Geometric K-homology and the Atiyah–Singer index theorem

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May 21, 2019

Thesis submitted for the degree of
Master of Philosophy
in
Pure Mathematics
at The University of Adelaide
Faculty of Engineering, Computer and Mathematical Sciences
School of Mathematical Sciences

Contents

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I acknowledge the support I have received for my research through the provision of an Australian Government Research Training Program Scholarship.

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Acknowledgements

First and foremost, the author acknowledges the invaluable assistance provided by his supervisors: Peter Hochs and Mathai Varghese.
For the questions too stupid even for them, I thank Hao Guo, Michael Hallam, Daniel John, Johnny Lim, John McCarthy, David Roberts and Matt Ryan. Your help and insightful comments were greatly appreciated.

Dedication

To Mum and Dad,
for your unending patience and support.

Abstract

This thesis presents a proof of the Atiyah–Singer index theorem for twisted Spin^c-Dirac operators using (geometric) K-homology. The case of twisted Spin^c-Dirac operators is the most important case to resolve, and will proceed as a corollary of the computation that the K-homology of a point is $\mathbb{Z}$. We introduce the topological index of a pair (M, E), $\mathrm{ind}_t(M, E) = (\mathrm{ch}(E) \cup \mathrm{Td}(M))[M]$ and the analytic index $\mathrm{ind}_a(M, E) = \dim(\ker D_E)^+ - \dim(\ker D_E)^-$ and show that they agree for a "test computation" on a pair of index 1. The main result is that both ind_a and ind_t are well-defined on classes $[(M, E)] \in K_0(\cdot)$ and that there exists a representative on each class for which the analytic and topological indices agree, proving the index theorem for twisted Spin^c-Dirac operators. We also present a description of an analogue the Atiyah–Singer index theorem when a compact Lie group action is introduced to (M, E) and an overview of the steps required prove this result.

Introduction

It is a truth universally acknowledged that a complicated theorem is in want of a simplified proof.

Famously, the "quadratic reciprocity" theorem of Legendre and Gauss has over 240. known proofs, with novel expositions being produced every few years since 1788 [Lem]. The quadratic reciprocity theorem is powerful because it is useful, but the measure of a theorem may be found in the diversity of its proofs. The index theorem of M.F. Atiyah and I.M. Singer was a landmark result in the nascent field of *index theory*, spurring interest in K-theory and leading to a new collaboration between physics and mathematics. The development of new proofs of the index theorem often lead to surprising results in other areas, and give new insight into the technicalities of the theorem.

What constitutes the components of an index theorem is often not entirely obvious, but typically it is presented as an equality between two invariants, typically an *analytic invariant*[1] of an operator f on a topological space X and a *topological invariant*[2] of X. The Euler characteristic of a manifold M is the typical example of a analytic invariant, and the total curvature is the typical example of a topological invariant. The classical example of an index theorem is the Gauss–Bonnet theorem on curvature, which is typically regarded as a result purely in the realm of differential geometry.

Theorem (Gauss–Bonnet index theorem). *Suppose M is a compact Riemannian manifold without boundary. Let $K(x) = k_1(x)k_2(x)$ be the product of the principal curvatures $k_1(x), k_2(x)$, for each $x \in M$. Then*

$$\int_M K dA = 2\pi\chi(M)$$

where $\chi(M)$ is the Euler characteristic of M.

Some explanation is required about this before we continue: the Euler characteristic is usually regarded as a topological invariant for 2-surfaces, satisfying

[1]meaning: preserved under addition of a compact operator
[2]meaning: preserved under homeomorphism

$\chi(M) = 2-2g$, for g the genus of M, but we can restate this in terms of the analytic index of $d+d^* : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)$, $\chi(M) = \dim\ker(d+d^*) - \dim\operatorname{coker}(d+d^*)$.

The first proof of this theorem was published by Pierre Bonnet in the mid-19th century, although Carl Friedrich Gauss had produced unpublished proofs before that. With the benefit of hindsight, we now know that this is one of the early examples of an *index theorem*.

Motivating problems; differential operators

The principal objects of study in differential geometry are manifolds and vector bundles over manifolds, and it is well known in physics that manifolds are of enormous importance for describing physicals systems: the Minkowski metric on $\mathbb{R}^4$ describes space-time. An area of more immediate practical purpose in physics is the theory of (and search for solutions to) differential equations. Many physical phenomena are described by a differential equation, perhaps most famous are Newton's second law of motion and the diffusion of heat through a physical system. Finding solutions to differential equations is typically very difficult. We can usually find a solution iteratively using a computational method, but this may be computationally expensive and in any event, is much less preferable to an exact solution. The theory of differential equations is more fruitfully presented as the study of differential operators. The example of a differential operator that we are all familiar with is the standard derivative, acting on smooth functions on $\mathbb{R}$. The link between differential equations and manifolds and vector bundles is in the form of sections of vector bundles, and differential operators typically act on the sections of a given vector bundle (the author highly recommends [Lee13] as an introduction to the topic). Index theory is all about using this link to study the relationship between differential operators and vector bundles. In particular, we would like to study those differential operators that are so-called *elliptic* operators. A little introduction to differential operators is required.

When we calculate the "principal symbol" of a differential operator (Definition 1.2.14) the operator is said to be elliptic if the principal symbol is invertible whenever a (supplied) cotangent vector ξ is non-zero (Definition 1.2.19). The symbol is important because for a given elliptic operator D, the invertibility of the symbol is sufficient to conclude that D has a finite dimensional kernel and cokernel. Indeed elliptic differential operators D, D' with the same principal symbol satisfy $\dim\ker D - \dim\operatorname{coker} D = \dim\ker D' - \dim\operatorname{coker} D'$. The benefit of this approach is the symbol is coarse enough to be used in a wide variety of situations, but still captures the relevant index-theoretic information.

The index theorem

The Atiyah–Singer index theorem and more modern results rely on the work of Alexander Grothendieck, who introduced a novel cohomology theory, called K-theory[3]. Grothendieck formulated his extension to Riemann–Roch in the language of morphisms of varieties, which became famous as Grothendieck–Riemann–Roch. Atiyah and Hirzebruch applied this construction to vector bundles (locally free sheaves) over a compact manifold M, creating topological K-theory. Atiyah and Singer then used it to great effect when they proved their theorem on the index of elliptic operators.

Theorem (Atiyah–Singer, Theorem 2.12 of [AS68c])**.** *Suppose M is a closed smooth manifold with smooth complex vector bundles $E, F \to M$. Suppose D is an elliptic differential operator acting between smooth sections of E and F, $D : \Gamma^\infty(E) \to \Gamma^\infty(F)$. Then*

$$\int_M \mathrm{ch}(D)\,\mathrm{Td}(M) = \dim\ker D - \dim(\Gamma^\infty(F)/\operatorname{im} D).$$

Some explanation of this theorem: the quotient $(\Gamma^\infty(F)/\operatorname{im} D)$ is the cokernel of D, $\mathrm{ch}(D)$ is the Chern character[4] of D and $\mathrm{Td}(M)$ is the Todd class of the complexification of TM. The "Chern character of D" is used here purely to illustrate that the topological index does indeed depend on D, which is not as clear in 2.12 of [AS68c].

Index theory encompasses an enormous body of work and it would be impossible to summarise the progress made since 1963 in an introduction with any degree of completeness. The more prominent results are the work of Atiyah, Patodi and Singer with their Atiyah–Patodi–Singer index theorem [APS75, Theorem 3.10], which establishes the index theorem on manifold with boundary and linked the index with the spectral invariants of D. Another notable result is the establishment of the index theorem in the case when M is merely a topological manifold (Teleman, [Tel84]). For us in particular, the introduction of a homology version of K-theory is worth mentioning. There are two main flavours of this homology theory: geometric "Baum–Douglas" K-homology and analytic K-homology. These two homology theories are non-trivially equivalent ([BHS07]) and indeed, the proof of the index theorem via K-homology usually proceeds via this equivalence, for which the interested reader may refer to [BD82b] and the related papers [BD82a, BD82c] (they deal with extensions of independently interesting results from [BD82a]). We will not be discussing analytic K-homology except in the context of this equivalence. More recently, there has been work proving the index theorem using only geometric

[3] the K is from the German word *Klasse* - "class"

[4] a modified version of the classical Chern character

K-homology, as in [BvE18]. We provide a short explanation of the differences between the two approaches. The work of Baum and Douglas in [BD82b] introduces the general definition of geometric (they call it "topological" - the nomenclature having not yet been completely settled) K-homology of a suitable space X in the form of equivalence classes of triples (M, E, f), where $f : M \to X$ is a smooth map. The main result is that there is a direct isomorphism between analytic and geometric K-homology, and that the index theorem of Atiyah and Singer proceeds as a result of this computation. Indeed, "it seems increasingly evident that index theory achieves both its greatest simplicity and maximum elegance in the framework of K-homology" and that with some generalisation "all index theorems known to us appear to fit into this framework" (both from [BD82a], in the introduction). A proof of the index theorem using only geometric K-homology has long been "generally accepted as valid" ([BvE18, "Introduction"]) but a proof did not exist in the literature until as recently as 2016 ([BvE18, Bv16][5]). The key remark is that it is no longer necessary to consider even general geometric K-homology. Indeed, the novelty (although the authors do not claim the result is novel) of this approach is the use of the geometric K-homology of a point. In the case that X is a point we can exclude the smooth map f from our triple (all functions with codomain a point are equivalent) and we reduce to pairs (M, E). This simplifies computations and produces a more elegant computation of the index theorem of Atiyah and Singer.

Main result; an overview of the proof

The aim is to present a complete description of the work done in [BvE18], substituting some of the arguments for the bordism invariance of the index with [Hig91] and using a partition of unity rather than Proposition 17 in [BvE18] for the proof that the topological index is preserved by bundle modification. The main application is the use of the K-homological techniques to prove new varieties of index theorems. In particular, we can (with some small effort) introduce the action by a compact Lie group G to Theorem 1.4.13 and formulate a new type of index theorem. A brief outline of some of the technical details are as follows. The proof of Theorem 1.4.13 will proceed as a natural consequence of the computation $K_0(\{\text{point}\}) \cong \mathbb{Z}$. The group $K_0(\cdot)$ consists of equivalence[6] classes with representatives (M, E), where M is a compact, even dimensional smooth manifold and $E \to M$ a complex vector bundle over M (among other things). The idea of the proof is that there are two homomorphisms from $K_0(\cdot)$, called the *topological* and *analytic* index

[5] the first (originally uploaded in to the arχiv in 2016) discusses the proof for a twisted Dirac operator and the second extends to all elliptic operators by the commutativity of a particular diagram

[6] the equivalence relation is defined in Definition 3.1.7

- $\text{ind}_t : K_0(\cdot) \to \mathbb{R}, \quad [(M, E)] \mapsto \text{ch}(E) \cup \text{Td}(M)[M]$
- $\text{ind}_a : K_0(\cdot) \to \mathbb{Z}, \quad [(M, E)] \mapsto \dim(\ker D_E)^+ - \dim(\ker D_E)^-$

discussed in Definition 1.4.9 and below Definition 1.2.11, respectively. Here, D is the Dirac operator of M (Definition 1.2.4) and D_E is the operator twisted by the bundle E (Definition 1.2.11). It turns out that the analytic index is actually an isomorphism into $\mathbb{Z}$ and that on a generator of $K_0(\cdot)$, $\text{ind}_t = \text{ind}_a = 1$. The equivalence relation for $K_0(\cdot)$ is constructed in such a way as to ensure that for a given pair (M, E) there is always an equivalence $[(M, E)] = [(S^n, q\beta)]$, where β is the so-called "Bott generator"[7] vector bundle, for $q \in \mathbb{Z}$ relying on M, E. Both the topological and analytic indices agree on $(S^n, q\beta)$ and have index q, providing an exact correspondence. In Chapter 4 we also establish that the topological and analytic indices are well-defined on K-homology classes, and most of the technical details are present in this chapter.

When we introduce a compact Lie group G, we must reformulate the relations of K-homology to accommodate for this G-action. The conclusion is largely the same, but the analytic and topological indices require some modification. The example pair is no longer (S^n, β) but now becomes $(S^n, \beta \otimes [V])$ for a particular equivalence class $[V]$ of a finite-dimensional irreducible representation V. The G action is trivial on $\beta \to S^n$ and, (for a fixed $g \in G$) the topological and analytic indices involve $[V](g)$ i.e. the trace of the representation associated to V evaluated at g. This is an outline of a proof using K-homology of Theorem 3.9 in [AS68c] for twisted Spin^c-Dirac operators.

[7] Definition 2.1.1

Chapter 1

The Dirac operator and the index theorem; ind$(D) < \infty$

This chapter serves to introduce the necessary background information to define the primary object of study: the Spin^c-Dirac operator on a manifold M and the related Atiyah–Singer index theorem for twisted Spin^c-Dirac operators. We cover the definition of the groups $\mathrm{Spin}(n), \mathrm{Spin}^c(n)$, the construction of a Spin^c structure for a manifold M, the related spinor bundles and twisted Spin^c-Dirac operators. For the sake of completeness we also include an abridged proof that the analytic index is well-defined on elliptic operators of order 1, giving firm foundation for the work done in the remaining chapters.

1.1 Clifford algebras and spinors

Let V be a vector space over a field $\mathbb{K}$, which we will usually take to be $\mathbb{R}$ or $\mathbb{C}$. Let B be a non-degenerate symmetric bilinear form on V. Let $Q(v) = B(v, v)$ and define $T(V) = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$ as the tensor algebra of V and $I(Q) \subset T(V)$ the two-sided ideal generated by

$$\{v \otimes v + \|v\|^2 : v \in V\}.$$

Set $C(Q) = T(V)/I(Q)$ and define $j = \pi \circ i$, where i is the natural embedding of the vector space V into $T(V)$ and π is the projection $\pi : T(V) \to I(Q)$. Then $j(v)^2 = -Q(v) \cdot 1$ for all $v \in V$.

Definition 1.1.1 (Clifford algebra)**.** *The pair $(C(Q), j)$ is called the Clifford algebra of V with respect to Q. Since j is canonical, we may write $C(Q)$ unambiguously.*

By the equation $B(v_1, v_2) = \frac{1}{2}Q(v_1+v_2)-Q(v_1)-Q(v_2)$ we lose no information about B by specifying only the quadratic form Q. If the basis $\{v_i\}_{i=1}^n$ is orthogonal with respect to the bilinear form B ($B(v_i, v_j) = 0$ if $i \neq j$) then the Clifford algebra $C(Q)$ is (linearly) generated by $1 \in \mathbb{K}$ and anti-symmetric products of the form $v_{i_1} \cdots v_{i_s}$, where $1 < i_1 < \cdots < i_s \leq l \leq n$ and $1 \leq l \leq n$.

Definition 1.1.2 ($\mathcal{C}_n, \mathcal{C}_n^c$)**.** *Let $\mathcal{C}_n$ be the real Clifford algebra of the form $Q(x_1, \ldots, x_n) = -x_1^2 - \cdots - x_n^2$ on $\mathbb{R}^n$ and $\mathcal{C}_n^c$ be the related complexified Clifford algebra of the form $Q_c(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$.*

Remark. The term "complexified" is no coincidence: there is an isomorphism $\mathcal{C}_n^c \cong \mathcal{C}_n \otimes_{\mathbb{R}} \mathbb{C}$ [Fri00, Corollary on page 11].

Definition 1.1.3 (Spin(n))**.** *The spin group is* $\mathrm{Spin}(n) = \{e_i \cdots e_{2j} \mid e_i \in \mathbb{R}^n, \|e_i\| = 1, j \in \mathbb{Z}_{\geq 0}\} \subset \mathcal{C}_n$*, where we take $j = 0$ to correspond to the single element 1.*

Remark. Spin(n) is the connected double cover for SO(n). For $n \geq 2$ this coincides with the universal cover of SO(n) and hence is unique.

Definition 1.1.4 (Spin$^c(n)$)**.** $\mathrm{Spin}^c(n) = (\mathrm{Spin}(n) \times U(1))/\mathbb{Z}_2$ *where we imagine $\mathbb{Z}_2 = \{(1, 1), (-1, -1)\}$ as a multiplicative subgroup of* $\mathrm{Spin}(n) \times U(1)$ *and have the equivalence relation $(g, z) \sim (-g, -z)$, for $g \in \mathrm{Spin}^c$ and $z \in U(1)$.*

Let $\rho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ be a group representation given by

$$\rho(g)(v) = gvg^{-1} \in \mathbb{R}^n \subset \mathcal{C}_n, v \in \mathbb{R}^n, \tag{1.1}$$

where we identify $\rho(g)$ with the linear map it induces in SO(n) and the multiplication of g on v is the multiplication of the Clifford algebra. This is the projection map that makes Spin(n) a double cover of SO(n) and we extend it to Spin$^c(n)$ by $\rho([g, z]) = \rho(g)$.

Definition 1.1.5 (Frame bundle)**.** *The frame bundle of a vector bundle $F \to M$, denoted $\mathcal{F}(F)$ is the principal* $\mathrm{GL}_n(\mathbb{R})$*-bundle with fibre at $p \in M$ that consists of ordered bases ("frames") of the fibre F_p.*

Equivalently, one can define the frame bundle fibre at p as linear isomorphisms $f : \mathbb{R}^n \to F_p$, because this is essentially just a choice of basis.

Definition 1.1.6 (Spin^c datum)**.** *Given a smooth manifold M and a smooth real vector bundle $F \to M$ a* Spin^c*-datum of F is a pair (P, η) consisting of*

- *a smooth principal right* Spin^c *bundle $P \to M$*

- *a smooth homomorphism* $\eta : P \to \mathcal{F}(F)$ *that is compatible with the representation* $\rho : \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$ *in the sense that the following diagram commutes*

$$\begin{tikzcd} P \times \mathrm{Spin}^c(n) \arrow[r] \arrow[d, "\eta\times\rho"] & P \arrow[d, "\eta"] \\ \mathcal{F}(F) \times \mathrm{GL}_n(\mathbb{R}) \arrow[r] & \mathcal{F}(F). \end{tikzcd}$$

Definition 1.1.7 (Isomorphism of data)**.** *Given a pair of* Spin^c *data* $(P, \eta), (P', \eta')$ *for a vector bundle* $\pi : F \to M$ *as in Definition 1.1.6 an isomorphism of* Spin^c *data is an isomorphism of the (associated) principal bundles* P, P' *that is compatible (commutes with) the maps* η, η'*.*

Definition 1.1.8 (Spin^c structure)**.** *An isomorphism class of data for* F *as in Definition 1.1.7 is a* Spin^c *structure on* F*.*

Remark. Spin^c structures are not unique; there can exist more than one isomorphism class of data.

Definition 1.1.9 ($\mathrm{Spin}^c(n)$ manifold)**.** *A* $\mathrm{Spin}^c(n)$ *manifold is a smooth* n*-manifold whose tangent bundle* TM *has a given* Spin^c *structure.*

If it is not necessary to specify the dimension n, we may instead write "Spin^c manifold"

Remark. A Spin^c structure on a manifold determines an orientation and a Riemannian metric via the orientation and Euclidean inner product on $\mathbb{R}^n$. When M is supplied with a Riemannian metric, we require that the Spin^c-induced metric agrees with the one provided.

It is also possible to reverse the Spin^c structure on the manifold, and it will become evident why this is important in Chapters 3 and 4.

Definition 1.1.10 (Quotient product)**.** *Given a pair of sets* X, Y *with a group action* G *on* $X \times Y$*, define* $X \times_G Y$ *to be the quotient*

$$X \times_G Y := (X \times Y)/\sim,$$

where for $(x, y), (x', y') \in X \times Y$ *we say* $(x, y) \sim (x', y')$ *if* $(x, y) = g \cdot (x', y')$*.*

Suppose that we have P, F as in Definition 1.1.6. We can act on P by an element of $\mathrm{Spin}^c(n)$ and likewise on $\mathbb{R}^n$ via the map defined in Eq. (1.1). We can write a right-action on $P \times \mathbb{R}^n$ as $(p, v) \cdot g = (pg, g^{-1}v)$, for $g \in \mathrm{Spin}^c(n)$, $p \in P$, $v \in \mathbb{R}^n$ and define the quotient $P \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n$ as the quotient of $P \times \mathbb{R}^n$ by this relation.

Definition 1.1.11 (Reversed structure)**.** *Given a datum (P, η) for F as in Definition 1.1.6, let $\mathcal{F}^-(F) = \mathcal{F}(F) \times_{\mathrm{GL}_n^+\mathbb{R}} \mathrm{GL}_n^-\mathbb{R}$, where we write $\mathrm{GL}_n^-\mathbb{R}$ to mean those matrices with negative determinant and $GL^+\mathbb{R}$, those with positive determinant. Suppose that $\pi : \tilde{\mathrm{O}}^-(n) \to \mathrm{O}^-(n)$ is the connected double cover of $\mathrm{O}^-(n)$, where $\mathrm{O}^-(n)$ is the collection of orthogonal matrices with determinant -1. The reversed datum of (P, η) is then written (P^-, η^-), where $P^- = P \times_{\mathrm{Spin}^c(n)} \tilde{\mathrm{O}}^-(n)$ and η^- is the map induced by taking $\eta \times \pi$ on pairs in $\mathcal{F}^-(F)$.*

Some commentary before we continue regarding this choice of "reversed" datum. In the original definition (that is, Lemma 3.1.5), we might have chosen to use $\mathcal{F}_+(F)$ (those frames which were positively oriented), which would make the obvious choice for the negative datum those frames which were negatively oriented (i.e. $\mathcal{F}_-(F)$). Since we did not just use those frames which are positively oriented in the datum, we need a bigger set for the negative datum, and the choice of $\mathcal{F}^-(F)$ is motivated primarily by the following fact. If we think of the frame bundle fibre at p as linear isomorphisms from $\mathbb{R}^n$ into F_p then the natural map $\mathcal{F}^-(F) \to \mathcal{F}(F)$ (which sends $[f, g] \in \mathcal{F}^-(F)$ to $f \circ g$) interchanges $\mathcal{F}_-(F)$ and $\mathcal{F}_+(F)$.

Remark. The compatibility of η^- with the representation $\rho : \mathrm{Spin}^c(n) \to SO(n)$ comes from the compatibility of η, so it is easy to see that (P^-, η^-) is a valid datum.

Proposition 1.1.12. *Let F, P as in Definition 1.1.6. There is an isomorphism $P \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n \cong F$.*

Proof. The isomorphism is

$$\varphi : [(p, v)] \mapsto [(\eta(p), v)] \in \mathcal{F}(F) \times_{\mathrm{GL}_n \mathbb{R}} \mathbb{R}^n \cong F,$$

for $v \in \mathbb{R}^n$ and $p \in P$. $\square$

Remark. For a given real vector bundle F we normally have the isomorphism $F = \mathcal{F}(F) \times_{\mathrm{GL}_n(\mathbb{R})} \times \mathbb{R}^n$ and the transition functions are invertible matrices. The Spin^c structure of F changes the transition functions from $\mathrm{GL}_n(\mathbb{R})$ to $\mathrm{Spin}^c(n)$ via Proposition 1.1.12 and we may write $F = P \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n$. In this sense, P may be described as the "structure bundle" of F.

Definition 1.1.13 (κ_n)**.** *Suppose $n = 2r$ is even and define $\kappa_n : \mathcal{C}_n^c \to \mathrm{End}(\mathbb{C}^{2^r})$ as the isomorphism $\mathcal{C}_n^c \cong \mathrm{End}(\mathbb{C}^{2^r})$ found in* [Fri00, Proposition on page 13.]. *If $n = 2r + 1$ is odd, then κ_n consists of an isomorphism $\mathcal{C}_n^c \cong \mathrm{End}(\mathbb{C}^{2^r}) \oplus \mathrm{End}(\mathbb{C}^{2^r})$ followed by projection onto the first factor.*

There are *many* isomorphic descriptions of $\mathrm{Spin}^c(n)$. The description used in Definition 1.1.4 has the advantage of being simple to write down, but hides a key fact we would like to use. Indeed, $\mathrm{Spin}^c(n)$ can be thought of as a subset of $\mathcal{C}_n^c$ ([BvE18, 2.3]) and κ_n provides a group representation (we will abuse notation and also call this κ_n) $\kappa_n : \mathrm{Spin}^c(n) \to \mathrm{End}(\mathbb{C}^{2^r})$ allowing us to define the spinor bundle.

Definition 1.1.14 (Spinor bundle)**.** *Suppose $F \to M$ is a real smooth* Spin^c *vector bundle over a smooth Riemannian manifold with structure bundle P as in Proposition 1.1.12, so that $F = P \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n$. Suppose $n = 2r$ or $n = 2r+1$. Then the spinor bundle of F is*

$$S_F = P \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r}.$$

Remark. Of particular note is the case when $F = TM$ is the tangent bundle of M. In this case, we do not write S_{TM} but instead S_M and say that S_M is the "spinor bundle of M".

Proposition 1.1.15. *Suppose E_1, E_2 are* Spin^c *vector bundles over a single smooth manifold X, which are not both odd dimensional. Then the direct sum $E_1 \oplus E_2$ has a* Spin^c *structure with spinor bundle $S_{E_1 \oplus E_2} = S_{E_1} \otimes S_{E_2}$.*

Proof. The statement is from [Hoc09, Lemma 13.6 on page 174]. □

We can define a Spin^c structure on the pullback of a vector bundle, but this requires the pullback of a principal bundle. A principal bundle is in particular a fibre bundle, and the pullback of a fibre bundle with principal bundle structure is simple: if $\pi_P : P \to M$ is a principal G-bundle over a smooth manifold M and $f : X \to M$ is a smooth map then $f^*P = \{(x,p) \mid f(x) = \pi_P(p)\}$ is a principal G-bundle with action $(x,p) \cdot g = (x, p \cdot g)$, for $x \in X, p \in P_x, g \in G$.

Proposition 1.1.16. *Suppose $E \to X$ is a* Spin^c *vector bundle over a manifold X and suppose that $f : Y \to X$ is a smooth map. Then f^*E has a* Spin^c *structure with spinor bundle $S_{f^*E} = f^*S_E$.*

Proof. The Spin^c structure of f^*E is induced from the Spin^c structure of E: if $E = P \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n$ then $f^*E = f^*P \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n$. Recall that pullbacks preserve rank, so there is no ambiguity in writing the rank of both bundles as $n = 2r$, or $n = 2r+1$. With this in mind, it suffices to prove that $S_{f^*E} = f^*P \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r}$ is the same as $f^*(P \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r}) = f^*S_E$. The association is

$$f^*P \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r} \ni [(y, p_1), z)] \mapsto (y, [p_1, z]) \in f^*(P \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r})$$

for $(y, p_1) \in f^*P, z \in \mathbb{C}^{2r}$. This is well-defined because the $\mathrm{Spin}^c(n)$ action on p_1 in $f^*P \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r}$ is the same as in $f^*(P \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r})$. □

Definition 1.1.17 (G-Spin^c datum)**.** *Suppose $F \to M$ is a real smooth vector bundle and there is a compact Lie group G acting on M and F is additionally a G-equivariant vector bundle. Then a G-*Spin^c *datum for F is a* Spin^c *datum (for F) (P, η) for which P is also a G-*Spin^c *principal bundle and the action by G on F, M is compatible with ρ, η in the sense that the diagram in Definition 1.1.6 accommodates the action by G on P.*

Proposition 1.1.18. *Let M be as in Definition 1.1.9 and suppose $P \to M$ is now a principal G-bundle for G a Lie group. Let Y be a manifold with a G-*Spin^c *structure. Let $S_M \to M$ be the spinor bundle of M and suppose $E \to Y$ is a G-equivariant* Spin^c *vector bundle with spinor bundle S_E. Then $P \times_G E \to P \times_G Y$ has a* Spin^c *structure with spinor bundle $S_{P\times_G E} = P \times_G S_E$.*

Proof. Let $P_E \to Y$ be a principal $G \times \mathrm{Spin}^c(n)$ bundle such that $E = P_E \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n$ i.e. P_E is the Spin^c-structure bundle for E. Then the Spinor bundle for E is $S_E = P_E \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r}$ by definition and $P \times_G E = P \times_G (P_E \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n)$. This is enough to completely determine the structure bundle for $P \times_G E$, i.e.

$$P \times_G S_E = P \times_G (P_E \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r}) = (P \times_G P_E) \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r} = S_{P\times_G E}.$$

□

Remark. We will use this construction in Chapter 4 and Definition 1.1.17 will be used again in Chapter 5.

Proposition 1.1.19. *Given a manifold Ω with boundary and* Spin^c *structure, we can construct a* Spin^c *structure on the boundary.*

The proof of this proposition requires the so-called "2-out-of-3 lemma" [BvE18, Lemma 5], which we will write below. First, suppose that we have two Spin^c bundles $F_1, F_2 \to M$ that have data $(P_1, \eta_1), (P_2, \eta_2)$, $\eta_j : P_j \to \mathcal{F}(F)$. We can easily form the Spin^c structure of $F = F_1 \oplus F_2$ via Proposition 1.1.15. The 2-out-of-3 lemma is a stronger statement: given datum for any two of the triple (F_1, F_2, F) we can recover the datum for the third and moreover, this agrees with (up to homotopy) the datum we began with.

Lemma 1.1.20 (2-out-of-3 lemma)**.** *Let F_1, F_2 be two smooth real vector bundles over M and assume a* Spin^c *datum exists for both F_1 and $F_1 \oplus F_2$. Then there exists a unique Spin^c structure for F_2 such that $F_1 \oplus F_2$ is the given one.*

Proof of Proposition 1.1.19. We inject the frame bundle of TM into the frame bundle of $T\Omega$,

$$j : \mathcal{F}(TM) \to \mathcal{F}(T\Omega\big|_M) \quad (v_1, v_2, \dots, v_n) \mapsto (\mathbf{n}, v_1, v_2, \dots v_n),$$

where $\mathbf{n}$ is the outward facing unit normal vector. The principal $\mathrm{Spin}^c(n)$ bundle P_M of M is the pre-image of P_Ω under this map in the sense that $P_M = \eta_\Omega^{-1}(j(\mathcal{F}(TM)))$, where η_Ω is the smooth homomorphism $\eta_\Omega : P_\Omega \to \mathcal{F}(T\Omega)$ as in Definition 1.1.6. If $\mathcal{N}_M$ is the normal bundle of M then we have the sum $TM \oplus \mathcal{N}_M = T\Omega\big|_M$, for which we have a structure for the normal bundle and the tangent bundle of Ω restricted to M and hence by Lemma 1.1.20, there is a structure for TM. $\square$

1.2 The Dirac operator

To seriously discuss the Dirac operator, we must spend time introducing Clifford multiplication, which serves as essentially the distinguishing characteristic of the Dirac operator when compared to other differential operators. In what follows, we assume that M is a smooth Spin^c manifold with Riemannian metric identifying the tangent and cotangent bundles. Of course, the metric is the one that arises naturally from the Spin^c structure, but we would like to make explicit that the tangent and cotangent bundles are not distinguished for the purposes of defining the Dirac operator.

