nuclear C*-algebra and let B be a σ-unital C*-algebra. By Theorem 16.1, there exist a C*-algebra D and a $*$-homomorphism $\mathbb{K}\otimes S^2B \xrightarrow{f} D$ such that $\mathrm{K}_*(D)$ is injective and $\mathrm{K}_*(f)$ is injective. By Lemma 6.53, this sequence is exact:

$$0 \to SD \xrightarrow{(0,\_)} C_f \to \mathbb{K}\otimes S^2B \to 0. \tag{60}$$

Its six term sequence with respect to KK is this:

$$\begin{tikzcd}
\mathrm{KK}_0(A,SD) \arrow[r, "\mathrm{KK}_0(\mathrm{id}_A,(0,_))"] & \mathrm{KK}_0(A,C_f) \arrow[r] & \mathrm{KK}_0(A,B) \arrow[d] \\
\mathrm{KK}_1(A,B) \arrow[u] & \mathrm{KK}_1(A,C_f) \arrow[l] & \mathrm{KK}_1(A,SD). \arrow[l, "\mathrm{KK}_1(\mathrm{id}_A,(0,_))"]
\end{tikzcd}$$

Definition 17.1. Define $\operatorname{coker}\mathrm{KK}_*(\mathrm{id}_A,(0,\_)) \xrightarrow{\delta} \mathrm{KK}_*(A,B)$ to be the inclusion following from the six term sequence.

Remark 17.2. It is not obvious that Definition 17.1 is independent of the choice of C*-algebra D and $*$-homomorphism $\mathbb{K}\otimes S^2B \xrightarrow{f} D$. We will first prove the UCT for D and f that satisfy the conditions of Definition 17.1. We will then show that a different choice of D and f gives an equivalent exact sequence of the form of sequence (41).

Theorem 17.3. Let A be a separable nuclear C*-algebra. If the UCT (Theorem 12.5) holds for all pairs (A,B) in which B is a σ-unital C*-algebra such that $\mathrm{K}_*(B)$ is injective, then the UCT holds for all pairs (A,B) in which B is a σ-unital C*-algebra.

Proof. Let B be a C*-algebra. By Theorem 16.1, there exist a C*-algebra D and a $*$-homomorphism $\mathbb{K}\otimes S^2B \xrightarrow{f} D$ such that $\mathrm{K}_0(D)$ and $\mathrm{K}_1(D)$ are injective and the group homomorphisms $\mathrm{K}_0(\mathbb{K}\otimes S^2B) \xrightarrow{\mathrm{K}_0(f)} \mathrm{K}_0(D)$ and $\mathrm{K}_1(\mathbb{K}\otimes S^2B) \xrightarrow{\mathrm{K}_1(f)} \mathrm{K}_1(D)$ are injective. Now f induces sequence (60). Its six term sequence with respect to K-groups is:

$$\begin{tikzcd}
\mathrm{K}_1(D) \arrow[r, "\mathrm{K}_0((0,_))"] & \mathrm{K}_0(C_f) \arrow[r] & \mathrm{K}_0(\mathbb{K}\otimes S^2B) \arrow[d, "\mathrm{K}_0(f)"] \\
\mathrm{K}_1(\mathbb{K}\otimes S^2B) \arrow[u, "\mathrm{K}_1(f)"] & \mathrm{K}_1(C_f) \arrow[l] & \mathrm{K}_0(D). \arrow[l, "\mathrm{K}_1((0,_))"]
\end{tikzcd}$$

The index and exponential maps in the diagram follow from Lemma 8.36. As $\mathrm{K}_0(f)$ and $\mathrm{K}_1(f)$ are injective, they split the diagram into two exact sequences:

$$0 \to \mathrm{K}_0(\mathbb{K}\otimes S^2B) \xrightarrow{\mathrm{K}_0(f)} \mathrm{K}_0(D) \xrightarrow{\mathrm{K}_1((0,\_))} \mathrm{K}_1(C_f) \to 0 \tag{61}$$

$$0 \to \mathrm{K}_1(\mathbb{K}\otimes S^2B) \xrightarrow{\mathrm{K}_1(f)} \mathrm{K}_1(D) \xrightarrow{\mathrm{K}_0((0,\_))} \mathrm{K}_0(C_f) \to 0. \tag{62}$$

It follows that $\mathrm{K}_1(C_f)$ is a quotient of $\mathrm{K}_0(D)$. A $\mathbb{Z}$-module is divisible if only if it is injective (See Lemma 4.23 and Lemma 4.24). A quotient of a divisible $\mathbb{Z}$-module is divisible. So $\mathrm{K}_1(C_f)$ is an injective $\mathbb{Z}$-module. Similarly, $\mathrm{K}_0(C_f)$ being a quotient of $\mathrm{K}_1(D)$ makes it injective.

