Diffeology, Groupoids & Morita Equivalence

M.Sc. Thesis in Mathematics

Nesta van der Schaaf

22nd June 2020

Supervisor:
Prof. dr. N.P. Landsman

Second reader:
Dr. I.T. Mărcuț

Radboud University

Abstract

This thesis consists of two parts. The first (Chapters **I** and **II**) is a thorough introduction to the theory of *diffeology*. We provide a ground-up account of the theory from the viewpoint of *plots* (in contrast to the sheaf-theoretic treatments). This includes a proof that the category **Diffeol** of diffeological spaces and smooth maps is complete and cocomplete, (locally) Cartesian closed, and a quasitopos. In addition, we treat many examples, including a detailed recollection of the classification of *irrational tori*.

The second part (Chapters **III** to **VI**) is a proposal for a framework of *diffeological Morita equivalence*. We give definitions of *diffeological groupoid actions, -bundles, and -bibundles*, generalising the known theory of Lie groupoids and their corresponding notions. We obtain a bicategory **DiffeolBiBund** of diffeological groupoids and diffeological bibundles. This has no analogue in the Lie theory, since we put no 'principality' restrictions on these bibundles. We then define a new notion of *principality* for diffeological (bi)bundles, and subsequently obtain a notion of Morita equivalence by declaring that two diffeological groupoids are equivalent if and only if there exists a *biprincipal bibundle* between them. Our main new result is the following: two diffeological groupoids are Morita equivalent if and only if they are *weakly equivalent* in the bicategory **DiffeolBiBund**. Equivalently, this means that a diffeological bibundle is *weakly invertible* if and only if it is biprincipal. This significantly generalises the original theorem in the Lie groupoid setting, where an analogous statement can only be made if we assume one-sided principality beforehand. As an application of this framework, we prove that two Morita equivalent diffeological groupoids have categorically equivalent *action categories*. We also prove that the property of a diffeological groupoid to be *fibrating* is preserved under Morita equivalence.

In a subsequent chapter, we propose an alternative framework for diffeological Morita equivalence using a *calculus of fractions*. We prove that the notion of Morita equivalence obtained in this way is identical to the one obtained from the bibundle theory. As a corollary, we prove that there is a *diffeomorphism* between the *orbit spaces* of two Morita equivalent diffeological groupoids. This generalises the well-known result from the Lie groupoid setting, where in general one only has a *homeo*morphism.

We then give a detailed construction of the *germ groupoid* of a space, and sketch a theory of *atlases*. To each atlas on a diffeological space we associate a *transition groupoid*, capturing the structure of the transition functions between the *charts*. We prove that two diffeological spaces are diffeomorphic if and only if their transition groupoids are Morita equivalent. This generalises earlier ideas of *orbifold atlases* to the diffeological setting.

Preface

"What is space?" This was once asked to me in a lecture on the history of quantum mechanics back in 2016. It caught me off guard. Mostly because it had never occurred to me, in any serious sense, to think about it. But it gave me a sense of mysterious excitement[1]. To me, it was in the same category of questions as *"what is a thing?"* or *"what is truth?"* And this thesis, even though not its point to provide an answer, is the closest opportunity I have had to touch upon it. Therefore, I would like to start the Introduction in Chapter **I** by sketching the landscape of *generalised smooth spaces*, which should lead naturally into the main topic of this thesis: *diffeology*.

There is something intriguing about the idea that there is an uttermost fundamental structure to it all. Not the form of differential equations or the definition of a smooth atlas, but the very structures that capture those ideas, and then the further structures that underlie those in turn. For me, it is exciting and maybe even comforting to imagine that such a thing exists. During my undergraduate lectures I was fairly close-minded in the way I estimated the broader context of the theories we encountered. The jump from metric spaces to topological spaces was logical, but to think that there is something beyond topology? Something beyond C*-algebras? Something beyond manifolds? Inside of those theories everything was well-behaved and pleasant, but beyond the boundaries of my ignorance I could not imagine anything else being necessary. Of course, the more I learn, the more these boundaries begin to vanish[2]. Now, I think a large part of what intrigues me about mathematics is the ways those boundaries can be broken.

One of these boundaries has been broken by diffeology, which extends the world of *differential topology*, reformulating what it means to be *smooth*. This will be the central notion in this thesis. In Chapter **II** we provide a detailed introduction to this theory. The main contribution of this thesis describes a generalisation of the theory of Lie groupoids and bibundles to the diffeological setting. In this we get a notion of *Morita equivalence* for diffeological groupoids. To read more on What this thesis is all about, please refer to Section **1.2**.

A short note on tangent structures and diffeology. The topic of this project started with groupoids, moved to tangent structures, and then went back to groupoids. Even though the results in this thesis do not relate to tangent structures directly, we nevertheless would like to make some remarks to document some findings that are not part of the main body of this thesis.

The motivation for this thesis has its origins in [**BFW13**] and [**Gł19**]. There, the authors consider embeddings of hypersurfaces in a 4-dimensional Lorentzian manifold that represent solutions to the initial value problem of the Einstein equations. Recalling from [**Gł19**, Section II.2.1], given a 3-manifold Σ, a Σ-*universe* is a certain equivalence class of proper embeddings $i : \Sigma \hookrightarrow (M, g)$ as space-like hypersurfaces into a Lorentzian 4-manifold. Two embeddings are equivalent when there exists an orientation-preserving isometry on the ambient Lorentzian manifold that sends the image of one embedding to the other. A pair of embeddings (i_1, i_0), where $i_1, i_0 : \Sigma \hookrightarrow (M, g)$, subject to a similar equivalence relation, then forms a groupoid of Σ-*evolutions*. The equivalence class of a pair of embeddings (i_1, i_0) is interpreted as a Cauchy development of the initial data that Σ represents. [**Gł19**] proves that the smooth structure on this groupoid defined in [**BFW13**] makes it into a diffeological groupoid.

For physical reasons we want to know the bracket structure of the associated algebroid of Σ-evolutions. These so-called *constraint brackets* determine the Einstein equations. Even though this 'algebroid' is calculated in [**BFW13**; **Gł19**], there is no general construction that associates a *"diffeological algebroid"* to an arbitrary diffeological groupoid. The first goal of this thesis was then to provide such a construction.

It became apparent quite quickly that the foundations for such a construction had yet to be laid. The definition of a Lie algebroid of a Lie groupoid depends heavily on the structure of the tangent bundle

[1] Not dissimilar to the impressions I got as a teenager when watching documentaries like *What the Bleep Do We Know!?*, containing quite mystical accounts of quantum mechanics, which in hindsight must have contributed to my choice of going into physics.

[2] One memorable glimpse of this realisation was during a differential geometry lecture by Gil Cavalcanti. The lecture was about Lie algebras, and the claim was made that the Lie algebra of a diffeomorphism group is the same as the space of vector fields. When a student asked why this was true, we were promptly reassured that the proof was beyond the scope of this course.

of a smooth manifold, which is not something that we have access to in the diffeological case. In fact, there seems to be no unambiguous notion of tangency on a diffeological space. However, as suggested in [**Gł19**, Section I.2.3], there is a way to obtain an algebroid-like object for diffeological groupoids, mimicking a technique that is used for Lie groupoids. Namely, [**SW15**] proves that the Lie algebroid of a Lie groupoid can be obtained as the Lie *algebra* of the *group of bisections*. In this sense the question of defining a diffeological algebroid is reduced to defining a diffeological Lie algebra associated to a diffeological group[3]. There has been little work done on this [**Les03**; **Lau11**], none of which treats the general case.

This leads us to the theory of *tangent structures*. It was Rosickỳ who first formalised this notion in [**Ros84**]. A lot of the differential geometric structure of smooth manifolds seemed to be encoded in certain properties of the tangent bundle functor $T : \mathbf{Mnfd} \to \mathbf{Mnfd}$, and Rosickỳ was able to condense them into a concise list of functorial and natural conditions. In the two papers [**CC14**; **CC16**] the theory is developed in a more modern account, and we refer the interested reader to those papers[4]. Why is this relevant to our discussion? It appears that the information needed to define a Lie algebra of a Lie group is already encoded in the tangent functor. Generally, the space of vector fields on an object in a tangent category carries a Lie bracket. Therefore, to any group object in such a category we can associate a Lie algebra. Recent work even suggests that we can get algebroids of internal groupoids [**Bur17**].

Before arriving at a theory of diffeological algebroids, it therefore became clear there was much more work to be done understanding the notion of a tangent structure on diffeological spaces. It was our next goal, then, to prove that one of the several notions of tangent bundles on diffeological spaces [**Vin08**; **CW14**] actually formed a tangent structure on **Diffeol**. I found that the most intriguing notion was the *internal tangent bundle*, which in its correct form was first defined in [**CW14**]. Let us explain what we mean by 'correct.' Already in [**Hec95**], Hector gave a definition of a tangent bundle $T^{\mathrm{H}}X$ on a diffeological space X. This was further developed in [**HMV02**], and also appearing in [**Lau06**]. It is lacking in an important way, however, as pointed out in [**CW14**, Example 4.3]. Namely, both the scalar multiplication and fibrewise addition on $T^{\mathrm{H}}X$ may fail to be smooth (showing that [**HMV02**, Proposition 6.6] and [**Lau06**, Lemma 5.7] do not always hold). The failure of smoothness prohibits Hector's tangent bundle from being a tangent structure. In [**CW14**] this defect was remedied using a technique that already appeared in [**Vin08**], and it is the resulting tangent bundle that we studied. It has also been studied further in [**CW17a**]. The result is a *diffeological vector pseudo-bundle* (cf. [**Per16**]) $\check{T}X \to X$, called the *internal tangent bundle*[5]. Based on this construction, we obtain a tangent functor $\check{T} : \mathbf{Diffeol} \to \mathbf{Diffeol}$, sending each smooth map $f : X \to Y$ to the *internal differential* $\check{\mathrm{d}}f : \check{T}X \to \check{T}Y$. I proved (which is elementary) that this functor forms an *additive bundle*, the elementary ingredients that comprise a tangent structure. In this sense, we have a reasonable contender for a tangent structure functor on the category of all diffeological spaces. My problem was that I could not prove it satisfied the axioms of a tangent structure! I already got stuck on proving that it preserves its own pullbacks. I tried imitating the construction for manifolds, where one can relate tangent spaces of embedded submanifolds and of preimages etc., but I found no straightforward generalisation of this to diffeology. It is not hard to write down a function from the tangent space of a fibred product to the fibred product of tangent spaces, but the problem is proving that this map is bijective. The difficulties are in part due to the fact that the behaviour of subspaces and tangent spaces can be a bit pathological: certain subspaces can have a higher-dimensional tangent space than the ambient space.

To circumvent these difficulties I tried to consider special classes of diffeological spaces. First, it was known that the internal tangent bundle on a diffeological group was particularly well-behaved [**CW14**]. The issue arises that $\check{T} : \mathbf{DiffeolGrp} \to \mathbf{Diffeol}$ needs to be an endofunctor. Dan Christensen suggested to me in an email that $\check{T}G$ could be equipped with a group structure by taking the internal differentials of the group operations of G. I did not end up pursuing this idea in the end.

Another nice class of diffeological spaces I considered was that of the *weakly filtered spaces*, studied first in [**CW17a**]. Their tangent behaviour is particularly nice, since their tangent spaces are

[3] The group of bisections has a natural diffeological structure.