Definition 1.2.1 (Clifford multiplication)**.** *The spinor bundle $S_M = P \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r}$ is equipped with a map $c : TM \to \mathrm{End}(S_M)$ called the Clifford multiplication defined by $c([p,v])([p,y]) = [p, \kappa_n(v)y]$, for $[p,v] \in P \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n = TM, [p,y] \in P \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r}$ and κ_n as in Definition 1.1.13.*

For a given fibre of TM, Clifford multiplication extends to products in $C(T_xM)$ by acting on a product $v \cdot w \in C(T_xM)$ as $c(v) \circ c(w)$.

Remark. It is not strictly necessary for Clifford multiplication to be defined on only the spinor bundle of a manifold (the spinor bundle of the tangent bundle). Indeed, we can write $c : E \to \mathrm{End}(S_E)$ for any bundle E, so long as it has a Spin^c structure.

Definition 1.2.2 (Clifford connection)**.** *Given a Spin^c manifold M with spinor bundle S_M, let $c(\omega) \in \mathrm{End}(S_M)$ be the grading operator of S_M, as in Lemma 1.2.8. Then a Clifford connection for M is a connection $\nabla : \Gamma^\infty(S_M) \to \Gamma^\infty(S \otimes T^*M)$ satisfying the following equality. $[\nabla, c(\omega)] = c(\nabla^{LC}\omega) : \Gamma^\infty(S) \to \Gamma^\infty(S)$, where ∇^{LC} is the Levi-Civita connection.*

Lemma 1.2.3. *Clifford connections exist.*

Proof. This is a consequence of the proposition on page 59 of [Fri00]. The definition is on page 57, and is written as "∇^A", although we will suppress the use of A, as we do not discuss the related principal bundle connection from which this notation originates. $\square$

Definition 1.2.4 (Dirac operator)**.** *The Dirac operator of* M *as in Definition 1.1.9 is* $D = c \circ \nabla : \Gamma^\infty(S_M) \to \Gamma^\infty(T^*M \otimes S_M) = \Gamma^\infty(TM \otimes S_M) \to \Gamma^\infty(S_M)$ *where* c *is Clifford multiplication and* ∇ *is the connection on spinor bundle* S_M *from Definition 1.2.2*

Formally we should say Spin^c-Dirac operator, but we will often suppress the mention of Spin^c. It is enlightening to see what happens when $M = \mathbb{R}^n$. The Dirac operator on $\mathbb{R}^n$ is

$$D = \sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}, \tag{1.2}$$

where matrices $\{A_j\}_{j=1}^n$ are $2^r \times 2^r$ complex matrices that are defined using an inductive procedure. Write r to mean the greatest integer less than or equal to $n/2$ i.e. $n = 2r$ for n even or $n = 2r+1$ for odd n. Suppose n is odd. If $n = 1$ then $A_1 = (-i)$. If $n > 1$ using matrices $A_1, \dots, A_n$ we can construct the matrices $\tilde{A}_1, \dots, \tilde{A}_{n+1}$.

Remark. The tilde is used to distinguish between the j^{th} matrix for n (which is A_j) and the j^{th} matrix for $n+1$ (which is $\tilde{A}_j$).

The matrices $\{\tilde{A}_j\}_{j=1}^{n+1}$ are

$$\tilde{A}_j = \begin{pmatrix} 0 & A_j \\ A_j & 0 \end{pmatrix} \text{ and } \tilde{A}_{n+1} = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}.$$

When n is even the process is simpler. $\tilde{A}_j = A_j$ and $A_{n+1} = \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}$, where I in the above matrices is the identity matrix of dimension 2^{r-1}. In either either case (bar $n = j = 1$) the trace is 0.

Lemma 1.2.5. *When* $n = 2r+1$ *is odd we have*

$$(-1)^j A_j = i^{r+1} A_1 \cdots \widehat{A_j} \cdots A_{2r+1}.$$

We use the convention $A_1, \dots, \widehat{A_j} \cdots A_{2r+1}$ to mean the list $A_1, \dots, A_{2r+1}$ with the j^{th} entry removed.

Lemma 1.2.6. *When* $n = 2r$ *is even we have*

$$i^r A_1 \cdots A_n = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

and hence $(i^r A_1 \cdots A_n)^2 = 1$.

Lemma 1.2.7. $A_j^2 = -I_{2^r}$

Proof of Lemmas 1.2.5 to 1.2.7. These are statements from [BvE18, 2. Dirac operator of $\mathbb{R}^n$]. □

Lemma 1.2.8 (Grading Lemma)**.** *Suppose M is an even dimensional* Spin^c *manifold with spinor bundle S_M and $e_1, \dots, e_n$ is an oriented orthonormal basis for T_xM with $n/2 = r$ and suppose $\omega = i^r e_1 e_2 \dots e_{n-1} e_n \in C(T_xM)$. Then there is a grading on each fibre of S_M given by the involution $c(\omega) : (S_M)_x \to (S_M)_x$.*

Proof. Clifford multiplication acts on the Spinor bundle S_M via κ_n. Because the matrices $\{A_j\}_{j=1}^n$ are just $\{\kappa_n(e_j)\}_{j=1}^n$, we have $c(\omega) = i^r A_1 \cdots A_n$. From Lemma 1.2.6 we know this matrix product squares to 1 and hence has eigenvalues ± 1, for each fibre of S_M. □

Definition 1.2.9 (Positive and negative spinor bundles)**.** *The* positive (S_M^+) *and* negative (S_M^-) *spinor bundles of M are the bundles obtained by taking fibrewise the positive and negative eigenspaces of the involution $c(\omega)$ given in Lemma 1.2.8.*

Lemma 1.2.10 (Clifford action decomposition)**.** *Suppose E_1 and E_2 are smooth* Spin^c *vector bundles over M of even rank with a fixed structure and suppose $E \cong E_1 \oplus E_2$ with* Spin^c *structure obtained from E_1 and E_1. Let $\xi_1 \in E_1$ and $\xi_2 \in E_2$. Define $\xi = f_E\xi_1, \xi_2) \in E_1 \oplus E_2$, where $f_E : E_1 \oplus E_2 \to E$ is the bundle isomorphism and $f_S : S_E \to S_{E_1} \otimes S_{E_2}$ is the spinor isomorphism. If c_1, c_2 and $c_{1+\oplus 2}$ are Clifford multiplication on the respective spaces, E_1, E_2, E and γ is the grading operator on S_1 as in Lemma 1.2.8, we have*

$$c_{1\oplus 2}(\xi) \circ f_S = f_S \circ c_1(\xi_1) \otimes 1 + \gamma \otimes c_2(\xi_2).$$

Proof. Let n, m be the rank of E_1 and E_2 and suppose they are even. Let $F = \kappa_n \otimes 1_{\mathcal{C}_m^c} + \gamma \otimes \kappa_m$ for κ_n, κ_m as in Definition 1.1.13. Let $r_n = n/2$ and $r_m = m/2$. There is a decomposition

$$\begin{array}{ccc} \mathcal{C}_{n+m}^c & \xrightarrow{\kappa_{n+m}} & \mathrm{End}(\mathbb{C}^{2^{r_n+r_m}}) \\ \downarrow \wr\!\! \| & & \downarrow \wr\!\! \| \\ \mathcal{C}_n^c \otimes \mathcal{C}_m^c & \xrightarrow{F} & \mathrm{End}(\mathbb{C}^{2^{r_n}}) \otimes \mathrm{End}(\mathbb{C}^{2^{r_m}}) \end{array}$$

and since Clifford multiplication has active ingredient κ, this is enough to conclude the result in the event that $E = E_1 \oplus E_2$ as sets (i.e. both f_S and f_E are the identity map). This decomposition can be obtained from the definition of κ in [Fri00] on page 14. We can write down a useful interpretation of f_S is (non-explicitly) as $f_S^{-1}([p_1, z_1] \otimes [p_2, z_2]) = [p, z]$ for $z \in \mathbb{C}^{2^{r_m + r_n}}$ satisfying $z = z_1 \otimes z_2 \in \mathbb{C}^{2^{r_n}} \otimes \mathbb{C}^{2^{r_m}}$, $p \in P_E, p_1 \in P_{E_1}, p_2 \in P_{E_2}$, where P_E, P_{E_1} and P_{E_2} are the structure bundles for E, E_1, E_2 respectively. This decomposition is due to Proposition 1.1.15, and extended linearly. We can then compute

$$\begin{aligned} f_S \circ c_{1\oplus 2}(\xi)([p,z]) &= f_S([p, \kappa_m(\xi_1)z_1 \otimes z_2 + \gamma(z_1) \otimes \kappa_m(\xi_2)]) \\ &= [p_1, \kappa_n(\xi_1)z_1] \otimes [p_2, z_2] + [p_1, \gamma(z_1)] \otimes [p_2, \kappa_m(\xi_2)z_2] \end{aligned}$$

which is exactly $(c_1(\xi_1) \otimes 1 + \gamma \otimes c_2(\xi)) \circ f_S([p_1, z_1] \otimes [p_2, z_2])$. □

We can also formulate an extension of the general Dirac operator of a manifold, which will be most useful when consider Theorem 1.4.13, later. Given a smooth complex rank vector bundle E over M (for M as in Definition 1.1.9) we can incorporate part of the information of E into the Dirac operator, so that we may use it with the index theorem. We would like to construct a map

$$D_E : \Gamma^\infty(S_M \otimes E) \to \Gamma^\infty(S_M \otimes E)$$

from $D = c \circ \nabla$ as in Definition 1.2.4. We need to extend both Clifford multiplication and the connection to E. This is quite easy for the Clifford part of D, we can simply decree that the Clifford multiplication on the E part is trivial. The connection is slightly more involved. Suppose now that the metric on M is Hermitian and the connection on E is a Hermitian connection, denoted ∇^E. The new connection on $S_M \otimes E$ is

$$\nabla^{S_M \otimes E} = \nabla \otimes 1_E + 1_{S_M} \otimes \nabla^E$$

and using this we can define the "twisted" Dirac operator.

Definition 1.2.11 (Twisted operator). *Given a pair (M, E) consisting of a smooth* Spin^c *manifold M (with Hermitian metric) and a smooth complex vector bundle $E \to M$ we can form the twisted Dirac operator D_E from the Dirac operator D of M. Suppose that ∇ is the connection on M that defines the Dirac operator $D = c \circ \nabla$ and suppose that ∇^E as above is the connection on E. We can then construct $D_E : \Gamma^\infty(S_M \otimes E) \to \Gamma^\infty(S_M \otimes E)$ defined by $D_E = (c \otimes 1_E) \circ \nabla^{S_M \otimes E}$.*

Remark. The word "twist" is a reference to the tensor product. Given two abstract bundles E, F, the bundle $E \otimes F$ is said to be the "twist" of E by F. Of course, since there is no distinguishing $E \otimes F$ from $F \otimes E$, so we only ever be twisting S_M by another bundle, to remain unambiguous.

Definition 1.2.12 (Differential operator). *Let M be an smooth n-manifold with vector bundles E_0 and E_1 and consider a linear map $D : \Gamma^\infty(M, E_0) \to \Gamma^\infty(M, E_1)$. We say that D is a differential operator of order k if and only if the commutator $[D, f]$ is an operator of order $k - 1$, for each function $f \in C^\infty(M)$. An operator is said to have order 0 if the commutator is 0, i.e. $[D, f] = 0$. We write $D \in$* Diff$^k(E_0, E_1)$ *if D is a differential operator between E_0 and E_1 of order k.*

This is not the standard definition (see [Ebe, Section 2.2 "Differential operators in general"]), but we prefer it because it is very easy to check if an operator is order 1.

Example 1.2.13. *The Dirac operator is an order 1 differential operator.*

Proof. The Dirac operator is the composition of the Clifford action with a connection. A connection is order 1 by the Leibniz rule, $[c \circ \nabla, f]s = c(df)s$, which is order 0, because the commutator $[c(df), g]s$ is 0. □

Definition 1.2.14 (Principal symbol of an operator)**.** *Let D be a differential operator of order k on a manifold M with between sections of the vector bundles E_0, E_1. Let $\pi : T^*M \to M$ be the cotangent bundle projection and fix $y \in M$, $\xi \in T_y^*M$, $e \in (E_0)_y$. Pick $f \in C^\infty(M)$ with $f(y) = 0$ and $df_y = \xi$. Choose a section $s \in \Gamma^\infty(M, E_0)$ with $s(y) = e_0$. The principal symbol of D is then the map*

$$\sigma_D(y, \xi) : (E_0)_y \to (E_1)_y$$

given by $\sigma_D(y, \xi)(e_0) = \frac{i^k}{k!} D(f^k s)(y) \in (E_1)_y$.

Some commentary on the above definition: σ_D accepts as inputs a covector ξ and a point $y \in M$, and with that data it becomes a mapping $(E_0)_y \to (E_1)_y$. We can also think of σ_D as a map $\sigma_D : \pi^* E_0 \to \pi^* E_1$ by applying $\sigma_D(y, \xi)$ to the second factor: $\sigma_D(\xi, e_0) = (\xi, \sigma_D(y, \xi)e_0)$. In this sense, σ_D is both a map between the pullbacks and a vector space homomorphism.

Lemma 1.2.15 (Lemma 2.2.12, page 22 of [Ebe])**.** *This definition does not depend on f or s.*

Lemma 1.2.16 (Lemma 2.2.20, page 24 of [Ebe])**.** *If D is an operator of order one, the symbol can be computed as the commutator $i[D, f]s$.*

Proposition 1.2.17. *The Dirac operator has principal symbol $ic(\xi)$.*

Proof. This proof is quite straightforward: the only thing we need is that the induced connection ∇ is actually a connection. Let ξ, f, s, y be as in Definition 1.2.14 and let D be the Dirac operator. We have

$$\begin{aligned} iD(fs)(y) &= ic(df \otimes s + f\nabla s)(y) \\ &= ic(df \otimes s)(y) \\ &= ic(df_y \otimes e) \\ &= ic(df_y)(e) = ic(\xi)(e). \end{aligned}$$

□

Lemma 1.2.18. *D_E has principal symbol $\sigma_D \otimes 1$.*

Proof. Much of the work was already done in Proposition 1.2.17. We need only isolate the symbol of the Dirac operator from the symbol of the twisted operator. Suppose as in the setting of Definition 1.2.14 we have a point $y \in M$ and smooth function $f \in C^\infty(M)$ satisfying $f(y) = 0$, with $df\big|_y = \xi$. The section s is now a section of both S_M and E, so we will write $s = s_1 \otimes s_E \in \Gamma^\infty(S_M \otimes E)$.

$$\begin{aligned} iD_E(fs)(e) &= i(c \otimes 1_E) \circ (\nabla^{S_M \otimes E})(fs)(y) \\ &= i(c \otimes 1_E) \circ (df \otimes s + f\nabla^{S_M \otimes E} s)(y) \\ &= i(c \otimes 1_E)(df \otimes s)(y) \quad \text{(because } f(y) = 0) \\ &= i(c(df)s_1 \otimes s_E)(y). \end{aligned}$$

This is the principal symbol evaluated at (y, ξ), which gives a homomorphism $ic(\xi) \otimes 1_E : (S_M)_y \otimes E_y \to (S_M)_y \otimes E_y$, which is exactly $\sigma_D \otimes 1_E$. □

Definition 1.2.19 (Elliptic differential operator)**.** *A differential operator D between bundles $E_0, E_1 \to M$ is said to be elliptic if the principal symbol of the operator is invertible (as a linear map $\sigma_D(y, \xi) : (E_0)_y \to (E_1)_y$) whenever $\xi \neq 0$.*

Before we can show the Dirac operator of a manifold M is an elliptic differential operator, we need the following lemma:

Lemma 1.2.20. *If M is a* Spin^c *manifold with spinor bundle S_M then $c(v)^2(e) = -\|v\|^2(e)$ for all $v \in T_yM$ and any $e \in (S_M)_y$.*

Proof. First, remember the identification of $TM \cong P \times_{\mathrm{Spin}^c} \mathbb{R}^n$ and note that if $[p, y] \in P \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r} = S_M$ then

$$c(v)([p, y]) = [p, \kappa_n(v)(y)].$$

We can compute the square of $c(v)$ as

$$\begin{aligned} c(v)^2([p, v]) &= c(v)(c(v)([p, v]) \\ &= c(v)([p, \kappa_n(v)(y)]) \\ &= [p, \kappa_n(v^2)(y)] \\ &= [p, \kappa_n(-\left\|v^2\right\|)(y)] \\ &= [p, -\left\|v^2\right\| \kappa_n(1)(y)] \\ &= [p, -\|v\|^2 y] \\ &= -\|v\|^2 [p, y], \end{aligned}$$

where $\|v\|^2 = g([p, v], [p, v])$ and g is the Riemannian metric on M induced by the Spin^c structure. □

Proposition 1.2.21. *The Dirac operator is an elliptic differential operator.*

Proof. We apply Lemma 1.2.20 to the composition $ic(\xi)(ic(\xi))(e)$and arrive at

$$ic(\xi)(ic(\xi))(e) = -ig(\xi,\xi)(e),$$

where g is the Riemannian metric associated to the manifold M. This allows us to write down the inverse of the symbol very explicitly, as

$$\sigma_D(y,\xi)^{-1}(e) = \frac{1}{ig(\xi,\xi)} ic(\xi)(e).$$

□

Remark. The twisted operator of D, D_E is an elliptic differential operator for the same reason D is.

1.3 An elliptic operator is Fredholm

Definition 1.3.1 (Fredholm operator)**.** *We say that a linear map (operator) $F : V \to W$ between vector spaces V, W is a Fredholm operator if the kernel of F and $W/\operatorname{im} F = \operatorname{coker}(F)$ are finite dimensional. The (Fredholm) index of F is then* $\operatorname{ind}(F) = \dim\ker F - \dim\operatorname{coker} F$.

Remark. The Fredholm index of F is essentially the analytic index ind_a discussed in the introduction, although formally $\operatorname{ind}_a(M,E)$ is the Fredholm index of the twisted operator in Definition 1.2.11.

Checking if an operator is Fredholm is often an onerous affair. To make this easier, we present the following (powerful) lemma.

Lemma 1.3.2 (Atkinson's lemma)**.** *A bounded linear operator $F : V \to W$ between separable Hilbert spaces V, W is Fredholm if and only if there exists a bounded linear operator $G : W \to V$ such that $FG - 1$ and $GF - 1$ are compact operators. G is called a parametrix for F.*

A compact operator is one for which the closure of the image of the unit ball is compact (there are many other equivalent definitions).

Proof of Lemma 1.3.2. The proof relies on a judicious choice of operators, see [Ebe, page 11, Theorem 1.5.1] for more detail. □

Fredholm operators are named after Erik Ivar Fredholm, a Swedish mathematician who is best known for his contributions to operator theory.

Definition 1.3.3 (L^2 inner product)**.** *Let E be a smooth vector bundle over a compact Riemannian manifold M with metric g. Suppose that E has a fibrewise Hermitian inner product and let $s_1, s_2 \in \Gamma^\infty(E)$. We define*

$$\langle s_1, s_2 \rangle_{L^2(E)} = \int_M \langle s_1(m), s_2(m) \rangle_{(E)_m} dm$$

and $L^2(E)$ to be the completion of $\Gamma^\infty(E)$ in this inner product. The metric g induces the volume form dm, which is (explicitly, in a chart) integration with respect to the Riemannian density $\sqrt{\det(g)}\, dx_1 \wedge \cdots \wedge dx_n$.

Definition 1.3.4 (Sobolev space of order 1)**.** *Let D be an elliptic differential operator of order 1 between the smooth complex vector bundles E_0, E_1 over a compact manifold M as in Definition 1.3.3, i.e. $D \in \mathrm{Diff}^1(E_0, E_1)$. The order 1 Sobolev space of sections of E_0 (defined by D) is the completion of $\Gamma^\infty(E_0)$ with the inner product*

$$\langle s_1, s_2 \rangle_{W^1_D(E_0)} = \langle s_1, s_2 \rangle_{L^2(E_0)} + \langle Ds_1, Ds_2 \rangle_{L^2(E_1)}$$

for $s_1, s_2 \in \Gamma^\infty(E_0)$. We denote this space $W^1_D(E_0)$.

This is not the standard approach for defining Sobolev spaces, but we show our Sobolev norm is equivalent to the typical Sobolev norm (and hence does not depend on the choice of elliptic operator D) in Lemma 1.3.6, below.

Remark. When it is necessary to emphasise the different constructions, we may write $W^1(E_0)$ to mean the Sobolev space in the standard sense, and add a subscript D as in Definition 1.3.4 to denote our new characterisation.

Lemma 1.3.5 (Gårding's inequality, a slight reformulation of 10.4.4 in [HR00])**.** *Let M, E be as in Definition 1.3.3 and suppose that $D \in \mathrm{Diff}^1(E, E)$ is a first order elliptic differential operator (for $E = E_0 = E_1$) as in Definition 1.3.4. Then there is a real positive constant c such that*

$$\|u\| + \|Du\| \geq c\,\|u\|_1 ,$$

for all $u \in W^1(E)$. The norm $\|\cdot\|$ is the ordinary L^2-norm and $\|\cdot\|_1$ is the standard Sobolev norm.

If there is any capacity for confusion we will write the subscript when using the L^2-norm, but the vast majority of all our norms will be L^2-norms.

Proof of Lemma 1.3.6. The reformulation is obtained from [HR00, 10.4.4] by taking M to be compact and $K = M$ to be the whole manifold. □

Lemma 1.3.6. *The norm induced by the inner product $\langle \cdot, \cdot \rangle_{W^1_D(E_0)}$ in Definition 1.3.4 is equivalent to the standard Sobolev norm $\|\cdot\|_1$.*

Proof. For any $a, b \in \mathbb{R}$ with $a, b \geq 0$ we have

$$a^2 + b^2 \leq (a+b)^2 \leq 2(a+b)^2$$

when $s \in \Gamma^\infty(E_0)$, $a = \|s\|_{L^2(E_0)}$ and $b = \|Ds\|_{L^2(E_1)}$ we have

$$\|s\|_{W^1_D(E_0)} \leq \|s\|_{L^2(E_0)} + \|Ds\|_{L^2(E_1)} \leq \sqrt{2}\,\|s\|_{W^1_D(E_0)}\,.$$

so the norm on $W^1_D(E_0)$ is equivalent to $\|s\|_{L^2(E_0)} + \|Ds\|_{L^2(E_1)}$. Since the operator D is bounded as map $D : W^1(E_0) \to L^2(E)$ there is a $c' > 0$ such that for all $s \in W^1(E_0)$ we have the inequality $\|Ds\|_{L^2(E_1)} \leq c'\,\|s\|_1$. The standard Sobolev norm of s is an upper bound for the L^2 norm of s by definition, so we have the inequality

$$\|s\|_{L^2(E_0)} + \|Ds\|_{L^2(E_1)} \leq (1+c')\,\|s\|_1\,.$$

By an application of Gårding's inequality (Lemma 1.3.5) to $\|s\|_{L^2(E_0)} + \|Ds\|_{L^2(E_1)}$ there is a $c > 0$ such that

$$c\,\|s\|_1 \leq \|s\|_{L^2(E_0)} + \|Ds\|_{L^2(E_1)}$$

and hence

$$c\,\|s\|_1 \leq \|s\|_{L^2(E_0)} + \|Ds\|_{L^2(E_1)} \leq (1+c')\,\|s\|_1$$

which is exactly norm-equivalence of our Sobolev norm and the standard Sobolev norm. □

Remark. It is now possible to omit D from $W^1_D(E_0)$, although we may include it to emphasise the characterisation of the Sobolev space used.

Fix an elliptic differential operator $D : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$ of order 1 as in Definition 1.3.4. By construction, this is bounded with respect to the Sobolev norm and L^2-norm, and $\Gamma^\infty(E_0)$ is dense in $W^1(E_0)$ and so it extends to

$$\bar{D} : W^1(E_0) \to L^2(E_1),$$

which is also bounded. This will be the operator that is Fredholm and later it will become apparent that $\ker \bar{D} = \ker(D)$ and $\operatorname{coker}(\bar{D}) = \operatorname{coker}(D)$.

Definition 1.3.7 (Formal adjoint)**.** *A formal adjoint of a linear operator $F : V \to W$ between inner product spaces is a map $F^* : W \to V$ satisfying $\langle Fv, w\rangle = \langle v, F^*w\rangle$, for all $v \in V, w \in W$.*

Theorem 1.3.8. *If $D : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$ is an (arbitrary) elliptic differential operator as in Definition 1.3.4, then* $\ker D^* \cong \operatorname{coker} D$.

Proof. See [Ebe, Theorem 3.7.4]. $\square$

To prove that an elliptic differential operator is a Fredholm operator, we need to review some results from functional analysis.

Definition 1.3.9 (Smoothing operator)**.** *An operator $Q : \Gamma^\infty(E_1) \to \Gamma^\infty(E_0)$ is said to be smoothing if it is of the form*

$$s \mapsto \left(m \mapsto \int_M \kappa(m, m')s(m')dm' \right)$$

where $\kappa \in \Gamma^\infty(\underline{\mathrm{Hom}}(E_0, E_1))$ is called the kernel of the operator and depends smoothly on m', m. dm is Riemannian volume form. $\underline{\mathrm{Hom}}(E_0, E_1)$ is the homomorphism bundle from E_0 to E_1.

There is a well-known identification between the sections of the homomorphism bundle and the bundle homomorphisms:

Lemma 1.3.10. *Let E, F be vector bundles over a manifold M. Then there is a ring isomorphism $\mathrm{Hom}(E, F) \cong \Gamma^\infty(\underline{\mathrm{Hom}}(E, F))$, where $\mathrm{Hom}(E, F)$ is the space of bundle homomorphisms from E to F and $\Gamma^\infty(\underline{\mathrm{Hom}}(E, F))$ is the space of sections of the homomorphism bundle.*

We don't want to distinguish between sections of the homomorphism bundle and homomorphisms. Indeed, we will swap between the characterisations as little commentary as possible.

Proof of Lemma 1.3.10. The map is $\Gamma^\infty(\underline{\mathrm{Hom}}(E, F)) \ni s \mapsto A(s) \in \mathrm{Hom}(E, F)$, where $(A(s)(e))(x) = s(x)(e)$, for $e \in E, x \in M$. $\square$

Theorem 1.3.11. *Let $D : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$ be an elliptic differential operator. Then there exists a bounded (with respect to L^2 norms) linear map $Q : \Gamma^\infty(E_1) \to \Gamma^\infty(E_0)$ such that*

$$S = QD - 1$$
$$T = DQ - 1$$

are smoothing operators.

Proof. A statement and proof of the theorem (in more general terms) is in [Shu87, Theorem 5.1]. $\square$

The aim is to prove that the operators S and T extend to compact operators $\bar{S}$ and $\bar{T}$, so that we can apply Lemma 1.3.2 and prove that $\bar{D}$ is Fredholm.

Theorem 1.3.12 (Rellich's lemma)**.** *The inclusion map $W^1_D(E_0) \hookrightarrow L^2(E_0)$ is a compact operator.*

Proof. See [Ebe, Theorem 3.6.3 (3)]. The theorem in [Ebe] is actually the inclusion mapping on the regular Sobolev space $W^1(E_0)$, rather than the one we have defined in Definition 1.3.4, although this is not a major obstacle. Because of Lemma 1.3.5, for any $s \in \Gamma^\infty(E_0)$ there exists constant $C > 0$ such that

$$\|s\|_1 \leq C(\|s\| + \|Ds\|) \leq \sqrt{2}C \|s\|_{W^1_D(E_0)}$$

and hence the identity map on $\Gamma^\infty(E_0)$ extends to a map from our characterisation $W^1_D(E_0)$ into the standard Sobolev space $W^1(E_0)$, $W^1_D(E_0) \hookrightarrow W^1(E_0)$. We can without injury compose this additional bounded map to get a compact inclusion $W^1_D(E_0) \hookrightarrow L^2(E_0)$. □

Lemma 1.3.13. *If a linear operator $A : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$ is smoothing then $D \circ A$ is also smoothing, for any differential operator D.*

Proof. Take the $\operatorname{rank}(E_0) = k$, $\operatorname{rank}(E_1) = l$ and $\dim(M) = n$. Suppose $m \in U$ and U is chart that trivialises E_0 and E_1. Then $E_1\big|_U \cong U \times \mathbb{R}^l$ and $E_0\big|_U \cong U \times \mathbb{R}^k$. We write,

$$D\big|_U = \sum_{|\alpha| \leq k} B_\alpha \frac{\partial^\alpha}{\partial x^\alpha}$$

for the operator D in its most general form, where (for $\alpha \in \mathbb{Z}^n_{\geq 0}$) B_α is a map $U \to M_{l \times k}(\mathbb{R})$.