Applying the covariant functors $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A), \_)$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A), \_)$ to sequences (61) and (62) gives these exact sequences (see Theorem 4.82 and recall Lemma 4.66):

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A), \mathrm{K}_0(B)) \xrightarrow{\mathrm{K}_0(f)\circ\_} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A), \mathrm{K}_0(D)) \xrightarrow{\mathrm{K}_1((0,\_))\circ\_} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A), \mathrm{K}_1(C_f)) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(A), \mathrm{K}_0(B)) \to 0$$

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A), \mathrm{K}_1(B)) \xrightarrow{\mathrm{K}_1(f)\circ\_} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A), \mathrm{K}_1(D)) \xrightarrow{\mathrm{K}_0((0,\_))\circ\_} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A), \mathrm{K}_0(C_f)) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(A), \mathrm{K}_1(B)) \to 0$$

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A), \mathrm{K}_0(B)) \xrightarrow{\mathrm{K}_0(f)\circ\_} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A), \mathrm{K}_0(D)) \xrightarrow{\mathrm{K}_1((0,\_))\circ\_} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A), \mathrm{K}_1(C_f)) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(A), \mathrm{K}_0(B)) \to 0$$

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A), \mathrm{K}_1(B)) \xrightarrow{\mathrm{K}_1(f)\circ\_} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A), \mathrm{K}_1(D)) \xrightarrow{\mathrm{K}_0((0,\_))\circ\_} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A), \mathrm{K}_0(C_f)) \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(A), \mathrm{K}_1(B)) \to 0.$$

In these sequences $\mathrm{K}_*(\mathbb{K}\otimes S^2B)$ has been replaced by $\mathrm{K}_*(B)$ (as a result of the Bott periodicity and stability). In terms of $\mathbb{Z}_2$-graded modules, the exactness implies:

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(B)) = \ker(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(D)) \xrightarrow{\mathrm{K}_*((0,\_))\circ\_} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_{*+1}(C_f))), \tag{63}$$

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(B)) = \mathrm{coker}(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_*(D)) \xrightarrow{\mathrm{K}_*((0,\_))\circ\_} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A), \mathrm{K}_{*+1}(C_f))). \tag{64}$$

Sequence (60) induces the following six term sequence with respect to KK-groups:

$$\begin{tikzcd}
\mathrm{KK}_1(A,D) \arrow[r, "\mathrm{KK}_0(\mathrm{id}_A,(0,_))"] & \mathrm{KK}_0(A,C_f) \arrow[r] & \mathrm{KK}_0(A,B) \arrow[d, "\mathrm{KK}_0(\mathrm{id}_A,f)"] \\
\mathrm{KK}_1(A,B) \arrow[u, "\mathrm{KK}_1(\mathrm{id}_A,f)"] & \mathrm{KK}_1(A,C_f) \arrow[l] & \mathrm{KK}_0(A,D). \arrow[l, "\mathrm{KK}_1(\mathrm{id}_A,(0,_))"]
\end{tikzcd}$$

The boundary maps follow from Lemma 8.36. In the diagram $\mathrm{KK}_*(A, \mathbb{K}\otimes S^2B)$ is replaced by $\mathrm{KK}_*(A,B)$ (again as a result of the Bott periodicity and stability). As D and C_f are injective, Theorem 15.1 applies to them. The following commutative diagram arises:

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(D)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(D)) \arrow[rr, "K_0((0,\_))\circ\_", "K_1((0,\_))\circ\_"'] & & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(C_f)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(C_f)) \\
\mathrm{KK}_1(A,D) \arrow[u, "\gamma(A,D)", "\cong"'] \arrow[rr, "\mathrm{KK}_0(\mathrm{id}_A,(0,\_))"] & & \mathrm{KK}_0(A,C_f) \arrow[u, "\cong", "\gamma(A,C_f)"'] \arrow[d] \\
\mathrm{KK}_1(A,B) \arrow[u, "\mathrm{KK}_1(\mathrm{id}_A,f)"] & & \mathrm{KK}_0(A,B) \arrow[d, "\mathrm{KK}_0(\mathrm{id}_A,f)"'] \\
\mathrm{KK}_1(A,C_f) \arrow[u] \arrow[d, "\gamma(A,C_f)"', "\cong"] & & \mathrm{KK}_0(A,D) \arrow[ll, "\mathrm{KK}_1(\mathrm{id}_A,(0,\_))"] \arrow[d, "\cong"', "\gamma(A,D)"] \\
\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(C_f)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(C_f)) & & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(D)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(D)). \arrow[ll, "K_1((0,\_))\circ\_"', "K_0((0,\_))\circ\_"]
\end{tikzcd}
\tag{65}
$$

The square in the middle is the six term sequence with respect to KK induced by sequence (60). Note that $\gamma(A,D)$ and $\gamma(A,C_f)$ are isomorphisms. Two elements of $\mathrm{KK}_*(A,C_f)$ are mapped to the same element in $\mathrm{KK}_*(A,B)$ whenever they differ an element of $\ker(\mathrm{KK}_*(A,C_f) \to \mathrm{KK}_*(A,B))$. In an exact sequence, the kernel is the image of the previous arrow. So in this case, $\ker(\mathrm{KK}_*(A,C_f) \to \mathrm{KK}_*(A,B)) = \mathrm{im}\,\mathrm{KK}_*(\mathrm{id}_A,(0,\_))$. We conclude that the six term sequence of sequence (60) induces an injective group homomorphism $\mathrm{coker}\,\mathrm{KK}_*(\mathrm{id}_A,(0,\_)) \to \mathrm{KK}_*(A,B)$.

The elements of $\mathrm{KK}_*(A,B)$ are mapped to $\mathrm{KK}_*(A,D)$ by $\mathrm{KK}_*(\mathrm{id}_A,f)$. By definition, the map $\mathrm{KK}_*(A,B) \xrightarrow{\mathrm{KK}_*(\mathrm{id}_A,f)} \mathrm{im}\,\mathrm{KK}_*(\mathrm{id}_A,f)$ is surjective. In an exact sequence, the image of an arrow is the kernel of the next arrow. So in this case, $\mathrm{im}\,\mathrm{KK}_*(\mathrm{id}_A,f) = \ker\mathrm{KK}_{*+1}(\mathrm{id}_A,(0,\_))$. Hence the exactness of the six term sequence induces the following exact sequences (with the obvious inclusions and quotients):

$$0 \to \mathrm{coker}\,\mathrm{KK}_0(\mathrm{id}_A,(0,\_)) \xrightarrow{\delta} \mathrm{KK}_0(A,B) \to \ker\mathrm{KK}_1(\mathrm{id}_A,(0,\_)) \to 0 \tag{66}$$

$$0 \to \mathrm{coker}\,\mathrm{KK}_1(\mathrm{id}_A,(0,\_)) \xrightarrow{\delta} \mathrm{KK}_1(A,B) \to \ker\mathrm{KK}_0(\mathrm{id}_A,(0,\_)) \to 0. \tag{67}$$

By definition, the left-hand morphisms are the δ from Definition 17.1. In (63) and (64) we find the kernel and cokernel of the top and bottom horizontal arrows of diagram (65). From the isomorphisms $\gamma(A,D)$ and $\gamma(A,C_f)$ follows $\ker\mathrm{KK}_*(\mathrm{id}_A,(0,\_)) = \ker(\mathrm{K}_*((0,\_))\circ\_)$ and $\mathrm{coker}\,\mathrm{KK}_*(\mathrm{id}_A,(0,\_)) = \mathrm{coker}(\mathrm{K}_*((0,\_))\circ\_)$. Applying (63) and (64), the sequences (66) and (67) become:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)) \xrightarrow{\delta} \mathrm{KK}_0(A,B) \xrightarrow{\gamma} \\ \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B)) \to 0$$

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B)) \xrightarrow{\delta} \mathrm{KK}_1(A,B) \xrightarrow{\gamma} \\ \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)) \to 0.$$

The map δ defined in Definition 17.1 is precisely the map $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) \to \mathrm{KK}_*(A,B)$ from sequences (66) and (67). We will now show that the right-hand morphism coincides with $\gamma(A,B)$.