[4] Beware that in those articles they write the composition of arrows in "diagrammatic order," whereas we adopt the standard notation for composition. So where we would write $f \circ g$ or fg to mean "f after g," they write gf.

[5] I propose the notation '$\check{T}$' of an inverted hat, pointing *inward*. [**CW14**] uses the notation 'T^{dvs}'.

1-representable. This means that every tangent vector can be represented by the velocity of a curve in the underlying space, something that is true for smooth manifolds but not for arbitrary diffeological spaces. I was hoping to use this to simplify calculations. (This already excludes spaces like the cross in the plane.) This class is the same as the *L-type* ones introduced by Leslie in [**Les03**]. Note that Laubinger's PhD thesis [**Lau06**] also defines L-type spaces, but not correctly. Again, we need to ensure that the tangent functor $\check{T}$ is an endofunctor, which in this case would amount to $\check{T}X$ being weakly filtered whenever X is. [**Lau06**] attempts to prove this, but does not seem to be correct.

The recently published [**ADN20**] offers a new perspective of the internal tangent bundle as a *section of a cosheaf*. This potentially allows us to extend the internal tangent bundle to a broader class of sheaves, since this construction is independent on the underlying set of points of a diffeological space. We do not know if this point of view can help answer the question about the existence of the internal tangent structure.

As I understand it now, the fundamental obstruction to constructing an *internal tangent structure* seems to lie in the failure of commutation of limits and colimits. On the one hand, we have the internal tangent spaces, which are defined as *co*limits. On the other hand, the definition of a tangent structure requires the tangent functor $\check{T}$ to preserve its own pullbacks. Therefore, for the internal tangent bundle functor to satisfy this demand, we need to have a commutation of these specific limits and colimits. The question of commuting limits and colimits is complicated, and it seems that there is no straightforward solution to the problem of when they do or do not commute. For lack of an answer to this problem, there seem to be two routes to take: either we modify the definition of a tangent structure, or we need to find an alternative notion of *tangency* on diffeological spaces, that is either not defined in terms of colimits, or is better behaved[6].

Around February of 2020, I put forward a suggestion to Klaas Landsman to study a Hilsum-Skandalis category of diffeological groupoids. From that meeting onwards, I started studying diffeological groupoids and bibundles instead, the result of which is this thesis!

[6] Actually, there is a *third* route: we abandon *diffeology*, and look for a notion of space that better supports a notion of tangency. A fundamental reason that the internal tangent space is defined as a colimit seems to be the fact that a diffeology is defined using the *"maps in"* approach, i.e., a diffeology is defined in terms of maps defined on Euclidean domains $U \subseteq \mathbb{R}^m$ *into* a set X. This means that the tangent space structure of the Euclidean domains has to be *pushed forward* onto X, and hence the colimit. There are other notions of generalised smooth spaces, some of which we will discuss in Section **1.1**, that take the *"maps out"* approach. For these, to transfer the tangent space structure of Euclidean spaces, we need to *pull back*, hence a *limit*, instead of a colimit. Since limits commute with limits, the fundamental obstruction described above would vanish.

It is with pleasure and humility that I can finally present this thesis. Writing it has been a labour of love for me, even (perhaps especially) under lockdown during the Sars-CoV-2 pandemic, and I hope this finds you in good health. I am satisfied with it to a good extent; there is more I would have liked to add, clarify, polish, prove, confirm, apply... But I have learned a lot; about what it might feel like to do research, about my own interests in mathematics, about diffeology, about how to approach the writing process, about all of the things I wish I could have done differently, and about the many things I wish I knew more about! In any case, I hope that you will find something useful down there. We present motivation for the main subject in Section **1.2.2**. An extensive outline of the contents and our results can be found in the bird's eye view **1.2.3**, at the end of Chapter **I**. Enjoy!

Acknowledgements. This thesis has benefited from help, tips, and hints from various people during various stages of the project. This includes many people who were kind and patient enough to respond to my (repeated) questions through email: Dan Christensen, Robin Cockett, Jonathan Gallagher, Patrick Iglesias-Zemmour, and Seth Wolberts.

Thanks are also due to Ioan Mărcuț. First: for being my first teacher in differential topology, and second: for being the second reader.

Special thanks to Walter van Suijlekom, who guided me through a project that—to a surprising extent—turned out to be related to this thesis, and for letting me speak at his noncommutative geometry reading seminar on several occasions.

I am grateful to Klaas Landsman, who supervised this project. I thank him for the advice he has given me, helping me navigate the various twisted roads and ideas that traverse this mathematical landscape. Without this advice I would have surely trodden off the path more than I already have. I greatly enjoyed having the freedom to give my own direction to this project, I hope that it turned out alright. I also thank him for thorough feedback on the manuscript, which lead to many improvements.

Another special thanks goes out to Jan Głowacki. Not only did this project grow out of a motivation that is based on his thesis [**Gł19**], but also for the many conversations we had—on mathematics and otherwise—and for all the fun we had sparring in front of a blackboard.

Lastly, I want to thank my fellow student colleagues. Thanks to both generations of students in the *master room,* who made my time there very enjoyable. I miss our weekly *MasterMath* day trips to Amsterdam and Utrecht where we learned all sorts of cool things! A special thanks to the original group from Eindhoven. Together we made the jump to Radboud, and I'm very grateful that we did. I have fond memories of all those physics lectures from all those years ago. Those were the days! An extra special thanks to Derryk for hogging the coffee machine with me.

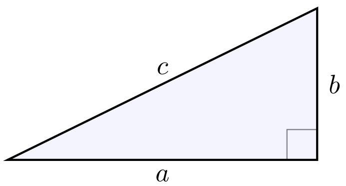
Final thanks to Emmily for asking me about quotient stacks.

This thesis was typeset using XƎLaTeX, based on the MiKTeX repository, and edited in TeXstudio. The title page uses the EB Garamond font, and is based on the `rutitlepage` package. The figures in Chapter **IV** were programmed in Ti*k*Z using `TikZEdt`. Please contact me at my email address:
`nestavanderschaaf@gmail.com` or `n.schaaf@student.ru.nl`.

Contents

Chapter I
Introduction

Geometry (from the Ancient Greek *ge(o)-*, "earth," and *-metria*, "measuring"), the study of *distance*, *angles* and *size*, has existed since ancient times. One of the most famous mathematical theorems, the *Pythagorean Theorem*, known since at least c. 500 BC, is a geometrical one:



$$a^2 + b^2 = c^2.$$

Some two hundred years later, Euclid published *The Elements*, c. 300 BC, containing the first axiomatisation of geometry, and in particular of the notion of *"space."* It is fair to say that, with his work, Euclid ushered in an age of mathematical thinking with axioms and rigorous proofs that still sets the standard to this day. His was a logical, *synthetic* approach to geometric reasoning. Two millennia later, in the first half of the 17th century, René Descartes introduced the *analytic* method to geometry: that of reasoning with *coordinate systems*, which now permeates modern mathematics and physics. It is to them we pay tribute when using the words *Euclidean space* and *Cartesian coordinate system*.

Over in the 19th century, we find the start of the story of the first modern conception of a *manifold* in Riemann's Habilitationsschrift [**Rie54**]. The modern definition of an arbitrary-dimensional manifold in terms of an atlas is credited to Whitney [**Whi36**], although earlier modern forms occurred for Riemann surfaces sometime earlier in the work of Weyl [**Wey14**]. Other sources credit [**WV32**]. Of course, the actual turn of events is far more gradual as the definition of a manifold did not spring out of nowhere. Historical accounts are in [**Die09**; **Fré18**; **Sch99**].

1.1 A panoramic view of the landscape of generalised smooth spaces

Let us jump now to the mid-20th century, where our story truly starts. The concept of *manifold* was now starting to be well-understood, supported by rigorous foundations of analysis and topology. Cartan defined notions of fibre bundles with connections, capturing both Riemann's ideas of curved spaces, and Klein's homogeneous spaces. The ideas of Chern subsequently elevated Cartan's work to global intrinsic geometry, leading to the modern way of thinking in differential geometry [**Yau06**].

But that is not the end of the story. There was more to be discovered, beyond the realm of manifolds. In the following we would like to sketch a brief history of these developments. These remarks must be taken with a grain of salt, since the author was neither there when they happened, nor is expert enough to understand their origins. The outline we describe below is based on various snippets and remarks of other authors. These are mainly: [**BH11**; **BIKW17**; **IZ13**; **IZ17**; **LS86**; **Wik20**], but other online resources such as the nLab and The n-Category Café have also been helpful.

1.1.1 A need for new spaces in geometry

In the mid-20th century, algebraic topologists started looking for alternatives to the category **Top** of topological spaces and continuous maps. This was no doubt in part motivated by the growing influence of *category theory* since the early 1940s. Under this pressure, the categorical properties of **Top** were pushed into the spotlight, and its cracks were showing. People were finding **Top** to be an 'inconvenient' category in which to do topology. One of the main reasons for this was that there is no canonical topology on the spaces of continuous functions, and **Top** is therefore not *Cartesian closed*[7] (Definition **A.5**). We recall that, if a category **C** is Cartesian closed, this means that for any two objects $A, Y \in \mathrm{ob}(\mathbf{C})$, there

[7] Which unfortunately means that $\mathbf{Top} \notin \mathbf{Topos}$.

is a third object $Y^A \in \mathrm{ob}(\mathbf{C})$ called the *exponent*, which is typically the collection of all arrows $A \to Y$, such that there is a natural bijection between two types of arrows:

$$\frac{X \longrightarrow Y^A}{X \times A \longrightarrow Y}.$$

In **Top**, Y^A is (or rather, should be) the function space $C(A, Y)$ of all continuous functions $A \to Y$. But there is no canonical topology on such a space, generally, so Y^A lies outside of the category. This is quite unsatisfactory when doing, for instance, homotopy theory, where one is studying exactly those kinds of spaces. In order to study the homotopy of a topological space, one had to, in a sense, step *out* of **Top** to perform any serious study in the first place, which was considered a flaw. Certainly as bare sets the spaces of continuous functions seemed to lose some of the topological information that was there. Besides that, it would not be possible to study the homotopy of function spaces themselves, such as loop spaces. These types of concerns were voiced by Brown [**Bro63**; **Bro64**], and later also more explicitly by Steenrod [**Ste67**], who came up with a notion of a *convenient category* of topological spaces. Such a category of spaces should be Cartesian closed, and closed under several other natural categorical operations. These concerns can be summarised by the quote [**Mac71**, Section VII.8]:

> *"All told, this suggests that in* **Top** *we have been studying the wrong mathematical objects. The right ones are the spaces in* **CGHaus**[8]*."*

Grothendieck faced similar problems in algebraic geometry. In the late 1950s he introduced *schemes*, first published in his famous *Éléments de Géométrie Algébrique* [**Gro60**], to generalise algebraic varieties, in part motivated to provide an encompassing framework in which to solve the Weil conjectures, but since providing the foundations of modern algebraic geometry. His approach was fundamentally different from the one described in the quote above. Whereas the algebraic topology-motivated approach by Brown, Steenrod, and others, was to *reduce* the category **Top** to a subcategory with nicer spaces, Grothendieck's approach was exactly the opposite: he introduced a *larger* class of spaces, flexible enough to encapsulate all of the desired categorical constructions. In doing so, one inevitably ends up with spaces that appear pathological from the viewpoint of the old class of spaces. There is a loss of structure, in a sense[9]. But this loss of structure is sometimes worth the pay-off, and can even lead to more fundamental insights, as was certainly the case for Grothendieck's approach. The success of schemes over varieties led to the following motto[10]:

> *"It's better to have a good category with bad objects than a bad category with good objects."*

Starting in the 1960s, these ideas started spreading to differential geometry as well. The categorical influence on differential topology and -geometry started already in the decade prior with the work of Charles Ehresmann [**Ehr59**], who introduced topological- and Lie groupoids, and more generally, the idea of *internalisation* (see also [**Pra07**; **Ehr07**]). This is a general concept where mathematical structures can be defined inside of a specified category[11]. This idea allowed for the analysis of smooth categorical structures, and led to the application of category-theoretic techniques into differential geometry (and to that end, Ehresmann established the journal *Cahiers de Topologie et Géométrie Différentielle Catégoriques* in 1957). Being the foundation for modern differential geometry, **Mnfd** was consequently pushed into the categorical spotlight as well, right next to **Top**, and it also had some cracks to show. The category **Mnfd** of finite-dimensional (second countable Hausdorff) smooth manifolds is generally even *worse* behaved than **Top**, since it is not even (co)complete. It is not Cartesian closed, either. Although (to be fair), it is possible to define on a function space $C^\infty(M, N)$ a structure of an *infinite*-dimensional manifold (such as Hilbert-, Banach- or Fréchet manifolds, or the infinite-dimensional manifolds of [**KM97**]), but then the spaces of smooth maps on *those* become ever more hard to work

[8] The category of compactly generated Hausdorff spaces.

[9] One can liken this to the concept of *strength* of a formal set of axioms. The more axioms you define, the more restricted your theory becomes, but the more theorems you can prove. In a theory with fewer axioms, the less you can prove, but the more objects you allow. A perfect example of this is the distinction between groups and *abelian* groups.

[10] This quote is sometimes attributed to Grothendieck himself, but there seems to be no concrete source, see [**Mos13**].

[11] For instance, a group can be internalised into the category **Top** of topological spaces, which gives the notion of a topological group; a group internal to the category **Mnfd** of smooth manifolds is a Lie group, etc.

with, and are still not always Cartesian closed. But there are other problems to speak of, the main one of which many would consider to be *singular quotients*. It is not hard to come up with an example of an equivalence relation on a manifold whose quotient space has no canonical smooth structure. And yet quotients appear as important constructions, such as orbit spaces of group actions, leaf spaces of foliations, or fibres of a bundle. Yet a third shortcoming is the existence of smooth structures on subspaces of manifolds. Generally the set of points in a space where two smooth functions coincide does itself not have a canonical smooth structure (i.e., **Mnfd** does not have equalisers). A simple example of this fact is that the cross $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$ in $\mathbb{R}^2$ is not a smooth manifold. This prevents us from constructing spaces such as pullbacks, which are particularly important to define a smooth version of the notion of a groupoid[12]. We therefore identify three main needs:

1. Infinite-dimensional spaces (Cartesian closure);
2. Quotient spaces (colimits);
3. Subspaces (limits).

All of this is captured in the following quote by Stacey [**Sta10a**]:

> *"Manifolds are fantastic spaces. It's a pity that there aren't more of them."*

In a most general sense, we can phrase the problem as follows:

Wishlist 1.1. We want a (co)complete Cartesian closed category **Spaces** of *"nice smooth spaces,"* such that there exists a canonical fully faithful embedding $\mathbf{Mnfd} \hookrightarrow \mathbf{Spaces}$ of finite-dimensional smooth manifolds.

In the rest of this section, let us discuss some of the responses to this need for new spaces in geometry. In our recollection of the story, we will distinguish between two main schools. The first is the family of approaches just mentioned, largely inspired by the work of Lawvere (and to some extent Grothendieck), resulting in synthetic differential geometry and related theories, which we might dub the *categorical generalisations*. The others can collectively be termed as the *set-based generalisations* of differential topology, which includes the approaches by Sikorski [**Sik67**], Chen [**Che77**], and of course: diffeology [**Sou80**]. Although, the distinction between these two categories might not be as sharp as here portrayed, for the purposes of exposition this will be a useful separation. Our focus here will definitively be on the set-based approaches, although we would like to take a detour through the ideas of Lawvere first.

Why Cartesian closedness? Perhaps the decisive spark, where the *"good categories over good spaces"* idea was inserted firmly into the field of differential geometry, happened during Lawvere's 1967 *Chicago lectures*, titled *Categorical Dynamics* [**Law67**], in which he outlined a programme for the axiomatisation and formalisation of geometry using category theory, leading to the field that is now known as *synthetic differential geometry*. These ideas have gained much esteem since then, and many people have taken up the task of developing them. A modern textbook account is [**Koc06**].

One of Lawvere's critical insights was that it is not just the specific incarnation of the spaces themselves that is important, but rather the properties of the collective *algebra of spaces*. By that we mean not the properties of function algebras such as $L^2(X)$, $C(X)$ or $C^\infty(X)$, but rather the laws of mappings between the spaces themselves; their *categorical* properties. In the introduction to [**Koc06**], Kock quotes Lawvere:

> *"In order to treat mathematically the decisive abstract general relations of physics, it is necessary that the mathematical world picture involve a cartesian closed category $\mathcal{E}$ of smooth morphisms between smooth spaces."* [**Law80**, Section 1]

[12] And we will see in Chapter **IV** that the definition of a Lie groupoid reflects this fact.

Why does Lawvere so ardently insist on a category that is Cartesian closed? In the introduction of [**LS86**], he gives an elementary physical argument, which we shall here adopt. For a lighter introduction we refer to [**LS09**, Session 30.2].

Suppose that we want to describe, in a most general sense, the movement of a physical body in space and time. There will be three main ingredients: a space E describing the spatial dimensions of our system, a space I describing the temporal interval, and a space B representing the physical body itself, all three of which we suppose live in a category **Spaces**. To each point (let us assume there are points) $b \in B$ of the physical body, and to each point $t \in I$ in time, the motion associates a point $p(b,t) \in E$ in space. In all, this is represented by an arrow

$$B \times I \xrightarrow{p} E.$$

Now, if there is some scalar quantity $E \to R$ (where R is to be thought of as a *real numbers object*), then the composition $B \times I \to E \to R$ describes the dynamics of that quantity along the evolution of the motion.

On the other hand, the *centre of mass* no longer depends on a single given point $b \in B$, but rather on the configuration of the body as a whole. The space of all physical configurations (even those physically impossible ones) is E^B, representing the collection of all arrows $B \to E$. That we can associate to each physical configuration a specified centre of mass is represented by the existence of an arrow (generally obtained by integral calculus)

$$E^B \xrightarrow{\text{integration}} E.$$

To calculate the evolution of the centre of mass corresponding to the motion, the arrow p needs to be reinterpreted as an arrow

$$I \longrightarrow E^B,$$

which to every point in time $t \in I$ associates the physical configuration $p(-,t) : B \to E$, describing the movement of the body as a whole. The evolution of the centre of mass is then determined by the composition $I \to E^B \to E$.

Lastly, we have a space E^I of allowed (read: smooth) "paths" traversing E. Generally, to any path in E^I we can by way of differentiation obtain its velocity as a path in $(TE)^I$, where TE is an abstract tangent bundle. This is represented by the existence of an arrow

$$E^I \xrightarrow{\text{differentiation}} (TE)^I.$$

In order to calculate the velocity of the motion, we should be able to determine the path that each point $b \in B$ traces in space as parametrised by time. Again, a point $(b,t) \in B \times I$ does not contain the information to determine this, since the velocity depends rather on an interval of time, and not on a snapshot. The two equivalent descriptions $B \times I \to E$ and $I \to E^B$ of p are therefore of little use here. Instead, we need to describe the motion as yet a third arrow

$$B \longrightarrow E^I,$$

associating to each point $b \in B$ the path $p(b,-) : I \to E$ in space. The velocity of the body traversing this movement is then encoded in the composition $B \to E^I \to (TE)^I$.

Thus it seems there is, from elementary considerations, necessarily a conceptual equivalence between the following three sorts of arrows:

- $B \times I \to E$ for quantities that depend only on the positions of parts of the body.
- $I \to E^B$ for quantities that depend on the configuration of the body as a whole.
- $B \to E^I$ for quantities that depend on the movement of (parts of) the body.

[**Law80**] gives concrete examples of physical quantities for each of these three descriptions of a physical motion. It is clear that none of these three arrows contain less or more information than the others, yet they are each necessary in their own right for certain calculations. In other words, **Spaces** should be *Cartesian closed*. Lawvere [**LS86**] argues that this ability to interchange between these realisations of p is

> *"[...]obviously more fundamental for phrasing general axioms and concepts of continuum physics than is the precise determination of the concepts of spaces-in-general (of which E, $[I]$, B are to be examples), yet these transformations are not possible for the commonest such determinations (for example Banach manifold)."*

Taking Cartesian closedness as a fundamental axiom, it is not much of a further leap to arrive at *toposes*. The central objects in synthetic differential geometry are then *smooth toposes*: universes of generalised smooth spaces, containing a distinguished object that behaves like the real number line, in which one can reason rigorously about geometry using infinitesimals. In this abstract theory, it begs the question what types of spaces actually fit the mould of synthetic differential geometry. Many examples of this can be found in the book [**MR91**].

Synthetic differential geometry does not deal per se with objects that are of geometric origin. Besides, the complicated topos-theoretic framework could be distracting to those who just want to focus on geometry. There is a need to stay close to the intuitions of classical differential topology, and yet to be able to deal with infinite-dimensional objects and singular quotients *as if they were smooth manifolds* (and not *as if they were objects in a topos*). This leads us to:

The set-based theories. What we mean by *set-based generalisation* is, roughly, a theory of generalised smooth spaces that relies on putting some form of *smooth structure* on a bare *set*. (This may or may not prerequire a topological structure.) We do this to distinguish these ideas from synthetic differential geometry, and other theories such as the categorical approaches using stacks [**BX11**], the algebro-geometric approaches of C^∞-schemes [**Joy12**], the categorical atlas approach [**Los94**], noncommutative geometry [**Con94**], the sheaf approach of 'abstract differential geometry' [**Mal98**; **AE15**], or the (higher) sheaf-theoretic approaches [**Sch20**] (which we discuss in Section **1.1.3**). Whereas the synthetic approach focuses on the bare axiomatics of differential geometry, in which in principle the particular definition of space is irrelevant, the set-based approaches start with explicit definitions of smooth structure, each in its own way trying to provide an answer to Wishlist **1.1**. As Stacey points out in the introduction to *Comparative Smootheology* [**Sta11**], each of the main set-based approaches was developed with a specific goal in mind. This distinguishes them from Lawvere's approach, which was to determine the axiomatic underpinnings of *all* differential geometry. As for the set-based theories, we can interpret them as trying to push the boundaries of classical differential geometry, to see which assumptions for fundamental geometric theorems are essential or not, and in which ways they can be extended. One of the first approaches in the set-based style seems to be [**Smi66**]. Smith's approach is an investigation to what extent the de Rham Theorem can be generalised beyond the scope of finite-dimensional smooth manifolds. The connection between the two schools is that the need for well-behaved infinite-dimensional spaces often amounts to the need for Cartesian closure. In that sense Lawvere's philosophy described above is not lost here. We identify the following five set-based smooth theories, listed in approximate chronological order of publication:

- Smith spaces [**Smi66**],
- Sikorski spaces [**Sik67**; **Sik71**],
- Chen spaces [**Che73**; **Che75**; **Che77**],
- Diffeological spaces [**Sou80**; **Sou84**],
- Frölicher spaces [**Frö82**; **FK88**].

