Geometry of the momentum map

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1 The momentum map

We will define a notion of 'reduction' of a symplectic manifold by an action of a Lie group. This procedure is called *symplectic reduction*, and it involves a momentum map. This is a map from the symplectic manifold to the dual of the Lie algebra of the group acting on it.

Historically, momentum maps and symplectic reduction appeared in many examples from classical mechanics before they were defined in general, by Kostant

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[18] and Souriau [26] around 1965. The conserved quantities linear and angular momentum for example are special cases of momentum maps. In classical mechanics, the phase space of a system is a symplectic manifold[1]. The dynamics is generated by a function on the phase space, called the Hamiltonian. This function has the physical interpretation of the total energy of the system. A symmetry of the physical system is an action of a group on the phase space, which leaves the symplectic form and the Hamiltonian invariant. We shall only study the symmetry reduction of symplectic manifolds, although the reduction of Hamiltonian functions is an interesting subject.

In this and the following sections, (M, ω) denotes a connected symplectic manifold. We shall use the letter G to denote a general Lie group, and we will write K for a compact connected Lie group. Their Lie algebras are denoted by $\mathfrak{g}$ and $\mathfrak{k}$, respectively. All manifolds, Lie groups and maps are assumed to be C^∞.

1.1 Definition of the momentum map

Let (M, ω) be a symplectic manifold. Suppose that G acts smoothly on M, and let

$$G \times M \to M, \quad (g, m) \mapsto gm = a(g)m$$

be the map defining the action. We also consider the corresponding infinitesimal action

$$\mathfrak{g} \times M \to TM, \quad (X, m) \mapsto X_m = A(X)_m,$$

where by definition,

$$X_m := \left.\frac{d}{dt}\right|_{t=0} \exp(tX) \cdot m \in T_m M.$$

In this way, every element X of $\mathfrak{g}$ defines a vector field in M, which is also denoted by X, or sometimes by X_M to avoid confusion. The map $X \mapsto A(X)$ is an *anti*homomorphism from $\mathfrak{g}$ to the Lie algebra $\mathrm{Vect}(M)$ of smooth vector fields on M, just as the map $g \mapsto a(g)$ defines an antirepresentation of G in the space of smooth functions on M.

We denote the connected component of G containing the identity element by G^0.

Lemma 1.1. *The symplectic form ω is invarant under the action of G^0, if and only if the one-form $X \lrcorner \omega$ is closed for all $X \in \mathfrak{g}$.*

The symbol '$\lrcorner$' denotes contraction of differential forms with vector fields.

Proof. The form ω is G^0-invariant if and only if the Lie derivative $\mathcal{L}_X \omega$ equals zero for all $X \in \mathfrak{g}$. By Cartan's formula,

$$\begin{aligned} \mathcal{L}_X \omega &= d(X \lrcorner \omega) + X \lrcorner d\omega \\ &= d(X \lrcorner \omega), \end{aligned}$$

[1]We will deal with symplectic manifolds rather than the more general Poisson manifolds.

because ω is closed. □

Definition 1.2. Suppose that G acts on the symplectic manifold (M, ω), leaving ω invariant. The action is called *Hamiltonian*[2] if there is a smooth, equivariant map

$$\mu : M \to \mathfrak{g}^*,$$

such that for all $X \in \mathfrak{g}$,

$$\boxed{d\mu_X = -X \lrcorner \omega.} \tag{1}$$

Here the function μ_X is defined by

$$\mu_X(m) = \langle \mu(m), X \rangle,$$

for $X \in \mathfrak{g}$ and $m \in M$.

The map μ is called a *momentum map*[3] of the action.

Remark 1.3. A momentum map is assumed to be equivariant with respect to the *coadjoint action* Ad^* of G on $\mathfrak{g}^*$:

$$\langle \mathrm{Ad}^*(g)\xi, X \rangle = \langle \xi, \mathrm{Ad}(g^{-1})X \rangle,$$

for all $g \in G$, $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. The corresponding infinitesimal action ad^* of $\mathfrak{g}$ on $\mathfrak{g}^*$ is given by

$$\langle \mathrm{ad}^*(X)\xi, Y \rangle = \langle \xi, -[X, Y] \rangle,$$

for all $X, Y \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$.

Remark 1.4 (Uniqueness of momentum maps)**.** If μ and ν are two momentum maps for the same action, then for all $X \in \mathfrak{g}$,

$$d(\mu_X - \nu_X) = 0.$$

Because M is supposed to be connected, this implies that the difference $\mu_X - \nu_X$ is a constant function, say c_X, on M. By definition of momentum maps, the constant c_X depends linearly on X. So there is a an element $\xi \in \mathfrak{g}^*$ such that

$$\mu - \nu = \xi.$$

By equivariance of momentum maps, the element ξ is fixed by the coadjoint action of G on $\mathfrak{g}^*$. In fact, given a momentum map the space of elements of $\mathfrak{g}^*$ that are fixed by the coadjoint action parametrises the set of all momentum maps for the given action.

An alternative definition of momentum maps can be given in terms of Hamiltonian vector fields and Poisson brackets:

[2]Sometimes an action is called 'strongly Hamiltonian' if it satisfies our definition of 'Hamiltonian'. The term 'Hamiltonian' is then used for actions that admit a momentum map which is not necessarily equivariant. Because we only consider equivariant momentum maps, we omit the word 'strongly'.

[3]Some authors prefer the term 'moment map'.

Definition 1.5. Let $f \in C^\infty(M)$. The *Hamiltonian vector field* $H_f \in \mathrm{Vect}(M)$ associated to f, is defined by the relation

$$df = -H_f \lrcorner \omega.$$

The *Poisson bracket* of two functions $f, g \in C^\infty(M)$ is defined as

$$\{f, g\} := \omega(H_f, H_g) = H_f(g) = -H_g(f) \in C^\infty(M).$$

The Poisson bracket is a Lie bracket on $C^\infty(M)$, and has the additional derivation property that for all $f, g, h \in C^\infty(M)$:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}. \tag{2}$$

The vector space $C^\infty(M)$, equipped with the Poisson bracket, is called the *Poisson algebra* $C^\infty(M, \omega)$ of (M, ω). A *Poisson manifold* is a manifold, together with a Lie bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ that satisfies (2). Poisson manifolds admit a canonical foliation whose leaves are symplectic submanifolds.

The property (1) can now be rephrased as

$$H_{\mu_X} = X. \tag{3}$$

It can be shown that a map μ satisfying (3) is equivariant if and only if it is an *anti-Poisson map*, which in this special case means that

$$\{\mu_X, \mu_Y\} = -\mu_{[X,Y]}.$$

Indeed, the implication 'equivariant $\Rightarrow$ anti-Poisson' can be proved as follows. If μ is equivariant, then for all $X, Y \in \mathfrak{g}$, $m \in M$,

$$\begin{aligned}
\{\mu_X, \mu_Y\}(m) &= X(\mu_Y)(m) \qquad \text{(by (3))} \\
&= \left.\frac{d}{dt}\right|_{t=0} \langle \mu(\exp tX \cdot m), Y\rangle \\
&= \left.\frac{d}{dt}\right|_{t=0} \langle \mathrm{Ad}^*(\exp tX)\mu(m), Y\rangle \\
&= \langle \mu(m), \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp -tX)Y\rangle \\
&= -\mu_{[X,Y]}(m).
\end{aligned}$$

□

1.2 Examples of Hamiltonian actions

Example 1.6 (Cotangent bundles)**.** Let N be a smooth manifold, and let $M := T^*N$ be its cotangent bundle, with projection map

$$\pi : T^*N \to N.$$

The *tautological 1-form* τ on M is defined by

$$\langle \tau_\eta, v \rangle = \langle \eta, T_\eta \pi(v) \rangle,$$

for $\eta \in T^*N$ and $v \in T_\eta M$. The one-form τ is called 'tautological' because for all 1-forms α on N, we have

$$\alpha^* \tau = \alpha.$$

Here on the left hand side, α is regarded as a map from N to M, along which the form τ is pulled back.

Let $q = (q_1, \ldots q_d)$ be local coordinates on an open neighbourhood of an element n of N. Consider the corresponding coordinates p on T^*N in the fibre direction, defined by

$$p_k = \frac{\partial}{\partial q_k}.$$

Then one has

$$\tau = \sum_k p_k \, dq_k.$$

The 2-form

$$\sigma := d\tau = \sum_k dp_k \wedge dq_k \tag{4}$$

is a symplectic form on M, called the *canonical symplectic form.*

Suppose G acts on N. The induced action of G on M,

$$g \cdot \eta := (T_n g^{-1})^* \eta,$$

for $g \in G$, $\eta \in T^*_n N$, is Hamiltonian, with momentum map

$$\mu_X = X \lrcorner \tau,$$

for all $X \in \mathfrak{g}$. Explicitly:

$$\mu_X(\eta) := \langle \eta, X_{\pi(\eta)} \rangle,$$

for $X \in \mathfrak{g}$ and $\eta \in T^*N$.

Proof. First note that the tautological 1-form is invariant under the action of G on M: for all $g \in G$, $\eta \in T^*N$ and $v \in T_\eta M$, we have

$$\begin{aligned}(g^*\tau)_\eta &= \tau_{g\cdot\eta} \circ T_\eta g \\ &= (g \cdot \eta) \circ T_{g\cdot\eta}\pi \circ T_\eta g \\ &= \eta \circ T_\eta (g^{-1} \circ \pi \circ g) \\ &= \eta \circ T_\eta \pi,\end{aligned}$$

because the projection map of the cotangent bundle is G-equivariant.

Hence $\mathcal{L}_X \tau = 0$ for all $X \in \mathfrak{g}$, which by Cartan's formula implies that

$$d(X \lrcorner \tau) + X \lrcorner d\tau = 0,$$

so that

$$d(X \lrcorner \tau) = -X \lrcorner \sigma.$$

This means that

$$\mu_X = X \lrcorner \tau$$

is the X-component of a momentum map. $\square$

Example 1.7 (Complex vector spaces)**.** Let M be the vector space $\mathbb{C}^n$, equipped with the Hermitian inner product

$$H(z, \zeta) = \langle z, \zeta \rangle := \sum_{k=1}^{n} z_k \bar{\zeta}_k,$$

for $z, \zeta \in \mathbb{C}^n$. Writing $B := \mathrm{Re}(H)$ and $\omega := \mathrm{Im}(H)$, we obtain

$$H = B + i\omega,$$

with B a Euclidean inner product, and ω a symplectic form on the vector space $\mathbb{C}^n$. By identifying each tangent space of M with $\mathbb{C}^n$, we can extend ω to a syplectic form on the manifold M. In the coordinates q and p on M, defined by

$$z_k = q_k + ip_k,$$

the form ω is equal to the form σ from Example 1.6, given by (4). (Note that $M \cong T^*\mathbb{R}^n$.)

Consider the natural action of the group

$$K := U_n(\mathbb{C})$$

on M. Let $\mathfrak{k}$ be the Lie algebra

$$\mathfrak{k} := \mathfrak{u}_n(\mathbb{C}) = \{X \in M_n(\mathbb{C}); X^* = -X\} \tag{5}$$

of K. A momentum map for the action of K on M is given by

$$\begin{aligned} &\mu : M \to \mathfrak{k}^* \\ &\mu_X(z) := iH(Xz, z)/2, \end{aligned} \tag{6}$$

where $X \in \mathfrak{k}$ and $z \in M$.

Proof. Equivariance of the map μ follows from the fact that K by definition preserves the metric H.

For all $X \in \mathfrak{k}$, $z \in M$ and $\zeta \in T_zM = \mathbb{C}^n$, the map μ given by (6) satisfies

$$\langle (d\mu_X)_z, \zeta \rangle = i\left(H(X\zeta, z) + H(Xz, \zeta)\right)/2. \tag{7}$$

Now $X \in \mathfrak{k}$, so by (5),

$$\begin{aligned} H(X\zeta, z) &= -H(\zeta, Xz) \\ &= -\overline{H(Xz, \zeta)}. \end{aligned}$$

Therefore, (7) equals

$$\begin{aligned} i\left(-\overline{H(Xz,\zeta)} + H(Xz,\zeta)\right)/2 &= i(2i\omega(Xz,\zeta))/2 \\ &= -\omega(Xz,\zeta). \end{aligned}$$

The vector field in M generated by an element $X \in \mathfrak{k}$ is given by

$$X_z = \left.\frac{d}{dt}\right|_{t=0} (e^{tX}z) = \left(\left.\frac{d}{dt}\right|_{t=0} e^{tX}\right) z = Xz,$$

for $z \in M$. We conclude that

$$(d\mu_X)_z = -X_z \lrcorner \omega_z$$

for all $X \in \mathfrak{k}$ and $z \in M$, which completes the proof. □

Example 1.8, as well as Example 1.13, is due to Kirillov in his 1962 thesis [13].

Example 1.8 (Restriction to subgroups)**.** Let $H < G$ be a Lie subgroup, with Lie algebra $\mathfrak{h}$. Let

$$i : \mathfrak{h} \hookrightarrow \mathfrak{g}$$

be the inclusion map, and let

$$p := i^* : \mathfrak{g}^* \to \mathfrak{h}^*$$

be the corresponding dual projection.

Suppose that G acts on M in a Hamiltonian way, with momentum map

$$\mu : M \to \mathfrak{g}^*.$$

Then the restricted action of H on M is also Hamiltonian. A momentum map is the composition

$$M \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{p} \mathfrak{h}^*.$$

Proof. Let $X \in \mathfrak{h} \hookrightarrow \mathfrak{g}$. Then

$$\begin{aligned} d\left((p \circ \mu)_X\right) &= d(\mu_{i(X)}) \\ &= -i(X) \lrcorner \omega \\ &= -X \lrcorner \omega. \end{aligned}$$

□

Remark 1.9. An interpretation of Example 1.8 is that the momentum map is functorial with respect to symmetry breaking. For example, consider a physical system of N particles in $\mathbb{R}^3$ (Example 1.15). If we add a function to the Hamiltonian which is invariant under orthogonal transformations, but not under translations, then the Hamiltonian is no longer invariant under the action of the Euclidean motion group G. However, it is still preserved by the subgroup $O(3)$ of G. In other words, the G-symmetry of the system is broken into an $O(3)$-symmetry. By Example 1.8, angular momentum still defines a momentum map, so that it is still a conserved quantity.

Example 1.10 (Unitary representations)**.** Example 1.7 may be generalised to arbitrary finite-dimensional unitary representations of compact Lie groups.

Let V be a finite-dimensional complex vector space, with a Hermitian inner product H. Consider the manifold $M := V$, with the symplectic form $\omega := \mathrm{Im}(H)$.

Let K be a compact Lie group, and let

$$\rho : K \to U(V, H)$$

be a unitary representation of K in V. This action of K on M is Hamiltonian, with momentum map

$$\mu_X(v) = iH(\rho(X)v, v)/2,$$

for $X \in \mathfrak{k}$ and $v \in V$.

Proof. Consider Example 1.7, and apply Example 1.8 to the subgroup $\rho(K)$ of $U(V, H)$. □

Example 1.11 (Invariant submanifolds)**.** Let (M, ω) be a symplectic manifold, equipped with a Hamiltonian action of G, with momentum map

$$\mu : M \to \mathfrak{g}^*.$$

Let $N \subset M$ be a submanifold, with inclusion map

$$i : N \hookrightarrow M.$$

Assume that the restricted form $i^*\omega$ is a symplectic form on N (i.e. that it is nondegenerate). Suppose that N is invariant under the action of G. Then the action of G on N is Hamiltonian. A momentum map is the composition

$$N \overset{i}{\hookrightarrow} M \overset{\mu}{\to} \mathfrak{g}^*.$$

Proof. Let $X \in \mathfrak{g}$. Then $(\mu \circ i)_X = \mu_X \circ i$, so

$$\begin{aligned} d(\mu \circ i)_X &= d(\mu_X \circ i) \\ &= d\mu_X \circ Ti, \end{aligned} \tag{8}$$

where $Ti : TN \hookrightarrow TM$ is the tangent map of i. By definition of μ, (8) equals

$$-(X \lrcorner \omega) \circ Ti = -X \lrcorner (i^*\omega).$$

□

The following example is a preparation for the study of a class of Hamiltonian actions on holomorphic submanifolds of projective space in Section 1.4.

Example 1.12 (Invariant submanifolds of complex vector spaces)**.** Let V be an n-dimensional complex vector space, with a Hermitian inner product H. Then as before, $\omega := \mathrm{Im}(H)$ is a symplectic form on V.

Write $V^\times := V \setminus \{0\}$. Let $L^\times$ be a holomorphic submanifold of $V^\times$ that is invariant under $\mathbb{C}^\times$.

Let K be a compact Lie group, and let

$$\rho : K \to U(V, H)$$

be a unitary representation of K in V. We consider the action of K on $V^\times$ defined by ρ. Assume that $L^\times$ is invariant under this action. Then the restricted action of K on $L^\times$ is Hamiltonian, with momentum map

$$\mu_X(l) := i(H(\rho(X)l, l)/2,$$

for $X \in \mathfrak{g}$, $l \in L^\times$.

Proof. Consider Example 1.10, and apply Example 1.11 to the invariant submanifold $L^\times$. □

Example 1.13 (Coadjoint orbits)**.** Let G be a connected Lie group. Fix an element $\xi \in \mathfrak{g}^*$. We define the bilinear form $\omega_\xi \in \mathrm{Hom}(\mathfrak{g}^{\otimes 2}, \mathbb{R})$ by

$$\omega_\xi(X, Y) := -\langle \xi, [X, Y] \rangle,$$

for all $X, Y \in \mathfrak{g}$. It is obviously antisymmetric, so it defines an element of $\mathrm{Hom}(\bigwedge^2 \mathfrak{g}, \mathbb{R})$.

Let G_ξ be the stabiliser group of ξ with respect to the coadjoint action:

$$G_\xi := \{g \in G; \mathrm{Ad}^*(g)\xi = \xi\}.$$

Let $\mathfrak{g}_\xi$ denote the Lie algebra of G_ξ:

$$\begin{aligned} \mathfrak{g}_\xi &= \{X \in \mathfrak{g}; \mathrm{ad}^*(X)\xi = \xi\} \\ &= \{X \in \mathfrak{g}; \omega_\xi(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}, \end{aligned} \tag{9}$$

by definition of ω_ξ. By (9), the form ω_ξ defines a symplectic form on the quotient $\mathfrak{g}/\mathfrak{g}_\xi$.