See the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) \arrow[r, "\mathrm{K}_*(f)\circ_"] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(D)) \\
\mathrm{KK}_*(A,B) \arrow[u, "\gamma(A,B)"] \arrow[r, "\mathrm{KK}_*(\mathrm{id}_A,f)"] & \mathrm{KK}_*(A,D). \arrow[u, "\gamma(A,D)"', "\cong"]
\end{tikzcd} \tag{68}$$

$\mathrm{KK}_*(\mathrm{id}_A,f)$ maps onto $\ker \mathrm{KK}_{*+1}(\mathrm{id}_A,(0,\_)) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ under $\gamma(A,D)$. The inclusion of $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ into $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(D))$ is given by $\mathrm{K}_*(f)\circ\_$. Hence the morphism connecting $\mathrm{KK}_*(A,B)$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$ in diagram (68) is $\gamma(A,B)$. It follows from the fact that γ is a morphism of functors (see Lemma 15.2). ☐

Remark 17.4. The group homomorphisms $K_*((0,\_))\circ\_$ and $\mathrm{KK}_*(\mathrm{id}_A,(0,\_))$ become odd instead of even by the application of Lemma 8.36. The removal of a suspension causes the reversal in grading of sequence (66) and (67).

Remark 17.5. Compared to the proof in [27], our proof is cleaner. In [27, Theorem 3.2] (Theorem 16.1), the created $*$-homomorphism does not exist. In reality, a double suspension and Bott periodicity are used. The resulting K-group homomorphism is the same. In [27, Theorem 4.1] (Theorem 17.3), the degree of the K- and KK-groups is lowered to get rid of the suspension. In our approach Bott periodicity has to be invoked only once in the two theorems. It is done at a point we no longer need the underlying $*$-homomorphisms.

Proof Theorem 12.5. Let $A \in \mathrm{Ob}(\mathsf{N})$ and let B be a σ-unital C*-algebra. If $\mathrm{K}_*(B)$ is an injective $\mathbb{Z}$-module, then $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) = 0$ and Theorem 15.1 shows that $\gamma(A,B)$ is an isomorphism. Hence if $\mathrm{K}_*(B)$ is injective, then sequence (41) is exact.

In case $\mathrm{K}_*(B)$ is not injective, then it might happen that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B)) \neq 0$. If δ is defined as in Definition 17.1, then Theorem 17.3 says that sequence (41) is exact. ☐

Theorem 17.6. A different choice of the C*-algebras D and $*$-homomorphisms $\mathbb{K}\otimes S^2B \xrightarrow{f} D$ such that $\mathrm{K}_*(D)$ is injective and $\mathrm{K}_*(f)$ is injective in Definition 17.1 does not change δ.

Proof. Suppose there exists another C*-algebra E and a $*$-homomorphism $\mathbb{K}\otimes S^2B \xrightarrow{g} E$ such that $\mathrm{K}_*(E)$ is injective and $\mathrm{K}_*(g)$ is injective, then the exists an exact sequence similar to (60):

$$0 \to SE \to C_g \to \mathbb{K}\otimes S^2B$$

By Lemma 6.48 and the fact that $\mathrm{K}_0(g)$ and $\mathrm{K}_1(g)$ are injective, the six term sequence with respect to the K-groups splits into two exact sequence (similar to sequences (61) and (62):

$$0 \to \mathrm{K}_0(\mathbb{K}\otimes S^2B) \xrightarrow{\mathrm{K}_0(g)} \mathrm{K}_0(E) \xrightarrow{\mathrm{K}_1((0,\_))} \mathrm{K}_1(C_g) \to 0,$$
$$0 \to \mathrm{K}_1(\mathbb{K}\otimes S^2B) \xrightarrow{\mathrm{K}_1(g)} \mathrm{K}_1(E) \xrightarrow{\mathrm{K}_0((0,\_))} \mathrm{K}_0(C_g) \to 0.$$

Similarly $\mathrm{K}_0(C_g)$ and $\mathrm{K}_1(C_g)$ are quotients of injective modules over a PID and therefore injective themselves. This gives $\mathrm{K}_0(\mathbb{K}\otimes S^2B)$ and $\mathrm{K}_1(\mathbb{K}\otimes S^2B)$ each two injective resolutions (of $\mathbb{Z}$-modules). By Corollary 4.37 injective resolutions of the same module are cochain homotopy equivalent. We will show what the implications are for sequence (61). The results for sequence (62) are similar.