What is their general idea, and how do we arrive at their technical definitions? The fundamental philosophy here is a categorical idea in disguise: that objects are characterised by the morphisms going in- and out of it. An elementary example of this idea is the celebrated *Yoneda Lemma*. If nothing more, then, a smooth structure should serve exactly as to determine which functions are smooth, and which ones are not.

The structure of a smooth manifold M determines which maps $\mathbb{R}^k \to M$ or $M \to \mathbb{R}^k$ are smooth. As it is well-known that a smooth map with multiple components is smooth if and only if each of its components is smooth, to distinguish the latter kind of map it suffices to determine the algebra of real-valued smooth functions[13] $C^\infty(M,\mathbb{R})$. The maps $\mathbb{R}^k \to M$ are actually also determined by the 1-dimensional smooth curves, due to the less well-known theorem by Boman:

Theorem 1.2 ([**Bom68**, Theorem 1])**.** *If a function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies that $f \circ u \in C^\infty(\mathbb{R})$ for all smooth curves $u : \mathbb{R} \to \mathbb{R}^n$, then f itself is smooth.*

So, together with Boman's Theorem, we have the following three equivalent characterisations of smooth functions [**BIKW17**, Section 1]:

Lemma 1.3. *A function $f : U \to V$ between open subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ is smooth if and only if at least one (and hence all) of the following three equivalent conditions are satisfied:*

1. *For each open subset $W \subseteq \mathbb{R}^k$ and smooth $h : W \to U$, the composition $f \circ h : W \to V$ is smooth.*
2. *For every real-valued smooth function $g \in C^\infty(V,\mathbb{R})$, the composition $g \circ f : U \to \mathbb{R}$ is smooth.*
3. *For every smooth curve $\gamma : \mathbb{R} \to U$, the composition $f \circ \gamma : \mathbb{R} \to V$ is smooth.*

This lemma tell us that the smoothness of the function $f : U \to V$ can be *probed* by smooth curves going in- and out of the domain and codomain. In other words, for the class of spaces that are of the form $U \subseteq \mathbb{R}^m$, the smoothness of maps defined on it are equivalently captured by its algebra of real-valued smooth functions $C^\infty(U,\mathbb{R})$, as by the space of smooth curves $C^\infty(\mathbb{R},U)$, as well as by the space of smooth functions $C^\infty(W,U)$, where W is allowed to range over all open subsets of the spaces $\mathbb{R}^k$ of varying dimension.

The fundamental idea is then this: to enlarge the class of objects $U \subseteq \mathbb{R}^m$ to an arbitrary set X, on which the smoothness of its functions are determined by three types of objects: real-valued functions, curves, and higher-dimensional curves. This also presents a fundamental shift compared to the way smoothness is typically defined for manifolds, where the smooth structure *determines* which functions are smooth (being its principal purpose), whereas here the idea is that a smooth structure is *determined by* exactly which functions are smooth. More precisely, this gives rise to three types of new structures:

- The first condition Lemma **1.3**(*1*) says that the smooth structure of U is determined by the spaces $C^\infty(W,U)$, of smooth functions on open subsets of $\mathbb{R}^k$ into it. On an arbitrary set X, we can then define a notion of smoothness by furnishing X with a family $\mathcal{D}_X$, containing exactly those functions of the form $W \to X$ which we deem to be *smooth*, subject to some natural consistency axioms. The notion of *smoothness* for a function $X \to Y$ between two such spaces is then almost verbatim as described in the first condition Lemma **1.3**(*1*): if and only if for all those maps $W \to X$ in the family $\mathcal{D}_X$, the composition $W \to X \to Y$ is 'smooth', meaning to be an element of $\mathcal{D}_Y$. This is exactly the idea of *diffeology*.
- The second condition Lemma **1.3**(*2*) says that the smooth structure of V is determined by the algebra $C^\infty(V,\mathbb{R})$ of real-valued smooth functions, and this fully determines which functions $U \to V$ are smooth. On an arbitrary set X we can then define a notion of smoothness by declaring a family $\mathcal{F}_X$ of exactly those real-valued functions $X \to \mathbb{R}$ that are supposed to be *smooth* (again, subject to some consistency axioms). And again, the notion of *smoothness* for a function $X \to Y$ between two such spaces can be copied from the lemma: if and only if for all $Y \to \mathbb{R}$ in $\mathcal{F}_Y$, the composition $X \to Y \to \mathbb{R}$ is in $\mathcal{F}_X$. This is exactly the idea behind *Sikorski spaces*.
- The third condition Lemma **1.3**(*3*), which exists because of Boman's Theorem **1.2**, says that the smooth structure of U is determined by its smooth curves $C^\infty(\mathbb{R},U)$. Combining this with the second condition Lemma **1.3**(*2*), the smooth structure of U is determined by the two spaces $C^\infty(\mathbb{R},U)$ and $C^\infty(U,\mathbb{R})$. We can define a notion of smoothness on an arbitrary set X by equipping it with two families of functions: one family $\mathcal{C}_X$ of would-be *smooth curves* $\mathbb{R} \to X$ of, and another family $\mathcal{F}_X$ of would-be *smooth real-valued functions*. This is the idea behind the notion of a *Frölicher space*.

[13] In fact, a smooth manifold M is already characterised by its algebra $C^\infty(M,\mathbb{R})$ of real-valued smooth functions (and some authors take this as a starting point to develop the theory [**Nes03**]).

The surprising result is that when one adopts one of these three new types of structures, one finds that they are much more general than manifolds. That is to say that there are spaces X that have, to go with the Sikorski structures as an example, an algebra of real-valued smooth functions $\mathcal{F}_X$ that *behaves like* the algebra of real-valued functions $C^\infty(M, \mathbb{R})$ on a smooth manifold, but does not have to be quite precisely a smooth manifold itself. This newfound generality comes from the fact that we now allow spaces X for which these different types of structures $\mathcal{D}_X$ and $\mathcal{F}_X$ no longer need to determine to each other. In a sense, it is a *letting-go* of Lemma **1.3**.

To summarise: the idea is that the smooth structure of a space X is simultaneously captured and defined by the smooth maps *into* it, and those *out of* it. In a broader setting, this leaves some room for interpretation to answer: maps *from where*, and *into what*? The general answer is that these should be the *model spaces* (or *test spaces*), spaces which have a definitive canonical smooth structure, which we want to transfer to X. For differential topology these model spaces are *Euclidean domains*: the open subsets $U \subseteq \mathbb{R}^n$, or some close variant thereof. Simply put, a smooth structure on X is determined by which smooth arrows in the following diagram are allowed to exist:

$$\begin{array}{ccc} \mathbb{R} & \searrow & \\ & & X \longrightarrow \mathbb{R}. \\ \mathbb{R}^m \supseteq U & \nearrow & \end{array}$$

This is a modern version of the Cartesian analytic method of describing a space by coordinates. Each of the main theories can be seen as a specific way of filling in the types of subsets $U \subseteq \mathbb{R}^m$ and types of arrows in this diagram. We see that diffeology takes the *"maps in"* approach, Sikorski took the *"maps out"* approach, and Frölicher spaces are defined in terms of a combination of the two. The detailed paper [**Sta11**] that we just mentioned describes an overarching framework in which all of the set-based theories can be unified through this idea of *test spaces.* Stacey provides concrete comparisons between the categories in terms of adjunctions. The paper [**BIKW17**] shows that Frölicher spaces are an 'intersection' (in a technical sense) of diffeological- and Sikorski spaces. In the next section we shall only describe a qualitative and conceptual comparison between the three approaches we have found above: Sikorski-, diffeological-, and Frölicher spaces. We refer the reader to the two previously mentioned papers for more details (also on Smith- and Chen spaces). Neither will we discuss the *S-manifolds* of Van Est [**vEs84**], the *V-manifolds* (now known as *orbifolds*) of Satake [**Sat56**], or *subcartesian spaces* [**Aro67**; **AS80**], since they live on a lower level of generality.

1.1.2 Conceptual Smootheology[14]

In this section we give a brief statement of the definition of Sikorski-, diffeological-, and Frölicher structures. The main reference is [**BIKW17**]. After stating the main definitions, we discuss some of their conceptual differences through examples.

In the late 1960s Sikorski introduced his *"differential structures"* [**Sik67**; **Sik71**] (what we shall call *Sikorski structures*). His notion followed from the observation that many of the properties of a smooth manifold are captured by its ring of smooth real-valued functions. When pushed much further, this idea evolves into the study of C^∞-schemes [**Joy12**]. The modern textbook account for Sikorski spaces is [**Śn13**].

Definition 1.4. A *Sikorski space* (also known in the literature as a *differentiable space*) is a topological space (X, τ) together with a non-empty family $\mathcal{F}$ of real-valued functions on X, called a *Sikorski structure*, such that:

1. *(Topological Compatibility)* The topology τ is the initial topology generated by the members of $\mathcal{F}$.

2. *(Smooth Compatibility)* If $f_1, \ldots, f_k \in \mathcal{F}$ and $F \in C^\infty(\mathbb{R}^k)$, then $F \circ (f_1, \ldots, f_k) \in \mathcal{F}$.

3. *(Locality)* If $f : X \to \mathbb{R}$ is a function such that, for every point $x \in X$, we can find an open neighbourhood $x \in U \subseteq X$ and an element $g \in \mathcal{F}$ such that $g|_U = f|_U$, then $f \in \mathcal{F}$.

[14]Cf. *Comparative Smootheology*, [**Sta11**].

We are to think of the family $\mathcal{F}$ as the space $C^\infty(X)$ of would-be smooth real-valued functions on X. Since the topology τ is determined by $\mathcal{F}$, a Sikorski space can equivalently be defined as a pair $(X, \mathcal{F}_X)$, consisting of a bare set X with a Sikorski structure $\mathcal{F}_X$ satisfying the second and third axioms, and equipping X with the initial topology generated by $\mathcal{F}_X$.

A function $F : (X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ between Sikorski spaces is called *Sikorski smooth* if for all $f \in \mathcal{F}_Y$ we have $f \circ F \in \mathcal{F}_X$. We shall denote the collection of all Sikorski smooth maps between X and Y by $C^\infty_{\text{Sik}}(X, Y)$. Equipping $\mathbb{R}$ with its standard Sikorski structure $\mathcal{F}_\mathbb{R} := C^\infty(\mathbb{R})$, we find that $\mathcal{F}_X = C^\infty_{\text{Sik}}(X)$, where the latter now includes smooth functions in this new sense.