Let

$$M_\xi := G/G_\xi \cong G \cdot \xi$$

be the coadjoint orbit through ξ. Note that for all $g \in G$, we have a diffeomorphism

$$G/G_{g\xi} \cong G/G_\xi,$$

induced by

$$h \mapsto ghg^{-1}, \qquad G \to G.$$

Hence the definition of M_ξ does not depend on the choice of the element ξ in its coadjoint orbit.

The tangent space

$$T_\xi M_\xi \cong \mathfrak{g}/\mathfrak{g}_\xi$$

carries the symplectic form ω_ξ. This form can be extended G-invariantly to a symplectic form ω on the whole manifold M_ξ. The form ω is closed, because it is G-invariant, and G acts transitively on M_ξ. This symplectic form is called the *canonical symplectic form* on the coadjoint orbit M_ξ.[4]

The coadjoint action of G on M_ξ is Hamiltonian. A momentum map is the inclusion

$$\mu : G \cdot \xi \hookrightarrow \mathfrak{g}^*.$$

Proof. For all $X \in \mathfrak{g}$ and $g \in G$, we have

$$\mu_X(g \cdot \xi) = \langle g \cdot \xi, X \rangle.$$

Because the action of G on M_ξ is transitive, the tangent space at the point $g \cdot \xi \in M_\xi$ is spanned by the values $Y_{g\xi}$ of the vector fields induced by elements $Y \in \mathfrak{g}$. For all such Y, we have

$$\begin{aligned}
\langle (d\mu_X)_{g\xi}, Y_{g\xi} \rangle &= \left.\frac{d}{dt}\right|_{t=0} \mu_X\left(\exp(tY) \cdot (g \cdot \xi)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \langle g \cdot \xi, \mathrm{Ad}\left(\exp(-tY)\right) X \rangle \\
&= \langle g \cdot \xi, [X, Y] \rangle \\
&= -\omega_{g\cdot\xi}(X_{g\cdot\xi}, Y_{g\cdot\xi}).
\end{aligned}$$

Hence

$$(d\mu_X)_{g\cdot\xi} = -(X \lrcorner \omega)_{g\cdot\xi}.$$

□

The following example plays a role in Example 1.15, and in the 'shifting trick' (Remark 1.22).

Example 1.14 (Cartesian products)**.** Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. Suppose that there is a Hamiltonian action of a group G on both symplectic manifolds, with momentum maps μ_1 and μ_2. The Cartesian product manifold $M_1 \times M_2$ carries the symplectic form $\omega_1 \times \omega_2$, which is defined as

$$\omega_1 \times \omega_2 := \pi_1^*\omega_1 + \pi_2^*\omega_2,$$

where $\pi_i : M_1 \times M_2 \to M_i$ denotes the projection map.

Consider the diagonal action of G on $M_1 \times M_2$,

$$g \cdot (m_1, m_2) = (g \cdot m_1, g \cdot m_2),$$

[4] In terms of Poisson geometry, coadjoint orbits are the symplectic leaves of the Poisson manifold $\mathfrak{g}^*$ (see the remarks below Definition 1.5).

for $g \in G$ and $m_i \in M_i$. It is easy to show that it is Hamiltonian, with momentum map

$$\mu_1 \times \mu_2 : M_1 \times M_2 \to \mathfrak{g}^*,$$
$$(\mu_1 \times \mu_2)(m_1, m_2) = \mu_1(m_1) + \mu_2(m_2),$$

for $m_i \in M_i$.

Example 1.15 (N particles in $\mathbb{R}^3$)**.** To motivate the term 'momentum map', we give one example from classical mechanics. It is based on Example 1.6 about cotangent bundles, and Example 1.14 about Cartesian products.

Consider a physical system of N particles moving in $\mathbb{R}^3$. The corresponding phase space is the manifold

$$M := \left(T^*\mathbb{R}^3\right)^N \cong \mathbb{R}^{6N}.$$

Let (q^i, p^i) be the coordinates on the ith copy of $T^*\mathbb{R}^3 \cong \mathbb{R}^6$ in M. We will write

$$q^i = (q^i_1, q^i_2, q^i_3),$$
$$p^i = (p^i_1, p^i_2, p^i_3),$$

and

$$(q, p) = \big((q^1, p^1), \dots, (q^N, p^N)\big) \in M.$$

Using Examples 1.6 and 1.14, we equip the manifold M with the symplectic form

$$\omega := \sum_{i=1}^N dp^i_1 \wedge dq^i_1 + dp^i_2 \wedge dq^i_2 + dp^i_3 \wedge dq^i_3.$$

The *Hamiltonian* of the system is a function h on M, which assigns to each point in the phase space the total energy of the system in that state. It determines the dynamics of the system as follows. A physical observable is a smooth function f on M. This function determines a smooth function F on $M \times \mathbb{R}$, by

$$\frac{\partial F}{\partial t}(m, t) = \{h, F(\cdot, t)\}(m) \tag{10}$$

and

$$F(m, 0) = f(m)$$

for all $m \in M$ and $t \in \mathbb{R}$. The value $F(m, t)$ of F is interpreted as the value of the observable f after time t, if the system was in state m at time 0. The condition (10) is often written as

$$\dot{f} = \{h, f\}.$$

Let G be the Euclidean motion group of $\mathbb{R}^3$:

$$G := \mathbb{R}^3 \rtimes O(3),$$

whose elements are

$$G := \{(v, A); v \in \mathbb{R}^3, A \in O(3)\},$$

with multiplication defined by

$$(v, A)(w, B) = (v + Aw, AB),$$

for all elements (v, A) and (w, B) of G. It acts on $\mathbb{R}^3$ by

$$(v, A) \cdot x = Ax + v,$$

for $(v, A) \in G$, $x \in \mathbb{R}^3$.

Consider the induced action of G on M. The physically relevant actions are those that preserve the Hamiltonian. In this example, if the Hamiltonian is preserved by G then the dynamics does not depend on the position or the orientation of the N particle system as a whole. In other words, no external forces act on the system.

By Examples 1.6 and 1.14, the action of G on M is Hamiltonian. As we will show below, the momentum map can be written in the form

$$\mu(q, p) = \sum_{i=1}^{N} (p^i, q^i \times p^i) \in \left(\mathbb{R}^3\right)^* \rtimes \mathfrak{o}(3)^* = \mathfrak{g}^*.$$

Note that the Lie algebra $\mathfrak{o}(3)$ is isomorphic to $\mathbb{R}^3$, equipped with the exterior product $\times$. We identify $\mathbb{R}^3$ with its dual (and hence with $\mathfrak{o}(3)^*$) via the standard inner product.

The quantity

$$\sum_{i=1}^{N} p^i$$

is the total linear momentum of the system, and

$$\sum_{i=1}^{N} q^i \times p^i$$

is the total angular momentum. By Noether's theorem, the momentum map is time-independent if the group action preserves the Hamiltonian. This formulation of Noether's theorem has an easy proof:

$$\begin{aligned} \dot{\mu}_X &= \{h, \mu_X\} \\ &= -H_{\mu_X}(h) \\ &= -X(h) \\ &= 0, \end{aligned}$$

by (3). In this example, this implies that the total linear and angular momentum of the system are conserved quantities.

Proof that μ is the momentum map from Examples 1.6 and 1.14. Let

$$\nu : M \to \mathfrak{g}^*$$

be the momentum map obtained using Examples 1.6 and 1.14. That is,

$$\nu_X(q,p) = \sum_{i=1}^{N} p^i \cdot X_{q^i},$$

for all $X \in \mathfrak{g}$ and $(q,p) \in M$. Here $\cdot$ denotes the standard inner product on $\mathbb{R}^3$, by which we identify T^*M with TM.

Let $Y, Z \in \mathbb{R}^3$, and consider the element

$$X = (Y, Z) \in \mathbb{R}^3 \times \mathbb{R}^3 \cong \mathfrak{g}.$$

Then

$$X_{q^i} = Y + Z \times q^i \in \mathbb{R}^3 \cong T_{q_i}\mathbb{R}^3 \hookrightarrow T_{q^i}M.$$

Therefore,

$$\begin{aligned}\mu_X(q,p) &= \sum_{i=1}^{N} (p^i, q^i \times p^i) \cdot (Y, Z) \\ &= \sum_{i=1}^{N} p^i \cdot Y + (q^i \times p^i) \cdot Z.\end{aligned}$$

Using the fact that

$$(q^i \times p^i) \cdot Z = p^i \cdot (Z \times q^i),$$

we conclude that

$$\begin{aligned}\mu_X(q,p) &= \sum_{i=1}^{N} p^i \cdot Y + p^i \cdot (Z \times q^i) \\ &= \sum_{i=1}^{N} p^i \cdot X_{q^i} \\ &= \nu_X(q,p).\end{aligned}$$

□

1.3 Symplectic reduction

Suppose (M, ω) is a connected symplectic manifold, equipped with a Hamiltonian G-action, with momentum map

$$\mu : M \to \mathfrak{g}^*.$$

Definition 1.16. A point $m \in M$ is called a *regular point* of μ if the tangent map

$$T_m\mu : T_mM \to \mathfrak{g}^*$$

is surjective. The set of regular points of μ is denoted by M_{reg}. We will write μ_{reg} for the restriction of μ to M_{reg}.

An element $\xi \in \mathfrak{g}^*$ is called a *regular value* of μ if all points in the inverse image $\mu^{-1}(\xi)$ are regular.

By the implicit function theorem, the subset

$$\mu_{\mathrm{reg}}^{-1}(\xi) \subset M$$

is a smooth submanifold, for all $\xi \in \mathfrak{g}^*$. Because the map μ is equivariant, the submanifold $\mu_{\mathrm{reg}}^{-1}(\xi)$ is invariant under the action of the stabiliser group G_ξ of ξ.

Proposition 1.17. *The defining relation*

$$d\mu_X = -X \lrcorner \omega,$$

for all $X \in \mathfrak{g}$, has the following consequences.

1. *A point $m \in M$ is a regular point of μ if and only if the Lie algebra of the stabiliser group G_m is zero. Hence the action of G_ξ on $\mu_{\mathrm{reg}}^{-1}(\xi)$ is locally free.*

2. *For all $\xi \in \mathfrak{g}^*$ and $m \in \mu_{\mathrm{reg}}^{-1}(\xi)$, the tangent space $T_m\left(\mu_{\mathrm{reg}}^{-1}(\xi)\right)$ is equal to the orthogonal complement with respect to ω_m of the space*

$$\{X_m; X \in \mathfrak{g}\} = T_m(G \cdot m).$$

3. *The kernel of ω_m on the space $T_m\left(\mu_{\mathrm{reg}}^{-1}(\xi)\right)$ equals*

$$T_m\left(\mu_{\mathrm{reg}}^{-1}(\xi)\right) \cap \left(T_m\left(\mu_{\mathrm{reg}}^{-1}(\xi)\right)\right)^{\perp} = \{X_m \in T_mM; X \in \mathfrak{g}_\xi\}.$$

   As before, $\mathfrak{g}_\xi$ denotes the stabiliser of $\xi \in \mathfrak{g}$, i.e. the subalgebra of elements $X \in \mathfrak{g}$ such that $\mathrm{ad}^*(X)\xi = \xi$.

Proof. 1. Let $m \in M$ be given. We claim that m is a regular point of μ if and only if the linear map

$$\varphi : \mathfrak{g} \to T_mM$$
$$X \mapsto X_m$$

is injective. Because the Lie algebra of the stabiliser group G_m equals

$$\mathrm{Lie}(G_m) = \ker(\varphi),$$

it equals zero precisely if φ is injective.

Let $X \in \mathfrak{g}$. We will show that the implication

$$X_m = 0 \quad \Longrightarrow \quad X = 0 \tag{11}$$

holds if and only if the tangent map $T_m\mu$ is surjective. Indeed, because the form ω is nondegenerate, we have

$$X_m = 0$$

if and only if

$$\omega_m(X_m, v) = 0$$

for all $v \in T_mM$. By the defining property of μ, this is equivalent to

$$(d\mu_X)_m = 0.$$

Consider the linear map

$$i_X : \mathfrak{g}^* \to \mathbb{R},$$

defined by pairing with X:

$$i_X(\eta) := \langle \eta, X \rangle.$$

Being a linear map, i_X is its own tangent map. Therefore,

$$\begin{aligned}(d\mu_X)_m &= (d(i_X \circ \mu))_m \\ &= i_X \circ T_m\mu.\end{aligned}$$

Hence $(d\mu_X)_m = 0$ if and only if i_X is zero on the image of $T_m\mu$. Therefore, the implication (11) holds for all $X \in \mathfrak{g}$ if and only if $T_m\mu$ is surjective.

2. Note that

$$\begin{aligned}T_m\left(\mu_{\mathrm{reg}}^{-1}(\xi)\right) &= \ker\left(T_m\mu_{\mathrm{reg}} : T_mM \to \mathfrak{g}^*\right) \\ &= \{v \in T_mM; \langle (d\mu_X)_m, v\rangle = 0, \ \forall X \in \mathfrak{g}\} \\ &= \{v \in T_mM; \omega_m(X_m, v) = 0, \ \forall X \in \mathfrak{g}\},\end{aligned}$$

by the defining property of μ.

3. Let v be an element of the intersection

$$T_m\left(\mu_{\mathrm{reg}}^{-1}(\xi)\right) \cap T_m\left(\mu_{\mathrm{reg}}^{-1}(\xi)\right). \tag{12}$$

Then by the second part of the proposition, there is an $X \in \mathfrak{g}$, such that

$$v = X_m,$$

and

$$T_m\mu_{\mathrm{reg}}(X_m) = 0.$$

Because μ is equivariant, we have

$$\begin{aligned}T_m\mu_{\mathrm{reg}}(X_m) &= \left.\frac{d}{dt}\right|_{t=0} \mu_{\mathrm{reg}}\left(\exp(tX)\cdot m\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}^*(\exp(tX))\xi \\ &= X_\xi.\end{aligned}$$

Hence v is in (12) if and only if there is a $X \in \mathfrak{g}$, for which

$$v = X_m$$

and

$$X_\xi = 0.$$

The stabiliser of ξ in $\mathfrak{g}$ equals

$$\mathfrak{g}_\xi = \{X \in \mathfrak{g} | X_\xi = 0\},$$

which completes the argument. □

Corollary 1.18. *Let $\xi \in \mathfrak{g}^*$, and consider the inclusion map*

$$i : \mu^{-1}_{\mathrm{reg}}(\xi) \hookrightarrow M.$$

Then the leaves of the null foliation of the form $i^*\omega$ are the orbits of the connected group G^0_ξ on $\mu^{-1}_{\mathrm{reg}}(\xi)$.

Proof. The tangent space to a G^0_ξ-orbit at a point m is equal to

$$T_m\left(G^0_\xi \cdot m\right) = \{X_m \in T_mM; X \in \mathrm{Lie}(G^0_\xi) = \mathfrak{g}_\xi\}.$$

By the third part of Proposition 1.17, this implies the statement. □

Theorem 1.19 (Marsden & Weinstein 1974 [20])**.** *Suppose (M, ω) is a connected symplectic manifold, equipped with a Hamiltonian action of G. Let $\xi \in \mathfrak{g}^*$ be a regular value of the momentum map $\mu : M \to \mathfrak{g}^*$. Suppose that the stabiliser G_ξ of ξ acts properly and freely on the submanifold $\mu^{-1}(\xi)$ of M, so that the orbit space*

$$\boxed{M^\xi := \mu^{-1}(\xi)/G_\xi}$$

is a smooth manifold.

Then there is a unique symplectic form ω^ξ on M^ξ such that

$$\boxed{p^*\omega^\xi = i^*\omega.}$$

Here p and i are the quotient and inclusion maps

$$\begin{array}{ccc} \mu^{-1}(\xi) & \overset{i}{\hookrightarrow} & M \\ \downarrow{\scriptstyle p} & & \\ \mu^{-1}(\xi)/G_\xi = M^\xi. & & \end{array}$$

(Note that p defines a principal G_ξ-bundle.)

Definition 1.20. The symplectic manifold (M^ξ, ω^ξ) from Theorem 1.19 is called the *symplectic reduction* or *reduced phase space* of the Hamiltonian action of G on M, at the regular value ξ of μ. If we do not specify at which value we take the symplectic reduction, we will mean the symplectic reduction at $0 \in \mathfrak{g}^*$. Because the stabiliser of 0 is the whole group G, we have

$$M^0 = \mu^{-1}(0)/G.$$

Remark 1.21. Suppose that $\xi \in \mathfrak{g}^*$ is not necessarily a regular value. Assume that $\mu_{\mathrm{reg}}^{-1}(\xi)$ is dense and $\mu^{-1}(\xi)$, and that the action of G on $\mu^{-1}(\xi)$ is proper, and free on $\mu_{\mathrm{reg}}^{-1}(\xi)$. Then we have the diagram

$$\begin{array}{ccccc} \mu_{\mathrm{reg}}^{-1}(\xi) & \hookrightarrow & \mu^{-1}(\xi) & \hookrightarrow & M \\ \downarrow & & \downarrow & & \\ M^\xi_{\mathrm{reg}} := \mu_{\mathrm{reg}}^{-1}(\xi)/G_\xi & \hookrightarrow & M^\xi. & & \end{array}$$

Here M^ξ_{reg} is a smooth symplectic manifold, equipped with the symplectic form ω^ξ. The orbit space M^ξ is a Hausdorff topological space. If the momentum map μ is proper, then $\mu^{-1}(\xi)$ and hence M^ξ is compact. Because M_{reg} is dense in M, this implies that M^ξ is a compactification of M^ξ_{reg}.

If we do not assume that G_ξ acts freely on $\mu_{\mathrm{reg}}^{-1}(\xi)$, it still acts locally freely by the first part of Proposition 1.17. If the component group G_ξ/G_ξ^0 is finite, then G_ξ acts on $\mu_{\mathrm{reg}}^{-1}(\xi)$ with finite stabilisers, so that M^ξ_{reg} is an *orbifold*. Equipped with the symplectic form ω^ξ, it is a symplectic orbifold. The orbifold singularities of M^ξ are relatively mild. The worst singular behaviour occurs at $M^\xi \setminus M^\xi_{\mathrm{reg}}$.

Remark 1.22 (The shifting trick)**.** The symplectic reduction of a Hamiltonian group action of G on (M, ω) at any regular value $\xi \in \mathfrak{g}^*$ of the momentum map can be obtained as a symplectic reduction at 0 of a certain symplectic manifold containing M by an action of G.