Now, choose a finite open cover of charts $\{U_j\}_{j=1}^r$ for M (M is compact) and require that each U_j is a trivialising neighbourhood for E_0, E_1. Take a smooth partition of unity $\{\psi_j\}_{j=1}^r$ subordinate to this cover. We can write $(As)(m)$ using this partition of unity. A is smoothing, so it is of the form described in Definition 1.3.9. We write

$$\begin{aligned}(As)(m) &= \sum_{j=1}^r \int_M \kappa(m, m')\psi_j(m')s(m')dm' \\ &= \sum_{j=1}^r \int_{U_j} \kappa(m, m')\psi_j(m')s(m')dm'.\end{aligned}$$

Next, apply D:

$$(DAs)(m) = \sum_{j=1}^r \sum_{|\alpha| \leq a} B^j_\alpha(m) \frac{\partial^\alpha}{\partial x^\alpha} \left(\int_{U_j} \kappa(m, m')\psi_j(m')s(m')dm' \right).$$

This integral converges absolutely (supp(ψ_j) is compact and κ is smooth), so the operator $\frac{\partial^\alpha}{\partial x^\alpha}$ commutes with the integral sign. We can commute the sums, because they are finite. Extend the integral over U_j to an integral over M by declaring that the integrand is 0 outside of U_j (this is fine, because the support of ψ_j is contained inside U_j). We arrive at the expression

$$\sum_{|\alpha|\leq a}\sum_{j=1}^{r}\int_M \psi_j(m')B^j_\alpha(m)\frac{\partial^\alpha}{\partial x^\alpha}\kappa(m,m')s(m')dm',$$

which at first doesn't really seem to make sense because we cannot differentiate sections using $\frac{\partial^\alpha}{\partial x^\alpha}$. However, because $m \in U_j$ is in a local trivialisation of the bundle E_0 and $\{U_j\}_{j=1}^r$ are coordinate charts for the manifold we have a function $\lambda_j : U_j \to V_j \subset \mathbb{R}^n$ that associates to $m \in U_j$ a point $x = \lambda(m) \in \mathbb{R}^n$. We differentiate with respect to the coordinates on $\mathbb{R}^n$, so it makes good sense to write $\frac{\partial^\alpha}{\partial x^\alpha}\kappa(m,m')$. We hope that $\kappa_D(m,m') := \sum_{|\alpha|\leq a}\sum_{j=1}^r \psi_j(m')B_\alpha(m)\frac{\partial^\alpha}{\partial x^\alpha}\kappa(m,m')$ is the smooth kernel that we are looking for, because then we arrive at the form required:

$$\int_M \kappa_D(m,m')s(m')dm'$$

which would show that $D \circ A$ is smoothing. We have to check that $\kappa_D(m,m')$ is

- a smooth map in m, m'
- a homomorphism between $(E_0)_{m'}$ and $(E_1)_m$.

It is clear that $\kappa_D(m,m')$ depends smoothly on m, m' because it is the sum of products that depend smoothly on m, m'. What remains to be shown is that $\kappa_D(m,m')$ really is a homomorphism from $(E_0)_{m'}$ to $(E_1)_m$. We can show this using the local trivialisations mentioned at the beginning of the proof. For a fixed $x \in \mathbb{R}^n$ there is a coordinate chart $\lambda_j : U_j \to \mathbb{R}^n$ that associates $x \in \mathbb{R}^n$ to a point $m \in U_j$. For this fixed x we can associate (local) bundle trivialisations τ_0 for E_0 and τ_1 for E_1. Consider the following diagram:

$$\begin{array}{ccc} \{x\}\times\mathbb{R}^k & \xrightarrow{B_\alpha} & \{x\}\times\mathbb{R}^l \\ \Big\downarrow{\scriptstyle(\tau_0)_m^{-1}} & & \Big\downarrow{\scriptstyle(\tau_1)_m^{-1}} \\ (E_0)_{\lambda_j^{-1}(x)} & \longrightarrow & (E_1)_{\lambda_j^{-1}(x)}. \end{array}$$

Using τ_0, τ_1 we can think of B_α as a map from $(E_0)_{\lambda_j^{-1}(x)} = (E_0)_m$ to $(E_1)_{\lambda_j^{-1}(x)} = (E_1)_m$. In the same way, we can consider $\frac{\partial^\alpha}{\partial x^\alpha}\kappa(m,m')$ as a map from $(E_0)_{m'}$ to $(E_0)_m$, and the composition of these two maps gives a homomorphism $(E_0)_{m'} \to (E_1)_m$. $\square$

Lemma 1.3.14. *A smoothing operator $A : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$ extends to a bounded operator $\bar{A} : L^2(E_0) \to L^2(E_1)$.*

Proof. This is a relatively straightforward calculation that relies on Cauchy-Schwarz. Let $A : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$ be a smoothing operator with smooth kernel κ. For $m, m' \in M$ define as $\|\kappa(m, m')\|$ the operator norm of $\kappa : (E_0)_{m'} \to (E_1)_m$ with respect to the metric inner product on the fibres. We compute, for $s \in \Gamma^\infty(E_0)$:

$$\begin{aligned}
\|As\|_{L^2}^2 &= \int_M \left\| \int_M \kappa(m, m') s(y) dm' \right\|^2 dm \\
&= \int_M \left(\int_M \|\kappa(m, m') s(m')\|_{(E_1)_m} dm' \right)^2 dm \\
&\leq \int_M \left(\int_m \|\kappa(m, m')\| \, \|s(m')\|_{(E_0)_{m'}} dm' \right)^2 dm \\
&\leq \int_M \left(\left(\int_M \|\kappa(m, m')\|^2 dm' \right)^{1/2} \left(\int_M \|s(m')\|^2_{(E_0)_{m'} dm'} \right)^{1/2} \right)^2 dm \\
&= \int_M \int_M \|\kappa(m, m')\|^2 \, dm \, dm' \, \|s\|^2_{L^2(E_0)} .
\end{aligned}$$

$\square$

Lemma 1.3.15. *A smoothing operator A on $\Gamma^\infty(E_j)$ extends to a bounded operator $\tilde{A} : L^2(E_j) \to W^1(E_j)$.*

Proof. The proof is straightforward. Let $s \in \Gamma^\infty(E_j)$. Then

$$\begin{aligned}
\|As\|^2_{W^1(E_j)} &= \|As\|_{L^2(E_j)} + \|DAs\|^2_{L^2(E_j)} \\
&\leq \underbrace{\|A\|^2 \cdot \|s\|^2_{L^2(E_j)}}_{\text{Lemma 1.3.14}} + \underbrace{\|DA\|^2 \cdot \|s\|^2_{L^2(E_j)}}_{\text{Lemma 1.3.13}} \\
&= \left(\|A\|^2 + \|DA\|^2 \right) \|s\|^2_{L^2(E_j)} .
\end{aligned}$$

$\square$

Proposition 1.3.16. *A smoothing operator A extends to compact operators*

- $\bar{A} : L^2(E_j) \to L^2(E_j)$
- $\tilde{A} : W^1(E_j) \to W^1(E_j)$.

Proof. By Lemma 1.3.15, A extends to a bounded operator $\bar{A} : L^2(E_j) \to W^1(E_j)$. Composing this with the inclusion map $W^1(E_j) \hookrightarrow L^2(E_j)$ (which is compact by Theorem 1.3.12) we see that $\bar{A}$ defines compact operator on $L^2(E_j)$. Similarly, $\tilde{A} = \bar{A}|_{W^1(E_j)}$ is the compact operator $W^1(E_j) \hookrightarrow L^2(E_j) \xrightarrow{\bar{A}} W^1(E_j)$. □

Proposition 1.3.17. *The operator $\bar{D} : W^1(E_0) \to L^2(E_1)$ is Fredholm.*

Proof. By Atkinson's lemma (Lemma 1.3.2), it suffices to show that there exists an operator $\bar{Q}$ for which the operators

$$\bar{T} = \bar{D}\bar{Q} - 1, \quad \bar{S} = \bar{Q}\bar{D} - 1$$

are compact. We first prove that $Q : \Gamma^\infty(E_1) \to \Gamma^\infty(E_0)$ extends to a bounded operator $\bar{Q} : L^2(E_1) \to W^1(E_0)$. This is relatively straightforward, given the work we've already done. Let Q be as in the conclusion of Theorem 1.3.11, then

$$\begin{aligned} \|Qs\|^2_{W^1(E_0)} &= \|Qs\|^2_{L^2(E_0)} + \|DQs\|_{L^2(E_1)} \\ &\leq \underbrace{\|Q\|^2 \|s\|^2_{L^2(E_0)}}_{\text{Q is bounded}} + \underbrace{\|(1+S)s\|^2_{L^2(E_1)}}_{\text{Theorem 1.3.11 and Lemma 1.3.14}}. \end{aligned} \tag{1.3}$$

Now, $\|(1+S)(s)\|_{L^2(E_1)} \leq \|s\|_{L^2(E_1)} + \|S\| \|s\|_{L^2(E_1)}$ so (Eq. (1.3)) $\leq 2(\|Q\|^2 + 1 + \|S\|^2) \|s\|^2_{L^2 E_1}$ and consequently Q extends to a bounded operator $\bar{Q} : L^2(E_1) \to W^1(E_0)$. By Proposition 1.3.16 the smoothing operators S, T extend to compact operators $\bar{S}$ and $\bar{T}$ on $L^2(E_1)$ and $W^1(E_0)$. We apply Atkinson's Lemma, and so $\bar{D}$ is a Fredholm operator. □

Theorem 1.3.18 (Elliptic regularity)**.** *Suppose that $D : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$ is an elliptic differential operator. Then if $s \in W^1_D(E_0)$ and $Ds \in \Gamma^\infty(E_1)$ then $s \in \Gamma^\infty(E_0)$.*

Proof. The details are in [Ebe, Section 3.5, Corollorary 3.5.2]. We have the same problem as in the discussion in the proof of Theorem 1.3.12. The original theorem in [Ebe] is only a statement about the ordinary Sobolev space, but it is again possible to make good sense of this for our situation. Suppose instead that the section s instead lies in the Sobolev space defined in Definition 1.3.4. Then by Lemma 1.3.5 s is also in $W^1(E_0)$ (i.e. the standard characterisation) and hence if Ds is smooth then s is also smooth, by the original application of elliptic regularity in Corollary 3.5.2 of [Ebe]. □

A notable example of elliptic regularity is the operator $\frac{\partial}{\partial \bar{z}}$. A continuously differentiable function $f : \mathbb{C} \to \mathbb{C}$ satisfying $\frac{\partial f}{\partial \bar{z}} = 0$ is infinitely differentiable.

Theorem 1.3.19. *An elliptic differential operator* $D : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$ *is Fredholm.*

Proof. We know from elliptic regularity (Theorem 1.3.18) that $\ker \bar{D} \subset \ker D$ and so $\ker D = \ker \bar{D}$. By the same reasoning, the result holds for D^*. Theorem 1.3.8 gives the equality $\ker D^* = \operatorname{coker} D$ and Proposition 1.3.17 tells us that $\bar{D}$ has finite dimensional kernel and cokernel. □

We have done a lot of the heavy lifting in the unproven lemmas and theorems. In particular, the elliptic regularity theorem and Rellich's lemma were key.

1.4 Characteristic classes and the index theorem

Building up the index theorem, and its K-homological proof requires a fair amount of work. The background information required to state the index theorem is somewhat laborious to perform, but we include it here for completeness.

1.4.1 Characteristic classes

Definition 1.4.1 (First Chern class)**.** *The first Chern class of a complex line bundle* L *is* $c_1(L) = \frac{i}{2\pi} F_\nabla$*, where* F_∇ *is the curvature of a connection* ∇ *on* L*.*

Remark. We could also define this class as the Euler class of the underlying real vector bundle obtained by discarding the complex structure, and this is the approach taken in [BT82], which has more information about characteristic classes generally.

Definition 1.4.2 (Chern character)**.** *Let* E *be a smooth complex rank* n *vector bundle over a smooth manifold* M *with connection* ∇*. The Chern character of* E *is*

$$\operatorname{ch}(E) = \left[\operatorname{tr}(\exp(\frac{i}{2\pi} F_\nabla))\right] \in H^*_{dR}(M),$$

where $F_\nabla := \nabla^2 \in \Omega^2(M, \operatorname{End}(E))$ *is the curvature of the connection* ∇ *on* E *and the exponential is a formal power series.*

Remark. These definitions do not depend on the choice of connection.

To define the A-hat genus, we must perform some setup. For any conjugation-invariant homogeneous polynomial $f : M_n(\mathbb{R}) \to \mathbb{R}$ of degree d, there is a symmetric and multilinear map $f_d : M_n(\mathbb{R}) \times \cdots \times M_n(\mathbb{R}) \to \mathbb{R}$ (d copies of $M_n(\mathbb{R})$) such that for all $a \in M_n(\mathbb{R})$ $f(a) = f_d(a, \ldots, a)$. Extend f_d to

$$\tilde{f}_d : (\Omega^*(M) \otimes M_n(\mathbb{R})) \times \cdots \times (\Omega^*(M) \otimes M_n(\mathbb{R})) \to \Omega^*(M)$$

by
$$\tilde{f}_d(\alpha_1 \otimes a_1, \ldots, \alpha_d \otimes a_d) = f_d(a_1, \ldots a_d)\alpha_1 \wedge \cdots \wedge \alpha_d,$$
for $\alpha_j \in \Omega^*(M)$ and $a_j \in M_n(\mathbb{R})$. We can extend $\tilde{f}_d$ multilinearly to all of $\Omega^*(M) \otimes M_n(\mathbb{R})$ by the multilinearity of f_d. Then $\tilde{f}$ is the evaluation of $\tilde{f}_d$ at the diagonal,
$$\tilde{f} : \Omega^*(M) \otimes M_n(\mathbb{R}) \to \Omega^*(M), \quad \tilde{f}(x) = \tilde{f}_d(x, \ldots, x).$$
If the polynomial is not homogeneous we can apply it separately to each homogeneous term. Let p be the polynomial $p : M_n(\mathbb{R}) \to \mathbb{R}$ given by
$$p(a) = \det\left(\frac{a/2}{\sinh(a/2)}\right)^{1/2}. \tag{1.4}$$

Definition 1.4.3 (A-hat genus)**.** *Given a real vector bundle $E \to M$ over a* Spin^c *manifold M, the A-hat genus of E, written $\hat{A}(E)$ is*
$$\hat{A}(E) = \tilde{p}(R)$$
where R is the curvature of a fixed connection on E, and $\tilde{p}$ is the map induced from the polynomial $\tilde{p}$ in (1.4)*, described immediately above.*

Remark. The A-hat genus of a manifold is by definition the A-hat genus of its tangent bundle: $\hat{A}(M) = \hat{A}(TM)$.

Lemma 1.4.4. *The A-hat genus is multiplicative over direct sum, in the sense that if E_1 and E_2 are* Spin^c *vector bundles over a manifold M then $\hat{A}(E_1 \oplus E_2) = \hat{A}(E_1)\hat{A}(E_2)$.*

Proof. This is due to a comment in [LM89] on page 138, for Spin^c manifolds X, Y of dimension divisible by 4, we have $\hat{A}(X \times Y) = \hat{A}(X) \times \hat{A}(Y)$. The full discussion may be found in Basic Construction 11.12 and Example 11.13 (pages 230 and 231) of the same text, and they are general statements about vector bundles rather than just the tangent bundles. □

In the typical sense of the word, the Todd class is a characteristic class that can be found for any complex vector bundle. We must instead now work with the Spin^c-Todd class, which we will call just the Todd class, but it is important to note that this is a different characteristic class to the standard Todd class.

Definition 1.4.5 (Spin^c determinant line bundle)**.** *Given a* Spin^c *bundle E of rank E over a smooth even dimensional manifold M, let* $\det : \mathrm{Spin}^c \to \mathbb{C}$ *be given by* $\mathrm{Spin}^c(n) \ni [p, z] \mapsto z^2$*. Then define a relation on $P \times \mathbb{C}$ by $(p, w) \sim (px^{-1}, \det(x)w)$ for all $p \in P_E$, $w \in \mathbb{C}, x \in \mathrm{Spin}^c(n)$. The* Spin^c *determinant line bundle, written L_E (or just L when E is unambiguous) is the quotient of $P \times \mathbb{C}$ by this relation,*
$$L_E = (P_E \times \mathbb{C})/\sim .$$

We will unambiguously omit the "Spin^c" from this definition, because we do not use the standard complex determinant line bundle.

Remark. Without squaring z in the above definition, the map det is not well-defined

Definition 1.4.6 (Spin^c-Todd class)**.** *Given E as in Definition 1.4.2 the* Spin^c *Todd class of $E \to M$ is*

$$\mathrm{Td}(E) = \exp(c_1(L)/2)\hat{A}(M)$$

where $c_1(L)$ is the first Chern class of the determinant line bundle of the structure bundle P for E and $\hat{A}(M)$ is the A-hat genus of M.

Proposition 1.4.7. *The Todd class is multiplicative over direct sum; if E_1 and E_2 are* Spin^c *vector bundles, then the direct sum $E_1 \oplus E_2$ has Todd class*

$$\mathrm{Td}(E_1 \oplus E_2) \cong \mathrm{Td}(E_1)\,\mathrm{Td}(E_2)$$

Proof. This is a consequence of Lemma 1.4.4. If L_E is as in Definition 1.4.6 is the determinant line bundle of $E = E_1 \oplus E_2$ then $L_E = L_{E_1} \otimes L_{E_2}$ and $c_1(L_E) = c_1(L_{E_1}) + c_1(L_{E_2})$. The exponential turns the sum into a product, so

$$\begin{aligned} \mathrm{Td}(E_1 \oplus E_2) &= e^{c_1(L_{E_1})/2 + c_1(L_{E_1})/2}\hat{A}(E_1 \oplus E_2) \\ &= e^{c_1(L_{E_1})/2}e^{c_1(L_{E_2})/2}\hat{A}(E_1 \oplus E_2) \\ &= e^{c_1(L_{E_1})/2}e^{c_1(L_{E_2})/2}\hat{A}(E_1)\hat{A}(E_2) \quad \text{(Lemma 1.4.4)} \end{aligned}$$

□

Definition 1.4.8 (Evaluation at the fundamental class)**.** *Let ω be a class in de Rham cohomology and let $[M]$ be the fundamental class of a smooth manifold M, which is a homology class. We can pair ω with $[M]$ and the pairing is called the evaluation of ω at M and is integration against ω over M,*

$$\omega[M] = \int_M \omega.$$

Remark. In the event that ω is of mixed degrees (i.e not only top degree), the integral is by definition integration of the top degree form(s).

Definition 1.4.9 (Topological index)**.** *Given a smooth vector bundle E as in Definition 1.4.2 the topological index of $E \to M$ is the evaluation* $\mathrm{ch}(E) \cup \mathrm{Td}(M)[M]$.

Note that the product $\cup$ here is the cup product of cohomology classes. The following gives some useful properties of the Chern character as it relates to the projection above. With the possible exception of Chern Fact 3, they are standard results.

Proposition 1.4.10 (Fast Chern facts)**.** *Suppose $E \to M$ is a smooth complex vector bundle of rank n with connection $\nabla : \Gamma^\infty(E) \to \Gamma^\infty(E \otimes T^*M)$.*

1. *If $E = \bigoplus_{j=1}^{\alpha} L_j$ then $\mathrm{ch}(E) = \sum_{j=1}^{n} \exp(c_1(L_j))$, for L_j the Chern roots of E.*

2. *The Chern character of the dual of E differs from the Chern character of E in component $2r$ by a factor of $(-1)^r$.*

Proposition 1.4.11 (Chern Fact 3)**.** *Suppose E as in Proposition 1.4.10. If E is the image of a projection $p : M \times \mathbb{C}^\alpha \to E$ ($\alpha > n$) then the Chern character of E is represented by a differential form whose component in dimension $2r$ is $\frac{i^r}{r!(2\pi)^r}\,\mathrm{tr}(p(dp)^{2r})$.*

Proof of Fact 1. Define as previously the curvature of the connection ∇ on E as F_∇. The definition of the Chern character is $\mathrm{ch}(E) = \mathrm{tr}(\exp(\frac{i}{2\pi}F_\nabla))$, which we expand as

$$\begin{aligned}\mathrm{tr}(\exp(\frac{i}{2\pi}F_\nabla)) &= \mathrm{tr}\left(\sum_{j=0}^{\infty}\left(\frac{i}{2\pi}\right)^j \frac{(F_\nabla)^j}{j!}\right)\\ &= \left(\sum_{j=0}^{\infty}\left(\frac{i}{2\pi}\right)^j \frac{1}{j!}\,\mathrm{tr}(F_\nabla^j)\right).\end{aligned}$$

It is clear we should investigate the $(F_\nabla)^j$ further. We can assign to E a projection mapping $p : M \times \mathbb{C}^\alpha \to E$ (it is always possible to do this) and write ps to mean a section of E that we have obtained by applying the projection map to a section s of $M \times \mathbb{C}^\alpha$. The connection applied to ps is then $\nabla ps = (p(d(ps))$ and its curvature is $F_\nabla(ps) = p(d(p(d(ps))))$. The placement of brackets here is key, and we would like to instead write the curvature as $p((dp)^2(ps))$ i.e. the application of the derivative of p twice to ps followed by the projection. Decompose $d(ps)$ into

$$d(ps) = ((dps)_1, \ldots, (dps)_\alpha)$$

where $(dps)_j$ is the j^{th} component of dps. For each $j \in \{1, \ldots, \alpha\}$ we have

$$\begin{aligned}(dps)_j &= d(ps)_j\\ &= d(\sum_k p_{jk}s_k)\\ &= \sum_k((dp_{jk}))s_k p_{jk} ds_k\\ &= ((dp)s)_j + (pds)_j.\end{aligned}$$

Thus, there is a Leibniz-like rule, $d(ps) = (dp)s + p(ds)$. We can use this to further investigate the curvature $F_\nabla = \nabla_E^2$. Take $s \in \Gamma^\infty(M \times \mathbb{C}^\alpha)$, i.e a smooth function $s : M \to \mathbb{C}^\alpha$. Then ps is a section of E by construction, so we may write

$$\begin{aligned}
\nabla_E^2(ps) &= \nabla_E(pd(ps)) \\
&= pd(pd(ps)) \\
&= pd(p((pd)s + pds)) \\
&= pd(p(dp)s + pds) \\
&= p((dp)(dp)s + pd((dp)s) + (dp)(ds) + pd^2 s)
\end{aligned}$$

The de Rham operator d satisfies $d^2 = 0$, $p^2 = p$ and $d((dp)s) = -(dp)(ds)$ so

$$\nabla_E^2(ps) = p((dp)(dp)s + pd((dp)s) + (dp)(ds) + pd^2 s) = p(dp)^2 s.$$

This computation will be used in the proof of Proposition 1.4.11 but it is important note now that if $E = E_1 \oplus E_2$ is the direct sum of two line bundles, then the projection p splits into $p_1 \oplus p_2$ and there is a corresponding splitting of the connection and hence curvature:

$$\operatorname{tr}(p(dp)^{2j}) = \operatorname{tr}(p_1(dp_1)^{2j}) + \operatorname{tr}(p_2(dp_2)^{2j})$$

and we note that because of this the definition of the Chern character splits as a sum

$$\operatorname{ch}(E) = \sum_{k=0}^{\infty} \frac{i^r}{k!2\pi^r}((F_{\nabla_1})^k + (F_{\nabla_2})^k)$$

which is exactly the definition of $\operatorname{ch}(E_1) + \operatorname{ch}(E_2)$. We have relied on the fact that the Chern character does not depend on the choice of connection used, so we may pick the connection on E that decomposes into a connection on E_1 and a connection on E_2. The cohomology class of $\frac{i}{2}F_\nabla$ is by definition the first Chern class of a line bundle, so if $E = \bigoplus_{j=1}^n L_j$ then the Chern character splits over each line bundle and we have the result

$$\operatorname{ch}(E) = \sum_{j=0}^{n} \exp(c_1(L_j)).$$

$\square$

Proof of Fact 2. The Chern class of the dual of a line bundle is the negative of the Chern class of the line bundle, so if $\{L_j\}_{j=1}^n$ are the Chern roots of E as in

Definition 1.4.2,

$$\begin{aligned}\mathrm{ch}(E^*) &= \sum_{j=1}^{n} \exp(-c_1(L_j)) \\ &= \sum_{j=1}^{n}\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} c_1(L_j)^r.\end{aligned}$$

This differs from the Chern character of E in component $2r$ by a factor of $(-1)^r$. $\square$

Proof of Fact 3. Continue where we left off in the proof of Fact 1. We have $F_\nabla = p(dp)^2$ and

$$\mathrm{ch}(E) = \mathrm{tr}\exp(\frac{i}{2\pi}F_\nabla) = \sum_{k=1}^{\infty}\left(\frac{i}{k!2\pi}\right)^k \mathrm{tr}(p(dp)^{2k})$$

which has component $\frac{i^r}{r!(2\pi)^r}\,\mathrm{tr}(p(dp)^{2r})$ in dimension $2r$. Note that this is the r^{th} term in the formal power series of the exponential, but the dimension doubles because of the $(dp)^2$ term. $\square$

Lemma 1.4.12. *The bundle $TS^{2r}\otimes\mathbb{C}$ is stably trivial and in particular,* $\mathrm{Td}(TS^{2r}\otimes\mathbb{C}) = 1$.

Proof. The trivial outward facing unit normal bundle $\mathcal{N}$ of S^{2r} provides the additional vector needed; $TS^{2r}\oplus\mathcal{N} = S^{2r}\times\mathbb{R}^{2r+1}$. The Todd class is multiplicative over direct sum so we have $1 = \mathrm{Td}((TS^{2r}\oplus\mathcal{N})\otimes\mathbb{C}) = \mathrm{Td}(TS^{2r})\cdot\mathrm{Td}(\mathcal{N}) = \mathrm{Td}(TS^{2r})$. $\square$

We are now ready to present the index theorem of Atiyah and Singer. A note before we begin. When discussing the index of D_E, we must preface it with the following comment. Formally, because twisted operator is self-adjoint, it has index 0. Fortunately, we can define an associated index that is more than sufficient for our purposes.

The spinor bundle S_M is graded via Lemma 1.2.8, and the operator D_E swaps the different gradings, so we can decompose the operator into the respective halves:

$$\begin{pmatrix} 0 & D_E\big|_{\Gamma^\infty(S_M^-\otimes E)} \\ D_E\big|_{\Gamma^\infty(S_M^+\otimes E)} & 0 \end{pmatrix}.$$

which we will denote by D_E^+ and D_E^-, respectively. The operator D_E^- is the formal adjoint of D_E^+ (by construction), and $\mathrm{ind}(D_E^+) = \dim\ker D_E^+ - \dim\ker(D_E^+)^*$.

We can abuse notation and write $\mathrm{ind}(D_E) = \dim\ker D_E^+ - \dim\ker D_E^-$. It is not common to discuss the "proper" index of D_E, which is always 0, and we certainly do not do it here.

Theorem 1.4.13 (Atiyah–Singer for twisted Spin^c-Dirac operators)**.** *Suppose M is an even dimensional compact* Spin^c *manifold without boundary and $E \to M$ is a smooth complex vector bundle over M. If $D_E : \Gamma^\infty(S_M \otimes E) \to \Gamma^\infty(S_M \otimes E)$ is the Dirac operator on M twisted by E, then*

$$\mathrm{ind}(D_E) = (\mathrm{ch}(E) \cup \mathrm{Td}(TM)[M].$$

It is possible to relax the conditions on D_E. As we mentioned in the introduction (Atiyah–Singer, 1963), we can instead require it to be merely elliptic rather than a twisted Spin^c-Dirac operator. The results for Dirac operators are the most important step, the problem for general elliptic differential operators can be reduced to the Dirac case using a commutative triangle, which the interested reader can read more on in [Bv16], but this extension is beyond the scope of our discussion here.

Chapter 2

$\mathbf{ch}(\beta)[S^{2r}] = \mathbf{ind}(D_\beta)$

The main result in this chapter is that that $1 = \mathrm{ch}(\beta)[S^{2r}] = \mathrm{ind}(D_\beta)$, and this computation will serve as a test case for the proof of Theorem 1.4.13 using Baum-Douglas K-homology. The index theorem of Atiyah and Singer is classically (i.e. in [AS68b]) presented as a statement about homomorphisms in K-theory. We take the same approach (although we are working with the homology equivalent): the analytic and topological indices are both group homomorphisms, and to show they are equal everywhere, it suffices to show that they agree on the generators of the group. This is discussed at length in Chapter 3 but the rough outline of this is that the analytic index is an isomorphism $K_0(\cdot) \to \mathbb{Z}$ and if it agrees with the topological index on a generator, they agree everywhere. The pair (S^n, β) is an example of a pair of index 1 and is also a generator of the group $K_0(\cdot)$. In this sense, Chapter 2 contains the most important computation we perform because all other computations reduce to index of the pair (S^n, β).

2.1 $\mathbf{ch}(\beta)[S^{2r}] = 1$

Remark. From this point on we will be working almost exclusively with the spinor bundle of S^n. With this in mind, we will introduce the convention of writing only S to mean S_{S^n}. Whenever writing the spinor bundle of a different manifold, we will be explicit.

Definition 2.1.1 (Bott generator vector bundle)**.** *The Bott generator vector bundle β is the dual of the positive spinor bundle $S^+_{S^n}$ on S^{2r}. The* Spin^c *structure of S^{2r} is the one it receives as the boundary of the unit ball.*

Recall that in Lemma 1.2.8 we could divide the spinor bundle S_M via the grading operator $c(\omega)$. This creates the so-called *positive* and *negative* spinor bundles, which we denote by S_M^+ and S_M^-, respectively. The positive spinor bundle of S^n can be imagined as a sub-bundle of the trivial bundle by using the projection

$$e : S^{2r} \to \mathrm{End}(\mathbb{C}^{2^r}) \quad e(t_1, \dots t_{2r+1}) = \frac{1}{2}\left(1 + i\sum_{j=1}^{2r+1} t_j A_j\right)$$

for $t_j \in \mathbb{R}$ satisfying $\sum_j^{2r+1} t_j^2 = 1$ and A_j as in Eq. (1.2).

Theorem 2.1.2. *If β is the Bott generator vector bundle on S^{2r} then* $\mathrm{ch}(\beta) \cup \mathrm{Td}(TS^{2r} \otimes \mathbb{C})[S^{2r}] = 1$.

This follows from the following lemma:

Lemma 2.1.3. $\mathrm{ch}(S^+)[S^{2r}] = (-1)^r$.

Proof of Theorem 2.1.2. Fast Chern Fact 2 (Proposition 1.4.10) provides the equality $\mathrm{ch}(\beta)[S^{2r}] = (-1)^r \mathrm{ch}(S^+)[S^{2r}]$ and from Lemma 2.1.3 we conclude that $\mathrm{ch}(\beta)[S^{2r}] = (-1)^r \mathrm{ch}(S^+)[S^{2r}] = 1$. The Todd class of S^{2r} is 1, and so is not seen when evaluating at the fundamental class. □

The proof of Theorem 2.1.2 will follow the outline in [BvE18, Propositions 6,7], with some of the missing gaps filled in.