In the early 1970s, Chen gave his first definition of a *"differentiable space"* (what we shall call *Chen spaces*) [**Che73**]. The definition was modified in [**Che75**], and the final definition (the one we state) was first published in [**Che77**, Definition 1.2.1]. Chen's motivation was to study the differential topology and cohomology of *loop spaces*, which lie outside of the reach of finite-dimensional manifolds.

Definition 1.5. A *Chen space* is a pair $(X, \mathcal{P})$, where X is a set and $\mathcal{P}$ is a family of maps into X defined on convex subsets of Euclidean spaces, such that:

1. *(Covering)* Every constant map $C \to X$ is in $\mathcal{P}$.
2. *(Smooth Compatibility)* If $\varphi : C \to X$ is in $\mathcal{P}$ and $h : D \to C$ is a smooth map between convex subsets in the usual sense, then $\varphi \circ h \in \mathcal{P}$.
3. *(Locality)* If $\varphi : C \to X$ is a function defined on a convex domain such that there is an open cover $(C_i)_{i \in I}$ of C, where each C_i is convex, and such that each restriction $\varphi|_{C_i}$ is in $\mathcal{P}$, then $\varphi \in \mathcal{P}$.

A function $f : (X, \mathcal{P}_X) \to (Y, \mathcal{P}_Y)$ between Chen spaces is called *Chen smooth* if we have $f \circ \varphi \in \mathcal{P}_Y$ for all $\varphi \in \mathcal{P}_X$.

We state Chen's definition because of its remarkable resemblance to the definition of a *diffeology*. The first published definition of a diffeological space is in [**Sou84**], although the concept had been defined four years before for groups in [**Sou80**]. Souriau's motivation was to study infinite-dimensional symplectomorphism groups in symplectic geometry, general relativity, and geometric quantisation. The definition of a *diffeological space* (Definition **2.2**) can be stated almost *verbatim* as that of a Chen space, replacing the word *"convex"* with *"open."* For the subtle differences between diffeology and Chen spaces we refer to the remarks in [**Sta11**, Section 6]. All of the proper terminology around diffeology will be introduced in Chapter **II**. The textbook account is [***Diffeology***].

Definition 1.6. Let X be a set. A *diffeology* on X is a collection $\mathcal{D}$ of functions $U \to X$, defined on Euclidean domains, satisfying the three axioms of a Chen space in Definition **1.5**, replacing the convex subsets with open subsets. The elements $\alpha \in \mathcal{D}$ of a diffeology are called *plots*, to distinguish them from arbitrary functions. A set X, paired with a diffeology $(X, \mathcal{D}_X)$, is called a *diffeological space*.

If we have a function $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ between diffeological spaces, we say it is *diffeologically smooth* if for all $\alpha \in \mathcal{D}_X$ we have that $f \circ \alpha \in \mathcal{D}_Y$, i.e., it sends plots to plots. We denote by $C^\infty_{\text{diff}}(X, Y)$ the space of all diffeologically smooth functions $X \to Y$. If we equip each Euclidean domain $U \subseteq \mathbb{R}^m$ with its natural diffeology, determined by all those smooth functions into it, then we can write $\mathcal{D}_X = \bigcup_U C^\infty_{\text{diff}}(U, X)$.

Out of these approaches, Frölicher spaces [**Frö82**] are closest to smooth manifolds. Frölicher spaces originated from a functional analytic angle, where Frölicher, Michor, Kriegl and others were motivated to develop a foundation for a Cartesian closed category of infinite-dimensional manifolds. This motivation started from the study of Banach manifolds, for which it was known that the smooth curves determine the smooth real-valued functions (thanks to a stronger version of Boman's Theorem). However, the category of Banach manifolds is not Cartesian closed. This led Frölicher and others to look for a more general framework in which Cartesian closedness could be achieved. This modern framework is essentially the one in [**KM97**]. Before we state the definition, we introduce some technical terminology:

Definition 1.7. Fix a set X, and let $\mathcal{F} \subseteq \mathrm{Hom}_{\mathbf{Set}}(X, \mathbb{R})$ be a family of real-valued functions on X. We then define

$$\Gamma\mathcal{F} := \{c : \mathbb{R} \longrightarrow X : \forall f \in \mathcal{F} : f \circ c \in C^\infty(\mathbb{R})\},$$

the collection of curves $\mathbb{R} \to X$ that compose with all elements of $\mathcal{F}$ into a smooth function on $\mathbb{R}$.

Let $\mathcal{C} \subseteq \mathrm{Hom}_{\mathbf{Set}}(\mathbb{R}, X)$ be a family of curves in X. We define

$$\Phi\mathcal{C} := \{f : X \longrightarrow \mathbb{R} : \forall c \in \mathcal{C} : f \circ c \in C^\infty(\mathbb{R})\},$$

the set of all real-valued functions that send the curves in $\mathcal{C}$ to smooth maps on $\mathbb{R}$.

Definition 1.8. A *Frölicher space* is a triple $(X, \mathcal{F}, \mathcal{C})$, where X is a set, $\mathcal{F}$ is a collection of real-valued functions on X, and $\mathcal{C}$ is a collection of functions of the form $\mathbb{R} \to X$, such that:

$$\Phi\mathcal{C} = \mathcal{F} \qquad \text{and} \qquad \Gamma\mathcal{F} = \mathcal{C}.$$

The compatibility of $\mathcal{F}$ and $\mathcal{C}$ ensures that there are three equivalent definitions of smoothness for functions $F : (X, \mathcal{F}_X, \mathcal{C}_X) \to (Y, \mathcal{F}_Y, \mathcal{C}_Y)$ between Frölicher spaces. These are:

1. For every $f \in \mathcal{F}_Y$ we have $f \circ F \in \mathcal{F}_X$.
2. For every $c \in \mathcal{C}_X$ and $f \in \mathcal{F}_Y$ we have $f \circ F \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$.
3. For every $c \in \mathcal{C}_X$ we have $F \circ c \in \mathcal{C}_Y$.

If F satisfies one, and hence all, of these conditions, then we call it *Frölicher smooth*.

Note how the three equivalent conditions for Frölicher smoothness resemble Lemma **1.3**. In that sense, one could say that Frölicher spaces capture most closely the smooth behaviour of manifolds. They are the most general notion of a smooth space for which Lemma **1.3** still holds.

Examples and counterexamples. As we have pointed out already, every smooth manifold can naturally be seen as an example of either of these generalised smooth spaces. It is the way they behave beyond the realm of manifolds that distinguishes them. Here are some of these examples, which can also be found in [**Sta11**; **Wat12**; ***Diffeology***; **BIKW17**]. The diffeological technology that underlies these examples will be developed in Chapter **II**.

The ideas behind these examples rely on the fact that each diffeology determines a Sikorski structure, and vice versa. Namely, equipping $\mathbb{R}$ with its natural diffeology, any diffeological space X gets a Sikorski structure defined by $C^\infty_{\text{diff}}(X)$, consisting of all diffeologically smooth real-valued functions on X. On the other hand, if X is a Sikorski space and we equip each open subset $U \subseteq \mathbb{R}^m$ with its natural Sikorski structure $C^\infty(U)$, then we get a diffeology $\mathcal{D}_X$, consisting of those Sikorski smooth functions $C^\infty_{\text{Sik}}(U, X)$. These two claims are proven in [**BIKW17**, Proposition 2.7]. It can be proved that ([**Wat12**, Lemmas 2.59 and 2.61]) these procedures respect quotients and subsets, respectively. In the following examples we will see that the two theories indeed behave differently with respect to these two constructions.

- On the Euclidean plane $\mathbb{R}^2$ we can define a diffeology consisting of those functions $U \to \mathbb{R}^2$ that factor locally through $\mathbb{R}$. Let us denote this space by $\mathbb{R}^2_{\text{wire}}$. This diffeology is called the *wire diffeology*, because every smooth map $U \to X$ looks locally like a wire. First of all, we note that this space is not diffeomorphic to ordinary Euclidean space, because the identity map $\mathrm{id}_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth, while $\mathrm{id}_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbb{R}^2_{\text{wire}}$ is not, since it cannot factor through $\mathbb{R}$. We construct the wire diffeology rigorously in Example **2.17**.

  The diffeology on $\mathbb{R}^2_{\text{wire}}$ determines a space of diffeologically smooth real-valued functions $\mathcal{F}_{\text{wire}} := C^\infty_{\text{diff}}(\mathbb{R}^2_{\text{wire}})$, which is a Sikorski structure on $\mathbb{R}^2$. Of course, it also has the standard Sikorski structure $\mathcal{F}_{\mathbb{R}^2} = C^\infty(\mathbb{R}^2)$. The standard Sikorski structure is actually contained in $\mathcal{F}_{\text{wire}}$, since the identity map $\mathrm{id}_{\mathbb{R}^2} : \mathbb{R}^2_{\text{wire}} \to \mathbb{R}^2$ is diffeologically smooth. What does it mean for a function f to be an element in $\mathcal{F}_{\text{wire}}$? It means that $f : \mathbb{R}^2_{\text{wire}} \to \mathbb{R}$ is diffeologically smooth, and by the definition of the wire diffeology, this just means that for every smooth curve $c : U \to \mathbb{R}$ the composition $f \circ c : U \to \mathbb{R}$ has to be smooth. But Boman's Theorem **1.2** then implies that $f : \mathbb{R}^2 \to \mathbb{R}$ has to be smooth! In other words, the Sikorski structure induced by the wire diffeology is just equal to the standard Sikorski structure on $\mathbb{R}^2$: $\mathcal{F}_{\text{wire}} = \mathcal{F}_{\mathbb{R}^2}$.

  This shows that there exists *non*-diffeomorphic diffeological structures ($\mathbb{R}^2 \not\cong \mathbb{R}^2_{\text{wire}}$), that are nevertheless indistinguishable as Sikorski- or Frölicher spaces.

- Any subset $A \subseteq X$ of a diffeological- or Sikorski space acquires a natural smooth structure itself. The diffeology $\mathcal{D}_{A\subseteq X}$ contains exactly those functions $U \to X$ in $\mathcal{D}_X$ whose image lies in A (Definition **2.51**). The Sikorski structure $\mathcal{F}_{A\subseteq X}$ contains those real-valued functions $g : A \to \mathbb{R}$ that locally extend to a smooth real-valued function on X, i.e., for every $x \in A$ there exists an open neighbourhood $x \in V \subseteq X$ and a smooth map $f \in \mathcal{F}$ such that $g|_{V\cap A} = f|_{V\cap A}$.

  Consider then the rational numbers $\mathbb{Q} \subseteq \mathbb{R}$. First, it gets the subset diffeology $\mathcal{D}_{\mathbb{Q}\subseteq\mathbb{R}}$. By the Intermediate Value Theorem, this diffeology only contains the plots that are locally constant. It follows that any real-valued function $\mathbb{Q} \to \mathbb{R}$ is diffeologically smooth, so the induced Sikorski structure is the trivial one: $\mathcal{F} = \mathrm{Hom}_{\mathbf{Set}}(\mathbb{Q}, \mathbb{R})$. If we alternatively start with the Sikorski subspace structure $\mathcal{F}_{\mathbb{Q}\subseteq\mathbb{R}}$, again the Intermediate Value Theorem says that each space $C^\infty_{\mathrm{Sik}}(U, \mathbb{Q})$ can only contain the locally constant functions, and therefore the diffeology that $\mathcal{F}_{\mathbb{Q}\subseteq\mathbb{R}}$ generates is equal to the subset diffeology $\mathcal{D}_{\mathbb{Q}\subseteq\mathbb{R}}$.