Indeed, let $M_\xi := G/G_\xi = G \cdot \xi$ be the coadjoint orbit of G through ξ (see Example 1.13). There is a diffeomorphism

$$M^\xi = \mu^{-1}(\xi)/G_\xi \cong \mu^{-1}(M_\xi)/G,$$

induced by the inclusion

$$\mu^{-1}(\xi) \hookrightarrow \mu^{-1}(M_\xi).$$

Next, consider the two symplectic manifolds

$$\begin{aligned} (M_1, \omega_1) &:= (M_{-\xi}, \omega_{-\xi}) \\ (M_2, \omega_2) &:= (M, \omega). \end{aligned}$$

On these symplectic manifolds, we have Hamiltonian G-actions, with momentum maps

$$\begin{aligned} \mu_1 : M_1 = G \cdot (-\xi) &\hookrightarrow \mathfrak{g}^* \\ \mu_2 = \mu : M_2 = M &\to \mathfrak{g}^*. \end{aligned}$$

Consider the Hamiltonian action of G on the Cartesian product $(M_1 \times M_2, \omega_1 \times \omega_2)$ (see Example 1.14). As we saw, a momentum map for this action is

$$\mu_1 \times \mu_2 : M_1 \times M_2 \to \mathfrak{g}^*,$$
$$(\mu_1 \times \mu_2)(m_1, m_2) := \mu_1(m_1) + \mu_2(m_2),$$

for $m_j \in M_j$. The symplectic reduction of the action of G on $M_1 \times M_2$ at the value 0 is equal to the symplectic reduction of M at ξ:

$$\begin{aligned}(\mu_1 \times \mu_2)^{-1}(0)/G &= \{(g \cdot (-\xi), m) \in M_{-\xi} \times M; g \cdot (-\xi) + \mu(m) = 0\}/G \\ &= \mu^{-1}(M_\xi)/G \\ &\cong \mu^{-1}(\xi)/G_\xi \\ &= M^\xi.\end{aligned}$$

This exhibits M^ξ as the symplectic reduction at zero of a Hamiltonian action.

In the situation of Example 1.6 about cotangent bundles, taking the symplectic reduction is a very natural procedure:

Theorem 1.23. *Consider Example 1.6. Suppose that the action of G on N is proper and free. Let $T^*(N/G)$ be the cotangent bundle of the (smooth) quotient N/G, equipped with the canonical symplectic form $\sigma_G = d\tau_G$. The symplectic reduction of (T^*N, σ) by the action of G is symplectomorphic to $(T^*(N/G), \sigma_G)$:*

$$\left((T^*N)^0, \sigma^0\right) \cong (T^*(N/G), \sigma_G).$$

Proof. The inverse image of $0 \in \mathfrak{g}^*$ of the momentum map

$$\mu : T^*N \to \mathfrak{g}^*$$

is

$$\mu^{-1}(0) = \{\eta \in T^*N; \text{ for all } X \in \mathfrak{g}, \langle \eta, X_{\pi(\eta)} \rangle = 0\}. \tag{13}$$

Because the action of G on N is free, $\mu^{-1}(0)$ is a sub-bundle of T^*N. And again because the action is free, the quotient $\mu^{-1}(0)/G$ defines a vector bundle

$$\mu^{-1}(0)/G \to N/G.$$

We claim that this vector bundle is isomorphic to the cotangent bundle

$$T^*(N/G) \xrightarrow{\pi_G} N/G.$$

Let

$$q : N \to N/G$$

be the quotient map. The transpose of the tangent map of q is a vector bundle homomorphism

$$Tq^* : T^*(N/G) \to T^*N.$$

It follows from (13) that the image of Tq^* is contained in $\mu^{-1}(0)$. Thus we obtain the diagram

$$\begin{array}{ccccc} T^*(N/G) & \xrightarrow{Tq^*} & \mu^{-1}(0) & \xhookrightarrow{i} & T^*N \\ & & \downarrow{\scriptstyle p} & & \\ & & \mu^{-1}(0)/G. & & \end{array}$$

We claim that the composition $p \circ Tq^*$ is the desired isomorphism. Indeed, it is obviously a homomorphism of vector bundles. And it is injective: if $\xi \in T^*(N/G)$ and

$$p\left(Tq^*\xi\right) = 0,$$

then there is a $g \in G$ such that

$$g \cdot Tq^*\xi = 0.$$

But g is invertible (it is actually the identity element if the action is free), so $Tq^*\xi = 0$. And Tq^* is injective, so $\xi = 0$.

The rank of the vector bundle $T^*(N/G)$ equals

$$\operatorname{rank} T^*(N/G) = \dim N - \dim G.$$

The rank of $\mu^{-1}(0)/G \to N/G$ equals the rank of $\mu^{-1}(0) \to N$, which is

$$\operatorname{rank} \mu^{-1}(0) = \dim N - \dim G = \operatorname{rank} T^*(N/G).$$

Hence $p \circ Tq^*$ is a homomorphism of vector bundles which is a fibre-wise isomorphism. So the bundles $T^*(N/G)$ and $\mu^{-1}(0)/G$ over N/G are isomorphic. In particular, the manifolds $T^*(N/G)$ and $\mu^{-1}(0)/G$ are diffeomorphic.

It remains to prove that

$$\left(p \circ Tq^*\right)^* \sigma^0 = \sigma_G. \tag{14}$$

By definition of σ^0, we have

$$p^*\sigma^0 = i^*\sigma.$$

So (14) is equivalent with

$$\left(i \circ Tq^*\right)^* \sigma = \sigma_G.$$

Now $\sigma_G = d\tau_G$, and

$$\begin{aligned} \left(i \circ Tq^*\right)^* \sigma &= \left(i \circ Tq^*\right)^* d\tau \\ &= d\left(\left(i \circ Tq^*\right)^* \tau\right). \end{aligned}$$

We shall prove that

$$\left(i \circ Tq^*\right)^* \tau = \tau_G,$$

which implies (14) by the above argument.

Let $\xi \in T^*(N/G)$, and $v \in T_\xi\left(T^*(N/G)\right)$. Then

$$\begin{aligned}
\left\langle \left((i \circ Tq^*)^* \tau\right)_\xi, v \right\rangle &= \langle \tau_{Tq^*(\xi)}, T_\xi(Tq^*)v\rangle \\
&= \langle Tq^*(\xi), T_{Tq^*(\xi)}\pi\left(T_\xi(Tq^*)v\right)\rangle \\
&= \langle \xi, Tq\left(T_\xi(\pi \circ Tq^*)v\right)\rangle \\
&= \langle \xi, T(q \circ \pi \circ Tq^*)v\rangle \\
&= \langle \xi, T_\xi \pi_G v\rangle \\
&= \langle (\tau_G)_\xi, v\rangle,
\end{aligned}$$

because

$$\pi_G = q \circ \pi \circ Tq^*.$$

□

Remark 1.24. In Theorem 1.23 it is assumed that the action of G on N is proper and free. If the action is only proper, consider the open subset

$$\tilde{N} \subset N$$

that consists of the points in N with trivial stabilisers in G. The action of G on $\tilde{N}$ is still proper, so that Theorem 1.23 applies:

$$\left(T^*\tilde{N}\right)^0 \cong T^*(\tilde{N}/G).$$

Let $\widetilde{T^*N}$ be the open subset of T^*N of points with trivial stabilisers in G. Then

$$T^*\tilde{N} \subset \widetilde{T^*N} \subset T^*N. \tag{15}$$

Example 1.25 (N particles in $\mathbb{R}^3$ revisited)**.** In Example 1.15, we considered a classical mechanical system of N particles moving in $\mathbb{R}^3$. We will now describe the symplectic reduction of the phase space $M = \left(T^*\mathbb{R}^3\right)^N$ of this system by the action of the Euclidean motion group $G = \mathbb{R}^3 \rtimes O(3)$.

First, consider the action on M of the translation subgroup $\mathbb{R}^3$ of G. By Example 1.8, the total linear momentum of the system defines a momentum map for this action. By Theorem 1.23,[5] the reduced phase space for this restricted action is

$$M^0 = \left(T^*\mathbb{R}^{3N}\right)^0 = T^*(\mathbb{R}^{3N}/\mathbb{R}^3).$$

Let V be the $(3N-3)$-dimensional vector space $\mathbb{R}^{3N}/\mathbb{R}^3$. As coordinates on V, one can take the difference vectors

$$q^i - q^j, \qquad i < j.$$

[5]The action is not proper, but it is free, and the quotient space is smooth. Theorem 1.23 actually applies in this slightly more general setting.

These coordinates satisfy the relations

$$(q^i - q^j) + (q^j - q^k) = q^i - q^k, \qquad i < j < k.$$

Other possible coordinates are

$$\bar{q}^i := q^i - \sum_{j=1}^{N} c_j q^j, \qquad i = 1, \ldots N,$$

for any set of coefficients $\{c_j\}$ with sum 1. The coordinates then satisfy the single relation

$$\sum_{i=1}^{N} c_i \bar{q}^i = 0.$$

A physically natural choice for the c_j is

$$c_j := \frac{m_j}{\sum_{k=1}^{N} m_k},$$

where m_j is the mass of particle j. The coordinates $\bar{q}^i$ are then related by

$$\sum_{i=1}^{n} m_i \bar{q}^i = 0.$$

Thus, the reduced phase space may be interpreted as the space of states of the N particle system in which the centre of mass is at rest in the origin.

Next, we consider the action on M of the connected component $G^0 = \mathbb{R}^3 \rtimes SO(3)$ of G. The action of G^0 on M is not free, and the momentum map μ has singular points. Indeed, a point $m = (q, p)$ in M is singular if and only if its stabiliser group G^0_m has dimension at least 1 (see the first part of Proposition 1.17). Note that G^0_m is the group of elements $(v, A) \in \mathbb{R}^3 \rtimes SO(3)$ such that

$$\begin{aligned} Aq + v &= q && (16) \\ Ap &= p. && (17) \end{aligned}$$

The group of elements (v, A) for which condition (16) holds is at least one-dimensional if and only if the vectors

$$q^1, \ldots, q^N \in \mathbb{R}^3$$

are *collinear*, i.e. they lie on a single line l in $\mathbb{R}^3$. This line does not have to pass through the origin, and some points q^i may coincide. Given such a q, the group of $A \in SO(3)$ for which (17) holds is at least one-dimensional if and only if all vectors p^i lie on the line through the origin, parallel to l. In other words, the set M_{sing} of singular points in M is the set of $(q, p) \in \mathbb{R}^{3N} \times \mathbb{R}^{3N}$, for which there exist vectors $b, v \in \mathbb{R}^3$, and sets of coefficients $\{\alpha^i\}$ and $\{\beta^i\}$, such that

$$q^i = b + \alpha^i v \qquad (18)$$

$$p^i = \beta^i v. \qquad (19)$$

The set of singular values of μ in $\mathfrak{g}^*$ is the image $\mu(M_{\text{sing}})$. By (18) and (19), this set equals

$$\begin{aligned}\mu(M_{\text{sing}}) &= \left\{\left(\Big(\sum_{i=1}^{N}\beta^i\Big)v, \Big(\sum_{i=1}^{N}\beta^i\Big)b \times v\right); \beta^i \in \mathbb{R}, b, v \in \mathbb{R}^3\right\} \\ &= \{(P, L) \in \mathbb{R}^3 \times \mathbb{R}^3; P \perp L\}.\end{aligned}$$

In particular, $(0, 0)$ is a singular value.

We now turn to the action of the whole group G. A point $m = (q, p)$ has nontrivial stabiliser in G if and only if the q^i are *coplanar* (i.e. they lie in a plane W in $\mathbb{R}^3$) and the p^i lie in the plane W_0 through the origin, parallel to W. Indeed, if $R \in O(3)$ is reflection in W_0, then

$$(q1 - Rq^1, R) \in G_m.$$

Note that q^1 may be replaced by any point in W, and that $R \notin SO(3)$.

Referring to Remark 1.24, we define $\widetilde{M} \subset M$ to be the open dense submanifold of points with trivial stabiliser in G. That is, $\widetilde{M}$ is the set of points $(q, p) \in M$ such that there is no plane W_0 in $\mathbb{R}^3$ containing the origin, and no vector $b \in \mathbb{R}^3$, such that for all i,

$$\begin{aligned} q^i &\in W := W_0 + b \\ p^i &\in W_0. \end{aligned}$$

The set of points q in $\mathbb{R}^{3N}$ on which G acts with trivial stabiliser is the set

$$\widetilde{\mathbb{R}}^{3N} = \{q \in \mathbb{R}^{3N}; \text{There is no plane in } \mathbb{R}^3 \text{ containing all } q^i.\}$$

So in this example, (15) becomes

$$T^*\widetilde{\mathbb{R}}^{3N} \subsetneq \widetilde{T^*\mathbb{R}^{3N}} \subsetneq T^*\mathbb{R}^{3N}.$$

Now by Theorem 1.23,

$$\begin{aligned}\left(T^*\widetilde{\mathbb{R}}^{3N}\right)^0 &= T^*\left(\widetilde{\mathbb{R}}^{3N}/G\right) \\ &= T^*\left(\widetilde{V}/O(3)\right).\end{aligned}$$

Here $\widetilde{V} := \widetilde{\mathbb{R}}^{3N}/\mathbb{R}^3$. The coordinates $q^i - q^j$ that we used on V reduce to the coordinates

$$\|q^i - q^j\|^2, \qquad i < j. \tag{20}$$

on $\widetilde{V}/O(3)$. There are $\binom{N}{2}$ of these coordinates, and the dimension of $\widetilde{V}/O(3)$ is $3N-6$. So if $N \geq 5$ (so that $\binom{N}{2} > 3N - 6$), then there are relations between the coordinates (20). In any case, it is clear that a function (observable) on the reduced phase space $\left(T^*\widetilde{\mathbb{R}}^{3N}\right)^0$ corresponds to a function on $T^*\widetilde{\mathbb{R}}^{3N}$ that only depends on the relative distances between the particles.

1.4 Smooth projective varieties

By using symplectic reduction, we will now show that a certain class of group actions on submanifolds of projective space are Hamiltonian. We prepared for this in Example 1.12.

Let V be a finite-dimensional complex vector space, with a Hermitian form $H = B + i\omega$. Consider the natural action of the unitary group $U(V, H)$ on V. As we saw in Example 1.7, this action is Hamiltonian, with momentum map $\mu : V \to \mathfrak{u}(V, H)^*$ given by

$$\mu_X(v) = iH(Xv, v)/2, \tag{21}$$

for $X \in \mathfrak{u}(V, H)$ and $v \in V$.

Proposition 1.26. *Consider the action of the circle group $U(1)$ on V defined by scalar multiplication. We consider the circle group $U(1)$ as a subgroup of the unitary group $U(V, H)$ by noting that*

$$U(1) \cong \{zI_V \in U(V, H); z \in \mathbb{C}, |z| = 1\}.$$

The Lie algebra of $U(1)$ is $\mathfrak{u}(1) = 2\pi i\mathbb{R}$. Let $1^* \in \mathfrak{u}(1)^*$ be the element defined by

$$\langle 1^*, 2\pi i\rangle = 1.$$

Then the symplectic reduction of V by $U(1)$ at the regular value -1^* of μ yields the projective space $\mathbb{P}(V)$, with symplectic form equal the the imaginary part of the Fubini-Study metric on $\mathbb{P}(V)$.

Note that we consider the symplectic reduction at the value -1^* instead of the symplectic reduction at 1^*, because the latter is empty.

Proof. By Examples 1.7 and 1.8, the action of $U(1)$ on V is Hamiltonian, with momentum map given by (21), for $X \in \mathfrak{u}(1) = 2\pi i\mathbb{R}$.

Note that

$$\begin{aligned}\mu^{-1}(-1^*) &= \{v \in V; \mu_{2\pi i}(v) = 1\} \\ &= \{v \in V; -\pi H(v, v) = -1\} \\ &= \{v \in V; \|v\|^2 = \frac{1}{\pi}\}.\end{aligned} \tag{22}$$

Here $\|\cdot\|$ denotes the norm on V coming from H. We conclude that the symplectic reduction of V by $U(1)$ at -1^* equals the space of $U(1)$-orbits in the sphere (22), which in turn equals $\mathbb{P}(V)$. □

Definition 1.27. The *standard symplectic form* on $\mathbb{P}(V)$ is *minus* the symplectic form obtained using Proposition 1.26. It is denoted by ω.

Proposition 1.28. *The standard symplectic form on* $\mathbb{P}(V)$ *has the property that*

$$\int_{\mathbb{P}(L)} \omega = 1$$

for any two-dimensional complex linear subspace L *of* V*, so that* $[\omega]$ *is a generator of the integral cohomology group* $H^2(\mathbb{P}(V);\mathbb{Z})$*.*

Symplectic forms whose integral over every two-dimensional submanifold is an integer play an important role in prequantisation (see Theorem 3.3 and Section 3.2).

Proof. We may assume that V is two-dimensional, and $L = V$.

Let α_n be the $(n-1)$-dimensional volume of the unit $(n-1)$-sphere in $\mathbb{R}^n$. It is given by

$$\alpha_n = \frac{2(\sqrt{\pi})^n}{\Gamma(\frac{n}{2})}.$$

In particular,

$$\begin{aligned} \alpha_2 &= 2\pi \\ \alpha_4 &= 2\pi^2. \end{aligned}$$

As we saw in (22), $\mu^{-1}(-1^*)$ is the 3-sphere in $\mathbb{C}^2$ of radius $\frac{1}{\sqrt{\pi}}$. The volume of a sphere of radius r does not depend on the metric used to define the sphere, as long as one uses the volume form associated to the metric. Therefore, the Euclidean volume of $\mu^{-1}(-1^*)$ is

$$\mathrm{vol}_3\left(\mu^{-1}(-1^*)\right) = \alpha_4 \left(\frac{1}{\sqrt{\pi}}\right)^3 = \frac{2\pi^2}{\left(\sqrt{\pi}\right)^3}.$$

The orbits of the action of $U(1)$ on $\mu^{-1}(-1^*)$ are great circles, so their 1-dimensional volume is

$$\mathrm{vol}_1(U(1)\text{-orbit}) = \alpha_2 \frac{1}{\sqrt{\pi}} = \frac{2\pi}{\sqrt{\pi}}.$$

Hence

$$\mathrm{vol}_2\left(\mu^{-1}(-1^*)/U(1)\right) = \frac{2\pi^2/(\sqrt{\pi})^3}{2\pi/\sqrt{\pi}} = 1.$$

Using the fact that the volume form defined by the Riemannian part $B = \mathrm{Re}(H)$ of H is equal to the (Liouville) volume form defined by its symplectic part $\omega = \mathrm{Im}(H)$, one can show that the volume of $\mathbb{P}(L)$ with respect to ω equals the Euclidean volume of $\mu^{-1}(-1^*)/U(1)$. □

To study actions on $\mathbb{P}(V)$ induced by unitary representations in V, we consider the following situation. Let G be a Lie group, and let $H, \tilde{H} < G$ be closed subgroups. Suppose G acts properly on a symplectic manifold (M, ω), and assume that the action is Hamiltonian, with momentum map $\mu : M \to \mathfrak{g}^*$. Suppose that the restricted actions of H and $\tilde{H}$ on M commute.