Proof of Lemma 2.1.3. We know from Chern Fact 3 (Proposition 1.4.11) that the Chern character is represented by a differential form with component in dimension $2r$ given by:

$$\frac{i^r}{r!(2\pi)^r}\,\mathrm{tr}(e(de)^{2r}).$$

We can write

$$(de)^{2r} = \left(\frac{i}{2}\right)^{2r}\left(\sum_{j_1,\dots,j_{2r}=1}^{2r+1} dt_{j_1} \wedge \cdots \wedge dt_{j_{2r}} A_{j_1} \cdots A_{j_{2r}}\right)$$

for A_j's as in (1.2). Because both the A_j's and the wedge product anti-commute (see [BvE18, 2.1]) and are arranged in the same way, the whole product $dt_{j_1} \wedge \cdots \wedge dt_{j_{2r}} A_{j_1} \cdots A_{j_{2r}}$ is invariant under swapping j_{k_1} and j_{k_2}. The product over $2r$ distinct things out of $2r+1$ things is the same as excluding one thing from $2r+1$ things. We arrive at

$$(de)^{2r} = \frac{i^{2r}(2r)!}{2^{2r}} \sum_{j=1}^{2r+1} dt_1 \wedge dt_2 \wedge \cdots \wedge \widehat{dt_j} \wedge \cdots \wedge dt_{2r+1} A_1 \cdots \widehat{A_j} \cdots A_{2r+1}.$$

Note the factor of $(2r)!$ because of the rearrangement required to collect the individual wedge products (that is to say, there are $(2r)!$ distinct copies of the product that excludes dt_1, $(2r)!$ that exclude dt_2, etc.).

Next, apply e

$$\begin{aligned} e(de)^{2r} &= \frac{1}{2}\left(1 + i\sum_{j=1}^{2r} t_j A_j\right)\frac{i^{r-1}(2r)!}{2^{2r}}\sum_{j=1}^{2r+1}(-1)^j A_j dt_1 \wedge dt_2 \wedge \cdots \wedge \widehat{dt_j} \wedge \cdots \wedge dt_{2r+1} \\ &= \frac{1}{2}\frac{i^{r-1}(2r)!}{2^{2r}}\sum_{j=1}^{2r+1}(-1)^j A_j dt_1 \wedge dt_2 \wedge \cdots \wedge \widehat{dt_j} \wedge \cdots \wedge dt_{2r+1} \\ &+ \left(i\sum_{j=1}^{2r+1} t_j A_j\right)\left(\frac{i^{r-1}(2r)!}{2^{2r+1}}\sum_{j=1}^{2r+1}(-1)^j A_j dt_1 \wedge dt_2 \wedge \cdots \wedge \widehat{dt_j} \wedge \cdots \wedge dt_{2r+1}\right). \end{aligned}$$

We have used Lemma 1.2.5 to turn $A_1 \cdots \widehat{A_j} \cdots A_{2r+1}$ into $(-1)^j A_j$. Now, $A_j^2 = -I_{2^r}$ (Lemma 1.2.7) and because A_j anti-commutes with A_k when $j \neq k$, $\operatorname{tr}(A_j A_k) = -\operatorname{tr}(A_k A_j) = 0$ so we have the result:

$$\begin{aligned} \operatorname{tr}(e(de^{2r})) &= \operatorname{tr}\left(\frac{i^r(2r)!}{2^{2r+1}}\sum_{j=1}^{2r+1}(-1)^j t_j A_j^2 dt_1 \wedge dt_2 \wedge \cdots \wedge \widehat{dt_j} \wedge \cdots \wedge dt_{2r+1}\right) \\ &= \frac{i^r(2r)!}{2^{r+1}}\sum_{j=1}^{2r+1}(-1)^{j-1} t_j dt_1 \wedge \cdots \wedge \widehat{dt_j} \wedge \cdots \wedge dt_{2r+1}. \end{aligned}$$

At this point we're almost there,

$$\frac{i^r}{r!(2\pi)^r}\operatorname{tr}(e(de)^{2r}) = \frac{i^{2r}(2r)!}{r!(2\pi)^r 2^{r+1}}\sum_{j=1}^{2r+1}(-1)^{j-1} t_j dt_1 \wedge \cdots \wedge \widehat{dt_j} \wedge \cdots \wedge dt_{2r+1}.$$

The Chern character evaluated at the fundamental class of S^{2r} is the integral $\int_{S^{2r}} \operatorname{ch}(S^+_{S^{2r}})$,

$$\frac{i^{2r}(2r)!}{(2\pi)^r 2^{r+1} r!}\int_{S^{2r}}\sum_{j=1}^{2r+1}(-1)^{j-1} t_j dt_1 \ldots \widehat{dt_j} \ldots dt_{2r+1} = \frac{i^{2r}(2r)!}{(2\pi)^r 2^{r+1} r!}\int_{B^{2r+1}}(2r+1) dt_1 \ldots dt_{2r+1}$$

by Stokes' theorem. The volume of the $(2r+1)$-ball of radius 1 is,

$$V_{2r+1} = \frac{2r! 2^{2r}\pi^r}{(2r+1)!},$$

and using it we can calculate the integral as

$$\begin{aligned}\frac{i^{2r}(2r)!}{(2\pi)^r 2^{r+1} r!} \int_{B^{2r+1}} (2r+1) dt_1 \dots dt_{2r+1} &= \frac{i^{2r}(2r)!}{(2\pi)^r 2^{r+1} r!} (2r+1) V_{2r+1} \\ &= i^{2r} = (-1)^r.\end{aligned}$$

For derivation of the volume, see [Joh]. The result in our case can be obtained by taking the formula in [Joh] and setting $n = 2r+1$ and the radius to 1 (it is called "r" in the reference). □

2.2 ind$(D_\beta) = 1$

Theorem 2.2.1. *If D is the Dirac operator of the even dimensional sphere S^n with the* Spin^c *datum it receives as the boundary of the unit ball in $\mathbb{R}^n$ then,*

$$\operatorname{ind}(D_\beta) = 1.$$

There are some preliminary steps we must take before we can prove this theorem. Let V be a finite dimensional vector space with some quadratic form Q defining a Clifford algebra $C(V,Q)$ and let S_r be the permutation group on r things. Define the isomorphism (see [LM89, page 11, equation 1.11] for more details) $\tilde{f} : \bigwedge^* V \to C(V,Q)$ by linearly extending the mapping

$$\tilde{f}(v_1 \wedge \cdots \wedge v_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \operatorname{sign}(\sigma) v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(r)}. \tag{2.1}$$

Remark. If V is the fibre of some bundle E over M, then we can abuse notation and write

$$\tilde{f} : \bigwedge^* E \to C(E) \tag{2.2}$$

if $\tilde{f}$ is defined for each $x \in M$ as a map $\tilde{f} : \bigwedge^* E_x \to C(E_x)$. We will be using this essentially exclusively for $E = TM$.

Lemma 2.2.2 (Useful fact)**.** *If $e_{i_1}, \dots e_{i_k}$ is an collection (with increasing index, say) of some orthonormal basis vectors of V as above, then*

$$\tilde{f}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = e_{i_1} \cdots e_{i_k}.$$

Proof. We can use anti-commutativity of the basis vectors to change every permuted product $e_{\sigma(i_1)} \cdots e_{\sigma(i_k)}$ back to $e_{i_1} \cdots e_{i_k}$, scaled by $(-1)^{b_\sigma}$, where b_σ is the number of pairwise swaps required to go from the permuted product back to the increasing one. The sign of the permutation is (-1) when b is odd and 1 when b is even, and the order of S_r is $r!$, so

$$\begin{aligned}\tilde{f}(e_{i_1} \wedge \cdots \wedge e_{i_k}) &= \frac{1}{r!} \sum_{\sigma \in S_r} \operatorname{sign}(\sigma)^2 e_{i_1} \cdots e_{i_k} \\ &= e_{i_1} \cdots e_{i_k}.\end{aligned}$$

□

Definition 2.2.3 (Contraction)**.** *Suppose X is a set and X^n is the set of ordered n-tuples of elements of X. Let $f : X^n \to Y$ be a map from X into any set Y. The contraction of f by a point $x \in X$ is $x \lrcorner f : X^{n-1} \to Y$ defined by*

$$(x \lrcorner f)(x_1, \ldots x_{n-1}) = f(x, x_1, \ldots, x_{n-1}).$$

The most common application of the above definition is the contraction of a differential n-form by a vector field, but we would like to use the notation more generally.

Lemma 2.2.4. *Choose $f \in C^\infty(M), y \in M$ and $s \in \Gamma^\infty(\bigwedge^* T^*M)$ with $f(y) = 0$ and $s(y) = e$. Let $\xi = df\big|_y$. If d is the de-Rham operator, then the principal symbol of d is $\sigma_d(\xi)(e) = i\xi \wedge e$ and in particular, $(\sigma_d(y,\xi) + \sigma_{d^*}(y,\xi))(e) = i\xi \wedge e - i\xi \lrcorner e$.*

Proof. We first note that from Lemma 1.2.15 the choice of e and ξ does not depend on f or s, and we include the mention of them in the statement of the lemma only as a reminder of the context. Now, principal symbol of the sum is

$$\sigma_{d+d^*}(y,\xi)(e) = \sigma_d(y,\xi)(e) + \sigma_{d^*}(y,\xi)(e) = \sigma_d(y,\xi)(e) + \sigma_d(y,\xi)^*(e)$$

and thus, we need only compute the principal symbol of d, which is a first order operator.

$$\begin{aligned}\sigma_d(y,\xi)(e) &= id(fs)(y) \\ &= i(df \wedge s + fds)(y) \\ &= idf\big|_y \wedge e = i\xi \wedge e.\end{aligned}$$

As a map, the dual of $\wedge$ is $\lrcorner$ i.e. $(\xi\wedge)^*(e) = \xi \lrcorner (e)$ and the dual of i is $-i$, so

$$\sigma_{d^*}(y,\xi)(e) = -i\xi \lrcorner e.$$

□

Lemma 2.2.5. *Let V be a vector space with a positive definite inner product identifying V and V^* and suppose $v \in V$ is fixed. If $\tilde{c}(v)$ is the map $x \mapsto (v \wedge x - v \lrcorner x)$ for $x \in \bigwedge^* V$ and $\tilde{f}$ is as in (2.1) then we have the following commutative diagram.*

$$\begin{array}{ccc} \bigwedge^* V & \xrightarrow{\tilde{f}} & C(V) \\ \big\downarrow{\scriptstyle \tilde{c}(v)} & & \big\downarrow{\scriptstyle v\cdot} \\ \bigwedge^* V & \xrightarrow{\tilde{f}} & C(V) \end{array}$$

Some commentary before we begin the proof. When M is a Spin^c manifold there is a Riemannian metric on M giving the isomorphism $T^*M \cong TM$. We would like to be able to identify co-vectors and vectors, so that if v is a co-vector and x is a vector we can treat x as a co-co-vector and evaluate v at x.

Proof of Lemma 2.2.5. The equality is $\tilde{f} \circ \tilde{c}(v) = v \cdot \tilde{f}(x)$. First, fix an orthonormal (with respect to the inner product) basis $\{e_i\}_{i=1}^{\dim V}$ for V and let $\{e^i\}_{i=1}^{\dim V}$ be the dual basis of V^*. For a collection of k basis vectors $e_{i_1}, \dots e_{i_k}$ and a covector e^j, we aim to show the following equality

$$e_j \tilde{f}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \tilde{f}(\underbrace{e_j \wedge (e_{i_1} \wedge \cdots \wedge e_{i_k})}_{\alpha} - \underbrace{e^j \lrcorner (e_{i_1} \wedge \cdots \wedge e_{i_k})}_{\beta}). \tag{2.3}$$

This is at the level of basis vectors, but because $\tilde{f}$ is linear (it is *not* an algebra homomorphism) the result holds for linear combinations. Now, we proceed in two cases. Let us first consider the scenario in which $e_j = e_{i_l}$ for some $l \in \{1, \dots, k\}$.

Evidently, $\alpha = 0$ as e_{i_l} occurs twice in α and for β we have $\beta = e^j \lrcorner (e_{i_1} \wedge \cdots \wedge e_{i_r})$:

$$\begin{aligned}
(e_j \lrcorner (e_{i_1} \wedge \cdots \wedge e_{i_k}))(e^{j_1}, \dots e^{j_{k-1}}) &= (e_{i_1} \wedge \cdots \wedge e_{i_k})(e^j, e^{j_1}, \dots, e^{j_{k-1}}) \\
&= (-1)^{l-1}(e_{i_1} \wedge \cdots \wedge e_{i_k})(e^{j_1}, \dots, e^{j_{l-1}}, e^j, e^{j_{l+1}}, \dots e^{j_{k-1}}) \\
&= (-1)^{l-1} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma) e_{\sigma(i_1)}(e^{j_1}) \cdots e_{\sigma(i_{l-1})}(e^{j_{l-1}}) e_{\sigma(i_l)}(e^j) \cdots \\
&\quad \cdots e_{\sigma(i_{l+1})}(e^{j_{l+1}}) \cdots e_{\sigma(i_k)}(e^{j_{k-1}}) \\
&= (-1)^{l-1}(\delta_{i_1 j_1}) \cdots (\delta_{i_{l-1} j_{l-1}})(\delta_{i_l j})(\delta_{i_{l+1} j_l})(\delta_{i_k j_{k-1}}) \\
&= (-1)^{l-1}(e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_l} \wedge \cdots \wedge e_{i_k})(e^{j_1}, \dots e^{j_{k-1}}),
\end{aligned}$$

i.e $e^j \lrcorner (e_{i_1} \wedge \cdots \wedge e_{i_k})$ is equal to $(e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_l} \wedge \cdots \wedge e_{i_k})$.

The second case occurs when $j \notin \{i_1, \dots i_k\}$. In this case we do not have $\alpha = 0$ but instead $\beta = e_j \lrcorner (e_{i_1} \wedge \cdots \wedge e_{i_r}) = 0$ (a similar computation to above) and $\alpha = (-1)^l e_{i_1} \wedge \cdots \wedge e_{i_l} \wedge e_j \wedge e_{i_{l+1}})$. Thus, for either of the two cases the right hand side of (2.3) is $e_j e_{i_1} \cdots e_{i_k}$. Now, by Lemma 2.2.2 for any collection of orthonormal basis vectors we have $\tilde{f}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = e_{i_1} \cdots e_{i_k}$ and hence,

$$e_j \tilde{f}(e_{i_1} \wedge \cdots \wedge e_{i_k}) = (-1)^l e_{i_1} \cdots \hat{e}_{i_l} \cdots e_{i_k}$$

which is exactly the right side of (2.3) under $\tilde{f}$. □

Before we can prove Theorem 2.2.1 we need to define an auxiliary operator $\tilde{D}_{S^*}$.

Definition 2.2.6 ($\tilde{D}_{S^*}$)**.** *Define $\tilde{D}_{S^*}$ by the commutativity of the following diagram.*

$$\begin{tikzcd}
\Gamma^\infty(\bigwedge^*(T^*S^n)) \arrow[r, "h"] \arrow[d, "d+d^*"] & \Gamma^\infty(S \otimes S^*) \arrow[d, "\tilde{D}_{S^*}"] \\
\Gamma^\infty(\bigwedge^*(T^*S^n)) \arrow[r, "h"] & \Gamma(S \otimes S^*).
\end{tikzcd}$$

where $h = c \circ \tilde{f}$.

Lemma 2.2.7. $\tilde{D}_{S^*}(\Gamma^\infty(S^+ \otimes (S^+)^*)) \subset \Gamma^\infty(S \otimes (S^+)^*)$

Proof. This follows from the compatibility of h with the odd/even and positive/negative gradings on the wedge products, we have

$$h\left(\bigwedge^{\substack{\text{even}\\ \text{odd}}}(T^*S^n)\right) = \mathrm{End}^\pm(S)$$

and

$$h\left(\bigwedge^{\pm}(T^*S^n)\right) = \mathrm{Hom}(S, S^\pm).$$

□

Now, the proof of Theorem 2.2.1 will proceed from the following propositions.

Proposition 2.2.8. *If $d : \Omega^*(S^n) \to \Omega^*(S^n)$ is the de Rham operator on forms on S^n then the principal symbol of $d + d^*$ corresponds to the principal symbol of D_{S^*}, in the sense that for every covector ξ and image covector $\xi' = h(\xi)$ we have $h \circ \sigma_{d+d^*}(\xi) = \sigma_{D_{S^*}}(\xi') \circ h$.*

Proposition 2.2.9. *$\tilde{D}_{S^*}$ has kernel $\mathbb{C} \cdot 1 \oplus \mathbb{C}c(\omega)$.*

Remark. Proposition 2.2.8 implies that $\tilde{D}_{S^*}$ the same index as D_{S^*} and, more importantly, that $\tilde{D}_\beta := \tilde{D}_{S^*}\Big|_{\Gamma^\infty(S^+ \otimes (S^+)^*)}$ has the same principal symbol as $D^+_{(S^+)^*} = D_\beta$.

Proof of Proposition 2.2.8. The principal symbol of D_{S^*} (i.e. the dual of the whole spinor bundle) is $\sigma_{D_{S^*}}(\xi) = ic(\xi) \otimes 1_{S^*}$. We know the principal symbol of $d + d^*$ from Lemma 2.2.4. We have the two principal symbols (evaluated at a generic e in some fibre of TS^n)

$$(\xi \wedge e - \xi \lrcorner\, e), \quad ic(\xi)(e). \tag{2.4}$$

We aim to find an isomorphism h such that $h \circ (\xi \wedge \ - \xi \lrcorner\ ) = c(\xi) \circ h$. To reduce complexity, we will consider this as a fibre-wise computation first and write V to mean a finite dimensional vector space, with an association $V = V^*$ similarly to $TM = T^*M$ for a Riemannian manifold M. For brevity, denote by $\tilde{c}(v)$ the operator $(v \wedge + v \lrcorner\ )$. We aim write down an h satisfying the following diagram:

$$\begin{tikzcd} \bigwedge V \arrow[r, "h"] \arrow[d, "\tilde{c}(v)"] & \mathrm{End}(\mathbb{C}^{2r}) \arrow[d, "c(v)\circ"] \\ \bigwedge V \arrow[r, "h"] & \mathrm{End}(\mathbb{C}^{2r}). \end{tikzcd} \tag{2.5}$$

The candidate mapping for h is (of course) the one we already have,

$$\begin{tikzcd} \bigwedge V \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{c}(v)"] & C(V) \arrow[r, "c"] \arrow[d, "v\cdot"] & \mathrm{End}(\mathbb{C}^{2^r}) \arrow[d, "c(v)\circ"] \\ \bigwedge V \arrow[r, "\tilde{f}"] & C(V) \arrow[r, "c"] & \mathrm{End}(\mathbb{C}^{2^r}) \end{tikzcd} \tag{2.6}$$

where c is Clifford multiplication. The commutativity of this diagram is really what is meant by (2.5) and indeed the equivalence of the symbols in (2.4). Proving that the outer square of (2.6) commutes is the same as proving that each inner square commutes. The right square commutes because c is an algebra homomorphism and the left square commutes as a consequence of the computation in Lemma 2.2.5. To modify this to the statement in the lemma, extend $\tilde{c}(v)$ to a map $\tilde{c}(v) : \bigwedge^* TM \to C(TM)$ by decreeing that $\tilde{c}(v) : \bigwedge^* TM \to C(TM)$ is the mapping induced by taking $\tilde{c}(v) : \bigwedge^* T_mM \to \bigwedge^* T_mM$ on each fibre. This is the same process that extended $\tilde{f}$ in (2.2).

□

Before the proof of Proposition 2.2.9, recall the Hodge theorem for differential forms.

Theorem 2.2.10 (Hodge theorem)**.** *The dimension of the kernel of $dd^* + d^*d$ acting on the $k-$forms of a compact oriented manifold is the same as the dimension of k^{th} de Rham cohomology group.*

Proof. The statement comes from Theorem 6.11 of [War83]. □

Proof of Proposition 2.2.9. By definition, $\tilde{D}_{S^*}$ has the same index as $(d+d^*)$. The kernel of $d+d^*$ is the same as the kernel of $(d+d^*)^2 = dd^* + d^*d$ (d and d^* square to 0), and by the Hodge theorem applied to S^n in degree $k = n$ and degree $k = 0$ the de Rham cohomology of the n-sphere is $\mathbb{R}$. In the case that $k \neq 0, n$ the cohomology group is trivial. Hence, the (complex) dimension $\ker(d+d^*)$ is 1.

A local description of the top degree form on S^n is $e_1 \wedge \cdots \wedge e_n$ for a local orthonormal frame $\{e_i\}_{i=1}^n$ of $TM = T^*M$. Recall in Lemma 1.2.8 that if $\omega = i^r e_1 e_2 \cdot e_n$ then $c(\omega)$ provided the grading operator for the spinor bundle. Using Lemma 2.2.2 we can associate this to exactly ω in the Clifford algebra. Of course, there is a scaling by the scaling factor of i^r, but this is not seen when taking the complex span. The smooth function (i.e. 0-form) part of the de Rham cohomology remains unchanged. The conclusion of this is that the kernel of $\tilde{D}_{S^*}$ is

$$h(\mathbb{C}\cdot 1 \oplus \mathbb{C}\omega) = \mathbb{C}\cdot 1 \oplus \mathbb{C}c(\omega) = \mathbb{C} \oplus \mathbb{C}c(\omega).$$

□

Proof of Theorem 2.2.1. Define $\mathrm{End}(S^+) = \{a \in \mathrm{End}(S) \mid a\big|_{S^-} = 0, \operatorname{im} a \subset S^+\}$ and likewise for $\mathrm{End}(S^-)$. Then $\mathrm{End}(S)$ splits into four constituent parts,

$$\mathrm{End}(S) = \mathrm{End}(S^+) \oplus \mathrm{End}(S^-) \oplus \mathrm{Hom}(S^+, S^-) \oplus \mathrm{Hom}(S^-, S^+). \tag{2.7}$$

The kernel of $\tilde{D}_{S^*}$ is $\mathbb{C} \oplus \mathbb{C}c(\omega) \subset \mathrm{End}(S)$. We note that, $\ker(1 \pm c(\omega)) = S^\mp$ and $\mathrm{Hom}(S^+, S^-) \cap \mathbb{C}(1+c(\omega)) = 0 = \mathrm{Hom}(S^-, S^+) \cap \mathbb{C}(1+c(\omega))$. Splitting $\mathbb{C}\oplus\mathbb{C}c(\omega)$ on $\mathrm{End}(S)$ via (2.7) we have a decomposition of $\mathbb{C}\cdot 1 \oplus \mathbb{C}\cdot c(\omega)$ into

$$\underbrace{\mathbb{C}\cdot(1+c(\omega))}_{\subset \mathrm{End}(S^+)} \bigoplus \underbrace{\mathbb{C}\cdot(1-c(\omega))}_{\subset \mathrm{End}(S^-)}, \tag{2.8}$$

and we note that $\mathrm{ind}(\tilde{D}_\beta) = \dim(\ker \tilde{D}_{S^*} \cap \mathrm{End}(S^+)) - \dim(\ker \tilde{D}_{S^*} \cap \mathrm{Hom}(S^+, S^-))$. Now, $(\ker \tilde{D}_{S^*})^+ \cap \mathrm{End}(S^+) = \mathbb{C}(1+c(\omega))$, which gives $\dim(\ker \tilde{D}_\beta)^+ = 1$. Note that we do not see the homomorphism parts of (2.7) in (2.8) because there is nothing in $\mathbb{C}\cdot 1 \oplus \mathbb{C}c(\omega)$ that is a homomorphism from S^+ to S^- or vice versa (except for 0). Because of this, the negative part of the kernel of $\tilde{D}_\beta$ is $\ker(\tilde{D}_\beta)^- = \ker(\tilde{D}_{S^*}) \cap \mathrm{Hom}(S^+, S^-) = 0$. Hence, the index of $\tilde{D}_\beta$ is 1, which is the same as the index of D_β. □

Corollary 2.2.11 (The whole point of this chapter)**.** *When $M = S^n$ and $E = \beta$, Theorem 1.4.13 is true.*

Proof. By Lemma 1.4.12, $\mathrm{Td}(S^{2r}) = 1$ and hence, $(\mathrm{ch}(\beta) \cup \mathrm{Td}(S^{2r}))[S^{2r}] = 1 = \mathrm{ind}(D_\beta)$,

□

Chapter 3

K-homology; $(M, E) \sim (S^n, q\beta)$

This chapter serves to compute the K-homology of a point, and to provide an explicit series of steps to deduce the relation $(M, E) \sim (S^n, q\beta)$, which will be made more rigorous below. We show that the K-homology of a point is $\mathbb{Z}$, and that Theorem 1.4.13 is a consequence of this computation. The chapter concludes with a proof of Theorem 1.4.13 under the assumption that the analytic and topological indices are well-defined (Chapter 4 resolves this problem).

3.1 K-homology

Definition 3.1.1 (Pair isomorphism)**.** *Suppose (M, E) is a pair as in Definition 1.2.11: M is a smooth, compact, even dimensional* Spin^c *manifold M, and $E \to M$ is a smooth complex vector bundle. We say that (M, E) with datum (P, η) is isomorphic to (M', E') with datum (P', η') (and write $(M, E) \cong (M', E')$) if there is a diffeomorphism $\varphi : M \to M'$ that preserves the datum, in the sense that the pullback $(\varphi^* P', \varphi^* \eta')$ is isomorphic to (P, η) (as Spin^c data) and that the pullback $\varphi^* E'$ is isomorphic (as a complex vector bundle on M) to E.*

This definition is only an intermediate step for the definition of $\mathcal{K}$ immediately below, and also for the statement Definition 3.1.9, below.

Definition 3.1.2. *$\mathcal{K}$ is the set of all pairs (M, E) in Definition 3.1.1 modulo pair-isomorphism.*

When we write (M, E) without particular reference, we mean $(M, E) \in \mathcal{K}$, or possibly (M, E) as a representative of a class in $\mathcal{K}$, but this is not an important distinction to make, as every statement is valid up to pair-isomorphism. It will be made explicit when a pair is not an element of $\mathcal{K}$.

Definition 3.1.3 (Ball bundle, sphere bundle)**.** *If $E \to M$ is a vector bundle with fibre-wise norm $\|\cdot\|_x$ then*

- *$B(E) \to M$ is the fibre bundle with fibre over $x \in M$ given by*

$$B(E)_x = \{v \in E_x \mid \|v\|_x \leq 1\}.$$

- *$S(E) \to M$ is the fibre bundle with fibre over $x \in M$ given by*

$$S(E)_x = \{v \in E_x \mid \|v\|_x = 1\}.$$

Write $\underline{\mathbb{R}}^\alpha \to M$ to mean the trivial vector bundle of rank α on M, i.e. $\underline{\mathbb{R}}^\alpha = M \times \mathbb{R}^\alpha \to M$, and likewise for $\underline{\mathbb{C}}^\alpha$.

Definition 3.1.4 (ΣF)**.** *Let M be as in Definition 3.1.1 and suppose F is a $\mathrm{Spin}^c(n)$ vector bundle on M with structure bundle P as in Definition 1.1.6, with F having even fibre dimension $n = 2r$. Define $\pi : \Sigma F \to M$ as the fibre bundle with fibres oriented spheres of dimension n, $\Sigma F = P \times_{\mathrm{Spin}^c(n)} S^n$. The Spin^c structure on ΣF is fixed as the one it receives as the boundary of the unit ball in $(F \oplus \underline{\mathbb{R}})_x$, for each $x \in M$.*

Remark. The action of $\mathrm{Spin}^c(n)$ on S^n is via the canonical covering map projection $\pi_{\mathrm{Spin}(n)} : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ sending $[p, z]\,\mathrm{Spin}^c(n)$ to $\pi_{\mathrm{Spin}(n)}(p)$. $\mathrm{SO}(n)$ acts on S^n by considering S^n as a dense open subset of the one-point compactification $\mathbb{R}^n \cup \{pt\}$. It would also be possible to consider S^n as the typical subset of $\mathbb{R}^{n+1}$ and acting on it in the usual way, but stereographic projection provides an exact correspondence between these actions, so the difference is largely academic.

The following lemma illustrates the local structure of ΣF, but it will not be very useful until Chapter 4.

Lemma 3.1.5. *Suppose that $\Sigma F = P \times_{\mathrm{Spin}^c(n)} S^n$ is as in Definition 3.1.4 and U is a trivialising neighbourhood for $P \to M$. Then*

$$\Sigma F\big|_{\pi^{-1}(U)} \cong U \times S^n.$$

Proof. Suppose (U, τ) is a trivialisation for $P\big|_U$. The isomorphism is

$$\Sigma F\big|_{\pi^{-1}(U)} \ni [p, x] \mapsto (u, g \cdot x) \in U \times S^n \tag{3.1}$$

for $\tau(p) = (u, g) \in U \times \mathrm{Spin}^c(n)$. □

Definition 3.1.6 (β_F)**.** *Suppose ΣF is as in Definition 3.1.4, and β is the Bott generator vector bundle as in Definition 2.1.1. Then define by β_F the vector bundle $\beta_F = P \times_{\mathrm{Spin}^c(n)} \beta \to \Sigma F$ with projection $\beta_F \ni [p, b] \mapsto [p, x] \in \Sigma F, p \in P, \beta \in \beta_x$.*

Definition 3.1.7 ($K_0(\cdot)$)**.** *Suppose $\mathcal{K}$ is as in Definition 3.1.2. Define $K_0(\cdot)$ as $\mathcal{K}/\sim$, where the equivalence relation $\sim$ is generated by*

- *Direct sum - disjoint union (Definition 3.1.8),*
- *Bordism (Definition 3.1.9),*
- *Vector bundle modification (Definition 3.1.11).*

Addition of equivalence classes is disjoint union

$$(M,E)+(M',E')=(M\sqcup M',E\sqcup E').$$

The additive inverse is $-[(M,E)]=[(-M,E)]$, where we use $-M$ to denote M with the reversed Spin^c *structure, from Definition 1.1.11.*

Remark. By *generated by*, we mean that any finite sequence of the three actions (not necessarily including all of them) is necessary and sufficient to establish equivalence.

Definition 3.1.8 (Direct sum - disjoint union)**.** *We say that $(M,E)\sqcup(M,E')\in\mathcal{K}$ is equivalent to $(M,E\oplus E')\in\mathcal{K}$ via direct sum - disjoint union.*

Definition 3.1.9 (Bordism)**.** *We say $(M,E)\in\mathcal{K}$ is bordant to $(M',E')\in\mathcal{K}$ if there exists a pair (W,F) consisting of a compact odd-dimensional Spin^c manifold with boundary ∂W and a smooth $\mathbb{C}$ vector bundle F over W such that $(\partial W,F\big|_{\partial W})$ is pair-isomorphic (in the sense of Definition 3.1.1) to $(M,E)\sqcup(-M',E')$.*

Remark. The pair (W,F) is not in $\mathcal{K}$ because W is not even-dimensional.