  This shows that there are *non*-diffeomorphic Sikorski structures ($\mathcal{F}_{\mathbb{Q}\subseteq\mathbb{R}} \neq \mathrm{Hom}_{\mathbf{Set}}(\mathbb{Q}, \mathbb{R})$), that define the same diffeological structure $\mathcal{D}_{\mathbb{Q}\subseteq\mathbb{R}}$.

- We can think of the subspace of $\mathbb{R}^2$ containing the coordinate axes in two ways. First, as literally a subset $E = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$, and the other as a wedge product $W = (\mathbb{R}_1 \sqcup \mathbb{R}_2)/\sim$, where $\sim$ identifies the two origins of the different copies of $\mathbb{R}$. The set W gets a natural diffeology of those maps $U \to W$ that factor through $\mathbb{R}_i \hookrightarrow \mathbb{R}_1 \sqcup \mathbb{R}_2 \to W$, where $i \in \{1, 2\}$ and the last map is the canonical quotient map. (Technically, this diffeology is a combination of quotient- and disjoint union diffeologies, cf. Section **2.2**.) On the other hand, E gets the natural Sikorski subset structure $\mathcal{F}_{E\subseteq\mathbb{R}^2}$. Each of these defines a notion of smooth curve and smooth real-valued functions on the respective spaces. In this way we get triples

  $$(W, \mathcal{C}_W, \mathcal{F}_W) := (W, C^\infty_{\mathrm{diff}}(\mathbb{R}, W), C^\infty_{\mathrm{diff}}(W)) \qquad \text{and} \qquad (E, \mathcal{C}_E, \mathcal{F}_E) := (E, C^\infty_{\mathrm{Sik}}(\mathbb{R}, E), \mathcal{F}_{E\subseteq\mathbb{R}^2}).$$

  Now, [**BIKW17**, Example 5.1] proves that, with this equipment, $(E, \mathcal{C}_E, \mathcal{F}_E)$ is a Frölicher space, while $(W, \mathcal{C}_W, \mathcal{F}_W)$ is *not*. (The proof resembles what we discuss in Example **2.18**.)

- An extraordinary example that singles out diffeology from the rest is the *irrational torus*. We discuss this example in great detail in Section **2.3**. The irrational torus can be defined as the quotient $T_\theta := \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z})$ of the additive subgroup $\mathbb{Z} + \theta\mathbb{Z} := \{n + \theta m : n, m \in \mathbb{Z}\}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is an irrational real number. It is well-known by *Kronecker's Density Theorem* (see e.g. [**BS06**]) that this is a dense subgroup, so the quotient topology on T_θ is trivial (Proposition **2.78**). The Topological Compatibility Axiom of a Sikorski structure then forces it to be trivial: it contains only the constant real-valued functions (cf. Proposition **2.80** for a more detailed proof). This also forces the Frölicher structure to be trivial. However, the diffeology of T_θ is decidedly *non*-trivial, which can be seen from its classification in Theorem **2.81**. In fact, the diffeology is so rich that it can distinguish whether θ is the solution to a quadratic equation with integer coefficients or not [**IZ17**]. Moreover, its smooth fundamental group is $\pi_1(T_\theta) \cong \mathbb{Z} \times \mathbb{Z}$, while its topological fundamental group is of course entirely trivial.

The last example shows that there are objects whose topology can be very poor (even trivial), while their smooth structure is very rich! This suggests also that a topology should not necessarily be a prerequirement for a definition of a smooth structure, challenging the philosophy of

"smooth object = topological object + extra structure,"

of which smooth manifolds are the perfect example. In terms of the comparison between the types of smooth structures, one takeaway is this: diffeology treats quotients well (even those with trivial topological structure), while Sikorski spaces seem to preserve more information of subspaces. For the use of Frölicher spaces we should emphasise the obvious upside of being closer to manifolds, making it so that there is much more structure to work with (and hence more to prove), while losing some freedom of generality. In the end, is up to the individual geometer and their goals to make a decision which path to choose.

Even though diffeology is able to reproduce the classification of noncommutative tori (Theorem **2.81**), we speculate that in other situations the Sikorski definition of smooth spaces may have implications in a noncommutative geometry sense as well, since it is so based around working with algebras of coordinate functions. However, so far, we are not aware of any work in this direction.

1.1.3 Smooth sets and beyond

From now on we shall focus our attention on *diffeology* (let us not forget the first word in the title of this thesis!). We will motivate the choice of diffeology more below, but before we do that, we take another small detour along a different path in the landscape of generalised smooth spaces. Looking at the definition of a diffeology above, it smells like something *sheafy* is going on. To see this, let us introduce the notation of **Eucl** to mean the category consisting of all Euclidean domains $U \subseteq \mathbb{R}^m$, and smooth maps between them (in the usual sense). Then each diffeological space $(X, \mathcal{D}_X)$ determines a *presheaf* $\overline{X} : \mathbf{Eucl}^{\mathrm{op}} \to \mathbf{Set}$, a *contravariant functor* from **Eucl** to **Set**, sending Euclidean domains $U \subseteq \mathbb{R}^m$ to $C^\infty_{\mathrm{diff}}(U, X)$, and with $\overline{X}(h) := - \circ h$. However, looking at the three axioms presented in Definitions **1.5** and **1.6**, there is clearly more structure present than just that of a presheaf. In fact, looking closely, we can recognise the *sheaf axiom*. We briefly recall the definition of sheaves on a *site* [**MM94**, Section III.2]:

Definition 1.9. Let $\mathbf{C}$ be a category with pullbacks. A *coverage* (also known as a *Grothendieck pretopology*) is a function Cov, assigning to every object $C \in \mathrm{ob}(\mathbf{C})$ a collection $\mathrm{Cov}(C)$ consisting of *covering families* $(f_i : U_i \to C)_{i \in I}$, satisfying the following axioms:

1. If $g : D \to C$ is an isomorphism, then $(g : D \to C) \in \mathrm{Cov}(C)$ is a covering family for C.
2. If $(f_i : U_i \to C)_{i\in I} \in \mathrm{Cov}(C)$ is a covering family for C, and $g : D \to C$ is an arrow, then the family of pullbacks $(\mathrm{pr}_2 : g^*U_i = U_i \times_U D \to D)_{i \in I} \in \mathrm{Cov}(D)$ is a covering family for D.
3. If $(f_i : U_i \to C)_{i\in I} \in \mathrm{Cov}(C)$ is a covering family for C, and for each $i \in I$ we have a covering family $(g_{ij} : V_{ij} \to U_i)_{j \in J_i} \in \mathrm{Cov}(U_i)$, then the compositions $(f_i \circ g_{ij} : V_{ij} \to C)_{i\in I, j \in J_i} \in \mathrm{Cov}(C)$ form a covering family for C.

A category $\mathbf{C}$ equipped with a coverage is called a *site*.

Definition 1.10. Consider now a presheaf $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ on a site $\mathbf{C}$, and a covering family $(f_i : U_i \to C)_{i \in I}$ of the object $C \in \mathrm{ob}(\mathbf{C})$. A *compatible family* is a collection $(x_i)_{i\in I}$ of elements $x_i \in F(U_i)$, such that for every two arrows $a : D \to U_i$ and $b : D \to U_j$ satisfying $f_i \circ a = f_j \circ b$ we have $F(a)(x_i) = F(b)(x_j)$.

A presheaf F is called a *sheaf* if, for every compatible family $(x_i)_{i \in I}$ of a covering family $(f_i : U_i \to C)_{i\in I}$, there exists a *unique* element $x \in F(C)$ such that $F(f_i)(x) = x_i$ for all $i \in I$.

To summarise: a site is a category in which it makes sense to talk about the notion of a *cover* $(U_i)_{i\in I}$ for its objects C. A compatible family contains elements $x_i \in F(U_i)$ of the presheaf, such that when evaluated along compatible arrows they coincide. The *sheaf axiom* says that the elements of a compatible family can always be *"glued together"* into a unique element $x \in F(C)$. The prototypical example of a site is the *category of open sets* $\mathscr{O}(X)$ of a topological space (X, τ), consisting of the open subsets $U \in \tau$ as objects, and inclusions $V \hookrightarrow U$ of open subsets as arrows. On $\mathscr{O}(X)$ we have a canonical coverage, namely the one assigning to each open set $U \in \tau$ the collection $\mathrm{Cov}(U)$ of all open covers of U. The prototypical example of a sheaf is the functor $F : \mathscr{O}(X)^{\mathrm{op}} \to \mathbf{Set}$, sending each open set $U \in \tau$ to the space of continuous real-valued functions $F(U) := C(U, \mathbb{R})$. The fact that continuous functions $f : U \to \mathbb{R}$ are determined uniquely by their local behaviour, i.e. can be obtained through gluing of the restricted functions $f|_{U_i} : U_i \to \mathbb{R}$ along an open cover, ensures that F is a sheaf. A sheaf on a site is thus, roughly speaking, a means of gluing together consistent local data into a unique global object.

Let us now unpack what these notions mean in the case of the presheaf $\overline{X} : \mathbf{Eucl}^{\mathrm{op}} \to \mathbf{Set}$, corresponding to a diffeological space $(X, \mathcal{D}_X)$. First, the category **Eucl** gets a coverage Cov by assigning to each $U \subseteq \mathbb{R}^n$ the collection $\mathrm{Cov}(U)$ consisting of open covers of U, just as for the category of open subsets of a topological space. A covering family then consists of a collection $(f_i : U_i \hookrightarrow U)_{i\in I}$ of open subsets $U_i \subseteq U$, where each f_i is the smooth inclusion function, and such that $\bigcup_{i \in I} U_i = U$. With this coverage, **Eucl** is called the *Euclidean site*. To each element U_i of the open cover, the presheaf $\overline{X}$ associates the diffeologically smooth functions $\overline{X}(U_i) = C^\infty_{\text{diff}}(U_i, X)$. A compatible family $(x_i)_{i\in I}$ then consists of smooth functions $x_i : U_i \to X$, defined on open subsets of U. If $a : D \to U_i$ and $b : D \to U_j$ are smooth functions between Euclidean domains, then the equation $f_i \circ a = f_j \circ b$ just means that they agree on the intersection $U_i \cap U_j \subseteq U$. The compatibility of the family then boils down to the equations $x_i|_{U_i\cap U_j} = x_j|_{U_i\cap U_j}$, for all $i, j \in I$. The *sheaf axiom* then says that, to each compatible family, there exists a *unique* smooth function $x : U \to X$ such that $x|_{U_i} = x_i$. That this holds is guaranteed by the Axiom of Locality in Definition **1.6**. This shows that $\overline{X}$ is not just a presheaf, but that it is a full-fledged *sheaf* on the Euclidean site. Although Chen did not use the language of sheaves explicitly, the definition of Chen spaces appears to be the first time that the idea of sheaves was used to understand the concept of a smooth space (the refinements he made to his definition in the sequence of publications [**Che73**; **Che75**; **Che77**] was each a step closer to capturing the notion of a sheaf). We refer to [**Gł19**] for a more detailed description of diffeological spaces as sheaves.