By Example 1.8, the action of $\tilde{H}$ on M is Hamiltonian, with momentum map

$$\tilde{\mu} : M \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{\tilde{i}^*} \tilde{\mathfrak{h}}^*,$$

where $\tilde{i} : \tilde{\mathfrak{h}} \hookrightarrow \mathfrak{g}$ is the inclusion map. Consider the symplectic reduction $\tilde{M}^\xi$ of the action of $\tilde{H}$ on M, at the regular value $\xi \in \tilde{\mathfrak{h}}^*$ of $\tilde{\mu}$. Because the actions of H and $\tilde{H}$ on M commute, the action of H on M induces an action of H on the symplectic reduction $\tilde{M}^\xi$.

Proposition 1.29 (Commuting actions)**.** *The action of H on $\tilde{M}^\xi$ is Hamiltonian, with momentum map $\mu' : \tilde{M}^\xi \to \mathfrak{h}^*$ induced by the composition*

$$\tilde{\mu}^{-1}(\xi) \overset{\iota}{\hookrightarrow} M \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{i^*} \mathfrak{h}^*,$$

where $\iota : \tilde{\mu}^{-1}(\xi) \to M$ and $i : \mathfrak{h} \to \mathfrak{g}$ are the inclusion maps.

Proof. We may assume that $H = G$, for otherwise we can apply Example 1.8.

Our claim is that for all $X \in \mathfrak{g}$,

$$d(\mu'_X) = -X_{\tilde{M}^\xi} \lrcorner \tilde{\omega}.$$

Here the symplectic form $\tilde{\omega}$ is determined by

$$\iota^*\omega = \pi^*\tilde{\omega}.$$

The maps ι and π are the inclusion and quotient maps in

$$\begin{array}{ccc} \tilde{\mu}^{-1}(\xi) & \overset{\iota}{\hookrightarrow} & M \\ \downarrow{\scriptstyle \pi} & & \\ \tilde{\mu}^{-1}(\xi)/\tilde{H}_\xi & & \end{array}$$

(see Theorem 1.19). Because the linear map

$$\pi^* : \Omega(\tilde{M}^\xi) \to \Omega(\tilde{\mu}^{-1}(\xi))$$

is injective, it is enough to prove that

$$\pi^*\left(d(\mu'_X)\right) = -\pi^*\left(X_{\tilde{M}^\xi} \lrcorner \tilde{\omega}\right). \tag{23}$$

Note that

$$\begin{aligned} \pi^*\left(d(\mu'_X)\right) &= d\left(\pi^*\mu'_X\right) \\ &= d\left(\mu_X \circ \iota\right) \\ &= \iota^* d\mu_X \\ &= \iota^*\left(-X_M \lrcorner \omega\right) \\ &= -X_{\mu^{-1}(\xi)} \lrcorner \iota^*\omega \\ &= -X_{\mu^{-1}(\xi)} \lrcorner \pi^*\tilde{\omega} \\ &= -\pi^*\left(X_{\tilde{M}^\xi} \lrcorner \tilde{\omega}\right), \end{aligned}$$

which proves (23). □

Theorem 1.30. *Let K be a compact Lie group, and let*

$$\rho : K \to U(V, H)$$

be a unitary representation of K in V. Write $V^\times := V \setminus \{0\}$, and let $L^\times \subset V^\times$ be a $\mathbb{C}^\times$-invariant, K-invariant holomorphic submanifold of $V^\times$. Consider the principal $\mathbb{C}^\times$-bundle

$$L^\times \to M := L^\times/\mathbb{C}^\times \subset \mathbb{P}(V).$$

The action of K on M induced by the action of K on $L^\times$ is Hamiltonian, with momentum map

$$\mu_X(m) = \frac{H(\rho(X)l, l)}{2\pi i H(l, l)}, \tag{24}$$

for $X \in \mathfrak{k}$, $l \in L$, and $m := \mathbb{C}^\times \cdot l$.

Proof. Suppose that $K = U(V, H)$ and that $L = V$. Otherwise, apply Examples 1.8 and 1.11. Note that complex submanifolds of complex symplectic manifolds are always symplectic (see Example 3.17).

We apply Proposition 1.29 about commuting actions with $G = H = U(V, H)$ and $\tilde{H} = U(1)$. The proposition then yields that the action of $U(V, H)$ on $\mathbb{P}(V) = \tilde{\mu}^{-1}(-1^*)/U(1)$ is Hamiltonian, with momentum map

$$\begin{aligned} \mu'_X\,(U(1) \cdot v) &= \mu_X(v) \\ &= iH(X \cdot v, v)/2, \end{aligned} \tag{25}$$

for all $X \in \mathfrak{u}(V, H)$ and $v \in V$ with $\|v\|^2 = \frac{1}{\pi}$.

Now note that for all $v \in V$ with $\|v\|^2 = \frac{1}{\pi}$, (25) equals

$$\frac{H(X \cdot v, v)}{2\pi i H(v, v)}, \tag{26}$$

and that (26) does not depend in the length of v. Hence (24) is a well-defined momentum map. ☐

Remark 1.31. Let (M, ω) be a symplectic manifold, equipped with a symplectic action of a compact Lie group K. Suppose that there is a principal $\mathbb{C}^\times$-bundle $L^\times \to M$, and that there is an action of K on $L^\times$ compatible with the action of K on M. Suppose that $L^\times$ can be embedded as a holomorphic submanifold of a complex vector space V (avoiding $0 \in V$). This induces an embedding of M into $\mathbb{P}(V)$. Assume that there is a Hermitian form H on V that induces the symplectic form ω on M by the construction we gave above. Suppose that there is a unitary representayion ρ of K in V, such that the action of K on $L^\times$ is the restriction of ρ to $L^\times$. This is called a *linearisation* of the action of K on (M, ω), using the $\mathbb{C}^\times$-bundle $L^\times$. If a linearisation exists, then Theorem 1.30 applies, but formula (24) for the momentum map depends on the choices made.

Suppose that ω_1 and ω_2 are symplectic forms on M, and that K acts on M, preserving both forms. Assume that the K-action on (M, ω_i) can be linearised, using $\mathbb{C}^\times$-bundles $L_i^\times$, for $i = 1, 2$, and denote the resulting momentum maps by μ_1 and μ_2, respectively. Then the action of K on the symplectic manifold $(M, \omega_1 + \omega_2)$ can be linearised using the $\mathbb{C}^\times$-bundle $L_1 \otimes L_2 \to M$, with momentum map $\mu_1 + \mu_2$. Likewise, the action of K on the symplectic manifold $(M, n\omega)$ can be linearised, using the $\mathbb{C}^\times$-bundle $L^{\otimes n}$, with momentum map $\mu_n := n \cdot \mu$, for all $n \in \mathbb{N}$.

2 Convexity theorems

In the case of a Hamiltonian action of a compact Lie group on a symplectic manifold (both assumed to be connected), the image of a momentum map has a very nice description. A Weyl chamber in the dual of an infinitesimal maximal torus is a fundamental domain for the coadjoint action of the compact Lie group on the dual of its Lie algebra. Therefore, the intersection of the momentum map image with a Weyl chamber determines this image completely, because the image of a momentum map is invariant under the coadjoint action. The intersection of the momentum map image with a Weyl chamber turns out te be a convex polyhedron (Theorem 2.27).

The proof of Theorem 2.27 makes heavy use of Morse theory, which we review in Subsection 2.1.

2.1 Morse theory

Let M be a compact manifold, and let $f \in C^\infty(M)$ be a real valued smooth function on M.

Definition 2.1. A point $c \in M$ is a *critical point* of f if $(df)_c = 0$ in T_c^*M. The *critical locus* of f is the set $\mathrm{Crit}(f)$ of critical points of f.

Let $c \in M$ be a critical point of f. The *Hessian* of f at c is the symmetric bilinear form

$$\mathrm{Hess}_c(f) : T_cM \times T_cM \to \mathbb{R},$$

defined by

$$\mathrm{Hess}_c(f)(X_c, Y_c) := X\left(Y(f)\right)(c),$$

for vector fields X, Y on M. It can be proved that at critical points, the Hessian is indeed well-defined and symmetric.

A critical point c of f is said to be *nondegenerate* if the Hessian of f at c is a nondegenerate symmetric bilinear form.

A function $f \in C^\infty(M)$ is called a *Morse function* if the critical locus $\mathrm{Crit}(f)$ of f is discrete in M, and all critical points of f are nondegenerate.

Note that if f is a Morse function, the critical locus of f is a closed discrete subset of the compact manifold M, so that f has only finitely many critical points.

Pick a Riemannian metric B on M, and let c be a critical point of f. Then, by abuse of notation, the nondegenerate bilinear form $\mathrm{Hess}_c(f)$ defines an invertible linear endomorphism $\mathrm{Hess}_c(f)$ of T_cM by

$$B_c(\mathrm{Hess}_c(f)v, w) = \mathrm{Hess}_c(f)(v, w),$$

for all $v, w \in T_cM$. Because $\mathrm{Hess}_c(f)$ is a symmetric bilinear form, the corresponding endomorphism of T_cM is symmetric with respect to B_c.

Let p_c be the number of positive eigenvalues of $\mathrm{Hess}_c(f) \in \mathrm{End}(T_cM)$, and let q_c be the number of negative eigenvalues, counted with multiplicity. Then by nondegeneracy, $p_c + q_c = \dim(M)$. The pair (p_c, q_c) is the *signature* of $\mathrm{Hess}_c(f)$.

Let F_t be the flow of the vector field $\mathrm{grad}(f)$, which is defined by

$$\langle df, X\rangle = B(\mathrm{grad}(f), X) \in C^\infty(M),$$

for all vector fields X on M.

Definition 2.2. For $c \in \mathrm{Crit}(f)$, the *stable manifold* S_c of $\mathrm{grad}(f)$ at c is defined as

$$S_c := \{m \in M; \lim_{t\to-\infty} F_t(m) = c\}.$$

Lemma 2.3. *The dimension of S_c equals p_c.*

Proof. Let $x = (x_1, \ldots, x_n)$ be a system of local coordinates around c, such that $x(c) = 0$. Because $\mathrm{grad}(f)(c) = 0$, the gradient of f can be expressed in these coordinates as

$$\begin{aligned}
\mathrm{grad}(f)(x) &= \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\right) \\
&= \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}(0) \cdot x + \mathcal{O}(\|x\|^2) \\
&= \mathrm{Hess}_0(f) \cdot x + \mathcal{O}(\|x\|^2),
\end{aligned}$$

by Taylor's theorem. By the Morse lemma (Milnor [22], Lemma 2.2), the coordinates x can be chosen so that

$$\mathrm{grad}(f)(x) = \mathrm{Hess}_0(f) \cdot x.$$

Then, in the coordinates x, the flow F_t of $\mathrm{grad}(f)$ is given by

$$F_t(x) = \left(e^{t\,\mathrm{Hess}_0(f)}\right) \cdot x.$$

The eigenvalues of the matrix $e^{t\,\mathrm{Hess}_0(f)}$ are precisely $e^{t\lambda}$, where λ is an eigenvalue of $\mathrm{Hess}_0(f)$. Let $x \in \mathbb{R}^n$ be an eigenvector of $e^{t\,\mathrm{Hess}_0(f)}$, corresponding to the eigenvalue $e^{t\lambda}$. Then

$$\lim_{t\to-\infty} F_t(x) = \lim_{t\to-\infty} e^{t\lambda}x = 0$$

if and only if $\lambda > 0$. This shows that locally around x, S_c is a submanifold of dimension p_c. But since the flow F_t is a diffeomorphism for fixed t, any point in S_c has a neighbourhood which is diffeomorphic to an open subset of $\mathbb{R}^{p_c}$. □

Theorem 2.4 (Morse)**.** *Let f be a Morse function on M. Then the manifold M decomposes as*

$$M = \coprod_{c \in \mathrm{Crit}(f)} S_c. \tag{27}$$

See Milnor [22]. Here S_c is a cell of dimension p_c, by Lemma 2.3.

Definition 2.5. The decomposition of Theorem 2.4 is called the *Morse decomposition* of M.

Remark 2.6. Suppose we are in the ideal situation that p_c is even, for all critical points c of f. Then all boundary maps in the cellular complex corresponding to the Morse decomposition of M are zero. For $k \in \{0, 1, 2, \dots\}$, we define the number

$$b_k := \#\{c \in \mathrm{Crit}(f); p_c = k\}.$$

By assumption, $b_k = 0$ for all odd k. And because all boundary maps in the cellular complex (27) are zero, the dimensions of the cohomology spaces of M are given by

$$\dim H^k(M) = b_k.$$

For compact and connected M, we have $b_0 = \dim(H^0(M)) = 1$ and $b_n = \dim(H^n(M)) = 1$. In other words, f has a unique local maximum and a unique local minimum.

Example 2.7. Let M be the two-sphere

$$M = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}.$$

Let $f \in C^\infty(M)$ be the height function

$$f(x, y, z) = z.$$

The critical points of f are $(0, 0, 1)$ and $(0, 0, -1)$. The point $(0, 0, 1)$ is a local maximum, so that $p_{(0,0,1)} = 0$ and $q_{(0,0,1)} = 2$. Its antipode $(0, 0, -1)$ is a local minimum, so $p_{(0,0,-1)} = 2$ and $q_{(0,0,-1)} = 0$. Here we are in the ideal situation referred to in Remark 2.6.

Example 2.8. Let M be the two-torus

$$M = \left\{ (x, y, z) \in \mathbb{R}^3; \left\| (x, y, z) - \frac{(2x, 2y, 0)}{(x^2 + y^2)^{1/2}} \right\| = 1 \right\}.$$

The central circle of M is the circle of radius 2 around the origin of the (x, y)-plane, and M consists of all points in $\mathbb{R}^3$ at distance 1 to this circle.

Let $f_1 \in C^\infty(M)$ be the height function along the first coordinate:

$$f_1(x, y, z) = x.$$

The critical points of f_1 are

point	(p, q)	nature of crit. pnt.
(3, 0, 0)	p=0,q=2	local maximum
(1, 0, 0)	p=q=1	saddle point
(-1, 0, 0)	p=q=1	saddle point
(-3, 0, 0)	p=2,q=0	local minimum.

Let $f_2 \in C^\infty(M)$ be the height function along the third coordinate:

$$f_2(x, y, z) = z.$$

The critical locus of f_2 consists of two circles:

$$\mathrm{Crit}(f_2) = \{(x, y, x) \in \mathbb{R}^3; x^2 + y^2 = 4 \text{ and } z = \pm 1\}.$$

Hence f_2 is not a Morse function, because its critical locus is not discrete. However, f_2 is a *Morse-Bott function*:

Definition 2.9. Let M be a compact manifold, and let f be a smooth function on M. Then f is a *Morse-Bott function* if

1. $\mathrm{Crit}(f)$ is a smooth submanifold of M
2. For all $c \in \mathrm{Crit}(f)$, the Hessian $\mathrm{Hess}_c(f)$ induces a nondegenerate bilinear form on the quotient space $T_cM/T_c\,\mathrm{Crit}(f)$.

Let f be a Morse-Bott function, and let C be a connected component of $\mathrm{Crit}(f)$. For each $c \in C$, we define the integer p_c as the number of positive eigenvalues of the invertible endomorphism of T_cM/T_cC defined by the Hessian of f at c. The number q_c is defined as the number of negative eigenvalues of this endomorphism. Then $p_c + q_c$ is the dimension of the quotient space T_cM/T_cC, which implies that

$$p_c + q_c + \dim C = \dim M.$$

By Lemma 2.10 below, the numbers p_c and q_c do not depend on the choice of $c \in C$. Hence we can write $p_C := p_c$ and $q_C := q_c$, and we define the *signature of* $\mathrm{Hess}(f)$ *at the connected component* C as the pair (p_C, q_C).

Lemma 2.10. *Let X be the space of invertible symmetric real $n \times n$ matrices. Let*

$$A : [0, 1] \to X \tag{28}$$

$$A \mapsto A_t \tag{29}$$

be a continuous curve in X. Then the number of positive eigenvalues of A_t does not depend on t. The same holds for the number of negative eigenvalues of A_t.

Proof. The eigenvalues of A_t are real, and the set of eigenvalues of A_t depends continuously on t. The eigenvalue zero does not occur for any t, so the numbers of positive and negative eigenvalues are independent of t. □

Definition 2.11. Let $f \in C^\infty$ be a Morse-Bott function, and let C be a connected component of its critical locus. The *stable manifold* S_C of $\mathrm{grad}(f)$ at C is defined as

$$S_C := \{m \in M | \lim_{t \to -\infty} F_t(m) \subset C\}.$$

The limit in the definition above should be interpreted in the following way. For $\varepsilon > 0$, and $x \in M$, we write

$$B_\varepsilon(x) := \{y \in M; d(x,y) \leq \varepsilon\},$$

where d denotes the geodesic distance. If $I \subset \mathbb{R}$ and $m \in M$, then we define

$$F_I(m) := \{F_t(m); t \in I\}.$$

Now the subset $\lim_{t \to -\infty} F_t(m)$ of M is defined as

$$\lim_{t \to -\infty} F_t(m) := \{x \in M; \forall \varepsilon > 0, \forall A \in \mathbb{R}, B_\varepsilon(x) \cap F_{(-\infty,A]} \neq \emptyset\}.$$

It has been proved by Duistermaat that $\lim_{t \to -\infty} F_t(m)$ actually consists of a single point if $f = \|\mu\|^2$, where μ is the momentum map of a Hamiltonian action (see Lerman [19]).

The following generalisation of Theorem 2.4 was proved by Bott in [4].

Theorem 2.12 (Morse-Bott)**.** *Let $f \in C^\infty(M)$ be a Morse-Bott function, and let $\mathcal{C}$ be the set of connected components of its critical locus. Then*

$$M = \coprod_{C \in \mathcal{C}} S_C.$$

Remark 2.13. In Morse-Bott theory, the ideal situation is the case where the number $\dim S_C$ is even, for all connected components C of $\mathrm{Crit}(f)$. If M is connected, then f attains a local minimum at a unique component C.

Indeed, if C is a connected component of $\mathrm{Crit}(f)$, then f has a local minimum at C if and only if $q_C = 0$. The dimension of the stratum S_C is then equal to

$$\dim S_C = \dim C + p_C = \dim M.$$

The first equality is an analogue of Lemma 2.3, and the second follows because $q_C = 0$.