Lemma 3.1.10. *The $[(-M,E)]$ in Definition 3.1.7 satisfies, $[(M\sqcup -M,E\sqcup E)]=0\in K_0(\cdot)$.*

Proof. Including orientations, $(M\sqcup -M,E\sqcup E)$ is the boundary of $(M\times[0,1],E\times[0,1])$, and so it bordant to the trivial element in $K_0(\cdot)$. □

Definition 3.1.11 (Bundle modification)**.** *Suppose that F, ΣF over M are as in Definition 3.1.4 and β_F as in Definition 3.1.6. Then we say $(\Sigma F,\beta_F\otimes\pi^*E)\in\mathcal{K}$ is related to the pair $(M,E)\in\mathcal{K}$ by bundle modification.*

Lemma 3.1.12. *Given a pair (M,E) as in Definition 3.1.9 and another pair $(N,F)\in\mathcal{K}$ the pair (M,E) is bordant to $(M,E)\sqcup(\partial N,F\big|_{\partial N})$.*

Proof. The bordism is provided by $(M\times[0,1]\sqcup N,E\times[0,1]\sqcup F)$. □

Remark. Because the index is preserved by bordism (proven in Chapter 4), this implies that the topological index of the pair $(\partial N,F\big|_{\partial N})$ is 0.

3.2 $(S^n, q\beta)$

We endeavour to show that for any pair $(M,E) \in \mathcal{K}$ we can write $[(M,E)] = [(S^n, q\beta)] \in K_0(\cdot)$, for some $q \in \mathbb{Z}, n \in \mathbb{N}$ depending on M and E.

Definition 3.2.1 ($q\beta$)**.** *When $q \geq 0$ we interpret $q\beta$ as the direct sum of q copies of β and when $q < 0$, $|q|$ copies of $\beta^\vee := (S^-_{S^n})^*$.*

Lemma 3.2.2 (Weak sphere lemma)**.** *Given a pair (M,E) as above there exists a vector bundle $F \to S^{2r}$ such that $(M,E) \sim (S^{2r}, F)$.*

This is called the *weak* sphere lemma because later we will see that we can specify F more explicitly.

Proof of Lemma 3.2.2. We must dive into the relations of K-homology. By the Whitney embedding theorem [Lee13, Theorem 6.15] we can embed M into $\mathbb{R}^{2r}$, for some sufficiently large $r \in \mathbb{N}$. First, consider the normal bundle $\nu := \{(m,v) \in M \times \mathbb{R}^{2r} \mid v \perp T_m M\}$. By the 2-out-of-3 principle of Lemma 1.1.20 applied to the exact sequence

$$0 \to TM \to M \times \mathbb{R}^{2r} \to \nu \to 0$$

a Spin^c orientation is determined for the normal bundle. A direct application of bundle modification ($\nu = F$) yields

$$(M,E) \sim (\Sigma\nu, \beta_\nu \otimes \pi^* E). \quad \text{(Step 1)}$$

Our first goal is to construct an explicit bordism between $\Sigma\nu$ and S^{2r}. Because M is compact, we can scale our embedding to be contained in the interior of the unit ball. Let $B' = B(\nu \oplus \underline{\mathbb{R}}) \to M$ and $S' = S(\nu \oplus \underline{\mathbb{R}}) \to M$. The aim is to show that the ball bundle $B(\nu \oplus \underline{\mathbb{R}})$ (which has boundary $\Sigma\nu$) identifies with a compact tubular neighbourhood of the embedding of M into $\mathbb{R}^{2r+1}$. Using the local diffeomorphism (choose v to be very small) $(m,v) \mapsto m + v$ we can identify ν with a compact tubular neighbourhood of (the embedding of) M in $\mathbb{R}^{2r}$. We will be using this association without particular reference for the remained of the proof, as it tends to obfuscate. The inclusion $\mathbb{R}^{2r} \hookrightarrow \mathbb{R}^{2r+1}$ gives an identification between the ball bundle $B(\nu \oplus \underline{\mathbb{R}})$ and a compact tubular neighbourhood of M in $\mathbb{R}^{2r+1}$.

Now, write

$$\nu \oplus \underline{\mathbb{R}} = P \times_{\mathrm{Spin}^c(2r)} \mathbb{R}^{2r+1}, \quad B(\nu \oplus \underline{\mathbb{R}}) = P \times_{\mathrm{Spin}^c(2r)} B^{2r+1}.$$

The boundary of is $\partial B(\nu \oplus \mathbb{R}) = P \times_{\mathrm{Spin}^c(2r)} S^{2r}$ and the $\mathrm{Spin}^c(2r)$ acts on by $\partial B(\nu \oplus \mathbb{R})$ in the same way it does on $\Sigma\nu$, which is the same way as in Definition 3.1.4 and the remark below it.

Denote by Ω the unit ball in $\mathbb{R}^{2r+1}$ with the interior of B' removed. By construction Ω has boundary that is the disjoint union of $\Sigma\nu$ and S^{2r}. The union is disjoint because of the scaling factor applied to M to ensure it is contained *within* the interior of the unit ball, and $\Sigma\nu$ is then entirely disjoint from the boundary S^{2r}. By the bundle extension lemma (Lemma 3.2.3, below) applied to B', Ω, $S' = B' \cap \Omega$ there is a vector bundle L over B' such that $\beta_\nu \otimes \pi^* E \oplus L\big|_{S'}$ (in the statement of the lemma, the bundle $\beta_\nu \otimes \pi^* E$ is "E") extends to a vector bundle F over Ω. By Lemma 3.1.12 we then have the equivalence in K-homology:

$$(\Sigma\nu, \beta_\nu \otimes \pi^* E) \sim (\Sigma\nu, \beta_\nu \otimes \pi^* E) \sqcup (\Sigma\nu, L\big|_{S'}) \quad \text{(Step 2)}$$

Step 3 is a straightforward application of direct sum-disjoint union

$$(\Sigma\nu, \beta_\nu \otimes \pi^* E) \sqcup (\Sigma\nu, L\big|_{S'}) \sim (\Sigma\nu, (\beta_\nu \otimes \pi^* E) \oplus L\big|_{S'}) \quad \text{(Step 3)}$$

The final step will draw on the bordism Ω,

$$(\Sigma\nu, (\beta_\nu \otimes \pi^* E) \oplus L\big|_{S'}) \sim (S^{2r}, F\big|_{S^{2r}}). \quad \text{(Step 4)}$$

□

Lemma 3.2.3 (Bundle extension lemma). *Let the unit ball in $\mathbb{R}^{2r}$ be the union of two compact sets B', Ω and let $S' = B' \cap \Omega$ be their intersection. If E is a complex vector bundle on S' then there exists a complex vector bundle L on B' such that $E \oplus L\big|_{S'}$ extends to Ω, in the sense that there exists another vector bundle R on Ω for which the restriction of R to S' is $E \oplus L\big|_{S'}$.*

Before we begin the proof of Lemma 3.2.3 we must make a short detour into the cohomological realm and in particular, K-theory. K-theory plays an outsized role in the proof of Lemma 3.2.5, but it is largely hidden behind the heavy machinery of cohomology. They key step is in Lemma 3.2.4 which allows us to jump from the F we found in Lemma 3.2.2 to the Bott bundle β and indeed, provides us with a fairly explicit way of proving the index theorem of Atiyah and Singer. We would not like to spend too much time on the details (the reader is directed to [Hatb] for a far better treatise on introductory cohomology and [Hata] for some of the K-theoretic details) but the summary is as follows.

Given X, A, B with X being the union of the interiors of A and B, there is [Hatb, page 203 for the cohomology version, 149 for the homology version] a long exact sequence in cohomology (with coefficients in a group G)

$$\cdots \to H^n(X; G) \to H^n(A; G) \oplus H^n(B; G) \to H^n(A \cap B; G) \to H^{n+1}(X; G) \to \cdots$$

which is called the Mayer-Vietoris sequence. There is [GBVF01, exercise on page 127] an analogue of this sequence for K-theory, which is only six terms (because of the famous *Bott periodicity*) and we must make use of this in the proof of Lemma 3.2.3.

Proof of Lemma 3.2.3. We have the Mayer-Vietoris sequence when $n = 0$:

$$\cdots \to K^0(\Omega) \oplus K^0(B') \to K^0(S') \to K^1(B' \cup \Omega) = 0 \to \cdots .$$

Exactness of this sequence implies that the map $\alpha : K^0(\Omega) \oplus K^0(B') \to K^0(S')$ is surjective. Thus, for a vector bundle $[E] \in K^0(S')$ there exists a pair $(l, m) \in \mathbb{Z} \times \mathbb{Z}$ such that

$$[F] - [\underline{\mathbb{C}}^m] \in K^0(\Omega), \quad [\tilde{L}] - [\underline{\mathbb{C}}^l] \in K^0(B'),$$

such that $\alpha(([F] - [\underline{\mathbb{C}}^m]), ([\tilde{L}] - [\underline{\mathbb{C}}^l])) = ([F\big|_{S'}] - [\underline{\mathbb{C}}^m]) - ([\tilde{L}\Big|_{S'}] - [\underline{\mathbb{C}}^l]) = [E]$. This gives the equality in K-theory

$$[E] + [\tilde{L}\Big|_{S'}] - [\underline{\mathbb{C}}^k] = [F\big|_{S'}] - [\underline{\mathbb{C}}^m],$$

which implies that there exists an isomorphism $E \oplus \tilde{L}\Big|_{S'} \oplus \underline{\mathbb{C}}^{k+l} \cong F\big|_{S'} \oplus \underline{\mathbb{C}}^{m+l}$ for some k. Now, $L = \tilde{L} \oplus \underline{\mathbb{C}}^{k+l} \to B'$ is a bundle over B' and $\left(F \oplus \underline{\mathbb{C}}^{m+l}\right)\big|_{S'} \cong E \oplus L\big|_{S'}$. Finally, $\left(F \oplus \underline{\mathbb{C}}^{m+l}\right)$ is a bundle over Ω and so we have the required extension. $\square$

Lemma 3.2.4 (Bott equivalence lemma)**.** *If E be a complex vector bundle on S^{2r}, then there exists a non-negative integers l, m and an integer q such that*

$$E \oplus \underline{\mathbb{C}}^l \cong q\beta \oplus \underline{\mathbb{C}}^m.$$

Proof. Recall from Definition 3.2.1 at the beginning of the section that when $q < 0$ we define $q\beta = |q|\beta^\vee$. We know that $K^0(S^{2r}) = \mathbb{Z}[\mathbb{C}] \oplus \mathbb{Z}[\beta]$ and so if E is a smooth complex vector bundle over S^{2r} it defines a K-theory class $[E] \in K^0(S^{2r})$ and hence there are $k, q \in \mathbb{Z}$ such that $[E] = k[\mathbb{C}] + q[\beta]$. There are four cases we have to consider.

1. $q, k \geq 0$
2. $q \geq 0, -k > 0$
3. $-q > 0, k \geq 0$
4. $-q, -k > 0$

If $k, q \geq 0$ then since equality in K-theory is given by stable equivalence of vector bundles, we know that there exists $m \in \mathbb{Z}_{\geq 0}$ such that

$$E \oplus \underline{\mathbb{C}}^m \cong \underline{\mathbb{C}}^k \oplus q\beta \oplus \underline{\mathbb{C}}^m.$$

If $-k > 0, q \geq 0$ we have $[E] + k[q] = q[\beta]$ and there exists an $m \in \mathbb{Z}_{\geq 0}$ such that

$$E \oplus \underline{\mathbb{C}}^{m+|k|} = q\beta \oplus \underline{\mathbb{C}}^m.$$

If $-q > 0, k \geq 0$, we have the initial K-theoretic expression $[E] = k[\underline{\mathbb{C}}] + q[\beta]$, which gives $[E] - q[\beta] = k[\underline{\mathbb{C}}]$. We conclude that there exists an $m \in \mathbb{Z}_{\geq 0}$ such that

$$E \oplus |q|\beta \oplus \underline{\mathbb{C}}^m \cong \underline{\mathbb{C}}^{|k|} \oplus \underline{\mathbb{C}}^m.$$

By adding $\beta^\vee$ to both sides and noting that $(\beta \oplus \beta^\vee)^* = (S^+ \oplus S^-)^* = \underline{\mathbb{C}}^{2^r}$, because it is the restriction of a trivial bundle on $\mathbb{R}^{2r+1}$ to the sphere (this is due to a comment on page 106 of [BvE18]) we get

$$\begin{aligned} E \oplus \underline{\mathbb{C}}^{2^r} \oplus \underline{\mathbb{C}}^m &\cong E \oplus |q|(\beta \oplus \beta^\vee) \oplus \underline{\mathbb{C}}^m \\ &\cong \underline{\mathbb{C}}^{|k|} \oplus \underline{\mathbb{C}}^m \oplus |q|\beta^\vee. \end{aligned} \tag{3.2}$$

If $-k, -q > 0$ then we must combine the cases when $-k > 0$ and $-q > 0$. We have the initial K-theoretic equation $[E] = k[\underline{\mathbb{C}}] + q[\beta]$, which gives $[E] - q[\beta] - k[\underline{\mathbb{C}}] = 0$. We conclude that there exists an $m \in \mathbb{Z}_{\geq 0}$ such that

$$E \oplus |q|\beta \oplus \underline{\mathbb{C}}^{|k|} \oplus \underline{\mathbb{C}}^m \cong \underline{\mathbb{C}}^m$$

so finally we have

$$E \oplus |q|\underline{\mathbb{C}}^{2^r} \oplus \underline{\mathbb{C}}^m \cong \underline{\mathbb{C}}^{|k|} \oplus \underline{\mathbb{C}}^m \oplus |q|(S^-)^*.$$

□

Lemma 3.2.5 (Strong sphere lemma)**.** *Given a pair (M, E) there exists an $r \in \mathbb{N}$ and $q \in \mathbb{Z}$ such that $(M, E) \sim (S^{2r}, q\beta)$.*

Proof. We continue where we left off in the weak sphere lemma (Lemma 3.2.2) with the relation $(M, E) \sim (S^{2r}, F)$. Choose q, l, m as in Lemma 3.2.4 and consider the following chain of K-equivalences

$$\begin{aligned} (M, E) &\sim (S^{2r}, F) \sqcup (S^{2r}, \underline{\mathbb{C}}^l) \quad \text{(Lemma 3.1.12)} \\ &\sim (S^{2r}, F \oplus \underline{\mathbb{C}}^l) \quad \text{(direct sum-disjoint union)} \\ &\sim (S^{2r}, q\beta \oplus \underline{\mathbb{C}}^m) \quad \text{(Lemma 3.2.4)} \\ &\sim (S^{2r}, q\beta) \sqcup (S^{2r}, \underline{\mathbb{C}}^m) \quad \text{(direct sum-disjoint union)} \\ &\sim (S^{2r}, q\beta) \quad ((S^{2r}, \underline{\mathbb{C}}^l) \sim (\emptyset, \emptyset), \text{via bordism}) \end{aligned}$$

□

Remark. This is one of the great triumphs of K-homology. The theory allows us to reduce computations on a large class (compact, even dimension, Spin^c, without boundary) of manifolds with (smooth, complex) vector bundles to computations on a sphere with a particular bundle.

A little bit of housekeeping before we get to the main result. Recall in Chapter 2 (specifically, in the proof of Theorem 2.2.1) we constructed $\tilde{D}_{S^*}$ that had the same principal symbol as $d + d^*$ and hence the same index. The proof that $\operatorname{ind} D_\beta = 1$ did not entirely reveal the role of S^+ (versus say, S^-), in $\beta = (S^+)^*$. The following lemma will illustrate how this grading-reversal affect the index.

Lemma 3.2.6. *The index of $D_{\beta^\vee}$ is -1.*

Proof. The positive part of the kernel is 0 because $\ker(\tilde{D}_{S^*}) \cap \operatorname{Hom}(S^-, S^+) = 0$ and the negative part is $\ker(\tilde{D}_{S^*})^- \cap \operatorname{End}(S^-) = \mathbb{C}(1 - c(\omega))$, so the index is -1. This is analogous to what we saw in the proof of Theorem 2.2.1 except with S^+ replaced with S^- in the appropriate (i.e. where S^+ does not change to S^- when swapping gradings) locations. □

Remark. We do not yet know if the analytic index of a pair (M,E) is well-defined on K-homology classes. This will be resolved in Chapter 4.

Theorem 3.2.7 (The analytic index is an isomorphism)**.** *The map*

$$\operatorname{ind}_a : K_0(\cdot) \to \mathbb{Z}, \quad [(M,E)] \mapsto \operatorname{ind}(D_E)$$

is an isomorphism of abelian groups.

Proof. Assume that $\operatorname{ind}_a(M,E) = \operatorname{ind}(D_E) = 0$ for a pair (M,E). Suppose that $q \geq 0$. We have the relation

$$(M,E) \sim (S^{2r}, q\beta)$$

and so $\operatorname{ind} D_{q\beta} = 0$ (assuming the analytic index is well-defined on K-homology classes). Direct sum - disjoint union allows us to decompose $(S^{2r}, q\beta)$ into $\bigsqcup_{j=1}^q (S^{2r}, \beta)$ and

$$0 = \operatorname{ind}(D_{q\beta}) = \operatorname{ind}_a(\bigsqcup_{j=1}^{q} (S^{2r}, \beta)) = q \operatorname{ind}(D_\beta) = q.$$

This proves injectivity in the non-negative case. Suppose that instead q is negative. By Lemma 3.2.6 above the index of the pair $(S^{2r}, \beta^\vee)$ is -1 and we have $\operatorname{ind}_a(S^{2r}, q\beta) = \bigsqcup_{j=1}^q \operatorname{ind}_a(S^{2r}, \beta^\vee) = |q|(-1) = q$. Note that we have actually constructed a surjection - given $q \in \mathbb{Z}$, $(S^{2r}, q\beta)$ has index q. □

We are now ready to prove Theorem 1.4.13:

Proof of Theorem 1.4.13. We have the two homomorphisms of abelian groups,

$$\begin{aligned} \mathrm{ind}_a : K_0(\cdot) \to \mathbb{Z}, &\quad (M, E) \mapsto \dim(\ker D_E)^+ - \dim(\ker D_E)^- \\ \mathrm{ind}_t : K_0(\cdot) \to \mathbb{R}, &\quad (M, E) \mapsto (\mathrm{ch}(E) \cup \mathrm{Td}(M))[M] \end{aligned}$$

and it suffices to show that they agree for one example of index 1 (as then they agree on $2 = 1 + 1$ and so on). By Lemma 1.4.12 $\mathrm{Td}(TS^{2r} \otimes \mathbb{C}) = 1$ and by Theorem 2.1.2 $\mathrm{ch}(\beta)[S^{2r}]$ is 1, while the analytic index of (S^{2r}, β) is also 1. $\square$

Remark. It is an immediate consequence of the theorem that that topological index is integral. Of course, because we have seen only one calculation of the topological index (and that index was integral) we do not have any reason to believe the topological index *isn't* integral, but it is not too hard to believe $\mathrm{ch}(E) \cup \mathrm{Td}(M)[M]$ is not *prima facie* forced to be an integer.

Chapter 4

Invariance of the index

This chapter serves to verify that the homomorphisms ind_a and ind_t in the proof of Theorem 1.4.13 are well-defined i.e. that that analytic and topological indices do not depend on the choice of representative of the class $[(M,E)] \in K_0(\cdot)$. The broad strokes of this outline are taken from [BvE18], although for specific computations in the the case of bordism invariance of the analytic index we will refer to some computations given in the enlightening paper [Hig91] by Nigel Higson.

4.1 Analytic index

A comment on notation before we begin. The analytic index on a K-homological pair (M,E) is by definition the Fredholm index of the Dirac operator of M twisted by E, $\mathrm{ind}_a(M,E) = \mathrm{ind}\, D_E$. We will alternate between $\mathrm{ind}\, D_E$ and $\mathrm{ind}_a(M,E)$ where it is appropriate to do so, without comment.

4.1.1 Under direct sum - disjoint union and bordism

Theorem 4.1.1 (Invariance under direct sum - disjoint union)**.** *The analytic index is invariant under direct sum - disjoint union. If $(M,E),(M,E')$ are as in Definition 3.1.8 then*

$$\mathrm{ind}_a((M,E) \sqcup (M,E')) = \mathrm{ind}(D_{E\oplus E'}).$$

Proof. We commit a mild sin with the notation $(M,E) \sqcup (M,E')$ because it is not of the form $(A,B) \in \mathcal{K}$, leading us to ask: how does one evaluate $\mathrm{ind}_a : \mathcal{K} \to \mathbb{Z}$ at $(M,E) \sqcup (M,E')$? What we really mean when writing $(M,E) \sqcup (M,E')$ is $(M \sqcup M, E \sqcup E)$, where $E \sqcup E'$ has the bundle structure of the disjoint union of each bundle. The sections of $E \sqcup E'$ are $\Gamma^\infty(E) \oplus \Gamma^\infty(E')$ and the spinor bundle of $M \sqcup M$ is the direct sum $S_M \oplus S_M$. Thus, we have

$$\Gamma^\infty(S_{M\sqcup M} \otimes (E \oplus E')) = \Gamma^\infty(S_M \otimes E) \oplus \Gamma^\infty(S_M \otimes E').$$

The kernel of $D_{E\sqcup E'}$ splits in the same way and $\mathrm{ind}(D_{E\sqcup E'}) = \mathrm{ind}(D_E) + \mathrm{ind}(D_{E'})$, which is exactly the index of $D_{E\oplus E'}$. □

Theorem 4.1.2 (Invariance under bordism)**.** *Let $(M_1, F_1), (M_2, F_2)$ be as in Definition 3.1.7 and related by a single bordism. Then the index the Dirac operator on M_1 twisted by F_1 is the same as the index of the operator of M_2 twisted by F_2.*

The theorem will exist as a corollary of the following result:

Proposition 4.1.3 (Boundary index)**.** *Assume that $M = \partial W$ is a smooth compact even dimensional Spin^c manifold that is the boundary of a compact Spin^c manifold W and $F \to M$ is a smooth complex vector bundle with $F = E\big|_{\partial W}$ for $E \to W$ another smooth complex vector bundle. Furthermore, suppose that the* Spin^c *structure that M receives is the one inherited as the boundary of W (see discussion after Definition 2.1.1). Then the index of the Dirac operator of M twisted by F is 0.*

To see why this would be useful, remember that the index is a additive over disjoint union. Assume that ∂W is a disjoint union $M_1 \sqcup (-M_2)$ as in the definition of bordism. Denote by F_1 the restriction of F to M_1 and F_2 the restriction of F to $-M_2$. The additivity of the index gives $\mathrm{ind}(D_F) = \mathrm{ind}(D_{F_1}) + \mathrm{ind}(D_{F_2}) = 0$. The reversal of the orientation of the second of manifold has the practical effect of multiplying $\mathrm{ind}(D_{F_2})$ by -1. We arrive at

$$\mathrm{ind}(F) = 0 = \mathrm{ind}(D_{F_1}) - \mathrm{ind}(D_{F_2})$$

which is bordism invariance of the analytic index. The proof of Proposition 4.1.3 requires some intermediary steps. Assume that we can form a collar of M near W, so that locally the boundary is diffeomorphic to $(-1, 0] \times M$. Define $W^+ = W \cup M \times (0, \infty)$ and by D the Dirac operator on W^+ and D_F the Dirac operator D twisted by F (see Definition 1.2.11). The following lemma will be useful in determining which operators are compact.

Lemma 4.1.4. *Let ψ be a compactly supported endomorphism on $S_W^+ \otimes F$ with* $\mathrm{supp}(\psi) \subset U$*, for $U \subset W$ a relatively compact open set. Then if D_F is the* Spin^c*-Dirac operator for W twisted by F, the operator $\psi \circ (D_F \pm i)^{-1}$ is a compact operator.*

Proof. This follows from a small extension of Theorem 1.3.12. For $E_0 \to M$ as in the setting of Theorem 1.3.12 and an open subset U of M, define $W_D^1(E_0\big|_U)$ to be the completion of $\Gamma_c^\infty(E_0\big|_U)$ (compactly supported sections) in the Sobolev norm. This sits naturally inside $\Gamma_c^\infty(E_0)$ by extending as 0 outside U. The extension of the lemma is then that if U is additionally relatively compact then the inclusion map $W_D^1(E_0\big|_U) \hookrightarrow L^2(E_0)$ is compact. We will need to use this inclusion to get a compact operator, but it is not worth emphasising too much beyond the comment in the proof.

The composition then becomes

$$L^2(S_{W^+} \otimes F) \xrightarrow[\text{(bounded)}]{(D_F+i)^{-1}} W^1_{D_F}(S_{W^+} \otimes F) \xrightarrow[\text{(bounded)}]{\psi} W^1_{D_F}((S_{W^+} \otimes F)\big|_U) \xhookrightarrow[\text{(compact)}]{} L^2(S_{W^+} \otimes F)$$

and the composition of a bounded operator with a compact operator is compact, so $\psi \circ (D_F + i)^{-1}$ is compact. $\square$

Now, choose a $\phi \in C^\infty(W)$ such that $\phi = 0$ on W and $\phi = 1$ on $M \times [1, \infty)$ (ϕ is a bump function of some sort, although it is not compactly supported). The important (bounded) operator is

$$F_W = 1 - 2i\phi(D_F + i)^{-1}\phi : L^2\left(S_{W^+} \otimes F\big|_{M\times[0,\infty)}\right) \to L^2\left(S_{W^+} \otimes F\big|_{M\times[0,\infty)}\right). \tag{4.1}$$

Lemma 4.1.5. *F_W is a Fredholm operator, having parametrix $G = 1 + 2i\phi(D_F - i)^{-1}\phi$.*

Proof. The aim is to first show that $F_W G - 1$ is a compact operator. This will naturally extend to the statement that $GF_W - 1$ is compact and hence F_W is Fredholm by Atkinson's lemma.

$$\begin{aligned} F_W G - 1 &= 2i\phi(D_F - i)^{-1}\phi - 2i\phi(D_F + i)^{-1}\phi + 4\phi(D_F + i)^{-1}\phi^2(D_F - i)^{-1}\phi \\ &= 2\phi(\underbrace{i(D_F - i)^{-1} - i(D_F + i)^{-1}}_{\alpha} + 2(D_F + i)^{-1}\phi^2(D_F - i)^{-1})\phi \end{aligned} \tag{4.2}$$

Now, the commutator $[a^{-1}, b]$ (for purely symbolic a, b) satisfies

$$a^{-1}b = ba^{-1} + [a^{-1}, b] = ba^{-1} + a^{-1}[b, a]a^{-1}$$

and notice that $[D_F - i, \phi] = c(d\phi)$. By Lemma 4.1.4 the composition of $c(d\phi)$ with $(D_F \pm i)^{-1}$ is compact, so $\mathcal{K} = a^{-1}[b, a]a^{-1}$ is compact. Thus, we can commute $a^{-1} = (D_F \pm i)^{-1}$ and $b = \phi^2$ so long as we add $\mathcal{K}$. We can combine the difference $\alpha = i(D_F - i)^{-1} - i(D_F + i)^{-1}$ into a single expression using the identity

$$(a - b)^{-1} - (a + b)^{-1} = (-2b)(a^2 - b^2)^{-1}$$

which follows from the fact that when a, b commute we have $((a+b)(a-b))^{-1} = (a^2-b^2)^{-1}$. Thus,

$$\begin{aligned} F_W G - 1 &= 2\phi(-2(D_F^2+1)^{-1} + 2\phi^2(D_F+i)^{-1}(D_F-i)^{-1})\phi + \mathcal{K} \\ &= 2\phi(-2(D_F^2+1)^{-1} + 2\phi^2(D_F^2+1)^{-1})\phi + \mathcal{K}. \end{aligned} \tag{4.3}$$

At this point we should note that because $1-\phi$ is compactly supported $1-\phi^2$ is also, which means that (4.3) is sum of $\mathcal{K}$ and the composition of a bounded operator and a compact operator and hence is compact. The other composition is $GF_W - 1$, but computation is completely analogous: the only difference is the ordering of the composition and we know that this unchanged up to addition of a compact operator. □

The proof of Proposition 4.1.3 will follow from the following three lemmas.

Lemma 4.1.6. *Let V be another manifold that satisfies the same conditions as W in Proposition 4.1.3. If F_V is the map,*

$$F_V = 1 - 2i\phi(D_F+i)^{-1}\phi : L^2\left(S_{V^+} \otimes F\big|_{M\times[0,\infty)}\right) \to L^2\left(S_{V^+} \otimes F\big|_{M\times[0,\infty)}\right)$$

which is exactly the map F_W, except we replace W by V, then the analytic index of F_V is the same as the index of F_W.

This will allow us to make the choice of a specific manifold which has M as boundary. Indeed, we make this choice in Lemma 4.1.8.

Lemma 4.1.7. *When W is as in Proposition 4.1.3 is compact,* $\operatorname{ind}(F_W) = 0$.

Lemma 4.1.8. *When W as above is replaced by $V = M \times (-\infty, 0]$ (the same replacement as in Lemma 4.1.6) so that $V_+ = M \times \mathbb{R}$, the index of F_V equals the index of the twisted Dirac operator D_M^E, the Dirac operator of M twisted by $E = F\big|_M$.*

Proof of Theorem 4.1.2. By Lemma 4.1.6, we can consider any manifold that has M as its boundary and the index will remain the same. Choose V as in Lemma 4.1.8 and by Lemma 4.1.7, $\operatorname{ind} F_V = 0$. Finally, Lemma 4.1.8 gives $\operatorname{ind}(D_E^M) = \operatorname{ind} F_V = 0$, completing the proof of Theorem 4.1.2. □

Remark. We are emphasising the "M" part of D_E^M because M is only the boundary of W, whereas usually D_E is the Dirac operator of the whole space (i.e. W) twisted by E. Of course, D twisted by E does not make sense (E is not a bundle over W) but the emphasis will prevent confusion.