By Theorem 4.36, there exist group homomorphisms $\mathrm{K}_0(D) \xrightarrow{\psi_0} \mathrm{K}_0(E)$, $\mathrm{K}_1(C_f) \xrightarrow{\psi_1} \mathrm{K}_1(C_g)$ such that $\psi_0 \circ \mathrm{K}_0(f) = \mathrm{K}_0(g)$ and $\psi_1 \circ \mathrm{K}_1((0,\_)_f) = \mathrm{K}_1((0,\_)_g) \circ \psi_0$. Likewise, there exist group homomorphisms $\mathrm{K}_0(E) \xrightarrow{\chi_0} \mathrm{K}_0(D)$, $\mathrm{K}_1(C_g) \xrightarrow{\chi_1} \mathrm{K}_1(C_f)$ such that $\chi_0 \circ \mathrm{K}_0(g) = \mathrm{K}_0(f)$ and $\chi_1 \circ \mathrm{K}_1((0,\_)_g) = \mathrm{K}_1((0,\_)_f) \circ \chi_0$. By Corollary 4.37 there exist group homomorphisms $\mathrm{K}_0(D) \xrightarrow{\eta_0} \mathrm{K}_0(\mathbb{K} \otimes S^2 B)$ and $\mathrm{K}_1(C_f) \xrightarrow{\eta_1} \mathrm{K}_0(D)$ such that $\mathrm{K}_0(f) \circ \eta_0 + \eta_1 \circ \mathrm{K}_1((0,\_)_f)\mathrm{K} = \mathrm{id}_{\mathrm{K}_0(D)} - \chi_0 \circ \psi_0$ and $\mathrm{K}_1((0,\_)_f) \circ \eta_1 = \mathrm{id}_{\mathrm{K}_0(C_f)} - \chi_1 \circ \psi_1$.
To show the fact that both injective resolutions give the same map δ, we make the following diagram:

$$\begin{tikzcd}
 \mathrm{KK}_*(A,D) \arrow[r, "\mathrm{KK}_{*+1}(\mathrm{id}_A,(0,_)_f)"] \arrow[d, shift left, "\cong"] & \mathrm{KK}_{*+1}(A,C_f) \arrow[r] \arrow[d, shift left, "\cong"] & \mathrm{KK}_{*+1}(A,B) & \\
 \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_0(D)) \arrow[r, "\mathrm{K}_*((0,_)_f)\circ"] \arrow[u, shift left, "\gamma(A,D)"] \arrow[d, shift right, "\psi_0\circ_"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_1(C_f)) \arrow[r] \arrow[u, shift left, "\gamma(A,C_f)"] \arrow[d, shift right, "\psi_1\circ_"'] & \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_0(B)) \arrow[r] \arrow[u, "\delta_f"'] \arrow[d, "\mathrm{id}_{\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_0(B))}"] & 0 \\
 \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_0(E)) \arrow[r, "\mathrm{K}_*((0,_)_g)\circ"] \arrow[u, shift right, "\chi_0\circ_"'] \arrow[d, shift right, "\gamma(A,E)"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_1(C_g)) \arrow[r] \arrow[u, shift right, "\chi_1\circ_"'] \arrow[d, shift right, "\gamma(A,C_g)"'] & \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_0(B)) \arrow[r] \arrow[d, "\delta_g"'] & 0 \\
 \mathrm{KK}_*(A,E) \arrow[r, "\mathrm{KK}_{*+1}(\mathrm{id}_A,(0,_)_g)"] \arrow[u, shift right, "\cong"'] & \mathrm{KK}_{*+1}(A,C_g) \arrow[r] \arrow[u, shift right, "\cong"'] & \mathrm{KK}_{*+1}(A,B). &
\end{tikzcd}$$

In this diagram only the relevant objects are included. $\mathrm{K}_0(\mathbb{K} \otimes S^2 B)$ is abbreviated to $\mathrm{K}_0(B)$. The diagram does not commute, but $\psi_1 \circ \chi_1$ and $\mathrm{id}_{\mathrm{K}_1(C_f)}$ differ by an element of $\operatorname{im} \mathrm{K}_1((0,\_)_f)$. As a result of this $\gamma(A,C_f)^{-1} \circ \chi_1 \circ \psi_1 \circ \gamma(A,C_f)$ differs only from $\mathrm{id}_{\mathrm{KK}\,*+1(A,C_f)}$ by an element of $\operatorname{im} \mathrm{KK}_{+1}(\mathrm{id}_A,(0,\_)_f)$. Hence δ_f coincides with δ_g. The same argument applies to sequence (62). □

Remark 17.7. Theorem 17.6 rules out a situation as in Remark 4.55. In that remark we showed that $0 \to \mathbb{Z}/3\mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$ and $0 \to \mathbb{Z}/3\mathbb{Z} \xrightarrow{\cdot 6} \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$ are non-equivalent extensions. The objects and the right-hand morphism are the same, only the left-hand morphism differs.