If there are two or more strata S_C of maximal dimension, then they must be joined by a stratum of codimension 1. If all strata are even-dimensional, this is impossible. Hence the function f has a local minimum at only one component C of $\mathrm{Crit}(f)$.

In [15], Kirwan developed a generalisation of Morse-Bott theory. This generalisation makes it possible to deal with functions whose critical loci are not necessarily smooth.

Definition 2.14. A smooth function f on a compact manifold M is called a *Morse-Kirwan function*[6] if it has the following properties.

1. The critical locus of f is the disjoint union of finitely many closed subsets $C \subset \mathrm{Crit}(f)$, on each of which f is constant. (For functions with well-behaved critical loci, one can take the subsets C to be the connected components of $\mathrm{Crit}(f)$.)

2. For every such C, there is a submanifold Σ_C of M containing C, with orientable normal bundle in M, such that

   (a) the restriction of f to Σ_C takes its minimal value on C,

   (b) for all $m \in \Sigma_C$, the Hessian $\mathrm{Hess}_m(f)$ is positive semidefinite on $T_m\Sigma_C \subset T_mM$, and there is no subspace of T_mM strictly containing $T_m\Sigma_C$, on which $\mathrm{Hess}_m(f)$ is positive semidefinite.

A submanifold Σ_C as above, is called a *minimising submanifold* for f along C.

If f is a Morse-Bott function, then it has the properties of a Morse-Kirwan function, with $\Sigma_C = S_C$.

We set

$$q_C := \dim \left(T_mM/T_m\Sigma_C\right),$$

for a $m \in C$. Note that q_C is precisely the number of negative eigenvalues of $\mathrm{Hess}(f)$ on C.

Kirwan's analogue of the Morse-Bott stratification is the following (see Kirwan [15], Theorem 10.4).

Theorem 2.15. *Let M be a compact Riemannian manifold, and let $f \in C^\infty(M)$ be a Morse-Kirwan function. Suppose that the gradient of f is tangent to the minimising manifolds Σ_C.*

Then

$$M = \coprod_C S_C,$$

where as before, the stable manifolds S_C are defined by

$$S_C := \{m \in M | \lim_{t \to -\infty} F_t(m) \subset C\},$$

with F_t the flow of $\mathrm{grad}\, f$. The strata S_C are smooth, and the minimising manifolds Σ_C are open neighbourhoods of C in S_C.

If the number $\dim S_C$ is even for all C, then f has a local minimum along a unique set C. (This is a generalisation of Remark 2.13.)

[6] This is not standard terminology. Kirwan herself used the term 'minimally degenerate function'.

2.2 The Abelian convexity theorem

Let (M, ω) be a connected, compact symplectic manifold. Let T be a connected, compact, Abelian Lie group. Then $T \cong \mathfrak{t}/\Lambda$, where $\Lambda := \ker \exp$ is the unit lattice of T. Let $T \times M \to M$ be a Hamiltonian action, with momentum map $\mu : M \to \mathfrak{t}^*$.

Definition 2.16. A *convex polytope* P in a vector space V is a compact subset of V which is given as the intersection of a finite number of closed half spaces. A *polytope* P in V is a finite union of convex polytopes. We will sometimes use the word *polyhedron* instead of the word polytope.

Theorem 2.17 (Abelian convexity theorem, Atiyah 1982 [1], Guillemin & Sternberg 1982 [6])**.** *The image $\mu(M)$ of M in $\mathfrak{t}^*$ is a convex polytope, with vertices contained in the set $\mu(M^T)$. Here M^T is the fixed point set of the action of T on M.*

Example 2.18. The real vector space $\mathrm{Herm}(n)$ of Hermitian $n \times n$ matrices carries a representation of $U(n)$, defined by conjugation. Let

$$\pi : \mathrm{Herm}(n) \to \mathrm{Diag}(n) \cong \mathbb{R}^n$$

be the projection onto the diagonal part:

$$\pi\left((a_{ij})_{i,j=1}^n\right) = (a_{ii})_{i=1}^n.$$

Then we have:

Theorem 2.19 (Horn 1954 [12])**.** *Let $D \in \mathrm{Herm}(n)$ be a diagonal matrix, and let $M := U(n) \cdot D$ be its conjugation orbit. Then the projection $\pi(M)$ is equal to the convex hull of the finite set*

$$\{\sigma(D) := \sigma D \sigma^{-1} | \sigma \in S_n \textit{ permutation matrix}\}.$$

This theorem fits in the situation of the Abelian convexity theorem. Indeed, note that

$$\mathrm{Lie}\,(U(n)) = \mathfrak{u}(n) = i\mathrm{Herm}(n) \cong \mathfrak{u}(n)^*,$$

and that the coadjoint action of $U(n)$ on $\mathfrak{u}(n)$ corresponds to conjugation in $i\mathrm{Herm}(n)$. Hence the conjugation orbit M corresponds to a coadjoint orbit in $\mathfrak{u}(n)$. By Example 1.13 it is a symplectic manifold, and the action of $U(n)$ is Hamiltonian. By Example 1.8, this action restricts to a Hamiltonian action of the subgroup of diagonal matrices in $U(n)$. The projection π is the transpose of the inclusion of the diagonal unitary matrices into the unitary matrices, so π is a momentum map.

The proof of the Abelian convexity theorem is based on Lemmas 2.20 – 2.22.

Lemma 2.20. *Let $m \in M$, and let $\mathfrak{t}_m$ be the stabiliser algebra of m. The image of the tangent map $T_m\mu : T_mM \to \mathfrak{t}^*$ is*

$$\mathrm{im}(T_m\mu) = \mathfrak{t}_m^\perp := \{\xi \in \mathfrak{t}^*; \langle \xi, X \rangle = 0 \textit{ if } X \in \mathfrak{t}_m\}.$$

Proof. For all $m \in M$, $v \in T_mM$ and $X \in \mathfrak{t}$, the defining relation of the momentum map μ implies that

$$\langle T_m\mu(v), X\rangle = \langle (d\mu_X)_m, v\rangle = -\omega(X_m, v).$$

Therefore, $X \in \operatorname{im}(T_m\mu)^\perp$ if and only if $X_m = 0$, which is to say that $X \in \mathfrak{t}_m$. So we have

$$\operatorname{im}(T_m\mu)^\perp = \mathfrak{t}_m \subset \mathfrak{t}.$$

Taking the annihilators of both sides in $\mathfrak{t}^*$, we obtain the desired result. □

Lemma 2.21. *There are only finitely many subalgebras of* $\mathfrak{t}$ *that occur as the stabiliser algebra of a point in* M*.*

Proof. We shall prove that every point $m \in M$ has a neighbourhood on which only a finite number of stabiliser groups (and hence stabiliser algebras) occur. Compactness of M then completes the argument.

Let $m \in M$ be given. Let T_m be the stabiliser group of m. Let U be an open neighbourhood of m such that all points in U are fixed by T_m.

The action of T_m on M can be linearised around m, in the sense that there exists a representation ρ of T_m in a finite-dimensional vector space V, and a diffeomorphism

$$\phi : U' \to V',$$

from a T_m-invariant open neighbourhood U' of m onto an open neighbourhood V' of 0 in V, such that ϕ intertwines ρ and the action of T_m on U'. The symplectic form ω_m on T_mM induces a symplectic form on V through the linear isomorphism $T_m\phi : T_mM \to V$. Thus the space V can be given the structure of a complex vector space.

Let U'' be an open neighbourhood of m, such that U'' is T_m-invariant, and $U'' \subset U \cap U'$. Identify the torus T_m with the torus

$$\mathbb{T}^k := \{(e^{2\pi i\alpha_1}, \ldots, e^{2\pi i\alpha_k}); \alpha_j \in \mathbb{R}\}.$$

Let $\{\lambda_1, \ldots, \lambda_n \in \mathbb{Z}^k\}$ be the set of weights of ρ. That is to say,

$$V = \bigoplus_{l=1}^n \mathbb{C}_{\lambda_l},$$

where $\mathbb{T}^k$ acts on $\mathbb{C}_{\lambda_l}$ by

$$(e^{2\pi i\alpha_1}, \ldots, e^{2\pi i\alpha_k}) \cdot z = e^{2\pi i\lambda_l\cdot\alpha} \cdot z,$$

where $z \in \mathbb{C}_{\lambda_l}$, and

$$\lambda_l \cdot \alpha := \sum_{p=1}^k \lambda_l^p \alpha_p.$$

Let $u \in U''$ be given. Write $v := \phi(u)$, and

$$v = \sum_{l=1}^{n} v_l,$$

with $v_l \in \mathbb{C}_{\lambda_l}$. Then an element $(e^{2\pi i \alpha_1}, \ldots, e^{2\pi i \alpha_k}) \in \mathbb{T}^k$ stabilises v if and only if for all $l \in \{1, \ldots n\}$:

$$\lambda_l \cdot \alpha \in \mathbb{Z} \quad \text{or} \quad v_l = 0.$$

Hence the stabiliser of v is determined if we know which v_l are zero. This implies that only 2^n subgroups of T_m can occur as the stabiliser of a point $u \in U''$. □

Lemma 2.22. *For every $X \in \mathfrak{t}$, the function $\mu_X \in C^\infty(M)$ is a Morse-Bott function, with even signatures along critical loci.*

Proof. Note that the critical points of μ_X are precisely the fixed points in M of the subgroup $T_X := \overline{\exp(\mathbb{R} \cdot X)}$ of T:

$$(d\mu_X)_m = 0 \quad \Leftrightarrow \quad (-X \lrcorner \omega)_m = 0 \quad \Leftrightarrow \quad X_m = 0.$$

Let m be a critical point of μ_X. As in the proof of Lemma 2.21, we consider a linearisation $\rho : T_X \to GL(V)$ of the action of T_X around m. Because the fixed point set of T_X in V is a linear subspace, the diffeomorphism ϕ defines a chart of M around m that locally exhibits the critical locus of μ_X as a smooth submanifold of M.

It remains to prove that the Hessian of μ_X is nondegenerate, transversally to $\mathrm{Crit}(\mu_X)$. As in the proof of Lemma 2.21, we transfer the symplectic form ω_m on T_mM to V. Furthermore, we equip V with the structure of a complex vector space, and with a Hermitian form H such that the symplectic form on V is the imaginary part of H. By the Darboux-Weinstein theorem (Guillemin & Sternberg [10], Section 22), the diffeomorphism

$$\phi : U' \to V' \subset V$$

can be chosen so that it preserves the symplectic forms on the spaces in question.

Because T_X is a torus, the representation ρ is automatically unitary with respect to H. By Example 1.10, a momentum map of the action of T_X on V is given by

$$\tilde{\mu}_X(v) = iH(\rho(X)v, v)/2,$$

$v \in V$. The momentum map $\tilde{\mu}$ corresponds to the momentum map μ up to addition of an element of $(\mathfrak{g}^*)^G$ (see Remark 1.4). So the Hessian of $\tilde{\mu}_X$ has the same eigenvalues as the Hessian of μ_X.

Note that the sesquilinear form

$$H_X := H(\rho(X)\cdot, \cdot)$$

is anti-Hermitian, because $\rho(X) \in \mathfrak{u}(V, H)$ is anti-Hermitian. So the eigenvalues of H_X are imaginary. Let $(v_1, \ldots, v_n)$ be a $\mathbb{C}$-basis of V such that

$$H_X(v_j, v_k) = -i\lambda_j \delta_{jk},$$

where $\lambda_j \in \mathbb{R}$, and δ_{jk} is the Kronecker delta. Let $z = (z_1, \ldots, z_n)$ be the complex coordinates on V defined by

$$v = \sum_{j=1}^{n} z_j v_j,$$

for all $v \in V$. Then

$$\tilde{\mu}_X(z) = \frac{1}{2} \sum_j \lambda_j |z_j|^2. \tag{30}$$

Hence in the real coordinates (x, y) defined by $z = x + iy$, the matrix of $\mathrm{Hess}(\mu_X)$ is given by

$$\mathrm{Hess}(\mu_X) = \begin{pmatrix} \lambda_1 & & & & \\ & \lambda_1 & & \emptyset & \\ & & \ddots & & \\ & \emptyset & & \lambda_n & \\ & & & & \lambda_n \end{pmatrix}. \tag{31}$$

Each eigenvalue λ_j occurs twice, because $|z_j|^2 = x_j^2 + y_j^2$.

We conclude that

$$v_j \in \ker \mathrm{Hess}(\tilde{\mu}_X)$$

if and only if $\lambda_j = 0$, which is equivalent to

$$v_j \in \ker d\mu_X,$$

by (30).

So $\mathrm{Hess}(\tilde{\mu}_X)$ is zero on $\mathrm{Crit}(\tilde{\mu}_X)$, and nondegenerate transversally to $\mathrm{Crit}(\tilde{\mu}_X)$.

The signatures of $\mathrm{Hess}(\tilde{\mu}_X)$ are even because every λ_j appears twice in (31).

□

Proof of the Abelian convexity theorem. Without loss of generality, we may assume that the action of T on M is locally free, i.e. M_{reg} is dense in M. For if the action is not locally free, then we replace T by the quotient group T/T_M, where T_M is the subgroup of T that stabilises all points in M. If the action is locally free, then $\mu(M)$ has nonempty interior, because $\mu(M_{\mathrm{reg}}) \subset \mu(M)^{\mathrm{int}}$.

We first claim that the boundary $\partial(\mu(M))$ contains an open dense subset that consists of a finite number of open subsets of affine subspaces of $\mathfrak{t}$. This explains the polyhedral nature of $\mu(M)$.

Let $m \in M$ be such that $\mu(m) \in \partial(\mu(M))$, and that $\partial(\mu(M))$ is smooth around $\mu(m)$. (Note that $\mu(M)$ is compact, hence closed, so that $\partial(\mu(M)) \subset \mu(M)$.) Then

$$T_{\mu(m)}\partial(\mu(M)) = \mathrm{Im}(T_m\mu) = \mathfrak{t}_m^{\perp},$$

by Lemma 2.20. By Lemma 2.21, there are only finitely many subspaces of $\mathfrak{t}$ that occur as stabilisers of points in M. So there are only finitely many subspaces of

$\mathfrak{t}$ that occur as tangent spaces to $\partial(\mu(M))$. This shows that $\partial(\mu(M))$ is locally affine.

To prove convexity of $\mu(M)$, we use the fact that μ_X is a Morse-Bott function, by Lemma 2.22. If $\mu(M)$ is not convex, then there is a direction $X \in \mathfrak{t}$ such that the subset of M where the height function μ_X has a local minimum, is not connected (see Figure 1). But this contradicts the fact that μ_X is a Morse-Bott

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Figure 1: The function μ_X has local minima on two separate sets.

function with even signatures (see Remark 2.13).

The claim about the vertices of $\mu(M)$ follows from Lemma 2.20. Indeed, the vertices of $\mu(M)$ are the points that lie on the intersection of a maximal number of hyperplanes that define $\mu(M)$. By Lemma 2.20, this implies that if $m \in M$ is a point that is mapped to a vertex, then $\mathfrak{t}_m = \mathfrak{t}$, hence m is fixed by T. $\square$

2.3 The nonabelian convexity theorem

Example 2.23. Let U be a compact, connected Lie group, and let $K < U$ be a closed subgroup. Let $\mathfrak{k} \subset \mathfrak{u}$ be their Lie algebras. Let $T < K$ be a maximal torus, with Lie algebra $\mathfrak{t}$. We identify $\mathfrak{t}^*$ with the subspace of $\mathfrak{k}^*$ consisting of the elements that are fixed by the coadjoint action of T on $\mathfrak{k}^*$. Let $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ be a closed Weyl chamber. Then $\mathfrak{t}^*_+$ is a strict fundamental domain for the coadjoint action of K on $\mathfrak{k}^*$: $\mathrm{Ad}^*(K)(\mathfrak{t}^*_+) = \mathfrak{k}^*$, and for all $\xi \in \mathfrak{k}^*$, there is at most one $k \in K$ such that $\mathrm{Ad}^*(k)\xi \in \mathfrak{t}^*_{++}$, with $\mathfrak{t}^*_{++}$ the interior of $\mathfrak{t}^*_+$.

Let M be a coadjoint orbit of U in $\mathfrak{u}^*$, equipped with the canonical symplectic form (see Example 1.13). Then the restricted coadjoint action of K on M is Hamiltonian, with momentum map

$$\mu : M \hookrightarrow \mathfrak{u}^* \to \mathfrak{k}^*$$

(see Examples 1.13 and 1.8).

Theorem 2.24 (Heckman 1982 [11])**.** *The intersection* $\mu(M) \cap \mathfrak{t}^*_+$ *is a convex polyhedron.*

This result was found on the basis of an ample supply of branching rule computations. The proof went by a classical limit of Kähler quantisation, i.e. by asymptotic behaviour of multiplicities of representations of compact Lie groups.

Guillemin and Sternberg found a more natural, and more general setting for this theorem to hold.

Theorem 2.25 (Guillemin & Sternberg 1984 [7])**.** *Let V be a finite-dimensional complex vector space, equipped with a Hermitian metric H. Let K be a compact Lie group, with maximal torus T. Let*

$$\rho : K \to U(V)$$

be a unitary representation of K in V.

Let $L^\times \subset V^\times$ be a holomorphic submanifold, invariant under the actions of $\mathbb{C}^\times$ and K. Let $M := L^\times/\mathbb{C}^\times \subset \mathbb{P}(V)$ be the corresponding projective variety. By Theorem 1.30, the action of K on M induced by ρ is Hamiltonian, with momentum map $\mu : M \to \mathfrak{k}^*$,

$$\mu_X(m) = \frac{H(\rho(X)l, l)}{2\pi i H(l, l)}$$

for all $X \in \mathfrak{k}$ and $m = \mathbb{C}^\times \cdot l \in M$.

Suppose that M is connected. Then $\mu(M) \cap \mathfrak{t}^*_+$ is a convex polyhedron, for all closed Weyl chambers $\mathfrak{t}^*_+$ in $\mathfrak{t}^*$.

Remark 2.26. Using Theorem 2.25, one can determine the whole image $\mu(M)$ of μ by noting that the Weyl chamber $\mathfrak{t}^*_+$ is a fundamental domain for the coadjoint action of K on $\mathfrak{k}^*$, so that

$$\mu(M) = \mathrm{Ad}^*(K)\left(\mu(M) \cap \mathfrak{t}^*_+\right).$$

We shall restrict ourselves to the case where the set M_{reg} of regular points of μ is dense in M. That is to say, the action is locally faithful.

The idea of Guillemin and Sternberg was the following.