Proof of Lemma 4.1.6. Let D_1 and D_2 be the twisted (by F) Dirac operators for V^+ and W^+.

$$\begin{aligned}\frac{1}{2i}(F_V - F_W) &= \phi(D_1+i)^{-1}\phi - \phi(D_2+i)^{-1}\phi \\ &= \phi(D_2+i)^{-1}((D_2+i)\phi - \phi(D_1+i))(D_1+i)^{-1}\phi \\ &= \phi(D_2+i)^{-1}(D_2\phi - \phi D_1)(D_1+i)^{-1}\phi\end{aligned}$$

Now, on $M\times[0,\infty)$ D_1 and D_2 are the same operator - W^+ and V^+ are indistinguishable on $M\times[0,\infty)$, so the expression $D_2\phi-\phi D_1$ is actually the commutator $[D_1,\phi]$. Recall that the Dirac operator is a first order operator (Example 1.2.13) and it has principal symbol $ic(d\phi)$. Conveniently, this is $[D_1,\phi]$ and $d\phi$ is a compactly supported, so by Lemma 4.1.4, $\frac{1}{2i}(F_V-F_W)$ is compact. □

Proof of Lemma 4.1.7. Our aim is to prove that F_W is a compact perturbation of the operator $1-2i(D_F+i)^{-1}=(D_F-i)(D_F+i)^{-1}$, which is a unitary automorphism and hence has index 0. For brevity, define $A=2i(D_F+i)^{-1}$. We have

$$\begin{aligned}F_W-(1-2i(D_F+i)^{-1}) &= \phi A\phi - A \\ &= \phi A\phi + (\phi-(1-\phi)A(\phi+(1-\phi)) \\ &= -((1-\phi)A(1-\phi)+\phi A(1-\phi)+(1-\phi)A\phi).\end{aligned}$$

Note that $(1-\phi)$ is compactly supported and so $(1-\phi)A, A(1-\phi)$ are compact operators by Lemma 4.1.4. □

Before we begin the proof of Lemma 4.1.8 there is setup to be done. The operator D^E_M is a map $\Gamma^\infty(S_M\otimes E)\to\Gamma^\infty(S_M\otimes E)$, but we would like to compare D^E_M with F_V. We can write $L^2(\mathbb{R})\otimes L^2(S_M\otimes E)\cong L^2(S_{V_+}\otimes F)$ using the isomorphism

$$L^2(\mathbb{R})\otimes L^2(S_M\otimes E)\ni\varphi\otimes s_E\mapsto s_F\in L^2(S_{V_+}\otimes F)$$

where $s_F(t,m)=\varphi(t)s_E(m)$. This isomorphism is really just a statement about the spinor bundles: the twisting is extraneous. We want to extend our previous definition of the grading operator from M to $M\times\mathbb{R}$. The idea is to define a new Clifford multiplication that takes into account this extra dimension. Define in the same way as previously a local oriented orthonormal frame $\{e_j\}_{j=1}^n$ of TM and write $\omega_M=i^k e_1\cdots e_n$ to mean their product in the Clifford algebra. Define the extension of Clifford multiplication as

$$c(e_j)=\begin{cases}c_M(e_j) & j\in\{1,\dots,n\}\\ c_M(\omega_M) & j=0\end{cases}$$

where $c_M : TM \to \mathrm{End}(S_M)$ is our already supplied Clifford multiplication on TM. We can now write down an expression for the operator D_F, namely

$$D_F = \sum_{j=0}^{n} c(e_j)\nabla_{e_j}^{S_{V_+}\otimes F} = c(e_0)\frac{d}{dt} + \sum_{j=1}^{n} c(e_j)\nabla_{e_j}^{S_{V_+}\otimes F}.$$

This can be rendered more usefully as

$$D_F = \frac{d}{dt}\otimes(-ic(\omega_M)) + 1\otimes D_E^M.$$

We can now compare D_E^M with F_V, by computing their indices.

Proof of Lemma 4.1.8. Write $F_V = 1 - 2i\phi(D_F+i)^{-1}\phi$ as before, and notice that F_V is a compact perturbation of the operator $U = 1 - 2i\phi(D_F+i)^{-1}$:

$$\begin{aligned} U - F_V &= \phi(D_F+i)^{-1}(1-\phi) \\ &= \phi(1-\phi)(D_F+i)^{-1} - (D_F+i)^{-1}(-c(d\phi))(D_F+i)^{-1} \end{aligned}$$

which is compact because $\phi(1-\phi)$ and $c(d\phi)$ have compact support, so we can apply Lemma 4.1.4. Define $\psi = 2\phi - 1$ and write

$$\begin{aligned} U &= (D_F+i)(D_F+i)^{-1} - 2i\phi(D_F+i)^{-1} \\ &= (D_F - i\psi)(D_F+i)^{-1}. \end{aligned}$$

Because $(D_F+i) : W_D^1(S_V\otimes F) \to L^2(S_V\otimes F)$ is a bijection, the kernel of U corresponds to the kernel of the operator $D_F - i\psi$ and the cokernel is the kernel of $(D_F - i\psi)^* = (D_F + i\psi)$. As earlier, obtain the splitting of the space of square-integrable sections, $L^2(S_{V_+}\otimes E) = L^2(\mathbb{R})\otimes L^2(S_M\otimes E)$. Let $K = \ker D_E^M$ and define by P the projection $P : L^2(\mathbb{R})\otimes L^2(S_M\otimes E) \to L^2(\mathbb{R})\otimes K$. Given $\zeta \in W_D^1(S_V\otimes F)$, if δ^2 is the smallest non-zero eigenvalue of D_F^2, then we have the following key estimate from Lemma 4.1.9 below:

$$\|(D_F \pm i\psi)\zeta\|^2 \geq \delta^2 \|(1-P)\zeta\|^2.$$

This is useful because it restricts the kernel of $D_F \pm i\psi$ to $L^2(\mathbb{R})\otimes K$ i.e. the kernel is a subset of $L^2(\mathbb{R})\otimes K$.

Now, when we restrict to $L^2(\mathbb{R})\otimes K$ the operator $D_F \pm i\psi$ is $\frac{d}{dt}\otimes(-ic(\omega_M))\pm i\psi$. If K_+ is the positive part of the kernel of D_E^M and K_- the negative part, then $(\frac{d}{dt}\otimes(-ic(\omega_M))\pm i\psi)f = 0$ is solved by the functions

$$f(t) = \begin{cases} \exp\left(\pm\int_0^t \psi(s)ds\right)v, & v\in K_+ \\ \exp\left(\mp\int_0^t \psi(s)ds\right)v, & v\in K_-. \end{cases}$$

Finally, $\psi \geq 0$ so of these only the negative exponentials are square-integrable. Thus, we have $\dim(K_-) = \dim\ker(D_F + i\psi)$ and $\dim(K_+) = \dim\ker(D_F - i\psi)$. Hence, $\operatorname{ind} D_E^M = \operatorname{ind}(D_F - i\psi)$. The operator $D_F - i\psi$ has the same index as U, which is a compact perturbation of F_V, so $\operatorname{ind} F_V = \operatorname{ind} D_E^M$. $\square$

Lemma 4.1.9. *Let δ be the smallest positive eigenvalue of D_F. Then for all $\zeta \in W^1_{D_F}(S_{V_+} \otimes F)$ the following estimate holds*

$$\|(D_F \pm i\psi)\zeta\|^2_{L^2} \geq \delta^2 \|(1-P)\zeta\|^2_{L^2}.$$

Proof. Let ζ be a smooth compactly supported section of $S_{V_+} \otimes F$. We will extend the result to $\zeta \in W^1_{D_F}(S_{V_+} \otimes F)$ after verifying that it is true in the smooth case. In the following, all norms are L^2-norms and the inner product is the L^2 inner product. Then $\|(D_F \pm i\psi)\zeta\|^2$ satisfies

$$\begin{aligned}\|(D_F \pm i\psi)\zeta\|^2 &= \langle (D_F \mp i\psi)(D_F \pm i\psi)\zeta, \zeta\rangle \\ &= \langle (D_E^M)^2\zeta, \zeta\rangle + \langle (ic(\omega)d/dt \pm i\psi)^*(-ic(\omega)d/dt \pm i\psi)\zeta, \zeta\rangle \\ &\geq \left\|D_E^M \zeta\right\|^2.\end{aligned}$$

The operator D_E^M does not initially make sense as something we can apply to ζ, (because ζ is a section of larger bundle than $S_M \otimes E$) but there is a way to make sense of $D_E^M\zeta$. The bundle S_{V_+} is essentially $\mathbb{R} \times S_M$ and the sections of S_{V_+} correspond to time-dependent sections of S_M, $\zeta(t,m) = (t, \tilde{\zeta}(t,n))$, say. In this way, D_E^M can act on these sections (for each $t \in \mathbb{R}$ separately) and so it makes good sense to write down $D_E^M\zeta$.

To complete the proof, we can write $\left\|D_E^M\zeta\right\| = \left\|D_E^M(P\zeta + (1-P)\zeta)\right\|$ and note that $D_E^M P\zeta = 0$, so we have $\left\|D_E^M\zeta\right\|^2 = \left\|D_E^M(1-P)\zeta\right\|^2$. We can decompose ζ into constituent vectors in each eigenspace and write $D_E^M(1-P)\zeta = D_E^M \sum_j (1-P)\zeta_j$, with ζ_j being the component of ζ in the eigenspace corresponding to eigenvalue δ_j. Remember that because eigenvectors across eigenspaces are orthogonal we can write $\left\|\sum_j \zeta_j\right\|^2 = \sum_j \|\zeta_j\|^2$. Finally, we notice that

$$\begin{aligned}\left\|D_E^M \sum_j (1-P)\zeta_j\right\|^2 &= \sum_{j=1} \delta_j^2 \|(1-P)\zeta_j\|^2 \\ &\geq \sum_{j=1} \delta^2 \|(1-P)\zeta_j\|^2 = \delta^2 \|(1-P)\zeta\|^2\end{aligned}$$

for $\delta_j = \delta$, the smallest positive eigenvalue. We can extend this to ζ in the Sobolev space $W^1_{D_F}(S_{V_+} \otimes F)$ using [Hig91, Theorem 1.1]. The theorem says is that if we have a pair $(\zeta,\xi) \in W^1_D(S_V \otimes F) \times L^2(S_V \otimes F)$ with $D_F\zeta = \xi$ then there is a sequence of smooth compactly supported sections ζ_n such that $\|\zeta - \zeta_n\|^2_{L^2} \to 0$ and also that $\|D_F\zeta_n - \xi\|^2_{L^2} \to 0$. Suppose we approximate ζ by a sequence $(\zeta_n)_{n\in\mathbb{N}}$ of smooth compactly supported sections of the bundle $S_{V_+} \otimes F$. Write $\zeta = \zeta - \zeta_n + \zeta_n$, we have the inequality

$$\|(\zeta - \zeta_n) + \zeta_n\|^2 \leq \|\zeta - \zeta_n\|^2 + \|\zeta_n\|^2 + 2\,\|\zeta - \zeta_n\|\,\|\zeta_n\|$$

which is the triangle inequality applied to $\|(\zeta - \zeta_n) + \zeta_n\|^2$. Apply this estimate to $\delta^2\,\|(1-P)(\zeta - \zeta_n + \zeta_n)\|^2$ to get

$$\begin{aligned} \delta^2\,\|(1-P)(\zeta - \zeta_n + \zeta_n)\|^2 &\leq \delta^2\,\|(1-P)(\zeta - \zeta_n)\|^2 + \delta^2\,\|(1-P)\zeta_n\|^2 + 2\,\|\zeta - \zeta_n\|\,\|\zeta_n\| \\ &\leq \delta^2\,\|(1-P)(\zeta - \zeta_n)\|^2 + \delta^2\,\underbrace{\|(D_F \pm i\psi)\zeta_n\|^2}_{\alpha} + 2\,\|\zeta - \zeta_n\|\,\|\zeta_n\|\,. \end{aligned}$$

Now, $\lim_{n\to\infty}\zeta_n = \zeta$ so both $\|(1-P)(\zeta - \zeta_n)\|^2$ and $\|\zeta - \zeta_n\|\,\|\zeta_n\|$ disappear in the limit. For the remaining term, because $D_F\zeta_n$ approximates ξ, the limit as n goes to infinity of α is just $\|(D_F \pm i\psi)\zeta\|^2$ i.e. as n goes to infinity we get

$$\delta^2\,\|(1-P)(\zeta)\|^2 \leq \|(D_F \pm i\psi)\zeta\|^2$$

for any $\zeta \in W^1_D(S_V \otimes F)$. □

4.1.2 Under bundle modification

Theorem 4.1.10 (Invariance under bundle modification)**.** *Suppose (M,E) is modified to yield $(\Sigma F, \beta_F \otimes \pi^*E)$ as in Definition 3.1.11. If D_E is the Dirac operator of M twisted by the vector bundle E and $D_{\beta_F\otimes\pi^*E}$ is the Dirac operator of ΣF twisted by $\beta_F \otimes \pi^*E$. Then*

$$\operatorname{ind}(D_E) = \operatorname{ind}(D_{\beta_F\otimes\pi^*E}).$$

The proof of Theorem 4.1.10 requires a fair amount of setup.

Definition 4.1.11 (Product operator)**.** *Let D_1 and D_2 be Dirac operators on even-dimensional* Spin^c *manifolds M_1, M_2 with spinor bundles S_1 and S_2 respectively, with γ being the grading operator on S_1. We define an operator on the product manifold:*

$$D_1\#D_2 = D_1 \otimes 1 + \gamma \otimes D_2 : \Gamma^\infty(S_1 \otimes S_2) \to \Gamma^\infty(S_1 \otimes S_2).$$

We would like this to be *the* Dirac operator on the product manifold, i.e. the one we receive when considering $M = M_1 \times M_2$ independently of the decomposition.

Proposition 4.1.12. *Suppose we fix $M = M_1 \times M_2$, and write D_M to mean the untwisted Dirac operator of M. Then $D_M = D_1 \# D_2$ and in particular,* $\operatorname{ind} D_M = \operatorname{ind}(D_1 \# D_2)$.

Proof. This follows from Lemma 1.2.10. Indeed, the definition of the Dirac operator of a manifold M is $c \circ \nabla$, where c is Clifford multiplication and ∇ is the Clifford connection on a S_M as described in Definitions 1.2.2 and 1.2.4. The Clifford multiplication distributes as in Lemma 1.2.10 and the Clifford connection ∇_M for S_M satisfies $\nabla_M = \nabla_{M_1} \otimes 1 + 1 \otimes \nabla_{M_2}$ where ∇_{M_1} is the connection for S_1 and ∇_{M_2} for S_2. $\square$

We could also appeal to their principal symbols being indistinguishable, but we also need to use the fact the $D_1 \# D_2$ really is a Dirac operator (and hence is self-adjoint) rather than just a statement about indices.

Lemma 4.1.13. *For D_1, D_2 as in Definition 4.1.11,* $\ker(D_1 \# D_2) = \ker D_1 \otimes \ker D_2$.

Proof. It is clear that if $s = s_1 \otimes s_2 \in \ker D_1 \otimes \ker D_2$ then $s \in \ker(D_1 \# D_2)$ so it remains only to show the opposite. Suppose that $s = s_1 \otimes s_2 \in \ker(D_1 \# D_2)$. The key fact about $D := (D_1 \# D_2)$ is that $D^2 = D_1^2 \otimes 1 + 1 \otimes D_2^2$. To see why this is the case, consider $(D_1 \# D_2)^2$,

$$\begin{aligned}(D_1 \# D_2)^2 s_1 \otimes s_2 = D_1^2 s_1 \otimes s_2 &+ D_1 \gamma s_1 \otimes D_2 s_2 \\ &+ \gamma D_1 s_1 \otimes D_2 s_2 + \gamma^2 s_2 \otimes D_2^2 s_2.\end{aligned}$$

Now, the Dirac operator swaps the grading of S_1, so $D_1 \gamma s_1 = -\gamma D_1 s_1$. Because $\gamma^2 = 1$ by definition (the grading is the decomposition of S_1 into $+1$ and -1 eigenspaces we get from $\gamma^2 = 1$), we get to

$$D_1^2 s_1 \otimes s_2 + D_1 \gamma s_1 \otimes D_2 s_2 + \gamma D_1 s_1 \otimes D_2 s_2 + \gamma^2 s_2 \otimes D_2^2 s_2 = D_1^2 s_1 \otimes s_1 + s_1 \otimes D_2^2 s_2.$$

Now, if $Ds = 0$ then (using L^2 norm), $\|D(s_1 \otimes s_s)\|^2 = 0$ and

$$\begin{aligned}\|D(s_1 \otimes s_s)\|^2 &= \langle (D_1 \# D_2)(s_1 \otimes s_2), D_1 \# D_2 (s_1 \otimes s_2) \rangle \\ &= \langle s_1 \otimes s_2, (D_1 \# D_2)^2 (s_1 \otimes s_2) \rangle,\end{aligned}$$

which is due to Dirac operators being formally self-adjoint. We have

$$\begin{aligned}\langle s_1 \otimes s_2, (D_1 \# D_2)^2 s_1 \otimes s_2 \rangle &= \langle s_1 \otimes s_2, D_1^2 s_1 \otimes s_2 + s_1 \otimes D_2^2 s_2 \rangle \\ &= \langle s_1 \otimes s_2, D_1^2 s_1 \otimes s_2 \rangle + \langle s_1 \otimes s_2, D_2^2 s_2 \rangle \\ &= \langle D_1 s_1, D_1 s_1 \rangle \cdot \langle s_2, s_2 \rangle + \langle D_2 s_2, D_2 s_2 \rangle \cdot \langle s_1, s_1 \rangle \\ &= \|D_1 s_1\|^2 \|s_2\|^2 + \|D_2 s_2\|^2 \|s_1\|^2 .\end{aligned}$$

The above is 0 only when s_1 s_2 are in the kernel of D_1 and D_2 respectively. $\square$

Proposition 4.1.14. *Suppose that we have D_1 and D_2 as in Definition 4.1.11., except now twisted by a vector bundle $E \to M = M_1 \times M_2$ that restricts to bundles E_1 and E_2 over M_1 and M_2. Suppose also that the index of D_2 is 1. Then* $\operatorname{ind} D_{M_1 \times M_2} = \operatorname{ind} D_1$.

Proof. From Lemma 4.1.13 we know that the kernel decomposes the kernel of D_1 and the kernel of D_2. The grading on the product $S_1 \otimes S_2$ is from the grading on the constituent spinor bundles S_1 and S_2:

$$(S_1 \otimes S_2)^+ = (S_1^+ \otimes S_2^+) \oplus (S_1^- \otimes S_2^-)$$
$$(S_1 \otimes S_2)^+ = (S_1^+ \otimes S_2^-) \oplus (S_1^- \otimes S_2^+).$$

This is due to the induced tensor grading: even parts correspond to tensors which have the same grading, odd to those which have opposite. If we intersect these positive and negative parts with the kernel of D, we get the decomposition of the kernel:

$$(\ker D)^+ = ((\ker D_1)^+ \otimes (\ker D_1)^+) \oplus ((\ker D_1)^- \otimes (\ker D_2)^-)$$
$$(\ker D)^- = ((\ker D_1)^+ \otimes (\ker D_2)^-) \oplus ((\ker D_1)^- \otimes (\ker D_2)^+).$$

Here we have used the notation $^+$ and $^-$ to denote the restriction to sections that are in the positive and negative parts of the spinor bundle. The index of D is by definition the index of the positive part, $\operatorname{ind} D := (\dim \ker D)^+ - (\dim \ker D)^-$, so

$$\begin{aligned} \dim(\ker D)^+ - \dim(\ker D)^- &= \dim(\ker D_1)^+ \cdot (\dim(\ker D_2)^+ - \dim(\ker D_2)^-) \\ &\quad - \dim(\ker D_1)^- \cdot (\dim(\ker D_2)^+ - \dim(\ker D_2)^-) \\ &= \dim(\ker D_1)^+ - \dim(\ker D_2)^-. \end{aligned}$$

□

This construction helps to motivate the construction of the # product, but it is still not yet clear how this will help us prove that the index is independent of bundle modification. Before we begin, there is some notational conventions that need to be discussed. In what follows, we will write $P_M : M \times S^n \to M$ for projection to M and likewise $P_{S^n} : M \times S^n \to S^n$ for projection to S^n. The restriction of P_M to a subset U of M will be denoted P_U.

Remark. The bundle $P_{S^n}^*\beta$ formally consists of triples $(u, x, b) \in U \times S^n \times \beta$ such that $x = \pi_\beta(b)$ (π_β is the bundle projection for β), but because b is in the fibre of β at x, if we supply $b \in \beta_x$ then we are also giving complete information about x. In light of this, we can write $U \times \beta = P_{S^n}^*\beta$, and likewise for $P_U^* E\big|_U = E\big|_U \times S^n$. We will alternate between these descriptions where appropriate, and only in the proof of Lemma 4.1.31 does it become more enlightening to use the triple to denote an element of $P^*_{S^n}\beta$ and $P^*_U E\big|_U$, because keeping track of the basepoints becomes important.

Recall that from Definition 3.1.4 we can construct a sphere-bundle ΣF from the bundle F over M. Then the modification (Definition 3.1.11) of (M, E) is $(\Sigma F, \pi^* E \otimes \beta_F)$.

Example 4.1.15 (Motivating example)**.** *When F as immediately above is $F = M \times \mathbb{R}^n$ (i.e. trivial), then $(\Sigma F, \pi^* E \otimes \beta_F) = (M \times S^n, E \boxtimes \beta)$ and in particular, $D_{\pi^* E \otimes \beta_F} = D_E \# D_\beta$. Hence, by Proposition 4.1.14 we have* $\operatorname{ind} D_E = \operatorname{ind} D_{\pi^* E \otimes \beta_F}$.

Proof. Recall that we can write $F = P \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n$. When F is trivial, P is also and, $\Sigma F = M \times S^n$ and the projection $\pi : M \times S^n \to M$ is projection to the first factor, giving $\pi^* E = P^*_M E$. For β_F we have $\beta_F = P \times_{\mathrm{Spin}^c(n)} \beta = M \times \beta$, which is $P^*_{S^n}\beta$. Since we have $(\Sigma F, \pi^* E \otimes \beta_F) = (M \times S^n, E \boxtimes \beta)$, the operator $D_{\pi^* E \otimes \beta_F}$ has the same symbol as $D_E \# D_\beta$ by Lemma 1.2.10. The index of D_β is 1, so in the case when F is trivial, the (analytic) index of (M, E) is invariant under vector bundle modification. $\square$

We aim to show that the index is invariant under bundle modification even when F is not a trivial bundle.

Definition 4.1.16 (τ)**.** *Let M, F be as in Definition 3.1.4, let P satisfy $F = P \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n$. Given an open subset U trivialising P, denote the trivialisation of P by the map $\tau : P\big|_U \to U \times \mathrm{Spin}^c(n)$.*

Our first use of τ is for the following local identifications.

$$U \times \mathrm{Spin}^c(n) \times_{\mathrm{Spin}^c(n)} S^n \to U \times S^n, \tag{4.4}$$

$$U \times \mathrm{Spin}^c(n) \times_{\mathrm{Spin}^c(n)} \beta \to U \times \beta, \tag{4.5}$$

which are given by (respectively) $(u, [g, x]) \mapsto (u, g \cdot x)$ and $(u, [g, b]) \mapsto (u, g \cdot b)$ for $u \in U, x \in S^n, g \in \mathrm{Spin}^c(n), b \in \beta_x$.

Definition 4.1.17 (ψ_τ and φ_τ)**.** *Let τ be as in Definition 4.1.16. Define ψ_τ and φ_τ by the commutativity of the following diagrams:*

$$\begin{tikzcd}
\beta_F\big|_{\pi^{-1}(U)} \arrow[rr, "\varphi_\tau"] \arrow[d, "\cong"] & & P^*_{S^n}(\beta) \arrow[d, "\cong"] \\
P\big|_U \times_{\mathrm{Spin}^c(n)} \beta \arrow[r, "\tau \times 1_\beta"] & U \times \mathrm{Spin}^c(n) \times_{\mathrm{Spin}^c(n)} \times \beta, &
\end{tikzcd}$$

$$\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\qquad\qquad\psi_\tau\qquad\qquad} & U \times S^n \\
\Big\downarrow \wr\| & & \Big\downarrow \wr\| \\
P\big|_U \times_{\mathrm{Spin}^c(n)} S^n & \xrightarrow{\tau \times 1_{S^n}} & U \times \mathrm{Spin}^c(n) \times_{\mathrm{Spin}^c(n)} \times S^n.
\end{array}$$

Remark. Both ψ_τ and φ_τ are well-defined because τ is $\mathrm{Spin}^c(n)$-equivariant. The map φ_τ is an isomorphism of vector bundles by construction.

Recall the setting of vector bundle modification: we have a pair (M, E) consisting a smooth compact even dimensional $\mathrm{Spin}^c(n)$ manifold with smooth complex vector bundle $E \to M$ as in Definition 3.1.7. The *bundle modification* (Definition 3.1.11) of the pair (M, E) given smooth real vector bundle $F \to M$ is the pair $(\Sigma F, \pi^* E \otimes \beta_F)$, where $\pi : \Sigma F \to M$ is the projection for ΣF and $\beta_F := P \times_{\mathrm{Spin}^c(n)} \beta \to \Sigma F$.

Definition 4.1.18 (φ_E)**.** *Let τ be as in Definition 4.1.16. Define φ_E as the map*

$$\varphi_E : \pi^* E\big|_{\pi^{-1}(U)} \to P_U^*(E\big|_U)$$

given by

$$\pi^* E\big|_{\pi^{-1}(U)} \ni ([p, x], e) \mapsto (u, g \cdot x, e) \in P_U^*(E\big|_U)$$

for $u \in U, p \in P_u, x \in S^n, e \in E_u, [p, x] \in (\Sigma F)_u$, if $\tau(p) = (u, g)$ for $g \in \mathrm{Spin}^c(n)$.

Lemma 4.1.19. *If φ_E is as in Definition 4.1.18 then φ_E is an isomorphism of vector bundles.*

Proof. The inverse is $(u, x, e) \mapsto ([p, x], e)$ such that $\tau(p) = (u, 1_{\mathrm{Spin}^c(n)})$. By construction, $p \in P_u$ and $e \in E_u$. The compatibility between the bases $\pi^{-1}(U)$ and $U \times S^n$ is given by ψ in Definition 4.1.17. □

Lemma 4.1.20. *Let ΣF be in the Definition 3.1.11. Then*

$$T\Sigma F \cong \pi^* TM \oplus (P \times_{\mathrm{Spin}^c(n)} TS^n).$$

Proof. A analogous version of this is in [Hoc09, Corollary 12.3 on page 158]. □

Lemma 4.1.21. *The isomorphism in Lemma 4.1.20 induces an isomorphism at the level of spinor bundles,*

$$G : S_{\Sigma F} \to \pi^* S_M \otimes (P \times_{\mathrm{Spin}^c(n)} S_{S^n}).$$

In particular this restricts to the local map

$$G\big|_{\pi^{-1}(U)} : S_{\Sigma F}\big|_{\pi^{-1}(U)} \to (\pi\big|_U{}^* S_M\big|_U) \otimes (P\big|_U \times_{\mathrm{Spin}^c(n)} S_{S^n}).$$

Proof. The statement that G restricts to $G\big|_{\pi^{-1}(U)}$ is immediate if G exists. We can use our previous results about the spinor bundle in Propositions 1.1.15, 1.1.16 and 1.1.18 to see why these bundles are isomorphic. From Proposition 1.1.15 we know that

$$S_{\pi^*TM\oplus P\times_{\mathrm{Spin}^c(n)}TS^n} = S_{\pi^*TM}\otimes S_{P\times_{\mathrm{Spin}^c(n)}TS^n},$$

and from Proposition 1.1.16 with $f=\pi:\Sigma F\to M$ we know

$$S_{\pi^*TM} = \pi^* S_M.$$

Finally Proposition 1.1.18 with $Y = S^n$, $E = TS^n$, and the Lie group being $\mathrm{Spin}^c(n)$ (acting on S^n via its canonical projection to $\mathrm{SO}(n)$) says that

$$S_{P\times_{\mathrm{Spin}^c(n)}TS^n} = P\times_{\mathrm{Spin}^c(n)} S_{S^n}.$$

Ordinarily, to show that a map between bundles is a bundle isomorphism, one must show the compatibility of the base manifolds. For a bundle $\pi_E : E \to M$ to be isomorphic to $\pi_F : F \to N$, we must have both a map $f : E \to F$ and a map $g : M \to N$ satisfying $g\circ\pi_E = \pi_F\circ f$. It is not necessary to show that the map between the base manifolds exists here, because this follows from the existence of such a compatible map for the isomorphism in Lemma 4.1.20, although this is hidden in the reference provided. $\square$

Definition 4.1.22 (C_τ)**.** *Let τ be as in Definition 4.1.16. Define*

$$C_\tau : \left(\pi\big|_U{}^* S_M\big|_U\right)\otimes\left(P\big|_U\times_{\mathrm{Spin}^c(n)} S_{S^n}\right)\to S_M\big|_U\boxtimes S_{S^n}$$

by

$$([p,x],s_1)\otimes[p,s_2]\mapsto(u,g\cdot x,s_1\otimes g\cdot s_2),$$

where $u\in U, s_1\in(S_M)_u, x\in S^n, p\in P_u, s_2\in\left(S_{S^n}\right)_x, \tau(p)=g$.

We can choose the same p in each part of the product $([p,x],s_1)\otimes[p,s_2]$ because the tensor product requires representatives to be in the same fibre - which are related by a group element by the definition of a principal bundle.

Definition 4.1.23 (φ_S)**.** *Let $G : S_{\Sigma F}\big|_{\pi^{-1}(U)}\to\pi\big|_U{}^*S_M\big|_U\otimes P\big|_U\times_{\mathrm{Spin}^c(n)} S_{S^n}$ be as in Lemma 4.1.21, then the commutativity of the following diagram defines φ_S*

$$\begin{tikzcd}
S_{\Sigma F}\big|_{\pi^{-1}(U)} \arrow[rr, "\varphi_S"] \arrow[dr, "G"'] & & S_M\big|_U\boxtimes S_{S^n} \\
& \pi\big|_U{}^*S_M\big|_U\otimes\left(P\big|_U\times_{\mathrm{Spin}^c(n)} S_{S^n}\right). \arrow[ur, "C_\tau"'] &
\end{tikzcd}$$

Definition 4.1.24 (Ψ_τ)**.**

$$\Psi_\tau : (S_{\Sigma F} \otimes \pi^* E \otimes \beta_F)\big|_{\pi^{-1}(U)} \to (S_M\big|_U \boxtimes S_{S^n}) \otimes P_U^*(E\big|_U) \otimes P_{S^n}^*(\beta)$$

is given by the product $\varphi_S \otimes \varphi_E \otimes \varphi_\tau$ of the three maps

$$\begin{aligned} \varphi_S &: S_{\Sigma F}\big|_{\pi^{-1}(U)} \to S_M\big|_U \boxtimes S_{S^n} \\ \varphi_E &: \pi^* E\big|_{\pi^{-1}(U)} \to P_U^*(E\big|_U) \\ \varphi_\tau &: \beta_F\big|_{\pi^{-1}(U)} \to P_{S^n}^*(\beta). \end{aligned}$$

For brevity, write $S_E\big|_U = S_M\big|_U \otimes E\big|_U$ and $S_\beta = S_{S^n} \otimes \beta$.