In fact, something stronger than Theorem 17.6 can be proven. The UCT is not just some extension, but a split extension! The power of this result is that for $A \in \mathrm{Ob}(\mathsf{N})$ and B σ-unital, $\mathrm{KK}_*(A,B)$ can be determined from the K-groups alone.

Theorem 17.8. Let $A \in \mathrm{Ob}(\mathsf{N})$ and let B be a σ-unital C*-algebra. Then sequence (41) in Theorem 12.5 splits.

Proof. See [27, Theorem 7.10]. □

Corollary 17.9. Let $A \in \mathrm{Ob}(\mathsf{N})$ and let B be a σ-unital C*-algebra. Then

$$\mathrm{KK}_0(A,B) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B))$$
$$\mathrm{KK}_1(A,B) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_0(B)) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(A),\mathrm{K}_1(B)) \oplus \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(A),\mathrm{K}_0(B)).$$

Proof. Sequence (41) can be separated into two parts; sequence (42) and sequence (43). Applying Lemma 4.8 to sequence (42) and sequence (43) finalises the lemma. □

Remark 17.10. In other words, for $A \in \mathrm{Ob}(\mathsf{N})$ and B σ-unital the group $\mathrm{KK}_*(A,B)$ is completely determined by $\mathrm{K}_*(A)$ and $\mathrm{K}_*(B)$.

In [27, Theorem 4.4] naturality of the UCT with respect to Kasparov products is proven. We will state this theorem as well with a short proof. For completeness we will add a similar theorem about $*$-homomorphisms. We don't need to be as elaborate as in [27, Theorem 4.4], because we already know a lot of properties of the six term sequences and of the Kasparov product.

Theorem 17.11 (Naturality w.r.t $*$-homomorphisms)**.** Let $A \in \mathrm{Ob}(\mathsf{N})$ and let B_1 and B_2 be σ-unital C*-algebras. Let $B_1 \xrightarrow{\beta} B_2$ be a $*$-homomorphism. Then the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B_1)) \arrow[r, "\delta(A,B_1)"] \arrow[d, "\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{id}_{\mathrm{K}_*(A)},\mathrm{K}_*(\beta))"'] & \mathrm{KK}_*(A,B_1) \arrow[r, "\gamma(A,B_1)"] \arrow[d, "\mathrm{KK}_*(\mathrm{id}_A,\beta)"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B_1)) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{id}_{\mathrm{K}_*(A)},\mathrm{K}_*(\beta))"] & 0 \\ 0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B_2)) \arrow[r, "\delta(A,B_2)"] & \mathrm{KK}_*(A,B_2) \arrow[r, "\gamma(A,B_2)"] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B_2)) \arrow[r] & 0. \end{tikzcd}$$

Let $A_1, A_2 \in \mathrm{Ob}(\mathsf{N})$ and let B be a σ-unital C*-algebra. Let $A_1 \xrightarrow{\alpha} A_2$ be a $*$-homomorphism. Then the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A_2),\mathrm{K}_*(B)) \arrow[r, "\delta(A_2,B)"] \arrow[d, "\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(\alpha),\mathrm{id}_{\mathrm{K}_*(B)})"'] & \mathrm{KK}_*(A_2,B) \arrow[r, "\gamma(A_2,B)"] \arrow[d, "\mathrm{KK}_*(\alpha,\mathrm{id}_B)"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A_2),\mathrm{K}_*(B)) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(\alpha),\mathrm{id}_{\mathrm{K}_*(B)})"] & 0 \\ 0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A_1),\mathrm{K}_*(B)) \arrow[r, "\delta(A_1,B)"] & \mathrm{KK}_*(A_1,B) \arrow[r, "\gamma(A_1,B)"] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A_1),\mathrm{K}_*(B)) \arrow[r] & 0. \end{tikzcd}$$