Let $M_{++} := \mu^{-1}(\mathfrak{t}^*_{++})$. Then M_{++} is a smooth, connected submanifold of M, invariant under T. It turns out that M_{++} is actually a symplectic submanifold. By Examples 1.8 and 1.11, the action of T on M_{++} is Hamiltonian, with momentum map

$$\mu_{++} : M_{++} \xrightarrow{\mu} \mathfrak{t}^*_{++} \subset \mathfrak{t}^*,$$

where $\mu_{++} := \mu|_{M_{++}}$.

Hence to determine $\mu(M_{++})$, is is sufficient to consider the action of T on M_{++}. This brings us back to the Abelian case. If M_{++} is compact (i.e. $\mu(M) \subset \mathrm{Ad}^*(K)\mathfrak{t}^*_{++}$), then we can apply the Abelian convexity theorem to deduce the desired result.

In general, the same arguments as those used in the proof of the Abelian convexity theorem can be used to show that

$$\overline{\mu_{++}(M_{++})} = \mu(M) \cap \mathfrak{t}^*_+$$

is a polyhedron. Indeed, the arguments about the polyhedral nature of $\mu(M)$ in the proof of the Abelian convexity theorem are purely local. Because the momentum map μ is proper (M is compact), it locally restricts to a map between compact sets.

It then remains to prove that $\mu(M) \cap \mathfrak{t}^*_+$ is convex. Guillemin and Sternberg proved this convexity by using Kähler quantisation and representation theory of compact groups (see Remark 3.23).

A more general nonabelian convexity theorem was proved by Kirwan.

Theorem 2.27 (Kirwan 1984 [15])**.** *Let M be a compact, connected symplectic manifold, K a compact Lie group, and $K \times M \to M$ a Hamiltonian action with momentum map $\mu : M \to \mathfrak{k}^*$. Let $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ be a closed Weyl chamber.*

Then $\mu(M) \cap \mathfrak{t}^*_+$ is a convex polyhedron.

Definition 2.28. The convex polyhedron $\mu(M) \cap \mathfrak{t}^*_+$ is called the *momentum polytope* of μ.

Guillemin and Sternberg's arguments for the polyhedral nature of $\mu(M)$ apply in this more general setting. Their argument for the convexity of $\mu(M)$ no longer applies, since Kirwan's theorem is a differential geometric statement, whereas Guillemin and Sternberg's theorem is complex geometric.

Kirwan's differential geometric proof of convexity of $\mu(M)$ is based on the following key lemma.

Lemma 2.29. *Let $(\cdot, \cdot)$ be an $\mathrm{Ad}^*(K)$-invariant inner product on $\mathfrak{k}^*$, and let $\| \cdot \|$ be the corresponding norm. Then the set $\mathrm{Min}\left(\|\mu\|^2\right)$ of points in M on which the function $\|\mu\|^2$ takes its minimal value is* connected*.*

Sketch of proof. The proof is based on the observation that the function $\|\mu\|^2$ is a Morse-Kirwan function, for which the strata S_C are symplectic submanifolds of M. (Kirwan [15], pp. 44–68.) Hence the numbers $\dim S_C$ are even, which implies that the function $\|\mu\|^2$ takes its minimal value on a unique set C_0 (Remark 2.13).

It turns out that the sets C may be chosen to be the connected components of the critical locus of $\|\mu\|^2$. (Lerman [19], Theorem 2.1.) In particular, the set C_0 on which $\|\mu\|^2$ is minimal, is connected. □

Corollary 2.30. *Let $\xi \in \mathfrak{t}^*_+$. Consider the function*

$$\psi : m \mapsto \|\mu(m) - \xi\|^2,$$

which assigns to a point $m \in M$ the distance squared from $\mu(m)$ to ξ. It takes its minimal value on a connected subset $\mathrm{Min}(\psi)$ of M.

Proof. We use the shifting trick (Remark 1.22). Let $(M_{-\xi}, \omega_{-\xi})$ be the coadjoint orbit of K through $-\xi$, equipped with its standard symplectic form (see Example 1.13). Consider the symplectic manifold

$$(\tilde{M}, \tilde{\omega}) := (M_{-\xi} \times M, \omega_{-\xi} \times \omega).$$

The action of K on $\tilde{M}$ is Hamiltonian, with momentum map

$$\tilde{\mu}\,(\zeta, m) = \mu(m) + \zeta$$

(see Example 1.14).

Let $\tilde{\psi}$ be the function on $\tilde{M}$ given by $\tilde{\psi}(\zeta, m) = \|\tilde{\mu}(\zeta, m)\|^2$. By Kirwan's key lemma (Lemma 2.29), it takes its minimal value on a connected subset $\mathrm{Min}(\tilde{\psi})$ of $\tilde{M}$.

Let $X \subset \tilde{M}$ be the subset

$$X := (\{-\xi\} \times M) \cap \mathrm{Min}(\tilde{\psi}).$$

We claim that

$$\{-\xi\} \times \mathrm{Min}(\psi) = X. \tag{32}$$

Indeed, note that by K-invariance of the norm $\|\cdot\|$, and by K-equivariance of μ,

$$\tilde{\psi}(-\mathrm{Ad}^*(k)\xi, n) = \|\mu(n) - \mathrm{Ad}^*(k)\xi\|^2 = \|\mu(k^{-1}n) - \xi\|^2,$$

for all $n \in M$ and $k \in K$. Hence if $m \in \mathrm{Min}(\psi)$, then for all n and k:

$$\tilde{\psi}(-\mathrm{Ad}^*(k)\xi, n) \geq \tilde{\psi}(-\xi, m),$$

so that $(-\xi, m) \in \mathrm{Min}(\tilde{\psi})$.

Conversely, if $(-\xi, m) \in \mathrm{Min}(\tilde{\psi})$, then for all $n \in N$:

$$\psi(n) = \tilde{\psi}(\xi, n) \geq \tilde{\psi}(-\xi, m) = \psi(m),$$

which completes the proof of (32).

It remains to show that the subset X of $\tilde{M}$ is connected. Let $(-\xi, m_0)$ and $(-\xi, m_1)$ be two elements of X. Because $\mathrm{Min}(\tilde{\psi})$ is connected, there is a curve

$$\gamma : [0, 1] \to \mathrm{Min}(\tilde{\psi}),$$

such that $\gamma(0) = (-\xi, m_0)$ and $\gamma(1) = (-\xi, m_1)$. We write

$$\gamma(t) = (\zeta_t, m_t).$$

Then $\zeta_0 = \zeta_1 = -\xi$.

Because the principal fibre bundle

$$K \to M_{-\xi}$$

is locally trivial, there is a curve k_t, $t \in [0, 1]$, such that for all t:

$$\zeta_t = -\mathrm{Ad}^*(k_t)\xi,$$

and $k_0 = e$.

Consider the curve

$$\delta : [0, 1] \to X,$$

defined by

$$\delta(t) = (-\xi, k_t^{-1} m_t).$$

Its image indeed lies in X, because for all t we have

$$\tilde{\psi}(\delta(t)) = \|\mu(k_t^{-1}m_t) - \xi\|^2 = \|\mu(m_t) + \zeta_t\|^2 = \tilde{\psi}(\zeta_t, m_t),$$

and by definition of γ, $(\zeta_t, m_t) \in \mathrm{Min}(\tilde{\psi})$.

The curve δ satisfies

$$\begin{aligned} \delta(0) &= (-\xi, m_0); \\ \delta(1) &= (-\xi, k_1^{-1}m_1). \end{aligned}$$

Note that k_1 is an element of $K_{-\xi}$, the stabiliser of ξ, which is connected. Therefore, the curve δ can be extended to a curve in X connecting $(-\xi, m_0)$ to $(-\xi, m_1)$. □

Proof of the nonabelian convexity theorem. We have already seen that $\mu(M) \cap \mathfrak{t}_+^*$ is a polyhedron, and it remains to prove that it is convex.

Suppose it is not. Then there is a point $\xi \in \mathfrak{t}_+^*$ and a radius r, such that the sphere around ξ of radius r is tangent to the boundary of $\mu(M) \cap \mathfrak{t}_+^*$ in more than one point (see Figure 2).

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Figure 2: The sphere of radius r around ξ touches $\mu(M_+)$ in two points.

For this ξ, the function

$$m \mapsto \|\mu(m) - \xi\|^2$$

attains its minimal value in two or more disctinct points. This is in contradiction with Corollary 2.30. □

Remark 2.31. Let $\xi \in \mu(M)$ be such that $\|\xi\|^2$ is the minimal value of the function $\|\mu\|^2$. It follows from Lemma 2.29 that for such ξ, the set

$$\mu^{-1}(\xi) = \mathrm{Min}(\|\mu\|^2)$$

is connected. One can prove that $\mu^{-1}(\xi)$ is in fact connected for arbitrary $\xi \in \mathfrak{k}^*$. This implies that $M^\xi = \mu^{-1}(\xi)/K_\xi$ is also connected.

Remark 2.32. The convexity theorem remains valid if we replace the assumption that M is compact by the assumption that $\mu : M \to \mathfrak{k}^*$ is *proper* (but M is still supposed to be connected). An important example is the following. Let G be a connected, real, linear, semisimple Lie group, and let $K < G$ be a maximal compact subgroup. The inclusion map $\mathfrak{k} \hookrightarrow \mathfrak{g}$ transposes to the projection $\mathfrak{g}^* \to \mathfrak{k}^*$.

Let $\xi \in \mathfrak{g}^*$, and let $M := G \cdot \xi$ be the coadjoint orbit through ξ. Then by Examples 1.13 and 1.8, the coadjoint action of K on M is Hamiltonian, with momentum map

$$\mu : M \hookrightarrow \mathfrak{g}^* \to \mathfrak{k}^*.$$

One can show that if the orbit M is a closed subset of $\mathfrak{g}^*$, then μ is proper.

Hence if M is closed, then $\mu(M) \cap C$ is a convex polyhedral region, which of course no longer needs to be compact.

3 Geometric quantisation

Geometric quantisation is a procedure that assigns a quantum mechanical phase space (Hilbert space) to a classical mechanical phase space (symplectic manifold). If the symplectic manifold is equipped with a suitable Hamiltonian group action, then the corresponding Hilbert space carries a representation of the group in question. The 'quantisation commutes with reduction' priciple (often abbreviated as '$[Q, R] = 0$') states that the geometric quantisation of the symplectic reduction of a symplectic manifold by a Hamiltonian group action is isomorphic to the space of invariant vectors in the geometric quantisation of the original manifold.

The geometric quantisation of a symplectic manifold will be realised as a space of sections of a certain line bundle. Later, geometric quantisation will be defined as the index of an elliptic differential operator. In special cases, this will reduce to the first definition. Note that the index of an operator is only a virtual Hilbert space (i.e. the formal difference of two Hilbert spaces).

3.1 Differential geometry of line bundles

Let M be a smooth manifold, and let $L \to M$ be a smooth complex line bundle over M. The space of smooth sections of L is denoted by $\Gamma^\infty(M, L)$. The space of smooth differential forms on M of degree k, with coefficients in L, is the space

$$\Omega^k(M; L) := \Gamma^\infty(M, \textstyle\bigwedge^k T^*M \otimes L).$$

Definition 3.1. A *connection* on L is a linear map

$$\nabla : \Gamma^\infty(M, L) \to \Omega^1(M; L)$$

such that for all $f \in C^\infty(M)$ and $s \in \Gamma^\infty(M, L)$,

$$\nabla(fs) = df \otimes s + f\nabla s. \tag{33}$$

The property (33) is called the *Leibniz rule* for ∇.

If $\langle \cdot, \cdot \rangle$ is a Hermitian metric on L, then a connection ∇ on L is called *unitary* if for all $s, t \in \Gamma^\infty(M, L)$,

$$d\langle s, t\rangle = \langle \nabla s, t\rangle + \langle s, \nabla t\rangle \in \Omega^1(M).$$

If X is a vector field on M, and ∇ a connection on L, their contraction is the *covariant derivative* ∇_X, defined by

$$\nabla_X : \Gamma^\infty(M, L) \to \Gamma^\infty(M, L), \tag{34}$$
$$\nabla_X s := X \lrcorner \nabla s, \tag{35}$$

for all sections $s \in \Gamma^\infty(M, L)$.

A connection ∇ on L can be extended uniquely to a linear map

$$\nabla : \Omega^k(M; L) \to \Omega^{k+1}(M; L),$$

such that for all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega(M; L)$, the following generalised Leibniz rule holds:

$$\nabla(\alpha \wedge \beta) = \alpha \wedge \nabla\beta + (-1)^k d\alpha \wedge \beta.$$

A consequence of this Leibniz rule is that the square of ∇,

$$\nabla^2 : \Omega^k(M; L) \to \Omega^{k+2}(M; L),$$

is a $C^\infty(M)$-linear mapping. Hence it is given by multiplication by a certain two-form.

Definition 3.2. The *curvature (form)* of a connection ∇ on L is the two-form

$$\omega \in \Omega^2_{\mathbb{C}}(M) := \Gamma^\infty(M, \textstyle\bigwedge^2 T^*M \otimes \mathbb{C})$$

such that for all $s \in \Gamma^\infty(M, L)$,

$$\nabla^2 s = 2\pi i \omega \otimes s. \tag{36}$$

An equivalent formulation of (36) is that for all vector fields X and Y on M, the $C^\infty(M)$-linear map

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} : \Gamma^\infty(M, L) \to \Gamma^\infty(M, L) \tag{37}$$

is multiplication by the function $2\pi i \omega(X, Y)$.

It turns out that ω is real, closed (the *Bianchi identity*), and that the cohomology class

$$[\omega] \in H^2_{\mathrm{dR}}(M)$$

is *integral*. That is, for all compact, two-dimensional submanifolds $S \subset M$,

$$\int_S \omega \in \mathbb{Z}.$$

Conversely, we have the following theorem. For a proof, see Woodhouse [25].

Theorem 3.3 (Weil)**.** *Let M be a smooth manifold, ω a real, closed two-form on M, with integral cohomology class $[\omega] \in H^2_{\mathrm{dR}}(M)$.*

Then there is a line bundle $L \to M$, with a Hermitian metric $\langle \cdot, \cdot \rangle$, and a unitary connection ∇ whose curvature form is ω.

Remark 3.4. Consider the Lie algebra $\mathrm{Vect}(M)$ of vector fields on M. By formula (37), the map

$$\begin{gathered} X \to \nabla_X, \\ \mathrm{Vect}(M) \to \mathrm{End}\left(\Gamma^\infty(M,L)\right), \end{gathered}$$

defines a representation of $\mathrm{Vect}(M)$ on $\Gamma^\infty(M,L)$ if and only if the curvature of ∇ vanishes.

Suppose M is oriented, and let ν be a volume form on M that is positive with respect to the given orientation. For $X \in \mathrm{Vect}(M)$, the *divergence* of X is defined as the function $\mathrm{div}(X)$ on M for which

$$\mathcal{L}_X(\nu) = \mathrm{div}(X)\nu.$$

The divergence-free vector fields on M form a Lie subalgebra $\mathrm{Vect}_0(M)$ of $\mathrm{Vect}(M)$.

Consider the pre-Hilbert space $\Gamma_c^\infty(M,L)$ of compactly supported smooth sections of L, with the inner product

$$(s,t) := \int_M \langle s(m), t(m)\rangle \nu,$$

for $s,t \in \Gamma_c^\infty(M,L)$. If the curvature of ∇ is zero, then the map $X \mapsto \nabla_X$ defines a representation of $\mathrm{Vect}_0(M)$ on $\Gamma_c^\infty(M,L)$. This representation is formally unitary, in the sense that for all $X \in \mathrm{Vect}_0(M)$,

$$\nabla_X^* = -\nabla_X.$$

Proof. Let $X \in \mathrm{Vect}_0(M)$. Then by definition,

$$\mathcal{L}_X \nu = 0.$$

Let $s,t \in \Gamma_c^\infty(M,L)$. Because the connection ∇ is unitary, we have

$$\begin{aligned} \mathcal{L}_X \langle s,t\rangle &= X(\langle s,t\rangle) \\ &= \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle. \end{aligned}$$

Since $\langle s,t\rangle\nu$ is a volume form, its Lie derivative $\mathcal{L}_X\left(\langle s,t\rangle\nu\right)$ is exact, so that its integral vanishes. Therefore,

$$\begin{aligned} 0 &= \int_M \mathcal{L}_X\left(\langle s,t\rangle\nu\right) \\ &= \int_M \left(\langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle\right)\nu + \int_M \langle s,t\rangle \mathcal{L}_X \nu \\ &= (\nabla_X s, t) + (s, \nabla_X t). \end{aligned}$$

□

3.2 Prequantisation

Let (M, ω) be a symplectic manifold. Recall Definition 1.5 of Hamiltonian vector fields and Poisson brackets. The map

$$f \mapsto H_f$$

is a Lie algebra homomorphism from the Poisson algebra $C^\infty(M, \omega)$ of (M, ω) to the Lie algebra $\mathrm{Vect}(M)$ of vector fields on M:

Lemma 3.5. *For all $f, g \in C^\infty(M)$,*

$$[H_f, H_g] = H_{\{f,g\}}.$$

Sketch of proof. In local Darboux coordinates (q, p), we have

$$\omega = \sum_k dp_k \wedge dq_k.$$

From the relation

$$-H_f \lrcorner \omega = df = \sum_k \frac{\partial f}{\partial q_k} dq_k + \frac{\partial f}{\partial p_k} dp_k$$

one deduces that

$$H_f = \sum_k \frac{\partial f}{\partial p_k} \frac{\partial}{\partial q_k} - \frac{\partial f}{\partial q_k} \frac{\partial}{\partial p_k}.$$

Therefore,

$$\{f, g\} = \sum_k \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} - \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k}.$$

The lemma now follws by a direct computation. □

The kernel of the map $f \mapsto H_f$ is the space of locally constant functions, which is isomorphic to $\mathbb{R}$ if M is connected. Its image is by definition the Lie subalgebra $\mathrm{Ham}(M, \omega)$ of Hamiltonian vector fields on M.

Example 3.6. Suppose $\dim M = 2$, and consider the local functions

$$f(q, p) = p^2/2$$
$$g(q, p) = q^2/2.$$

Then

$$\{f, g\}(q, p) = pq,$$

so that

$$H_{\{f,g\}}(p, q) = q\frac{\partial}{\partial q} - p\frac{\partial}{\partial p}.$$

On the other hand,

$$H_f(q,p) = p\frac{\partial}{\partial q},$$
$$H_g(q,p) = -q\frac{\partial}{\partial p},$$

so that

$$[H_f, H_g](p,q) = -p\frac{\partial}{\partial p} + q\frac{\partial}{\partial q}.$$

From now on, assume that the cohomology class $[\omega] \in H^2_{\mathrm{dR}}(M)$ is integral. By Theorem 3.3, there is a complex line bundle $L \to M$, with a Hermitian metric $\langle \cdot, \cdot \rangle$, and a unitary connection ∇, such that

$$\nabla^2 = 2\pi i \omega.$$

Definition 3.7. The triple $(L, \langle \cdot, \cdot \rangle, \nabla)$ is called a *prequantum line bundle* for (M, ω).