Lemma 4.1.25. *In Definition 4.1.24, $(S_M\big|_U \boxtimes S_{S^n}) \otimes P_U^*(E\big|_U) \otimes P_{S^n}^*(\beta)$ is*

$$(S_M \otimes E)\big|_U \boxtimes (S_{S^n} \otimes \beta) = S_E\big|_U \boxtimes S_\beta.$$

Proof. The rearrangement is

$$\mathcal{E} : (S_M\big|_U \boxtimes S_{S^n}) \otimes P_U^*(E\big|_U) \otimes P_{S^n}^*(\beta) \to \left((S_M \otimes E)\big|_U \boxtimes (S_{S^n} \otimes \beta)\right),$$

given by

$$((u, x, s_1 \otimes s_2)) \otimes (u, x, e) \otimes (u, x, b) \mapsto (u, x, (s_1 \otimes e) \otimes (s_2 \otimes b))$$

for $u \in U, x \in S^n, s_1 \in (S_M)_u, s_2 \in (S_{S^n})_x, e \in (E)_u, b \in (\beta)_x$. $\square$

Lemma 4.1.26. *Suppose that $J : (S_{\Sigma F} \otimes \pi^* E \otimes \beta_F)\big|_{\pi^{-1}(U)} \to \pi^{-1}(U)$ is the bundle projection of $(S_{\Sigma F} \otimes \pi^* E \otimes \beta_F)\big|_{\pi^{-1}(U)}$ over $\pi^{-1}(U)$ and likewise suppose that $K : S_E\big|_U \boxtimes S_\beta \to U \times S^n$ is the bundle projection for $S_E\big|_U \boxtimes S_\beta$ over $U \times S^n$. Then Ψ_τ is compatible with ψ_τ, in the sense that the following diagram commutes:*

$$\begin{tikzcd} (S_{\Sigma F} \otimes \pi^* E \otimes \beta_F)\big|_{\pi^{-1}(U)} \arrow[r, "\Psi_\tau"] \arrow[d, "J"] & S_E\big|_U \boxtimes S_\beta \arrow[d, "K"] \\ \pi^{-1}(U) \arrow[r, "\psi_\tau"] & U \times S^n. \end{tikzcd} \tag{4.6}$$

Proof. We have already determined all of the necessary components of Ψ_τ and ψ_τ to complete this proof. Suppose that $p \in P_u, x \in S^n, [p, x] \in (\Sigma F)_u, e \in E_u, s_1 \in (S_M)_u, s_2 \in (S_{S^n})_x,$ $b \in \beta_x$. Suppose additionally in the following that for G as in Lemma 4.1.21 and $s \in (S_{\Sigma F})_{[p,x]}$ we have $G(s) = ([p, x], s_1) \otimes [p, s_2] \in$

$\pi\big|_U{}^* S_M\big|_U \otimes (P\big|_U \times_{\mathrm{Spin}^c(n)} S_{S^n})$ and recall that the if we have a fixed τ as in the definition of ψ_τ we can define $\tau(p) = (u, g)$, for $g \in \mathrm{Spin}^c(n)$. Then,

$$\begin{aligned}\psi_\tau \circ J(s \otimes ([p, x], e) \otimes [p, b]) &= \psi_\tau([p, x]) \\ &= (u, g \cdot x).\end{aligned}$$

If we compare the other composition:

$$\begin{aligned}K \circ \Psi_\tau(s \otimes ([p, x], e) \otimes [p, b]) &= K \circ C_\tau(([p, x], s_1) \otimes [p, s_2]) \otimes \varphi_E([p, x], e) \otimes \varphi_\tau([p, b]) \\ &= K((u, g \cdot x, s_1 \otimes g \cdot s_2) \otimes (u, g \cdot x, e \otimes g \cdot b)) \\ &= (u, g \cdot x).\end{aligned}$$

□

Remark. The above proof also demonstrates that C_τ is a vector bundle isomorphism.

Corollary 4.1.27. *$\Psi_\tau : (S_{\Sigma F} \otimes \pi^* E \otimes \beta_F)\big|_{\pi^{-1}(U)} \to S_M\big|_U \boxtimes S_{S^n} \otimes P_U^*(E\big|_U) \otimes P_{S^n}^*(\beta)$ is an isomorphism.*

Proof. Because each tensor factor of Ψ_τ is an isomorphism, and the diagram (4.6) commutes, Ψ_τ is an isomorphism. □

Definition 4.1.28 (Φ_τ)**.** *Define*

$$\Phi_\tau : \Gamma^\infty\left((S_{\Sigma F} \otimes \pi^* E \otimes \beta_F)\big|_{\pi^{-1}(U)}\right) \to \Gamma^\infty\left((S_M \otimes E)\big|_U \boxtimes (S_{S^n} \otimes \beta)\right)$$

is given by $(\Phi_\tau s)(u, x) = \Psi_\tau(s(\psi_\tau^{-1}(u, x)))$, for $s \in \Gamma^\infty\left((S_{\Sigma F} \otimes \pi^* E \otimes \beta_F)\big|_{\pi^{-1}(U)}\right)$, $u \in U, x \in S^n$.

Lemma 4.1.29. *The map*

$$\Phi_\tau : \Gamma^\infty\left((S_{\Sigma F} \otimes \pi^* E \otimes \beta_F)\big|_{\pi^{-1}(U)}\right) \to \Gamma^\infty\left((S_M \otimes E)\big|_U \boxtimes (S_{S^n} \otimes \beta)\right)$$

in Definition 4.1.28 is an isomorphism.

Proof. It suffices to show that Φ_τ is a bijection, because Ψ_τ is a vector bundle isomorphism and ψ_τ provides the basepoint compatibility. We should remark at this point that there isn't very much to be done here: everything proceeds from Corollary 4.1.27 and the definition of ψ_τ. If $s_1, s_2 \in \left((S_{\Sigma F} \otimes \pi^* E \otimes \beta_F)\big|_{\pi^{-1}(U)}\right)$ satisfy $(\Phi_\tau s_1)(u, x) = (\Phi_\tau s_2)(u, x)$ then because both Ψ_τ and ψ_τ are isomorphisms, $s_1 = s_2$. Surjectivity follows from the existence of a right-inverse: if $s_3 \in \Gamma^\infty\left((S_M \otimes E)\big|_U \boxtimes (S_{S^n} \otimes \beta)\right)$ then $s_4 := \Psi_\tau^{-1} s_3 \psi_\tau \in \Gamma^\infty\left((S_{\Sigma F} \otimes \pi^* E \otimes \beta_F)\big|_{\pi^{-1}(U)}\right)$ is a section for which $\Phi_\tau(s_4) = s_3$. □

We should note that Φ_τ respects the grading on the sections of the spinor bundle because the map G in Lemma 4.1.21 does so, although this is not immediately obvious.

Definition 4.1.30. *Define*

$$\tilde{D}_{\pi^*E\otimes\beta_F}\Big|_{\pi^{-1}(U)} : \Gamma^\infty\left((S_{\Sigma F}\otimes\pi^*E\otimes\beta_F)\,|_{\pi^{-1}(U)}\right) \to \Gamma^\infty\left((S_{\Sigma F}\otimes\pi^*E\otimes\beta_F)\,|_{\pi^{-1}(U)}\right)$$

by

$$\tilde{D}_{\pi^*E\otimes\beta_F}\Big|_{\pi^{-1}(U)} = \Phi_\tau^{-1}\circ(D_E\big|_U \# D_\beta)\circ\Phi_\tau.$$

We aim to prove that this does not depend on the choice of trivialisation (U,τ) of $P\big|_U$. Denote by adding a $'$ to the subscript the maps that correspond to φ_E, φ_τ,φ_S except containing a different trivialisation τ'. Before we state precisely the next lemma, it is worth referring back to Lemma 4.1.25. Formally, we will need to conjugate the Ψ-part by $\mathcal{E}$ to make sense of the compositions in the proof of Lemma 4.1.31 although because this is only a re-arrangement we tend not to emphasise it.

Lemma 4.1.31. *Suppose $s = s_E\otimes s_\beta \in \Gamma^\infty(S_E\big|_U\otimes S_\beta), u\in U$ and $x\in S^n$. Suppose that we define $p=\tau^{-1}(u,1_{\mathrm{Spin}^c(n)})$ and define $g\in\mathrm{Spin}^c$ by $\tau'(p)=(u,g^{-1})$, i.e. $\tau'\circ\tau^{-1}(u,1_{\mathrm{Spin}^c(n)})=(u,g^{-1})$. Then*

$$\Psi_\tau\circ\Psi_{\tau'}^{-1}s(\psi'_\tau\circ\psi_\tau^{-1}(u,x)) = s_E\otimes g s_\beta((u,g^{-1}\cdot x)),$$

which is by definition the group action by $g\in\mathrm{Spin}^c(n)$ on a section $s\in\Gamma^\infty(S_E\big|_U\boxtimes S_\beta)$.

Proof. Common to all three components is the basepoint-changing isomorphism $\psi_{\tau'}\circ\psi_\tau^{-1}$,

$$\begin{aligned}\psi_{\tau'}\circ\psi_{\tau^{-1}}(u,x) &= ((\tau'\circ\tau^{-1})(u,1_{\mathrm{Spin}^c(n)}),x)\\ &= (u,[g^{-1},x])\\ &= (u,g^{-1}\cdot x).\end{aligned}\tag{4.7}$$

The final step is due to (4.4) and in what follows we will use (4.4),(4.5) without particular reference. For readability, we will compute $\Psi_\tau\circ\Psi_{\tau'}^{-1}$ component-wise. First, note that the restriction to $\pi^{-1}(U)$ of the global isomorphism $G: S_{\Sigma F}\to \pi^*S_M\otimes P\times_{\mathrm{Spin}^c(n)} S_{S^n}$ does not depend on τ and so does not appear in $\varphi_S\circ\varphi_{S'}^{-1}$. We can compute the inverse of $C_{\tau'}$ as

$$C_{\tau'}^{-1} : (u,x,s_1\otimes s_2)\mapsto([p,x],s_1)\otimes[p,s_2],$$

for generic $s_1 \in (S_M)_u$, $s_2 \in (S_{S^n})_x$ and $p \in P_u$, $\tau(p) = (u, 1_{\mathrm{Spin}^c(n)})$.

The trivialisations τ, τ' are $\mathrm{Spin}^c(n)$-equivariant so if we fix a g as in Lemma 4.1.31 we get

$$(u, 1_{\mathrm{Spin}^c(n)}) = g\tau' \circ \tau^{-1}(u, 1_{\mathrm{Spin}^c(n)}) = \tau' \circ \tau^{-1}(u, g),$$

which is exactly equivalent to $\tau \circ (\tau')^{-1}(u, 1_{\mathrm{Spin}^c(n)}) = (u, g)$. Define $p_1 = (\tau')^{-1}(u, 1_{\mathrm{Spin}^c(n)})$ and define the now fixed vectors $s_1 \in (S_M)_u$ and $s_2 \in (S_{S^n})_{g^{-1}\cdot x}$ by

$$s(u, g^{-1} \cdot x) = (u, g^{-1} \cdot x, s_1 \otimes s_2)$$

where s is a section of $S_M\big|_U \boxtimes S_{S^n}$. The basepoints are included to aid in comprehension. We compute explicitly $\varphi_S \circ \varphi_{S'}^{-1}(s(u, g^{-1} \cdot x))$,

$$\begin{aligned}
&\big(\varphi_S \circ \varphi_{S'}^{-1}((u, g^{-1} \cdot x, s_1 \otimes s_2)\big) \\
&= C_\tau(([p_1, g^{-1} \cdot x], s_1 \otimes s_2)) \\
&= (u, gg^{-1} \cdot x, s_1 \otimes g \cdot s_2) \\
&= (u, x, s_1 \otimes g \cdot s_2).
\end{aligned} \tag{4.8}$$

Remember that s_2 is in the fibre at $g^{-1} \cdot x$, so $g \cdot s_2$ is in the fibre at x. Next, consider $\varphi_E \circ \varphi_{E'}^{-1} : P_U^*(E\big|_U) \to P_U^*(E\big|_U)$ acting on $(u, g^{-1} \cdot x, e) \in P_U^*(E\big|_U)$ for a fixed $e \in E_u$,

$$\begin{aligned}
\varphi_E \circ \varphi_{E'}^{-1}(u, g^{-1} \cdot x, e) &= ([\tau(p), g^{-1} \cdot x], e) \\
&= (u, [g, g^{-1} \cdot x], e) = (u, x, e).
\end{aligned} \tag{4.9}$$

Finally, consider $\varphi_\tau \circ \varphi_{\tau'}^{-1} : P_{S^n}^*(\beta) \to P_{S^n}^*(\beta)$ acting on $(u, g^{-1} \cdot x, b) \in P_{S^n}^*(\beta)$ for a fixed $b \in \beta_{g^{-1}\cdot x}$,

$$\begin{aligned}
\varphi_\tau \circ \varphi_{\tau'}^{-1}(u, g^{-1} \cdot x, b) &= (u, [g, g^{-1} \cdot x], b) \\
&= (u, x, b).
\end{aligned} \tag{4.10}$$

If we combine Eqs. (4.7) to (4.10):

$$(\varphi_S \circ \varphi_{S'}^{-1} \otimes \varphi_E \circ \varphi_{E'} \otimes \varphi_\tau \circ \varphi_{\tau'}^{-1}) s(\psi'_\tau \circ \psi_\tau^{-1}(u, x)) = s_E \otimes g \cdot s_\beta(g^{-1}(u, x))).$$

$\square$

Lemma 4.1.32. *If (U, τ') is another trivialisation of $P\big|_U$, then the operator $D_E\big|_U \# D_\beta$ commutes with $\Phi_\tau \circ \Phi_{\tau'}^{-1}$.*

Proof. By Lemma 4.1.31, we have

$$\begin{aligned} D_E \# D_\beta \left(\Phi_\tau \circ \Phi_{\tau'}^{-1} s_E \otimes s_\beta\right) &= D_E \# D_\beta (s_E \otimes g \cdot s_\beta) \\ &= (D_E \# g \circ D_\beta)(s_E \otimes s_\beta) \end{aligned}$$

which is exactly $\Phi_\tau \circ \Phi_{\tau'}^{-1}(D_E \# D_\beta) s_E \otimes s_\beta$. The final line is because D_β is Spin^c-equivariant. □

Proposition 4.1.33. *The differential operator* $\tilde{D}_{\pi^* E \otimes \beta_F}\Big|_{\pi^{-1}(U)} = \Phi_\tau^{-1} \circ \big(D_E\big|_U \# D_\beta\big) \circ \Phi_\tau$ *does not depend on the choice of trivialisation* (U, τ)*, in the sense that if* (U, τ') *is another choice of trivialisation of* $P\big|_U$*, then*

$$\Phi_\tau^{-1} \circ \left(D_E\big|_U \# D_\beta\right) \circ \Phi_\tau = \Phi_{\tau'}^{-1} \circ \left(D_E\big|_U \# D_\beta\right) \circ \Phi_{\tau'}.$$

Proof. This follows from Lemma 4.1.32. Write

$$\begin{aligned} \Phi_{\tau'}^{-1} \circ \left(D_E\big|_U \# D_\beta\right) \circ \Phi_{\tau'} &= (\Phi_\tau^{-1} \circ \Phi_\tau) \circ \Phi_{\tau'}^{-1} \left(D_E\big|_U \# D_\beta\right) \Phi_{\tau'} \circ (\Phi_\tau^{-1} \circ \Phi_\tau) \\ &= \Phi_\tau^{-1} \circ (\Phi_\tau \circ \Phi_{\tau'}^{-1}) \circ \left(D_E\big|_U \# D_\beta\right) \circ (\Phi_{\tau'} \circ \Phi_\tau^{-1}) \circ \Phi_\tau. \end{aligned}$$

Now, because of Lemma 4.1.32 we can swap $\left(D_E\big|_U \# D_\beta\right)$ and $(\Phi_{\tau'} \circ \Phi_\tau^{-1})$ to get

$$\Phi_\tau^{-1} \circ (\Phi_\tau \circ \Phi_{\tau'}^{-1}) \circ (\Phi_{\tau'} \circ \Phi_\tau^{-1}) \circ \left(D_E\big|_U \# D_\beta\right) \circ \Phi_\tau = \Phi_\tau^{-1} \circ \left(D_E\big|_U \# D_\beta\right) \circ \Phi_\tau.$$

□

Proposition 4.1.34. $\ker \tilde{D}_{\pi^* E \otimes \beta_F} \cong \ker D_E \otimes \ker D_\beta$ *and the isomorphism respects the grading.*

Proof. We will show this is true locally and then rely on a lemma proven below to extend the result to a global statement. Because Φ_τ is an isomorphism and respects the grading, it suffices to check that the following relations hold

1. $\ker D_E\big|_U \otimes \ker D_\beta \subset \Phi_\tau \left(\tilde{D}_{\pi^* E \otimes \beta_F}\Big|_{\pi^{-1}(U)}\right)$

2. $\Phi_\tau \left(\ker \tilde{D}_{\pi^* E \otimes \beta_F}\Big|_{\pi^{-1}(U)}\right) \subset \ker D_E\big|_U \otimes \ker D_\beta.$

To show the first inclusion we need to check that an element $s_E \otimes s_\beta \in \ker D_E\big|_U \otimes \ker D_\beta$ satisfies $\Phi_\tau^{-1}(s_E \otimes s_\beta) \in \ker \tilde{D}_{\beta_F \otimes \pi^* E}\Big|_{\pi^{-1}(U)}$. We have

$$\begin{aligned}\tilde{D}_{\beta_F \otimes \pi^* E}\Big|_{\pi^{-1}(U)} \Phi_\tau^{-1}(s_E \otimes s_\beta) &= \Phi_\tau^{-1} D_E\big|_U \# D_\beta (s_E \otimes s_\beta) \\ &= \Phi_\tau^{-1}(s_0)\end{aligned}$$

where s_0 is the zero section, and so hence $\Phi_\tau^{-1} s_0$ is also.

We now consider the second point. Let $0 \neq s \in \Gamma^\infty\left((S_{\Sigma F} \otimes \pi^*(E) \otimes \beta_F)\big|_{\pi^{-1}(U)}\right)$ and suppose further that $s \in \ker \tilde{D}_{\beta_F \otimes \pi^* E}$. Write $\Phi_\tau(s) = s_E \otimes s_\beta$ and consider

$$\left(D_E\big|_U \# D_\beta\right) \Phi_\tau(s) = \tilde{D}_{\pi^* E \otimes \beta_F}\Big|_{\pi^{-1}(U)} s = 0.$$

By Lemma 4.1.36 below, Φ_τ does not depend on the choice of τ and so defines an isomorphism

$$\Phi : \ker \tilde{D}_{\pi^* E \otimes \beta_F} \cong \ker D_E \otimes \ker D_\beta.$$

$\square$

Remark. Because the index of a twisted operator is a difference of the graded parts of its kernel (Definition 1.2.11 and below it for details) Proposition 4.1.34 combined with the result of Proposition 4.1.14 implies that $\operatorname{ind} \tilde{D}_{\pi^* E \otimes \beta_F} = \operatorname{ind} D_E$.

Lemma 4.1.35. *Let $E \to M$ be a vector bundle. Then the identity section $1_E \in \Gamma^\infty(\underline{\mathrm{End}}(E))$ defined by $M \ni m \mapsto 1_{E_m} \in \mathrm{End}(E_m)$ corresponds to the identity endomorphism $1_E \in \mathrm{End}(E)$ given by $1_E(e) = e$, for $e \in E$.*

Proof. Denote by ζ the isomorphism given in Lemma 1.3.10. Then

$$\zeta(1_{S^+})(s) = (1_{S^+}(x))(s) = s,$$

for $s \in (S^+)_x$. $\square$

Lemma 4.1.36. *The local isomorphism*

$$\Phi_\tau : \ker \tilde{D}_{\pi^* E \otimes \beta_F}\Big|_{\pi^{-1}(U)} \to \ker D_E\big|_U \otimes \ker D_\beta$$

does not depend on the choice of τ when restricted to sections in the kernel of $\tilde{D}_{\pi^* E \otimes \beta_F}$ and hence extends to an isomorphism

$$\Phi : \ker \tilde{D}_{\pi^* E \otimes \beta_F} \cong \ker D_E \otimes \ker D_\beta.$$

Proof. Recall that from Lemma 4.1.31 that $\Phi_\tau \circ (\Phi_{\tau'})^{-1}$ acts as multiplication by an element of $\mathrm{Spin}^c(n)$ on $s_E \otimes s_\beta$ and in particular this is an action only on the s_β part. Now, the key fact about this is that the action of $\Phi_\tau \circ \Phi_{\tau'}^{-1}$ on the kernel of $D_E \# D_\beta$ is trivial, i.e. when $s_E \otimes s_\beta \in \Gamma^\infty(S_E) \otimes \ker D_\beta$,

$$\Phi_\tau \circ \Phi_{\tau'}^{-1}(s_E \otimes s_\beta) = s_E \otimes s_\beta.$$

The positive part of the kernel of D_β is $\mathbb{C}\cdot(1+c(\omega))$. This is an endomorphism on S^+, and $\mathbb{C}(1+c(\omega))\big|_{S^+} = \mathbb{C}(2\cdot 1_{S^+}) = \mathbb{C}\cdot 1_{S^+}$. Note that formally 1_{S^+} is not actually the identity map on S^+, it is the identity *section* i.e. it is the section that takes $x \in S^n$ and sends it to $1_{S_x^+} \in \mathrm{End}(S^+) \cong S^+ \otimes (S^+)^*$. We have not taken any effort to distinguish between these because as Lemma 1.3.10 makes clear, they are the same space. By Lemma 4.1.35 the identity section of the endomorphism bundle corresponds to the identity endomorphism in the endomorphism ring. Recall that when $T \in \mathrm{End}(S^+)$ and $g \in \mathrm{Spin}^c(n)$ the action by g on T is

$$g \cdot T = g \circ T \circ g^{-1}.$$

Notably, if $T = 1_{S^+} \in \mathrm{End}(S^+)$ then g does nothing at all. With this in mind,

$$\begin{aligned}\Phi_\tau \circ (\Phi_{\tau'})^{-1}(s_E \otimes s_\beta) &= s_E \otimes g \cdot s_\beta \\ &= s_E \otimes z \cdot g \cdot 1_{S_{S^+}} \\ &= s_E \otimes z \cdot 1_{S_{S^+}}.\end{aligned}$$

Where $z \in \mathbb{C}$ corresponds to our generic $s_\beta \in \ker D_\beta$ via $(\ker D_\beta)^+ \cong \mathbb{C}\cdot 1_{S^+}$. □

Lemma 4.1.37. *Suppose that $f = (f_1 \otimes f_2) \in C^\infty(U) \otimes C^\infty(S^n)$ and that $y = (y_1, y_2) \in U \times S^n$ with $f(y) = 0$. Write $s_E \otimes s_\beta = s \in \Gamma^\infty(S_E \boxtimes S_\beta)$ and suppose that $s(y) = (e_1, e_2) \in (S_E)_{y_1} \otimes (S_\beta)_{y_2}$. Then the principal symbol of $\Upsilon := D_E\big|_U \# D_\beta$ is*

$$\sigma_\Upsilon(df\big|_y)(e) = \sigma_{D_E|_U}(df_1\big|_{y_1})(e_1) \otimes s_\beta(y_2) + (c(\omega)s)(y) \otimes \sigma_{D_\beta}(df_2\big|_{y_2})(e_2).$$

Proof. This relies on the fact that a simple tensor $s_E \otimes s_\beta \in \Gamma^\infty(S_M \otimes E) \otimes \Gamma^\infty(S \otimes \beta)$ is linear over both entries.

$$\begin{aligned}\left(iD_E\big|_U \# D_\beta\right)(f s_E \otimes s_\beta) &= i(D_E\big|_U \otimes 1 + c(\omega) \otimes D_\beta)(f s_E \otimes s_\beta)(y) \\ &= iD_E\big|_U (f_1 s_E)(y_1) \otimes s_\beta(y_2) + (c(\omega)s_E)(y_1) \otimes i(D_\beta f_2 s_\beta)(y_2).\end{aligned}$$

Now, because $iD_E\big|_U(f_1 s_E)(y_1) = \sigma_{D_E|_U}(df_1\big|_{y_1})(s_E(y_1))$ and $i(D_\beta f_2 s_\beta)(y_2) = \sigma_{D_\beta}(df_2\big|_{y_2})(s_\beta(y_2))$ the end result can be rendered in a more palatable shorthand (omitting the cotangent vector),

$$\sigma_\Upsilon = \sigma_{D_E|_U} \otimes 1 + c(\omega) \otimes \sigma_{D_\beta}.$$

□

Lemma 4.1.38. *Let* $Y = (\beta_F \otimes \pi^* E)\big|_{\pi^{-1}(U)}$ *and* $\Upsilon = D_E\big|_U \# D_\beta$, *i.e. suppose* $\tilde{D}_Y = \tilde{D}_{(\beta_F \otimes \pi^* E)}\Big|_{\pi^{-1}(U)}$ *as in Definition 4.1.30. Then the principal symbol of* $\tilde{D}_Y$ *is*

$$\sigma_{\tilde{D}_Y}(\xi) = \Psi_\tau^{-1} \sigma_\Upsilon(\xi \circ T_{\psi_\tau(m)} \psi_\tau^{-1}) \Psi_\tau,$$

for all $m \in M, \xi \in T_m^* M$, *and for* ψ_τ, Ψ_τ *as in Definition 4.1.28.*

Proof. Recall from Lemma 1.2.16 that the we can compute the principal symbol of an operator of order 1 using a commutator. Now, fix an $s \in \Gamma^\infty((S \otimes \beta_F \otimes \pi^* E)\big|_{\pi^{-1}(u)})$ and a smooth function $f \in C^\infty(\pi^{-1}(U))$, with $f(m) = 0$ i.e. as in Definition 1.2.14. Then we can compute the principal symbol of $\tilde{D}_Y$,

$$\begin{aligned}\Phi_\tau^{-1}(\Upsilon \Phi_\tau(f \cdot s)(m) - f \Upsilon \Phi_\tau(s))(m) &= \Phi_\tau^{-1}\Big(\left(\psi_\tau^{-1}\right)^* f \Upsilon \Phi_\tau(s) + \sigma_\Upsilon(d(\psi_\tau^{-1})^* f) \Phi_\tau(s) \\ &\quad - (\psi_\tau^{-1})^* f \cdot \Upsilon \Psi_\tau(s)\Big)(m) \\ &= \left(\Phi_\tau^{-1} \sigma_\Upsilon(d(\psi_\tau^{-1})^* f) \Phi_\tau(s)\right)(m).\end{aligned}$$

Conjugation by Φ_τ corresponds to conjugating by Ψ_τ,

$$\begin{aligned}\left(\Phi_\tau^{-1} \sigma_\Upsilon(d(\psi_\tau^{-1})^* f) \Phi_\tau(s)\right)(m) &= \Psi^{-1} \sigma_\Upsilon(d(\psi_\tau^{-1})^* f) \Psi_\tau(s(\psi_\tau^{-1} \circ \psi_\tau(m))) \\ &= \Psi_\tau^{-1} \sigma_\Upsilon(d(\psi_\tau^{-1})^* f) \Psi_\tau(s(m))\end{aligned}$$

for a particular point $m \in \pi^{-1}(U)$. The de-Rham operator d satisfies (for a generic smooth function f on $\pi^{-1}(U)$),

$$d_{\psi_\tau(m)}((\psi_\tau^{-1})^* f) = df\big|_m \circ T_{\psi_\tau(m)} \psi_\tau^{-1}.$$

If $\xi \in T_m^*(\pi^{-1}(U))$ we conclude that

$$\sigma_{\Phi_\tau^{-1} \Upsilon \Phi_\tau}(\xi) = \Psi_\tau^{-1} \sigma_\Upsilon(\xi \circ T_{\psi_\tau(m)} \psi_\tau^{-1}) \Psi_\tau.$$

$\square$

Proposition 4.1.39. *The principal symbol of* $\tilde{D}_{\pi^* E \otimes \beta_F}$ *is the principal symbol of* $D_{\pi^* E \otimes \beta_F}$.