Proof. The right-hand squares of these diagrams commute, as the Kasparov product commutes with $*$-homomorphisms. See Lemma 11.48. The left-hand squares commute, because δ comes from the six term sequence of KK, see Definition 17.1. In the proof of Theorem 17.3 we found that $\operatorname{coker}\mathrm{KK}_*(\mathrm{id}_A,(0,\_)) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B))$. For $B_1 \xrightarrow{\beta} B_2$, let $x \in \operatorname{coker}\mathrm{KK}_*(\mathrm{id}_A,(0,\_))_{B_1}$, then

$$\gamma(A,B_2)(\mathrm{KK}_*(\mathrm{id}_A,\beta)(\delta(A,B_1)(x))) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{id}_{\mathrm{K}_*(A)},\mathrm{K}_*(\beta))(\gamma(A,B_1)(\delta(A,B_1)(x))) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{id}_{\mathrm{K}_*(A)},\mathrm{K}_*(\beta))(0) = 0.$$

Hence $\mathrm{KK}_*(\mathrm{id}_A,\beta)(\delta(A,B_1)(x))$ comes from $\operatorname{coker}\mathrm{KK}_*(\mathrm{id}_A,(0,\_))_{B_2}$. This way $\mathrm{KK}_*(\mathrm{id}_A,\beta)$ defines a group homomorphism from $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B_1))$ to $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B_2))$. This group homomorphism coincides with $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{id}_{\mathrm{K}_*(A)},\mathrm{K}_*(\beta))$. This follows from applying $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\_)$ to sequences (61) and (62) for both B_1 and B_2 and then connecting them with β and applying Theorem 4.36. Using the naturality of γ, it follows that the constructed group homomorphism coincides with $\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{id}_{\mathrm{K}_*(A)},\mathrm{K}_*(\beta))$. The construction for $A_1 \xrightarrow{\alpha} A_2$ is similar. □

Theorem 17.12 (Naturality w.r.t Kasparov products)**.** Let $A \in \mathrm{Ob}(\mathsf{N})$ and let B_1 and B_2 be σ-unital C*-algebras. Let $Y \in \mathrm{KK}_*(B_1,B_2)$. Then the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B_1)) \arrow[r, "\delta(A,B_1)"] \arrow[d, "\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{id}_{\mathrm{K}_*(A)},\,\_\otimes_{B_1}Y)"'] & \mathrm{KK}_*(A,B_1) \arrow[r, "\gamma(A,B_1)"] \arrow[d, "\_\otimes_{B_1}Y"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B_1)) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(\mathrm{id}_{\mathrm{K}_*(A)},\,\_\otimes_{B_1}Y)"] & 0 \\ 0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B_2)) \arrow[r, "\delta(A,B_2)"] & \mathrm{KK}_*(A,B_2) \arrow[r, "\gamma(A,B_2)"] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A),\mathrm{K}_*(B_2)) \arrow[r] & 0. \end{tikzcd}$$

Let $A_1, A_2 \in \mathrm{Ob}(\mathsf{N})$ and let B be a σ-unital C*-algebra. Let $X \in \mathrm{KK}_*(A_1, A_2)$. Then the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A_2), \mathrm{K}_*(B)) \arrow[r, "\delta(A_2,B)"] \arrow[d, "\mathrm{Ext}^1_{\mathbb{Z}}(X\otimes_{A_2}\text{-}, \mathrm{id}_{\mathrm{K}_*(B)})"'] & \mathrm{KK}_*(A_2, B) \arrow[r, "\gamma(A_2,B)"] \arrow[d, "X\otimes_{A_2}\text{-}"'] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A_2), \mathrm{K}_*(B)) \arrow[r] \arrow[d, "\mathrm{Hom}_{\mathbb{Z}}(X\otimes_{A_2}\text{-}, \mathrm{id}_{\mathrm{K}_*(B)})"] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_*(A_1), \mathrm{K}_*(B)) \arrow[r, "\delta(A_1,B)"] & \mathrm{KK}_*(A_1, B) \arrow[r, "\gamma(A_1,B)"] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_*(A_1), \mathrm{K}_*(B)) \arrow[r] & 0.
\end{tikzcd}$$

Proof. The right-hand squares of these diagrams commute, as the Kasparov product is associative. See Lemma 11.44. The left-hand squares commute, because δ comes from the six term sequence of KK. The construction of the left vertical morphism is similar to the proof of Theorem 17.11. $\square$

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