Definition 3.8. Let $(L, \langle \cdot, \cdot \rangle, \nabla)$ be a prequantum line bundle for (M, ω). Let $f \in C^\infty(M)$, and consider the linear operator $P(f)$ on $\Gamma^\infty(M, L)$, defined by

$$P(f) := \nabla_{H_f} - 2\pi i f. \tag{38}$$

It is called the *prequantisation operator* of the function f.

The linear map

$$P : C^\infty(M) \to \mathrm{End}\,(\Gamma^\infty(M, L))$$

defined by (38), is called *prequantisation*.

Theorem 3.9 (Kostant & Souriau, 1970)**.** *Prequantisation defines a Lie algebra representation of the Poisson algebra $C^\infty(M,\omega)$ on the vector space $\Gamma^\infty(M, L)$.*

Proof. The claim is that for all $f, g \in C^\infty(M)$,

$$[P(f), P(g)] = P(\{f, g\}).$$

This follows by a straightforward computation:

$$\begin{array}{rcll} P(f)P(g) & = & \nabla_{H_f}\nabla_{H_g} - 2\pi i f \nabla_{H_g} - 2\pi i g \nabla_{H_f} - 2\pi i H_f(g) - 4\pi^2 fg & \\ P(g)P(f) & = & \nabla_{H_g}\nabla_{H_f} - 2\pi i g \nabla_{H_f} - 2\pi i f \nabla_{H_g} - 2\pi i H_g(f) - 4\pi^2 gf & - \\ \hline [P(f), P(g)] & = & [\nabla_{H_f}, \nabla_{H_g}] - 2\pi i H_f(g) + 2\pi i H_g(f) & \\ & = & \nabla_{[H_f, H_g]} + 2\pi i \omega(H_f, H_g) - 2\pi i H_f(g) + 2\pi i H_g(f), & \end{array}$$

because $2\pi i \omega$ is the curvature of ∇,

$$= \nabla_{H_{\{f,g\}}} - 2\pi i \{f, g\},$$

by Lemma 3.5 and the definition of the Poisson bracket,

$$= P(\{f, g\}).$$

□

Remark 3.10. The representation of $C^\infty(M,\omega)$ on $\Gamma_c^\infty(M,L)$ is *formally unitary*, with respect to the inner product given by

$$(s,t) := \int_M \langle s,t\rangle \frac{\omega^n}{n!}, \tag{39}$$

where $s,t \in \Gamma_c^\infty(M,L)$, and $n = \dim M/2$. The form $\frac{\omega^n}{n!}$ is called the *Liouville volume form* on (M,ω).

More precisely, for $f \in C^\infty(M)$ we define the function $f^* \in C^\infty(M)$ by

$$f^*(m) = -\overline{f(m)}.$$

For $A \in \mathrm{End}\left(\Gamma_c^\infty(M,L)\right)$, we define A^* to be the adjoint of A with respect to the inner product (39). Then a computation shows that for all $f \in C^\infty(M)$,

$$P(f^*) = P(f)^*.$$

This notion of unitarity corresponds to the one in Remark 3.4 via the Lie algebra homomorphism $f \mapsto H_f$ from $C^\infty(M,\omega)$ to $\mathrm{Vect}(M)$.

Remark 3.11 (Equivariant prequantum line bundles)**.** Let $G \times M \to M$ be a smooth action of a Lie group G on M. A *linearisation* of this action is a G-equivariant line bundle $L \to M$. That is, a line bundle $L \to M$, equipped with an action of G, such that for all $m \in M$, $l \in L_m$ and $x \in G$,

1. $x \cdot l \in L_{x\cdot m}$,
2. $x : L_m \to L_{x\cdot m}$ is multiplication by a scalar.

Given such a linearisation, we define the *permutation representation* P of G in $\Gamma^\infty(M,L)$ by

$$\left(P(x)s\right)(m) := x \cdot s(x^{-1}m),$$

for $x \in G$, $s \in \Gamma^\infty(M,L)$ and $m \in M$. It is easy to check that $P(xy) = P(x)P(y)$ for all $x,y \in G$ so that P is indeed a representation. The associated infinitesimal Lie algebra representation is given by

$$P : \mathfrak{g} \to \mathrm{End}\left(\Gamma^\infty(M,L)\right) \tag{40}$$

$$\left(P(X)s\right)(m) = \left.\frac{d}{dt}\right|_{t=0} \left(P(\exp tX)s\right)(m). \tag{41}$$

Note that $P(X)$ is a first order differential operator on $\Gamma^\infty(M,L)$.

Now suppose that (M,ω) is a symplectic manifold, $[\omega]$ is an integral cohomology class, and $(L,\ \langle\cdot,\cdot\rangle,\ \nabla)$ is a prequantum line bundle. Suppose that $G \times M \to M$ is a Hamiltonian action, with momentum map μ.

Then the prequantum bundle $(L,\ \langle\cdot,\cdot\rangle,\ \nabla)$ is called *equivariant* if there exists a linearisation $G \times L \to L$ of the action, such that

$$\underbrace{\left.\frac{d}{dt}\right|_{t=0} P(\exp tX) =: P(X)}_{\text{permutation representation}} = \underbrace{P(-\mu_X) := -\nabla_{H_{\mu_X}} + 2\pi i \mu_X}_{\text{prequantisation representation}} \tag{42}$$

There is a minus sign in front of μ_X because the map $\mathfrak{g} \to \mathrm{Vect}(M)$ defined by $X \mapsto X_M$, is an *anti*homomorphism of Lie algebras.

3.3 Prequantisation and reduction

Let (M, ω) be a symplectic manifold, and let $G \times M \to M$ be a Hamiltonian action, with momentum map μ. Suppose that $0 \in \mathfrak{g}^*$ is a regular value of μ, and that the action of G on $\mu^{-1}(0)$ is proper and free. Recall the construction of the symplectic reduction M^0 as in Theorem 1.19:

$$\begin{array}{ccc} \mu^{-1}(0) & \xhookrightarrow{\; i \;} & M \\ \downarrow p & & \\ M^0 := \mu^{-1}(0)/G. & & \end{array}$$

The symplectic form ω^0 on M^0 is defined by

$$p^*\omega^0 = i^*\omega.$$

Suppose that $[\omega]$ is an integral cohomology class, and that there is an equivariant prequantum line bundle $(L, \langle \cdot, \cdot \rangle, \nabla)$. Consider the restriction $L|_{\mu^{-1}(0)}$ of L to the submanifold $\mu^{-1}(0)$ of M. Let $L^0 \to M^0$ be the line bundle such that

$$p^*L^0 = L|_{\mu^{-1}(0)}.$$

The Hermitian metric and connection on L induce a Hermitian metric $\langle \cdot, \cdot \rangle$ and a connection ∇^0 on L^0, such that $(L^0, \langle \cdot, \cdot \rangle, \nabla^0)$ is a prequantum line bundle over the symplectic manifold (M^0, ω^0).

We then have the diagram

$$\begin{array}{ccc} \Gamma^\infty(\mu^{-1}(0), L|_{\mu^{-1}(0)})^G & \xleftarrow{\; i^* \;} & \Gamma^\infty(M, L)^G \\ \cong \uparrow p^* & \swarrow & \\ \Gamma^\infty(M^0, L^0) & & \end{array} \qquad (43)$$

If we call the space $\Gamma^\infty(M, L)^G$ the (quantum) reduction of the prequantisation of (M, ω), symbolically

$$RP(M, \omega) := \Gamma^\infty(M, L)^G,$$

and $\Gamma^\infty(M^0, L^0)$ the prequantisation of the (classical) reduction of (M, ω), symbolically

$$PR(M, \omega) := \Gamma^\infty(M^0, L^0),$$

then diagram (43) yields the following relation.

Corollary 3.12. *There is a natural linear map*

$$\Gamma^\infty(M, L)^G \to \Gamma^\infty(M^0, L^0).$$

Or symbolically,

$$RP \to PR. \qquad (44)$$

Remark 3.13. If the Lie group G is compact, one can prove that the map $RP \to PR$ is surjective. But since it is defined by restriction of G-invariant sections of L to $\mu^{-1}(0)$, it is never injective (if μ is not the zero map). In other words, prequantisation does not commute with reduction. In the next section, we will define a notion of quantisation which does commute with reduction, in the case of compact Lie groups. This is the 'quantisation commutes with reduction' theorem of Guillemin and Sternberg (Theorem 3.20).

3.4 Quantisation

Definition 3.14. Let (V, ω) be a symplectic vector space of dimension $2n$. The symplectic form ω extends complex-linearly to the complexification $V \otimes \mathbb{C}$. A *polarisation* of $V \otimes \mathbb{C}$ is a complex Lagrangian subspace P of $V \otimes \mathbb{C}$. That is, $P^\perp = P$, where $P^\perp$ is the subspace of $V \otimes \mathbb{C}$ orthogonal to P with respect to ω.

The Hermitian form

$$\langle X, Y \rangle := -i\omega(X, \bar{Y}),$$

$X, Y \in P$, has kernel $P \cap \bar{P}$. Let (r, s) be the signature of the induced Hermitian form on $P/\left(P \cap \bar{P}\right)$. The polarisation P is called *real* if $P = \bar{P}$, so that P is the complexification of a real Lagrangian subspace of V. The polarisation P is called *Dolbeault* if $r + s = n$, i.e. $P \cap \bar{P} = 0$. If $r = n$ and $s = 0$, then P is called a *Kähler polarisation*.

Definition 3.15. Let (M, ω) be a symplectic manifold, and let P be a smooth subbundle of the complexified tangent bundle $TM \otimes \mathbb{C}$. Then P is called a *polarisation* of (M, ω) if it has the following properties.

1. The subspace $P_m \subset T_mM \otimes \mathbb{C}$ is a polarisation of $(T_mM \otimes \mathbb{C}, \omega_m)$ for all $m \in M$.

2. The signatures (r_m, s_m) are locally constant on M.

3. The subbundle P of $TM \otimes \mathbb{C}$ in *integrable*. That is, the space of sections of P is closed under the Lie bracket of vector fields.

Example 3.16. Let N be a manifold, and let M be the cotangent bundle T^*N, equipped with the standard symplectic form $\sigma = d\tau$ from Example 1.6. Let $P \subset TM \otimes \mathbb{C}$ be the subbundle

$$P := \ker T_{\mathbb{C}}\pi,$$

where $\pi : T^*N \to N$ denotes the cotangent bundle projection. Then P is a polarisation of (M, σ), called the *vertical* polarisation. Note that

$$P \cong TN \otimes \mathbb{C} \hookrightarrow TM \otimes \mathbb{C}.$$

Example 3.17 (Kähler polarisation)**.** Let M be a complex manifold, and let H be a Hermitian metric on TM. Let B and ω be the real and inaginary parts of H, respectively. The pair (M, H) is called a *Kähler manifold* if $d\omega = 0$. In that case (M, ω) is a symplectic manifold.

Let $J : TM \to TM$ be the complex structure on M. Then

$$B(\cdot, \cdot) = \omega(J\cdot, \cdot)$$

is a Riemannian metric on M. Because B and H are determined by ω and J, we may also denote the Kähler manifold (M, H) by (M, ω, J), or (M, ω) by abuse of notation.

The 2-form ω has degree $(1, 1)$ with respect to J, because the Hermitian metric H is complex linear in the first variable and antilinear in the second.

The *Kähler polarisation* of (M, ω) is the $-i$ eigenspace of J acting on $TM \otimes \mathbb{C}$:

$$P := \{JX - iX; X \in TM\}.$$

A function $f \in C^\infty(M)$ is holomorphic if and only if $Z(f) = 0$ for all $Z \in \Gamma^\infty(M, P)$.

Definition 3.18 (Quantisation I)**.** Let (M, ω) be a compact Kähler manifold, such that $[\omega]$ is an integral cohomology class. Let P be the Kähler polarisation of M, and let $(L, \langle\cdot, \cdot\rangle, \nabla)$ be a prequantum line bundle. Then the *geometric quantisation* of (M, ω) is the finite-dimensional vector space

$$Q(M, \omega) := \{s \in \Gamma^\infty(M, L); \nabla_Z s = 0 \text{ for all } Z \in \Gamma^\infty(M, P)\}.$$

Note that if a group G acts on the manifold M, and if the prequantum line bundle is equivariant, then the space $Q(M, \omega)$ carries a representation of G.

Remark 3.19. We can give the line bundle L the structure of a *holomorphic* line bundle, by saying that its space of holomorphic sections is $Q(M, \omega)$.

Consider the Dolbeault complex on M with coefficients in L:

$$0 \longrightarrow \Omega^{0,0}(M; L) \xrightarrow{\bar\partial \otimes 1_L} \Omega^{0,1}(M; L) \xrightarrow{\bar\partial \otimes 1_L} \ldots \xrightarrow{\bar\partial \otimes 1_L} \Omega^{0,n}(M; L) \longrightarrow 0.$$

Here n is the complex dimension of M. The zeroth cohomology space $H^{0,0}(M; L)$ is the space of holomorphic sections of L, which we defined to be $Q(M, \omega)$. Hence $Q(M, \omega)$ is not the zero space if the line bundle L is sufficiently positive.

Indeed, if $L \otimes \bigwedge^{0,n} TM$ is a positive line bundle, then by Kodaira's vanishing theorem all Dolbeault cohomology spaces $H^{0,k}(M; L)$ vanish for $k > 0$. Then the Hirzebruch-Riemann-Roch theorem expresses the number

$$\sum_{k=0}^{n} (-1)^k \dim H^{0,k}(M; L) = \dim H^{0,0}(M; L)$$

as the integral over M of a certain differential form. If L is positive enough, this number turns out to be nonzero.

3.5 Quantisation commutes with reduction

Let K be a compact Lie group, (M, ω) a Kähler manifold, and $K \times M \to M$ a *holomorphic* Hamiltonian action. Suppose that 0 is a regular value of the momentum map μ, and that K acts freely on the level set $\mu^{-1}(0)$.

Let G be the complexification of K. (As a manifold, $G \cong T^*K = K \times \mathfrak{k}^*$, and on the Lie algebra level $\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}$.) The group G is a complex reductive algebraic group. The action of K on M extends to a holomorphic action of G on M, which however need no longer be Hamiltonian.

Suppose that there is a K-equivariant prequantum line bundle $(L, \langle\cdot,\cdot\rangle, \nabla)$ on M. Then the action of G on M lifts to a holomorphic action of G on L.

The following beautiful theorem of Guillemin and Sternberg expresses that 'quantisation commutes with reduction', or '$[Q, R] = 0$'.

Theorem 3.20 (Guillemin & Sternberg 1982 [9])**.** *Let $Q(M, \omega)^K$ be the subspace of $Q(M, \omega)$ consisting of the K-invariant sections. Let (M^0, ω^0) be the symplectic reduction of M by the action of K (see Theorem 1.19). Then*

$$Q(M, \omega)^K \cong Q(M^0, \omega^0).$$

Taking the subspace of K-fixed vectors of a representation space of K may be interpreted as quantummechanical reduction. So Guillemin & Sternberg's theorem indeed states that 'quantum reduction after geometric quantisation equals geometric quantisation after symplectic reduction'. This can be visualised as follows:

$$\begin{array}{ccc} (M,\omega) & \overset{Q}{\longmapsto} & Q(M,\omega) \\ \downarrow R & & \downarrow R \\ (M^0,\omega^0) & \overset{Q}{\longmapsto} & Q(M^0,\omega^0) \cong Q(M,\omega)^K \end{array}.$$

Proof. The space of sections of the line bundle $L^0 \to M^0$ satisfies

$$p^*\Gamma^\infty(M^0, L^0) = \Gamma^\infty(\mu^{-1}(0), L|_{\mu^{-1}(0)})^K,$$

where

$$p : \mu^{-1}(0) \twoheadrightarrow \mu^{-1}(0)/K$$

is the quotient map. Let

$$i^* : \Gamma^\infty(M, L)^K \twoheadrightarrow \Gamma^\infty(\mu^{-1}(0), L|_{\mu^{-1}(0)})^K$$

be restriction of K-invariant sections of L to $\mu^{-1}(0)$. Consider the linear map

$$(p^*)^{-1} \circ i^* : \Gamma^\infty(M, L)^K \to \Gamma^\infty(M^0, L^0). \tag{45}$$

It follows from the definition of the connection ∇^0 on L^0 that $(p^*)^{-1} \circ i^*$ maps $Q(M, \omega)^K$ into $Q(M^0, \omega^0)$. It therefore makes sense to define the linear map

$$\psi : Q(M, \omega)^K \to Q(M^0, \omega^0)$$

as the restriction of (45). We claim that this is the desired isomorphism.

Injectivity of ψ. Let an $\mathrm{Ad}^*(K)$-invariant inner product on $\mathfrak{k}$ be given, and consider the function $\|\mu\|^2$ on M. The level set $\mu^{-1}(0)$ is a connected component of the critical locus of $\|\mu\|^2$. Let $S_{\mu^{-1}(0)}$ be its stable manifold. Since $\|\mu\|^2$ is a Morse-Kirwan function with even-dimensional strata (Kirwan [15], pp. 44–68), and has a local minimum at $\mu^{-1}(0)$, the stratum $S_{\mu^{-1}(0)}$ is an open dense subset of M (see Remark 2.13).

We claim that $G \cdot \mu^{-1}(0)$ is an open dense subset of M. We will prove this claim by showing that the open dense stratum $S_{\mu^{-1}(0)}$ is contained in $G \cdot \mu^{-1}(0)$.

By Lemma 3.21, we have

$$\operatorname{grad} \mu_X = JX,$$

for all $X \in \mathfrak{k}$. Let $\{X_i\}$ be an orthonormal basis of $\mathfrak{k}$. Then

$$\begin{aligned}\operatorname{grad} \|\mu\|^2 &= \operatorname{grad} \sum_i \mu_{X_i}^2 \\ &= \sum_i 2\mu_{X_i} \operatorname{grad} \mu_{X_i} \\ &= \sum_i 2\mu_{X_i} JX_i.\end{aligned}$$

Therefore, at every point $m \in M$ there is an $X \in \mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ such that

$$\operatorname{grad} \|\mu\|^2(m) = X_m.$$

We conclude that the stable manifold $S_{\mu^{-1}(0)}$ is indeed contained in $G \cdot \mu^{-1}(0)$.