The above therefore describes an assignment $\mathbf{Diffeol} \to \mathrm{Sh}(\mathbf{Eucl}) : X \mapsto \overline{X}$, of diffeological spaces to the sheaves on **Eucl**. But, it turns out that diffeological spaces are merely the *concrete* sheaves on **Eucl**. This means that, in a technical sense, the sheaf $\overline{X}$ is determined by the underlying set X. This is not surprising, since we *started* with a set-based object. It was first proven in [**BH11**] that the category **Diffeol** of diffeological spaces is in fact *equivalent* to cSh(**Eucl**), the category of concrete sheaves on **Eucl**:

$$\mathbf{Diffeol} \simeq \mathrm{cSh}(\mathbf{Eucl}).$$

(We elaborate more on this, and what concreteness means, in Section **2.7**.) It is also known, due to the nice local behaviour of smooth functions on manifolds, that there is an equivalence

$$\mathrm{cSh}(\mathbf{Eucl}) \simeq \mathrm{cSh}(\mathbf{Mnfd}),$$

between the concrete sheaves on **Eucl** and the concrete sheaves on **Mnfd**. This is true in spite of concreteness. Essentially, this means that we could as well have stated the definition of a diffeology by using arbitrary manifolds M, instead of just open subsets $U \subseteq \mathbb{R}^m$, the latter clearly being preferable for its simplicity. The realisation of diffeological spaces as concrete sheaves on **Eucl** brings with it some pleasant consequences: categories of concrete sheaves are generally well-behaved, they are known as *quasitoposes*. This is a category with properties not dissimilar to that of a topos, but slightly weaker. An immediate corollary of this equivalence is then that **Diffeol** is such a quasitopos, and compared to **Mnfd**, its categorical properties therefore appear far superior. The entire situation can be sketched as follows:

$$\begin{array}{ccccc}
\mathrm{cSh}(\mathbf{Mnfd}) & \simeq & \mathrm{cSh}(\mathbf{Eucl}) & \simeq & \mathbf{Diffeol} \\
\big\downarrow & & \big\downarrow & & \\
\mathrm{Sh}(\mathbf{Mnfd}) & \simeq & \mathrm{Sh}(\mathbf{Eucl}) & =: & \mathbf{SmoothSet}.
\end{array}$$

Here, a new category appears: the category **SmoothSet** of *smooth sets*, defined as the category of *all* sheaves on **Eucl**. This is categorically even nicer, and a question then arises: why not just take this category, which *does* form a genuine topos? An answer is that, *yes*, one can do this, leading to what is known as *cohesive topos theory*, which originated from Lawvere's ideas he developed in parallel to synthetic differential geometry [**Law91**]. A *cohesive topos* $\mathscr{E}$ is a category in which there is a notion of *"coherence"* (of *"hanging together"*) between the points of its objects. Technically, this is achieved by equipping a topos with an adjoint triple $\mathrm{Disc} \dashv U \dashv \mathrm{coDisc}$:

$$\mathscr{E} \;\; \begin{array}{c} \xleftarrow{\;\mathrm{Disc}\;} \\ \perp \\ \xrightarrow{\;\;U\;\;} \\ \perp \\ \xleftarrow[\mathrm{coDisc}]{} \end{array} \;\; \mathbf{Set},$$

where U represents a forgetful functor (into **Set**), and the other functors describe a way to equip any set with a *discrete* of *codiscrete* cohesive structure. All other types of cohesion must lie in between these two: discrete cohesion means that *none* of the points hang together, while codiscrete cohesion means *all* of the points hang together. We refer to [*n*L20] for a detailed exposition. For a familiar example, think of the discrete and indiscrete topologies on a set. In diffeology we have similar constructions (Definition 2.23), although neither of these are examples of a cohesive topos. So while **Diffeol** is not a cohesive topos, this new category **SmoothSet** does form an example of such a category. To get back to the question of which of these to choose, the following quote by Stacey [Sta10b] may be illuminating:

> *"The problem is that there are some ornery people who really like manifolds as they are, but sometimes have to work with things that are almost but not quite completely unlike manifolds. For these people, the further away from true manifolds they get, the more uncomfortable they feel. One of the biggest steps for such people is losing the underlying set. So diffeological spaces are a category in which those people can have most of the benefits of sheaves without having to discard their comfort blanket of something that still resembles manifolds in some way. So diffeological spaces are a convenient (yes, I use the word deliberately!) half-way house whereby those who have Seen The Light can still talk to those still quivering under their comfort blankets."*[15]

See also [Car10; Nik10]. Pushing the cohesive topos train of thought further, into the realm of higher categories, one arrives at *cohesive* $(\infty, 1)$*-toposes* and *cohesive homotopy type theory*. This has been developed by Schreiber [Sch20] and others. This produces a whole hierarchy of generalised smooth spaces, the bottom-end of which might be pictured as follows:

$$\begin{array}{ccccccccccccc} \mathbf{Eucl} & \hookrightarrow & \mathbf{Mnfd} & \hookrightarrow & \mathbf{Diffeol} & \hookrightarrow & \mathbf{SmoothSet} & \hookrightarrow & \mathbf{SmoothGrpd} & \hookrightarrow & \mathbf{Smooth\infty Grpd} & \hookrightarrow & \cdots ? \\ & & & & \wr\| & & \| & & & & & & \\ & & & & \mathrm{cSh}(\mathbf{Eucl}) & \hookrightarrow & \mathrm{Sh}(\mathbf{Eucl}) & \hookrightarrow & \mathrm{PSh}(\mathbf{Eucl}) & & & & \end{array}$$

Actually, as remarked in [*n*L19d], diffeology arises naturally from the cohesive theory. If one follows the philosophy of cohesive toposes, the natural cohesive topos for differential geometry is **SmoothSet**. This is because smooth sets are in a sense the most general type of nice smooth spaces built on Cartesian coordinate systems. That Cartesian coordinate systems themselves are appropriate as a foundation for physics and geometry is widely accepted but not often motivated directly. The fundamental premise here, as described in [*n*L19a, Section 1], is that *"The abstract worldline of any particle is modelled by the continuum real line* $\mathbb{R}$*."* Whether or not there is good reason to doubt this premise (as some do [Bae16]), for our purposes it will be sufficient to accept it. (If $\mathbb{R}$ is not smooth, then what is?) In a sense, the sheaves on Cartesian spaces then form the most general type of smooth objects that behave locally like Euclidean spaces in a consistent way[16] (see more at [*n*L19b]). If we then allow the conclusion that **SmoothSet** is the natural cohesive topos for differential geometry, diffeology actually follows naturally. Every cohesive topos has a canonical induced quasitopos of *concrete* spaces. The concrete spaces are induced in some sense by the subtopos of *codiscrete* objects. A codiscrete object is something like a topological space with its codiscrete (or indiscrete) topology, i.e., only containing the empty set and entire space. For diffeology, codiscreteness means *coarseness* (Definition 2.23). Lawvere refers to the passage of the entire cohesive topos to the subtopos of codiscrete spaces as *"pure Becoming"* [Law91, p.7]. For the cohesive topos **SmoothSet** of differential geometry, the concrete spaces are exactly the *diffeological spaces* [BH11, Proposition 24]. This shows that diffeology arises from very fundamental assumptions that underlie our geometric intuitions.

1.2 What this thesis is all about

Above, we hope to have portrayed an accurate sketch of (part of) the landscape of generalised smooth spaces. Out of all options, we choose here for *diffeology*. Our choice was initially made for pragmatic reasons, which we explain below. Particularly in relation to the discussion directly above, more is known about diffeology than about smooth sets. We mean this in a technical sense that more *can* be proven, but also in terms of existing literature. There also seems to be no definitive advantage of smooth sets

[15] We should emphasise that Stacey himself identifies as belonging to the latter group of people! Personally, my toes are poking out from under the blanket.

[16] Presheaves in PSh(**Eucl**) would be even more general, but their local behaviour might be inconsistent.

over diffeological spaces for the problem we are studying. That being said, we believe that a natural extension of this thesis would be to develop the theory (of Morita equivalence) also for smooth sets (and beyond). The theory of diffeological groupoids and their Morita equivalence seemed interesting enough, and was sufficiently close to the theory of Lie groupoids to appear to be of some significance and usefulness in that field. Nevertheless, it is possible to motivate the choice of diffeology even in hindsight, which we try to do below.

1.2.1 What diffeology offers

The Preface of the [***Diffeology***] textbook lists many examples suggesting a need for diffeology (each of which is in one form or another a reflection of one of the three shortcomings of manifolds we listed above): the irrational torus and its relation to quasiperiodic potentials in physics, orbifolds in relation to symplectic reduction, spaces of connections and differential forms in gauge theory, groups of symplectomorphisms in symplectic geometry and geometric quantisation, coadjoint orbits of diffeomorphism groups, etc.

"Diffeology did not spring up on an empty battlefield," to quote [***Diffeology***]. Diffeology was discovered around the same time as Rieffel's publication on *irrational rotation algebras* [**Rie81**], which was one of the motivations for noncommutative geometry. With the publication of [**DI83**], the first classification of diffeological irrational tori (Theorem **2.81**), we saw a hint that diffeology could be an alternative to noncommutative geometry. Diffeology offers a more comfortable alternative to the functional-analysis-heavy C*-algebraic framework[17]. The relation between the two theories has only recently begun to be explored [**KLMV14**; **Ber16**; **IZL18**; **IZP20**].

1.2.2 Why diffeological bibundles?

Our ultimate reason for choosing diffeology for this project was motivated by the work of [**BFW13**; **Gł19**], where diffeology is used to study general relativity. As we mentioned in the preface, the aim was at first to provide a more solid foundation for a theory of *"diffeological algebroids."* As this started to look unfeasible, given the author's lack of expertise and time and progress, eventually a decision was made to study a Hilsum-Skandalis category of diffeological groupoids. Having worked briefly with Walter van Suijlekom on a project in noncommutative geometry the year before, relating to Lie groupoids and orbifolds, there was already some familiarity with the notion of a bibundle. Having been intrigued in the meantime by the work of Jan Głowacki on diffeological groupoids in general relativity, the connection was subsequently made.

Rings, like groups, can be studied through their actions, i.e., their modules. Many important properties in fact turn out to be captured by their representation theory. In [**Mor58**], Morita introduced a notion of equivalence between rings that preserves the behaviour of their representation theory. Indeed, this is exactly the external definition: two rings R and S are equivalent if and only if their module categories R-**Mod** and S-**Mod** are equivalent. Morita proved that, for rings, this notion of equivalence is identical to the following internal definition: R and S are equivalent if and only if there exists an *equivalence bimodule* ${}_R\mathsf{E}_S$ between them. This later came to be known as *Morita equivalence*.