Proof. Recall from Proposition 1.2.17 that $\sigma_{D_{\pi^* E \otimes \beta_F}} = \sigma_{D_Y} = i c_{\Sigma F}(\xi) \otimes 1_{\pi^* E} \otimes 1_{\beta_F}$. From Lemmas 4.1.37 and 4.1.38 and we know that the symbol of $\tilde{D}_Y$ is $\Psi_\tau^{-1} \circ \sigma_\Upsilon \circ \Psi_\tau$. Define $\xi = (\xi_M, \xi_{S^n}) \circ T\psi_\tau$ as above. Then,

$$\begin{aligned}\Psi_\tau(i c_{\Sigma F}(\xi) \otimes 1_{\pi^*(E)} \otimes 1_{\beta_F}) &= \varphi_S(i c_{\Sigma F}(\xi)) \otimes \varphi_E \otimes \varphi_\tau \\ &= (i c_M(\xi_M) \otimes 1_S + c(\omega) \otimes c_{S^n}(\xi_{S^n}))\, \varphi_S \otimes \varphi_E \otimes \varphi_\tau \\ &= \sigma_\Upsilon(\xi) \Psi_\tau,\end{aligned}$$

which is exactly equivalent to $\Psi_\tau^{-1} \circ \sigma_{D_\Upsilon} \circ \Psi_\tau = \sigma_{\tilde{D}_Y}$. Note that the compatibility of φ_S with the Clifford multiplication is due to Lemma 1.2.10, where we remark that the C_τ part of φ_S commutes with Clifford multiplication because the multiplication by an element of $\mathrm{Spin}^c(n)$ on $s_2 \in S_{S^n} = P_{S^n} \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^{n/2}}$ acts only on the structure bundle (i.e. P_{S^n}) part of S_{S^n} and does not act on the complex part. □

Proof of Theorem 4.1.10. Proposition 4.1.34 implies that $\operatorname{ind} D_E = \operatorname{ind} \tilde{D}_{\pi^* E \otimes \beta_F}$ and Proposition 4.1.39 implies that $\operatorname{ind} \tilde{D}_{\pi^* E \otimes \beta_F} = \operatorname{ind} D_{\pi^* E \otimes \beta_F}$, giving

$$\operatorname{ind} D_{\pi^* E \otimes \beta_F} = \operatorname{ind} D_E.$$

□

4.2 Topological index

Recall that the Todd class of a manifold M is by definition the Todd class of the tangent bundle TM. The topological index of a pair (M, E) as in Definition 3.1.7 is $(\mathrm{ch}(E) \cup \mathrm{Td}(M))[M]$ and by Definition 1.4.8 this is exactly the integral

$$\int_M \mathrm{ch}(E) \cup \mathrm{Td}(M).$$

Theorem 4.2.1 (Invariance under direct sum - disjoint union)**.** *The topological index is preserved under direct sum - disjoint union. Suppose that $(M, E) \sqcup (M, E') \sim (M, E \oplus E')$. Then*

$$\int_M \mathrm{ch}(E) \cup \mathrm{Td}(M) + \int_M \mathrm{ch}(E') \cup \mathrm{Td}(M) = \int_M \mathrm{ch}(E \oplus E') \cup \mathrm{Td}(M).$$

Proof. The Chern character is additive across direct sum; $\mathrm{ch}(E \oplus E') = \mathrm{ch}(E) + \mathrm{ch}(E')$. □

Theorem 4.2.2 (Invariance under bordism)**.** *The topological index is preserved by bordism. If (M_1, F_1) is bordant to (M_2, F_2) then*

$$(\mathrm{ch}(F_1) \cup \mathrm{Td}(M_1))[M_1] = (\mathrm{ch}(F_2) \cup \mathrm{Td}(M_2))[M_2].$$

Proof. Recall previously in the proof of Theorem 4.1.2 that the index of a boundary was 0 was equivalent to the index was invariant under bordism. The same principle holds for the topological index. Suppose that $F \to X$ is a vector bundle on a manifold X that restricts to E on $\partial X = M$. The Chern character and Todd class commute with pullback, and $\mathrm{ch}\left(E\big|_{\partial X}\right) \cup \mathrm{Td}(\partial X)$ is the restriction to the boundary of $\mathrm{ch}\,(E) \cup \mathrm{Td}(X)$. Define by ι the inclusion mapping of $M = \partial X \hookrightarrow X$. Then,

$$\int_M \mathrm{ch}(E) \cup \mathrm{Td}(M) = \int_{\partial X} \mathrm{ch}(\iota^* F) \cup \mathrm{Td}(\iota^*(TX)).$$

The normal bundle $\mathcal{N}$ is trivial and $TX\big|_M = TM \oplus \mathcal{N}$, so

$$\begin{aligned}\int_{\partial X} \mathrm{ch}(\iota^* F) \cup \mathrm{Td}(\iota^* TX) &= \int_{\partial X} \iota^*(\mathrm{ch}(F) \cup \mathrm{Td}(TM)) \\ &= \int_X d(\mathrm{ch}\,(F) \cup \mathrm{Td}(X)) \\ &= 0.\end{aligned}$$

The second to last line is due to Stokes' theorem, and the final line is because $\mathrm{ch}(F), \mathrm{Td}(X)$ are closed forms. □

Theorem 4.2.3 (Invariance under bundle modification)**.** *Let E, M, β_F and $\pi : \Sigma F \to M$ be as in Definition 3.1.11. Then*

$$(\mathrm{ch}(E) \cup \mathrm{Td}(M))[M] = (\mathrm{ch}(\beta_F \otimes \pi^* E) \cup (\mathrm{Td}(\Sigma F))[\Sigma F].$$

Proof. We first note that the following proof is another example of remarkable efficacy of our computation in Chapter 2.

We can use a partition of unity to decompose this integral. Suppose that $\{U_j\}_{j=1}^l$ is a cover of M and $\{\chi_j\}_{j=1}^l$ is a partition of unity subordinate to this cover. Suppose also that these U_j are trivialising neighbourhoods for the sphere bundle $\pi : \Sigma F \to M$. Recall that in Definition 4.1.17 we had a trivialisation $\psi : \pi^{-1}(U) \to U \times S^n$. For each U_j in the cover, there is a corresponding $\psi_j : \pi^{-1}(U_j) \to U_j \times S^n$. Let $P_{U_j} : U_j \times S^n \to U_j$, $P_{S^n} : U_j \times S^n \to S^n$ be the projections to each factor. The decomposition due to the partition of unity is:

$$\int_{\Sigma F} \mathrm{ch}(\pi^* E \otimes \beta_F)\,\mathrm{Td}(\Sigma F) = \sum_{j=1}^{l} \int_{\pi^{-1}(U_j)} \pi^* \chi_j\,\mathrm{ch}(\pi^* E \otimes \beta_F)\,\mathrm{Td}(\Sigma F).$$

For a particular j we have

$$\begin{aligned}&\int_{\pi^{-1}(U_j)} \pi^* \chi_j\,\mathrm{ch}(\pi^* E \otimes \beta_F)\,\mathrm{Td}(\Sigma F) = \\ &\qquad \int_{U_j \times S^n} (\psi_j^{-1})^* \pi^* \chi_j (\psi_j^{-1})^*\,\mathrm{ch}(\pi^* E \otimes \beta_F\big|_{\pi^{-1}(U)})(\psi_j^{-1})^*\,\mathrm{Td}(\pi^{-1}(U)),\end{aligned}$$

which is due to the invariance of integrals under pullbacks of diffeomorphisms. Because ψ_j is a trivialisation of the bundle ΣF there is a commuting diagram, $\pi \circ \psi_j^{-1} = P_{U_j}$. We may write the integral above as

$$\int_{U_j \times S^n} P_{U_j}^* \chi_j (\psi_j^{-1})^* \operatorname{ch}(\pi^* E \otimes \beta_F\big|_{\pi^{-1}(U)})(\psi_j^{-1})^* \operatorname{Td}(\pi^{-1}(U)).$$

The aim is to pull back every part of the above integrand by either P_{U_j} or P_S^n, so that we can split it into integrals over U_j and S^n separately. We have the isomorphism $P_{S^n}^* \beta = (\psi_j^{-1})^* \beta_F\big|_{\pi^{-1}(U)}$ for each j, which is via the map

$$P_{S^n}^* \beta \to (\psi_j^{-1})^* \beta_F\big|_{\pi^{-1}(U_j)}, \quad (u, x, b) \mapsto (u, x, [p, b])$$

where $\pi_\beta(b) = x$, $\tau(p) = (u, 1_{\mathrm{Spin}^c(n)})$, for $u \in U_j, x \in S^n, b \in \beta$. Because $\tau(p) = (u, 1_{\mathrm{Spin}^c(n)})$, $\psi_j^{-1}(u, x) = [p, x]$. Using this we can further modify the integral above:

$$\int_{U_j \times S^n} P_{U_j}^* \chi_j P_{U_j}^* \operatorname{ch}(E) P_{S^n}^* \operatorname{ch}(\beta)(\psi_j^{-1})^* \operatorname{Td}(\pi^{-1}(U)).$$

The only remaining term is the Todd class of $\pi^{-1}(U)$. The key fact to overcoming this is the tangent map for the trivialisation (ψ_j^{-1}). $T(\psi_j^{-1})$ provides a trivialisation of $T\pi^{-1}(U_j)$, which is $T\Sigma F\big|_{\pi^{-1}(U_j)}$. This splits $T\pi^{-1}(U_j)$ into the direct sum, $T\pi^{-1}(U_j) = P_{U_j}^* TU_j \oplus P_{S^n}^* TS^n$, which is a local statement of Lemma 4.1.20.

$$\begin{aligned} &\int_{U_j \times S^n} P_{U_j}^* \chi_j P_{U_j} \operatorname{ch}(E\big|_{U_j}) P_{U_j}^* (\operatorname{Td}(U_j)) P_{S^n}^* \operatorname{ch}(\beta) P_{S^n}^* (\operatorname{Td}(S^n)) \\ &= \int_{U_j} \chi_j \operatorname{ch}(E\big|_{U_j}) \operatorname{Td}(U_j) \int_{S^n} \operatorname{ch}(\beta) \operatorname{Td}(S^n). \end{aligned}$$

Now, from Lemma 1.4.12 we know that the Todd class of S^n is 1 (n is even) and from Theorem 2.1.2 the Chern character of β evaluated at the fundamental class of S^n is 1. We conclude that

$$\int_{U_j} \chi_j \operatorname{ch}(E\big|_{U_j}) \operatorname{Td}(U_j) \int_{S^n} \operatorname{ch}(\beta) \operatorname{Td}(S^n) = \int_{U_j} \chi_j \operatorname{ch}(E\big|_{U_j}) \operatorname{Td}(U_j).$$

Finally, the whole integral may be written as a sum:

$$\begin{aligned} \operatorname{ch}(\pi^* E \otimes \beta_F) \cup \operatorname{Td}(\Sigma F))[\Sigma F] &= \sum_{j=1}^{l} \int_{U_j} \chi_j \operatorname{ch}(E\big|_{U_j}) \operatorname{Td}(U_j) \\ &= \sum_{j=1}^{l} \int_M \chi_j \operatorname{ch}(E) \operatorname{Td}(M) \\ &= (\operatorname{ch}(E) \cup \operatorname{Td}(M))[M], \end{aligned}$$

which is invariance of the topological index under vector bundle modification. $\square$

Chapter 5

A group-equivariant index theorem

We formulate a suitable notion of group-equivariant K-homology of a point and see how the introduction of a group action by a compact Lie group G changes the topological and analytic index. The aim is to present an outline of how a suitable analogue of Theorem 1.4.13 holds when we consider M as a G-manifold and $E \to M$ a G-equivariant vector bundle.

5.1 Group-equivariant K-homology

The structures of Spin^c manifolds adapt well to the introduction of a Lie group, and much of the introduction is merely in writing down where one must incorporate the action by G.

Definition 5.1.1 (G-Spin^c datum)**.** *Let M be as in Definition 1.1.17. If (P, η) is a G-Spin^c datum for TM then an isomorphism of G-Spin^c data is an isomorphism of the relevant Spin^c data except where the isomorphism respects the group action. A G-isomorphism class is then a G-Spin^c structure for TM, which is by definition a G-Spin^c structure for M.*

Remark. Specifying the decomposition $TM \cong P \times_{\mathrm{Spin}^c(n)} \mathbb{R}^n$ is equivalent to specifying a Spin^c structure, and this is also true in the case of the G-Spin^c structure. The action by G on TM or $S_M = P \times_{\mathrm{Spin}^c(n)} \mathbb{C}^{2^r}$ is an action on the P-part.

The K-homology of a point extends to the case when we introduce a group, because we can decree that the action of G on M lifts to an equivariant action on a smooth complex vector bundle $E \to M$. If it is important to emphasise the existence of the G-Spin^c-structure on (M, E) we may write $(M, E)_G$.

Definition 5.1.2 (Pair G-isomorphism)**.** *Suppose we have an isomorphism of pairs (M,E), (M',E') as in Definition 3.1.1 and both M, M' have G-*Spin^c *structures as in Definition 1.1.17, with E, E' being G-vector bundles. Suppose additionally that the isomorphism of those pairs satisfies the following conditions. The diffeomorphism $\varphi : M \to M'$ and bundle isomorphism $E \to E'$ respects the group action and preserves the G-*Spin^c *datum, in the sense that $(\varphi^* P', \varphi^* \eta')$ is G-isomorphic to (P, η) (as G-Spin^c data as in Definition 5.1.1) and that the pullback $\varphi^* E'$ is isomorphic (as a complex vector bundle on M) to E and the isomorphism respects the group action. Then we say that $(M,E)_G$ is now pair G-isomorphic to $(M',E')_G$*

Definition 5.1.3 ($\mathcal{K}_G$)**.** *Define analogously to Definition 3.1.2 the set $\mathcal{K}_G$ which consists of pairs all pairs $(M,E)_G$ modulo the G-respecting isomorphism described in Definition 5.1.2.*

Remark. Essentially, a pair $(M,E) \in \mathcal{K}_G$ is a pair in $\mathcal{K}$ that incorporates a G-Spin^c structure on M and the G-action is compatible with $E \to M$.

We can proceed with the incorporation of the G-action into the relations described in Definitions 3.1.8, 3.1.9 and 3.1.11.

Definition 5.1.4 (Direct sum - disjoint union)**.** *Suppose two pairs $(M,E')_G$ and $(M,E)_G$ are in $\mathcal{K}_G$. Then $(M,E)_G \sqcup (M,E')_G$ is related to $(M, E \oplus E')_G$ and the action by G on the direct sum $E \oplus E'$ is $g(e+e') = ge + ge'$, for $g \in G$, $e \in E_m$, $e' \in (E')_m$.*

Suppose that as in the setting of Definition 3.1.9 we have a pair (W,F) that satisfies $(\partial W, F\big|_{\partial W}) \cong (M,E) \sqcup (-M',E')$. We must first say what it means to get a G-Spin^c structure on the boundary of W. At each stage in the construction of the Spin^c structure on the boundary in Proposition 1.1.19, insert a group action and ensure that it respects the structure provided. The important fact is that we must require that the orbit of each connected component of the boundary is entirely contained within that connected component. In the case of W with boundary $(M,E) \sqcup (-M',E')$ we require that $G(M) \subset M$ and $G(M') \subset M'$.

Definition 5.1.5 (G-bordism)**.** *Let (W,F) be as in Definition 3.1.9 providing (without considering the G action) a bordism between (M,E) and (M',E'). Suppose additionally that W is a G-*Spin^c *manifold and that the two connected components of ∂W are preserved by the G action as above. Then if $(\partial W, F\big|_{\partial W})_G$ is pair G-isomorphic (in the sense of Definition 5.1.2) to $(M,E)_G \sqcup (-M',E')_G$ then $(W,F)_G$ is said to provide a G-bordism from $(M,E)_G$ to $(M',E')_G$.*

We remark that once again that the pair $(W,F)_G$ is not in $\mathcal{K}_G$.

Vector bundle modification incorporates a G-action in a similar way. Fix a G-Spin^c structure for a given vector bundle $F \to M$ as in Definition 3.1.11. This has as datum a principal Spin^c bundle P that also has a G-action and a G-respecting homomorphism η as in Definition 1.1.17. The datum defines a G-Spin^c structure for F. The associated bundle is $\Sigma F = P_F \times_{\mathrm{Spin}^c(n)} S^n$ which inherits the action by G on the P_F-part and is a G-Spin^c manifold because of it. The vector bundle β_F becomes G-Spin^c by the action on the P_F part.

Definition 5.1.6 (Vector bundle G-modification). *The bundle G-modification of a pair $(M, E)_G \in \mathcal{K}_G$ is the pair $(\Sigma F, \pi^* E \otimes \beta_F)$, where G acts on $\pi^* E \otimes \beta_F$ as described immediately above.*

The G-equivariant K-homology of a point defined similarly to the normal K-homology of a point.

Definition 5.1.7 (G-Equivariant K-homology of a point). *Define $K_0^G(\cdot)$ to be equivalences classes of pairs $(M, E)_G \in \mathcal{K}_G$ where the equivalence relation is the one generated by Definitions 5.1.4 to 5.1.6.*

In the same way that Theorem 1.4.13 was a consequence of the computation of the K-homology of a point, there is a notion of a G-index and a related *G-index theorem.*

5.1.1 Equivariant indices

Recall that given a group G there is the *representation ring* of G.

Definition 5.1.8 (Representation ring). *The representation ring of a compact group G is the Grothendieck group of formal differences of isomorphism classes of finite dimensional representations $\rho : G \to \mathrm{GL}(V)$ of G.*

Remark. It is an unfortunate consequence of inertia that it is standard to omit the representation $\rho_V : G \to \mathrm{GL}(V)$ and instead write or $[V] - [W]$ (or just $[V]$ when appropriate) to mean a class in $R(G)$. When it is necessary to speak of the representation specifically, it will not be omitted.

Definition 5.1.9 (Analytic G-index). *Suppose F is a G-equivariant Fredholm operator $F : H_1 \to H_2$ between representation spaces H_1 and H_2 of G. Then the analytic G-index of F is the element* $\mathrm{ind}_G(F) = [\ker F] - [\operatorname{coker} F] \in R(G)$.

Here we note that the representation for each part in the formal difference $[\ker F] - [\operatorname{coker} F]$ arises as map that sends $g \in G$ to $A \in \mathrm{GL}(V)$ (V is either $\ker F$ or $\operatorname{coker} F$) given by (for $h_1 \in H_1$) $As = g \cdot h$, and the action by g on classes in $\operatorname{coker} F$ is $g \cdot (h_2 + \operatorname{im} F) = g \cdot h_2 + \operatorname{im} F$. This action is well-defined, because if $h \in \operatorname{im} F$, then because F is equivariant (by definition) $g \cdot h \in \operatorname{im} F$.

Remark. Similarly to the non-equivariant case, we will define the analytic G-index on K_0^G-homology classes as the analytic G-index of the associated operator, $\operatorname{ind}^a_G((M,E)_G) := \operatorname{ind}_G(D_E)$.

In the event that we have a twisted operator D_E then the G-index becomes a formal difference of the graded parts as in Definition 1.2.11.

This new definition of an index corresponds with the old in the following sense: given a fixed $g \in G$, we can obtain a complex number from a formal difference by first evaluating at a group element $g \in G$ and taking the trace of the resultant linear map. Denote this composition $\operatorname{ind}_G(F)(g)$ i.e. $\operatorname{ind}(F)(g) = [\ker F](g) - [\operatorname{coker} F](g)$.

Lemma 5.1.10. $\operatorname{ind}_G(D_E)(g)$ *does not depend on the choice of group G which contains g. In particular, we may assume that G is the topological closure of the subgroup generated by $g \in G$.*

It is important to note that we may still evaluate the G-index at any group element before we made this assumption (i.e. at $\hat{g} \notin \overline{\langle g \rangle}$) but it allows us to write $\operatorname{ind}_{\overline{\langle \hat{g} \rangle}}(D_E)(\hat{g})$ rather than $\operatorname{ind}_{\overline{\langle g \rangle}}(D_E)(\hat{g})$, which will be useful later. From now on we will abuse notation and write G to mean the topological closure of the group generated by g.

Proof of Lemma 5.1.10. If G is a subgroup of some larger group G' then given a representation $\rho : G' \to \mathrm{GL}(V)$, we have $\rho\big|_G : G \to \mathrm{GL}(V)$ so $\rho\big|_G(g) = \rho(g)$ and hence they have identical trace. □

Remark. The topological closure is for technical purposes. It is of course possible to not include closure in the definition, but there are some issues that arise when considering particular pathological examples. The main problem we would like to avoid is the one that arises from irrational rotations: given a single irrational rotation generates a (nasty) subgroup of U(1), but the topological closure of said group is U(1), which is comparatively benign.

Lemma 5.1.11. *For any trivial G-space M there is an isomorphism*

$$K^0_G(M) \cong K^0(M) \otimes R(G).$$

Proof. The isomorphism is easy to write down but the details lie in establishing the ancillary details. It is $K^0(M) \otimes R(G) \ni [E] \otimes [V] \mapsto [E \otimes V] \in K^0_G(M)$. □

Remark. If $g \in G$ does not act trivially on M, then the fixed point set $M^g = \{x \in M \mid g \cdot x = x\}$ is a trivial G-space and hence we can write $K^0_G(M^g) \cong K^0(M^g) \otimes R(G)$. Note that this is only true because of the assumption that $G = \overline{\langle g \rangle}$. Without it, we can only conclude that M^g is $\overline{\langle g \rangle}$ invariant, which is (at worst) merely a subgroup of the original group in which g resides.

We can therefore evaluate the class $[E\big|_{M^g}] \in K_G^0(M^g)$ at a point $g \in G$ and take the trace of resulting $GL(V)$-part to get a complex number, $[E\big|_{M^g}](g) \in K(M^g) \otimes \mathbb{C}$. We can evaluate K-theory classes using the Chern character $\mathrm{ch} : K^0(M^g) \to H^*(M^g)$ and hence it makes good sense to consider $\mathrm{ch}([E\big|_{M^g}](g))$.

Definition 5.1.12 ($\bigwedge N_{\mathbb{C}}$)**.** *Let N be the union of normal bundles on each of the connected components of M^g. Define by $[\bigwedge N_{\mathbb{C}}]$ the class*

$$[\bigwedge N_{\mathbb{C}}] = \left[\bigoplus_j \bigwedge^{2j} N \otimes \mathbb{C}\right] - \left[\bigoplus_j \bigwedge^{2j+1} N \otimes \mathbb{C}\right] \in K_G^0(M^g).$$

Similarly to the Chern character, we can evaluate this K_G-theory class at an element $g \in G$ and obtain an element of $K^0(M^g) \otimes \mathbb{C}$. The final step is to note that because the Chern character is a homomorphism and the class $[\bigwedge N_{\mathbb{C}}](g)$ is an invertible element of the ring ([AS68a, Lemma 2.7]) it makes good sense to write down (non-explicitly) the (multiplicative) inverse

$$\frac{1}{\mathrm{ch}([\bigwedge N_{\mathbb{C}}]\,(g))} \in H^*(M^g) \otimes \mathbb{C}.$$

Definition 5.1.13 (Topological G-index)**.** *Given a fixed group element $g \in G$, the topological G-index of a pair $(M, E) \in \mathcal{K}_G$ evaluated at g is*

$$\mathrm{ind}_G^t(M, E)(g) = \int_{M^g} \frac{\mathrm{ch}([E\big|_{M^g}](g))\,\mathrm{Td}(TM^g)}{\mathrm{ch}([\bigwedge N_{\mathbb{C}}](g))}.$$

Remark. It does not make sense to consider $\mathrm{ind}_G^t(M, E)$ independently of a choice of group element g.

With this in mind we can formulate an appropriate analogue of Theorem 1.4.13 when introducing a group action.

Theorem 5.1.14 (Atiyah–Segal–Singer, Theorem 3.9 in [AS68c]/Theorem 2.12 in [AS68a])**.** *Suppose that (M, E) is a pair in $\mathcal{K}_G$. Then*

$$\mathrm{ind}_G^a(M, E)(g) = \int_{M^g} \frac{\mathrm{ch}([E\big|_{M^g}](g))\,\mathrm{Td}(TM^g)}{\mathrm{ch}([\bigwedge N_{\mathbb{C}}](g))}.$$

In the case that $G = \{e\}$ is trivial Theorem 5.1.14 reduces to Theorem 1.4.13. Indeed, when G is trivial, all representations are identical and the trace gives exactly the dimension of $(\ker D_E)^{\pm}$, so

$$\mathrm{ind}_G(D_E)(e) = ([(\ker D_E)^+] - [(\ker D_E)^-])(e) = \dim(\ker D_E)^+ - \dim(\ker D_E)^-.$$

For the topological index, let us first consider the numerator,

$$\mathrm{ch}([E\big|_{M^g}](g))\,\mathrm{Td}(TM^g).$$

The class $[E\big|_{M^e}] = [E \otimes \mathbb{C}] \in K^0_G(M)$ decomposes into $[E] \otimes [\mathbb{C}] \in K^0(M) \otimes R(G)$ via the isomorphism in Lemma 5.1.11, because G is trivial. The Chern character of this class evaluated at e is then $\mathrm{ch}([E\big|_{M^e}](e)) = \mathrm{ch}([E\big|_{M^e}]) \otimes [\mathbb{C}](e) = \mathrm{ch}(E) \otimes \dim_{\mathbb{C}} \mathbb{C} = \mathrm{ch}(E) \otimes 1$. The Todd class of TM^e is then $\mathrm{Td}(TM^e) = \mathrm{Td}(TM)$, so the numerator is just $\mathrm{ch}(E)\,\mathrm{Td}(TM)$.

The denominator is

$$\mathrm{ch}([\textstyle\bigwedge N_{\mathbb{C}}](g)),$$

and the normal bundle N is 0 because $TM^e = TM$. Hence, the the denominator is $\mathrm{ch}([\bigwedge N_{\mathbb{C}}](e)) = \mathrm{ch}(\mathbb{C}) \otimes \dim \mathbb{C} = 1$ (under the isomorphism of Lemma 5.1.11). In summary, when G is trivial we have:

$$\int_{M^g} \frac{\mathrm{ch}([E\big|_{M^g}](g))\,\mathrm{Td}(TM^g)}{\mathrm{ch}([\bigwedge N_{\mathbb{C}}](g))} = \int_M \mathrm{ch}(E)\,\mathrm{Td}(TM).$$

The idea of the proof

In the proof of Theorem 1.4.13 we saw that the result followed from the computation $K_0(\cdot) = \mathbb{Z}$. We will present an outline of a similar proof here for Theorem 5.1.14, although naturally there will be some modification. The K-homology is no longer $\mathbb{Z}$, but instead is $R(G)$ and the representation ring is isomorphic to $K_0^G(\cdot)$ via the analytic G-index.

Let $\hat{G}$ be the set of equivalence classes of irreducible representations of G. Two representations $\rho_1 : G \to GL_(V)$ and $\rho_2 : G \to GL_(W)$ of G are said to be equivalent if there is a linear isomorphism $\varphi : V \to W$ such that $\varphi \circ \rho_1(g) = \rho_2(g) \circ \varphi$ for all $g \in G$. We can write the representation ring of G as

$$R(G) = \{\bigoplus_{V \in \hat{G}} n_V[V] \mid n_V \in \mathbb{Z}, \text{only finitely many } n_V \text{ non-zero}\},$$

which is the free abelian group generated by the (equivalence classes of) irreducible representations.

Theorem 5.1.15 (A special case of Theorem 3.11 in [BOOSW10])**.**

$$\mathrm{ind}^a_G : K_0^G(\cdot) \to R(G)$$

is an isomorphism of groups.

Proof. Some clarifying comments for how this particular case is obtained: when $X = Y$ is a point the "natural isomorphism" α in [BOOSW10, Theorem 3.11] is the analytic G-index. $\square$

Remark. We can offer some justification for surjectivity: Given $[V] \in R(G)$ the candidate pair is $(M_V, E_V) = (S^{2r}, \beta \otimes V)$, where the integer r is the same one obtained in Lemma 3.2.5 and the action by G on S^{2r}, $\beta \otimes V \to S^{2r}$ is only on the V-part. Because of the computation in Theorem 2.2.1 the positive part of the kernel of $D_{\beta \otimes V}$ is $\ker D_\beta \otimes V = \mathbb{C} \otimes V = V$ and the negative part of the kernel is 0. Hence, the G-index of such a pair is $\mathrm{ind}^a_G(S^{2r}, \beta \otimes V)(g) = [V](g) = \mathrm{tr}(\rho_V(g))$.

We hypothesise the following conjecture, which the author thinks is probably true, but unfortunately did not have enough time to properly investigate.

Hypothesis 5.1.16. *The topological G-index is a well-defined group homomorphism on K^G-homology classes.*

Remark. This is clearly true for direct sum disjoint union, because the Chern character splits and the denominator is common to both integrals in $\mathrm{ind}^t_G((M, E) \sqcup (M, E'))$.

The two analytic and topological G-indices must now agree on a generator that can be found for each $(M, E) \in \mathcal{K}_G$. This is a similar to the proof of Lemma 3.2.5.

Proposition 5.1.17. *Suppose G is a compact Lie group acting on a K^G-homological pair $(M, E) \in \mathcal{K}_G$. Then for each $V \in \hat{G}$ there exists a pair (M_V, E_V) defining a class $[(M_V, E_V)] \in K^G_0(\cdot)$ that satisfies*

$$\mathrm{ind}^a_G([(M_V, E_V)]) = [V] = \mathrm{ind}^t_G([(M_V, E_V)]),$$

in the sense that $\mathrm{ind}^a_G(M_V, E_V)(g) = \mathrm{ind}^t_G(M_V, E_V)(g)$.

Proof. We have already seen in Theorem 5.1.15 that the analytic G-index is a well-defined isomorphism, and in the remark below saw that we can find a candidate pair for surjectivity without much difficulty. We now only need to verify that the topological G-index of $(S^n, \beta \otimes V)$ is $[V](g)$.

Our first aim is to show $\mathrm{ch}([\beta \otimes V](g)) = \mathrm{ch}(\beta) \otimes [V](g)$. We rely on Lemma 5.1.11 to decompose $[\beta \otimes V]$ into $\beta \otimes [V]$. The evaluation at G of the product is evaluation of the representation part, and the Chern character only sees the (non-equivariant) K-theory part.

The end result is a tensor product (interpreted as an actual product), $\mathrm{ch}(\beta) \cdot \mathrm{tr}(\rho_V(g))$. This completely extracts the representation from the pair $(S^{2r}, \beta \otimes V)$, so we can rely on the computation from Chapter 2 to compute the topological G-index,

$$\begin{aligned}\operatorname{ind}^t_G(S^n, \beta\otimes V)(g) &= \int_{(S^{2r})^g} \frac{\operatorname{ch}([\beta\otimes V](g))\operatorname{Td}(T(S^{2r})^g)}{\operatorname{ch}((\bigwedge N_{\mathbb{C}})[g])} \\ &= \int_{(S^{2r})^g} \operatorname{tr}(\rho_V(g))\frac{\operatorname{ch}(\beta)\operatorname{Td}(T(S^{2r})^g)}{\operatorname{ch}([\bigwedge N_{\mathbb{C}}])(g))}.\end{aligned}$$

Because G acts trivially on S^{2r} the denominator is 1 (the normal bundle to $T(S^n)^g = TS^n$ is 0) and $(S^{2r})^g = S^{2r}$. Thus the integral reduces to

$$\begin{aligned}\operatorname{tr}(\rho_V(g))\int_{S^{2r}}\operatorname{ch}(\beta)\operatorname{Td}(TS^{2r}) &= \operatorname{tr}(\rho_V(g)). \\ &= \operatorname{ind}^a_G(S^n,\beta\otimes V)(g),\end{aligned}$$

where $\int_{S^{2r}}\operatorname{ch}(\beta)\operatorname{Td}(TS^{2r}) = \operatorname{ind}_t(S^n,\beta) = 1$, from the computation in the proof of Theorem 2.1.2.

□

Proof of Theorem 5.1.14. By Theorem 5.1.15, if (M_V, E_V) as in Proposition 5.1.17, the classes $\{[(M_V, E_V)]\}_{V\in\hat{G}}$ generate $K_0^G(\cdot)$. Assuming Hypothesis 5.1.16 it makes good sense to consider the topological index on (M_V, E_V) and by Proposition 5.1.17 the G-indices agree on generators. □

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