If two sections $s, t \in Q(M, \omega)^K$ are mapped to the same section in $Q(M^0, \omega^0)$, then by injectivity of $(p^*)^{-1}$, we must have

$$i^*s = i^*t.$$

In other words, the restrictions of s and t to $\mu^{-1}(0)$ coincide. Because s and t are K-invariant, they are also G-invariant, so that they coincide on the dense subset $G \cdot \mu^{-1}(0)$ of M. So $s = t$.

Surjectivity of ψ. Let $\sigma \in Q(M^0, \omega^0)$ be given. Consider the section

$$s := p^*\sigma \in \Gamma^\infty(\mu^{-1}(0), L|_{\mu^{-1}(0)})^K.$$

We claim that s extends to a K-invariant section of L, defined on the whole manifold M. Then $i^*s = p^*\sigma$, which is to say that $\psi(s) = \sigma$.

To show that s extends to all of M, we first note that it extends G-invariantly to a holomorphic section defined on the open dense subset $G \cdot \mu^{-1}(0)$ of M. It is therefore enough to prove that s is bounded on $G \cdot \mu^{-1}(0)$.

To prove boundedness, we use Lemma 3.22: for all $X \in \mathfrak{k}$, we have

$$JX\left(\|s\|^2\right) = -4\pi\mu_X \|s\|^2. \tag{46}$$

This implies that the norm function $\|s\|^2$ on $G\cdot\mu^{-1}(0)$ is maximal on $\mu^{-1}(0)$. Indeed, the function μ_X is increasing in the direction JX, because $JX = \operatorname{grad}\mu_X$ (Lemma 3.21). So the right hand side of (46) is a nonpositive function on $G\cdot\mu^{-1}(0)$, from which we deduce that the function $\|s\|^2$ decreases away from $\mu^{-1}(0)$. We conclude that s is indeed a bounded section. □

Lemma 3.21. *For all $X \in \mathfrak{k}$, we have the following equality of vector fields on M:*

$$JX = \operatorname{grad}\mu_X.$$

Proof. Let $X \in \mathfrak{k}$, and let Y be a vector field on M. Then by definition of the Riemannian metric

$$B(\cdot,\cdot) = \omega(J\cdot,\cdot)$$

and the defining property of the momentum map μ, we have

$$\begin{aligned} B(JX,Y) &= \omega(J^2X,Y) \\ &= \langle -X \lrcorner \omega, Y\rangle \\ &= \langle d\mu_X, Y\rangle \\ &= B(\operatorname{grad}\mu_X, Y). \end{aligned}$$

□

Lemma 3.22. *Let $U \subset M$ be an open, K-invariant subset. Let $s \in \Gamma^\infty(U, L|_U)$ be a K-invariant local section of L, such that for all $Z \in \Gamma^\infty(U, P|_U)$,*

$$\nabla_Z s = 0.$$

(One might say that s is a local section in $Q(M,\omega)^K$.)

Let $X \in \mathfrak{k}$. Then

$$JX\left(\|s\|^2\right) = -4\pi\mu_X\|s\|^2.$$

Proof. Let s and X be as above. Because the connection ∇ is unitary, we have

$$JX\left(\|s\|^2\right) = \langle\nabla_{JX}s, s\rangle + \langle s, \nabla_{JX}s\rangle. \tag{47}$$

And since s is holomorphic:

$$\nabla_{(JX-iX)}s = 0,$$

we see that (47) equals

$$-2\operatorname{Im}\left(\langle\nabla_X s, s\rangle\right). \tag{48}$$

Next, note that s is K-invariant, so that

$$P(X)s = 0,$$

where P denotes the infinitesimal permutation representation of $\mathfrak{k}$ in $\Gamma^\infty(M,L)$. By definition of equivariant prequantum line bundles (42), this implies that

$$\left(-\nabla_X + 2\pi i\mu_X\right)s = 0.$$

Therefore, (48) equals

$$-2\,\mathrm{Im}\,(2\pi i\mu_X)\langle s,s\rangle) = -4\pi\mu_X\|s\|^2.$$

$\square$

Remark 3.23 (Quantisation commutes with reduction and convexity theorems)**.** Let K be a compact Lie group acting symplectically on a compact Kähler manifold (M,ω). Suppose that the action is Hamiltonian, free, and that it admits an equivariant prequantum line bundle L. Suppose that L is positive, so that the induced line bundles L^ξ over the reduced phase spaces M^ξ are also positive (see Guillemin & Sternberg [9]).

Let $Q(M,\omega)$ be the geometric quantisation of (M,ω) as in Definition 3.18. Then $Q(M,\omega)$ is a finite-dimensional representation of the compact Lie group K. We fix a maximal torus T of K, and a positive Weyl chamber $\mathfrak{t}^*_+$ inside $\mathfrak{t}^*$. The set of positive integral elements of $2\pi i\mathfrak{t}^*$ is denoted by P_+. The representation $Q(M,\omega)$ decomposes as

$$Q(M,\omega) = \bigoplus_{\lambda\in P_+}[Q(M,\omega):\pi_\lambda]\pi_\lambda.$$

Here the $[Q(M,\omega):\pi_\lambda]$ are nonnegative integers, and π_λ denotes the irreducible representation of K with highest weight λ. The 'product' $[Q(M,\omega):\pi_\lambda]\pi_\lambda$ is the direct sum of $[Q(M,\omega):\pi_\lambda]$ copies of π_λ.

Schur's lemma implies that

$$[Q(M,\omega):\pi_\lambda] = \dim\left(\mathrm{Hom}_K(Q(M,\omega),\pi_\lambda)\right).$$

Fix an element $\lambda\in P_+$, and let $\xi\in\mathfrak{t}^*$ be the element such that $\lambda = 2\pi i\xi$. Let (M_ξ,ω_ξ) be the coadjoint orbit of K in $\mathfrak{k}^*$ through ξ, equipped with its standard symplectic form (see Example 1.13). The Borel-Weil theorem (Duistermaat & Kolk [3], Theorem 4.12.5) states that

$$Q(M_\xi,\omega_\xi) = \pi_\lambda.$$

Now

$$\begin{aligned}\mathrm{Hom}_K(Q(M,\omega),\pi_\lambda) &= \left(Q(M,\omega)\otimes\pi_\lambda^*\right)^K\\ &= \left(Q(M,\omega)\otimes\pi_{-w_0\lambda}\right)^K\\ &= Q(M\times M_{-\xi},\omega\times\omega_{-\xi})^K.\end{aligned}$$

Here $w_0\in W$ is the longest Weyl group element.

By the Guillemin-Sternberg theorem (Theorem 3.20), the latter vector space is isomorphic to

$$Q\left((M\times M_{-\xi})^0,(\omega\times\omega_{-\xi})^0\right).$$

Applying the shifting trick (Remark 1.22), we conclude that

$$[Q(M,\omega):\pi_\lambda] = \dim Q(M^\xi,\omega^\xi). \tag{49}$$

Recall that (M^ξ, ω^ξ) is the symplectic reduction of (M, ξ) at ξ (see Marsden & Weinstein's theorem 1.19).

Because the line bundle L^ξ over M^ξ is positive, the Hirzebruch-Riemann-Roch theorem can be used to express the number (49) as the integral over M^ξ of a certain differential form (see Remark 3.19). Hence if $[Q(M,\omega) : \pi_\lambda]$ is nonzero, then necessarily M^ξ is nonempty. In other words, $\mu^{-1}(\xi) \neq \emptyset$, or, because μ is K-equivariant,

$$M_\xi \subset \mu(M).$$

The converse implication

$$M_\xi \subset \mu(M) \quad \Rightarrow \quad [Q(M,\omega) : \pi_\lambda] > 0 \tag{50}$$

is not valid for all λ. But for 'generic' integral λ, it does hold.

Next, consider the decomposition

$$Q(M, 2\omega) = \bigoplus_{\lambda \in P_+} [Q(M, 2\omega) : \pi_\lambda] \pi_\lambda.$$

We claim that for all $\lambda, \nu \in P_+$ such that $[Q(M,\omega) : \pi_\lambda] > 0$ and $[Q(M,\omega) : \pi_\nu] > 0$, we have

$$[Q(M, 2\omega) : \pi_{\lambda+\nu}] > 0.$$

Indeed, let s_λ and s_ν be elements of $Q(M,\omega)$, contained in a copy of π_λ and π_ν, respectively. Suppose that s_λ and s_ν are highest weight vectors in the respective irreducible representations. Then the section $s_\lambda s_\nu$ of $L \otimes L$, defined by

$$s_\lambda s_\nu(m) = s_\lambda(m) \otimes s_\nu(m),$$

is an element of $Q(M, 2\omega)$ (see Remark 1.31). It is nonzero, because s_λ and s_ν are nonzero and holomorphic, and M is connected. Furthermore, the section $s_\lambda s_\nu$ is annihilated by all positive root spaces in the Lie algebra $\mathfrak{k}$ of K, and the maximal torus $\mathfrak{t}$ acts on $s_\lambda s_\nu$ with weight $\lambda + \nu$. So the representation $Q(M, 2\omega)$ contains an irreducible subrepresentation of highest weight $\lambda + \nu$. In other words,

$$[Q(M, 2\omega) : \pi_{\lambda+\nu}] > 0.$$

As we noted above, the condition that $M_\xi \subset \mu(M)$ implies that $[Q(M,\omega) : \pi_\lambda] > 0$, for 'most' $\lambda \in P_+$. Let $\lambda, \nu \in P_+$ for which this implication holds be given. Write $\lambda = 2\pi i \xi$, $\nu = 2\pi i \zeta$, for $\xi, \zeta \in \mathfrak{t}^*$. For such λ and ν, we have found that $M_\xi \subset \mu(M)$ and $M_\zeta \subset \mu(M)$ implies that $[Q(M, 2\omega : \pi_{\lambda+\nu})] > 0$. Noting that 2μ is a momentum map for the action of K on $(M, 2\omega)$, and that $\frac{1}{2} M_{\xi+\zeta} = M_{\frac{\xi+\zeta}{2}}$, we conclude that

$$M_{\frac{\xi+\zeta}{2}} \subset \mu(M).$$

This statement can easily be generalised to the case where λ and ν are not necessarily integral, but positive rational linear combinations of elements of P_+. The set of such elements for which the implication (50) holds is an open dense subset of the set of all positive rational linear combinations of elements of P_+. This is a strong indication of the convexity of $\mu(M) \cap \mathfrak{t}^*_+$.

3.6 Generalisations

If the line bundle L is not sufficiently positive, then Guillemin and Sternberg's theorem may reduce to the equality $0 = 0$. To remedy this, we redefine quantisation as follows.

Definition 3.24 (Quantisation II)**.** Let (M, ω) be a compact Kähler manifold, suppose that $[\omega]$ is an integral cohomology class, and let $(L, \langle \cdot, \cdot \rangle, \nabla)$ be a prequantum line bundle. We define the *geometric quantisation* of (M, ω) as the virtual vector space

$$Q(M, \omega) := \sum_{k=0}^{n} (-1)^k H^{0,k}(M; L),$$

the alternating sum of the Dolbeault cohomology spaces of M with coeffiecients in L.

If the line bundle L is positive enough, then the definition of quantisation agrees with the previous one (see Remark 3.19).

Definition 3.24 may be reformulated in a way that makes sense even when the manifold M is not Kähler.

Definition 3.25 (Quantisation III)**.** Let (M, ω) be a compact symplectic manifold. Suppose that $[\omega]$ is an integral cohomology class, and let $(L, \langle \cdot, \cdot \rangle, \nabla)$ be a prequantum line bundle. Let J be an almost complex structure on TM such that $\omega(J \cdot, \cdot)$ is a Riemannian metric on M. Such a J always exists (see for example Guillemin, Ginzburg & Karshon [5], pp. 111-112).

As we noted before, the connection ∇ on L defines a differential operator

$$\nabla : \Omega^k(M; L) \to \Omega^{k+1}(M; L),$$

such that for all $\alpha \in \Omega^k(M)$ and $s \in \Gamma^\infty(M, L)$,

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla s.$$

Consider the projection

$$\pi^{0,*} : \Omega^*_{\mathbb{C}}(M; L) \to \Omega^{0,*}(M; L).$$

Define the differential operator

$$\bar{\partial}_L : \Omega^{0,q}(M; L) \to \Omega^{0,q+1}(M; L)$$

by

$$\bar{\partial}_L := \pi^{0,*} \circ \nabla.$$

The *Dolbeault-Dirac operator* is the elliptic differential operator

$$\bar{\partial}_L + \bar{\partial}_L^* : \Omega^{0,*}(M; L) \to \Omega^{0,*}(M; L).$$

The *geometric quantisation* of (M, ω) is defined as the virtual vector space

$$Q(M,\omega) := \ker\left((\bar{\partial}_L + \bar{\partial}_L^*) \,|_{\Omega^{0,\mathrm{even}}(M;L)} \right) - \ker\left((\bar{\partial}_L + \bar{\partial}_L^*) \,|_{\Omega^{0,\mathrm{odd}}(M;L)} \right).$$

In other words, $Q(M, \omega)$ is the index of the operator

$$\bar{\partial}_L + \bar{\partial}_L^* : \Omega^{0,\mathrm{even}}(M;L) \to \Omega^{0,\mathrm{odd}}(M;L).$$

Because this operator is elliptic and M is compact, the index is well-defined.

Remark 3.26 (Quantisation III for Kähler manifolds)**.** If M is a complex manifold, and L is a holomorphic line bundle over M, then we can define the elliptic differential operator

$$(\bar{\partial} + \bar{\partial}^*) \otimes 1_L : \Omega^{0,*}(M;L) \to \Omega^{0,*}(M;L) \tag{51}$$

as follows. Locally, one has

$$\Omega^{0,q}(U; L|_U) \cong \Omega^{0,q}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U, L|_U).$$

Here U is an open subset of M over which L trivialises, $\mathcal{O}(U)$ denotes the space of holomorphic functions on U, and $\mathcal{O}(U, L|_U)$ is the space of holomorphic sections of L on U. Because (by definition) $\bar{\partial} f = 0$ for holomorphic functions f, we can locally define the differential operator

$$\bar{\partial} \otimes 1_L : \Omega^{0,q}(U; L|_U) \to \Omega^{0,q+1}(U; L|_U),$$

by

$$\bar{\partial} \otimes 1_L(\alpha \otimes s) = \bar{\partial}\alpha \otimes s,$$

for all $\alpha \in \Omega^{0,q}(U)$ and $s \in \mathcal{O}(U, L|_U)$. These local operators patch together to a globally defined operator

$$\bar{\partial} \otimes 1_L : \Omega^{0,q}(M;L) \to \Omega^{0,q+1}(M;L),$$

from which we can define the operator (51) by

$$(\bar{\partial} + \bar{\partial}^*) \otimes 1_L := \bar{\partial} \otimes 1_L + (\bar{\partial} \otimes 1_L)^*.$$

If (M, ω) is a compact Kähler manifold that admits a prequantum line bundle $(L, \langle \cdot, \cdot \rangle, \nabla)$, then the Dolbeault-Dirac operator $\bar{\partial}_L + \bar{\partial}_L^*$ turns out to have the same principal symbol, and hence the same index, as the operator $(\bar{\partial} + \bar{\partial}^*) \otimes 1_L$. So for Kähler manifolds, Definition 3.25 may be rephrased as

$$Q(M,\omega) := \operatorname{index}\left((\bar{\partial} + \bar{\partial}^*) \otimes 1_L : \Omega^{0,\mathrm{even}}(M;L) \to \Omega^{0,\mathrm{odd}}(M;L) \right).$$

Lemma 3.27. *If (M, ω) is a Kähler manifold, then Definitions II and III of geometric quantisation agree.*

Proof. Note that

$$\begin{aligned} H^{0,k}(M;L) &= \ker\left(\bar\partial^k \otimes 1_L\right) / \operatorname{im}\left(\bar\partial^{k-1} \otimes 1_L\right) \\ &\cong \ker\left(\bar\partial^k \otimes 1_L\right) \cap \left(\operatorname{im}\left(\bar\partial^{k-1} \otimes 1_L\right)\right)^{\perp} \\ &= \ker\left(\bar\partial^k \otimes 1_L\right) \cap \ker\left(\bar\partial^{k-1} \otimes 1_L\right)^{*} \\ &= \ker\left(\left(\bar\partial^k + \left(\bar\partial^{k-1}\right)^{*}\right) \otimes 1_L\right), \end{aligned}$$

because the images of $\bar\partial^k$ and $\left(\partial^{k-1}\right)^*$ lie in different spaces.

We conclude that

$$\begin{aligned} H^{0,\mathrm{even}}(M;L) &= \bigoplus_{k \text{ even}} \ker\left(\bar\partial^k + \left(\bar\partial^{k-1}\right)^{*}\right) \otimes 1_L \\ &= \ker\left(\left(\bar\partial + \bar\partial^*\right) \otimes 1_L|_{\Omega^{0,\mathrm{even}}(M;L)}\right). \end{aligned}$$

On the other hand, note that

$$\begin{aligned} H^{0,\mathrm{odd}}(M;L) &= \bigoplus_{k \text{ odd}} \ker\left(\bar\partial^k + \left(\bar\partial^{k-1}\right)^{*}\right) \otimes 1_L \\ &= \ker\left(\left(\bar\partial + \bar\partial^*\right) \otimes 1_L|_{\Omega^{0,\mathrm{odd}}(M;L)}\right). \end{aligned}$$

□

If K is a compact Lie group acting on (M,ω) in a Hamiltonian way, then the space $Q(M,\omega)^K$ is controlled by Atiyah, Singer and Segal's K-equivariant index theorem [2]. The quantisation commutes with reduction theorem '$[Q,R]=0$' has been proved in this generality by Meinrenken & Sjamaar in [21], using the index theorem, together with the K-equivariant localisation theorem of Witten, Jeffrey and Kirwan. For a nice survey, see Sjamaar [24], Vergne [27] or Guillemin, Ginzburg & Karshon [5].

In the case of noncompact symplectic manifolds we have the following result, due to Paradan.

Theorem 3.28 (Paradan 2002 [23])**.** *Let (M,ω) be a (possibly noncompact) symplectic manifold. Let $K \times M \to M$ be a Hamiltonian action of the compact Lie group K. Assume that the momentum map $\mu : M \to \mathfrak{k}^*$ is proper, and that (M,ω) allows a K-equivariant prequantum line bundle. Suppose that the norm squared function $\|\mu\|^2$ (with respect to some* $\mathrm{Ad}^*(K)$*-invariant inner product on $\mathfrak{k}^*$) has a compact critical locus. Then the relation '$[Q,R]=0$' still holds.*

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