Some time later, Rieffel extended these ideas to the C*-realm [**Rie74**]. For technical reasons, the two definitions given above are not equivalent in this setting, and the definition of *strong Morita equivalence* of C*-algebras is now usually given in terms of equivalence bimodules. (Although, it can also be defined as isomorphism in a suitable category of C*-algebras and *C*-correspondences.*) Subsequently, the work of Hilsum and Skandalis [**HS87**] was motivated by providing a geometric realisation of earlier C*-algebraic results on the theory of foliations [**HS83**] in terms of groupoids. *Hilsum-Skandalis morphisms*, what we shall call *bibundles*, are then essentially an adaptation of the notion of bimodules in the setting of groupoids. It appeared that this new type of morphism between groupoids was more appropriate to study the geometry of foliations than the ordinary definition of a functor between groupoids. The first intrinsic definition of Morita equivalence of locally compact Hausdorff groupoids appears to be in [**MRW87**]. It is now a well-known folk theorem that two groupoids are Morita equivalent if and only

[17] *"Physicists run fast; if we want to stay close to them we need to jog lightly,"* quoting the Preface in [***Diffeology***].

if there exists an invertible Hilsum-Skandalis morphism between them. (A generalised version of this statement is our main theorem: Theorem **4.69**.)

The introduction of these notions turned out to be very fruitful, for several reasons. First, the structure of the C*-algebras of foliations was reflected by their corresponding (Lie) groupoids. Indeed, due to a construction first given in Renault's PhD thesis [**Ren80**], we since know that there exists a functorial assignment

$$C^* : \mathbf{LCHGrpd} \longrightarrow \mathbf{C^*Alg},$$

of locally compact Hausdorff groupoids with *Haar systems* (an appropriate generalisation of a *Haar measure* from groups to groupoids) and Hilsum-Skandalis maps to a suitable category of C*-algebras and bimodules (up to technical details, cf. [**Lan00**]). In particular, a foliation of a smooth manifold induces a *holonomy groupoid*, which is a Lie groupoid that captures in some sense the smooth paths along the leaves of the foliation. Then, it can be shown that the smooth structure of a Lie groupoid gives rise to a canonical Haar system, and in this way the above functor gives the C*-algebra corresponding to the foliation. Together with the aforementioned folk theorem, the functoriality here ensures that Morita equivalence of the holonomy Lie groupoids is preserved. With this, it became apparent that the theory of groupoids and their Morita equivalence could be used to supplement the C*-algebraic theory of foliations.

From an entirely different angle, there were attempts at applying the algebro-geometric notion of a *stack* to differential geometry. Out of these efforts evolved the notion of a *differentiable stack* [**BX11**], which is yet another type of generalised smooth space. These types of objects were motivated in part by the study of *orbifolds* [**Moe02**], spaces that are locally homeomorphic to quotients of Euclidean spaces by finite group actions. It so happened that these objects could be geometrically represented by *Lie groupoids*, up to *Morita equivalence* [**BX11**, Theorem 2.26]. The philosophy here is that a Lie groupoid (a perfectly good smooth object) is a *model* for its underlying (often singular) space of orbits, and Morita equivalence is a suitable equivalence that preserves the orbit spaces and the transversal geometry. Therefore, two Morita equivalent groupoids serve as the same model, or *atlas*, of the orbit space. In the particular case of foliation theory, the holonomy Lie groupoid of the foliation serves as a model for a singular leaf space (such as the irrational torus).

Now it appears that, since its birth in operator theory, the notion of Morita equivalence of groupoids has become an important notion in its own right. It is used further in Poisson geometry, where there exists a notion of Morita equivalence for symplectic groupoids [**Xu91**]. This connects also to the way that Morita equivalence has been used by Landsman [**Lan01a**; **Lan01b**; **Lan01c**; **Lan06**] in the context of quantization (where "Poisson manifolds are the classical analogue of C*-algebras," [**Xu91**]). Independently, notions of bibundles and Morita equivalence have appeared in topos theory [**Moe91**].

Below are some points motivating the study of diffeological groupoids and their Morita equivalence:

- Of all the set-based approaches to generalised smooth spaces, diffeology is one of the most developed, with an active research community and an excellent textbook [***Diffeology***]. In particular, the theory of diffeological groupoids has been known since [**Igl85**], so there was a solid foundation to build this work on. Besides, as we have pointed out, diffeology has sufficient implications about the classical world of differential topology so that a theory of Morita equivalence for diffeological groupoids could be of future use in the theory of Lie groupoids.

- Diffeological groupoids have recently been used in mathematical physics [**BFW13**; **Gł19**].

- As already mentioned, diffeology intrinsically deals well with quotients. In the sense that differentiable stacks are used to model spaces with singular quotients, diffeology could be a natural supplement in that regard. Instead of using Lie groupoids as representations for differentiable stacks, we could consider diffeological groupoids instead, leading to what we could call *"diffeological stacks."* A notion of Morita equivalence for diffeological groupoids would be essential for that. The relation between diffeology and stacks is beginning to be explored [**WW19**]. Since both diffeology and the theory of differentiable stacks deal well with objects such as the irrational torus, it would be interesting to see to what extent there is a relation between the two. More related research is [**KW16**]

- There is a mature theory of symplectic diffeology [***Diffeology***, Chapter 9]. In fact, one of Souriau's original motivations for diffeology was to study symplectomorphism groups. A notion of Morita equivalence between diffeological groupoids might extend Xu's definition of Morita equivalence between symplectic groupoids [**Xu91**] to the diffeological world.

- Recently, in [**GZ19**] a new notion for Morita equivalence for holonomy groupoids of *singular foliations* has been proposed. As the authors of that article suggest, this could fit naturally into a framework of diffeological Morita equivalence, but also remark "[t]he theory of Morita equivalence for diffeological groupoids has not been developed yet," [**GZ19**, p.3]. This also relates to work in preparation on the integration of *singular subalgebroids* [**AZ**] using diffeological groupoids. We hope the contents of this thesis can be of use here.

- In [**Ber16**; **IZL18**; **IZP20**], people have begun to explore the relation between diffeology and noncommutative geometry. There is already a bridge between the theory of Lie groupoids (differentiable stacks) and noncommutative geometry, by associating to each Lie groupoid its groupoid C*-algebra, which preserves Morita equivalence. This bridge is already being explored in [**IZL18**; **IZP20**] for the special classes of diffeological orbifolds and *quasifolds*. Our theory of Morita equivalence between diffeological groupoids would be a natural framework to capture the Morita equivalences described in those papers. Hence, it seems that a proper notion of Morita equivalence for diffeological groupoids is important to understand fully a relation between diffeology and noncommutative geometry. To complete this bridge, we need a functorial assignment

$$C^* : \mathbf{DiffeolGrpd} \longrightarrow \mathbf{C^*Alg},$$

  of a groupoid C*-algebra for an arbitrary diffeological groupoid. It is currently unknown if such a functor exists. Some constructions have been suggested for a special class of groupoids [**ASZ19**]. The main ingredient in the corresponding construction for locally compact Hausdorff groupoids is that one assumes the groupoid is equipped with a Haar system: a family of measures on the space of arrows of the groupoid. These measures are used to define a convolution algebra, which can be completed into a C*-algebra. For the construction of the functor above, we then seem to be missing a crucial ingredient: *diffeological measure theory*[18].

- In particular, it seems that the notion of Morita equivalence we propose can be used to unify the arguments in [**IZL18**; **IZP20**] about diffeological orbifolds and quasifolds. Classically, Satake defined orbifolds in terms of *atlases* [**Sat56**]. The modern way is to view orbifolds as Morita equivalence classes of proper étale Lie groupoids [**Moe02**; **Ler08**]. The reason for this is that such groupoids are locally isomorphic to *action groupoids* of finite groups on Euclidean domains [**Ler08**, Proposition 2.23]. Since action groupoids are smooth geometric models for the quotient space of their action, a proper étale Lie groupoid is a model for a space that looks locally like the quotient of a Euclidean space by a finite group action, i.e., orbifolds. Orbifolds have also been described in noncommutative geometry [**RV08**; **Har14**]. Since quotients of Euclidean spaces are perfectly fine diffeological spaces, orbifolds can be studied quite naturally in the setting of diffeology [**IKZ10**] (without using groupoids as models). It would be interesting to know the precise relation between the Lie groupoid, noncommutative geometry, and diffeology approaches to orbifolds.

[18] Also, as pointed out by Klaas Landsman, the category **C*Alg** is a (noncommutative) topological world. Given the dichotomy between topology and diffeology, as showcased by the irrational torus, it is unclear what the interpretation of such a functor would be. On the other hand, the irrational torus can be modelled by a Lie groupoid, either as the holonomy groupoid of the *Kronecker foliation*, or as an *action groupoid*. Neither of these groupoids are topologically trivial, and both return the irrational rotation C*-algebra. Is diffeology able to capture *noncommutative topological* information, while carrying no *topological* data at all?

1.2.3 A bird's-eye view

To help the reader find their ground in this thesis, we provide a short outline of the contents. We summarise the contents of each section, and highlight important results obtained in them. Some things that you might find:

1. A succinct, yet detailed introduction to the theory of diffeology (Chapter **II**). This includes a thorough discussion of limits and colimits (Section **2.2.6**) and functional diffeologies (Section **2.4**).

2. A proof, without sheaf technology, that **Diffeol** is a quasitopos (Theorem **2.118**).

3. A generalisation of the theory of Lie groupoid bibundles to the diffeological setting (Chapter **IV**).

4. A proof that this forms a general framework to study Morita equivalence of diffeological groupoids (Theorem **4.69**). This could lead to a more elegant treatment of diffeological orbifolds (and other such objects) in relation to both noncommutative geometry and the theory of stacks. We also obtain a genuine *diffeo*morphism between the orbit spaces of Morita equivalent groupoids Theorem **5.18**, whereas for Lie groupoids this is generally only a *homeo*morphism ([**CM18**, Lemma 2.19]).

5. A detailed construction of the *germ groupoid* of a space (Section **6.1**), and a rudimentary study of *atlases* and their groupoids of transition functions (Section **6.1.1**).

Below, we outline the contents of this thesis, highlighting its main results.

Diffeology. In Chapter **II** we provide a detailed account of the theory of diffeology *à la* [***Diffeology***] in terms of *plots* (Definition **2.2**). This textbook is indeed our main reference for this chapter, but we go deeper in some areas where necessary. In particular, as Chapter **II** proceeds, we provide a proof that the category **Diffeol** of diffeological spaces is a *quasitopos*. This result has been known at least since [**BH11**, Theorem 52], but we provide an account from a more down-to-earth view without using any sheaf theory. We chose this approach mainly because [**BH11**; **Gł19**] already contain excellent sheaf-theoretic proofs of this fact, but also because the later contents of this thesis are approached from this more down-to-earth view as well, and it would be jarring if we suddenly switched from sheaves to plots. It is well known that the category of sheaves on a site forms a topos ([**MM94**]). In that sense, the proof that **Diffeol** is a quasitopos from the sheaf point of view can be seen as a *top-down* approach, realising diffeological spaces into a setting which is inherently topos-theoretic. Our approach is more *ground-up*, where we start just from the Axioms of Diffeology, and get topos-like properties from there.

We start Chapter **II** with the definition of a *diffeological space*, and prove that smooth manifolds **Mnfd** form a full subcategory of **Diffeol** (Theorem **2.11**). This means that we can think of diffeology as a genuine extension of classical differential topology. In Section **2.2** we discuss categorical constructions such as products and coproducts, which will be used extensively later on, and in Section **2.2.6** we prove that **Diffeol** is complete and cocomplete (Theorem **2.72**). As an intermezzo, we give a detailed discussion of the *irrational torus* in Section **2.3**. Then, in Section **2.4**, we discuss the remarkable fact that the space $C^\infty(X, Y)$ of smooth maps between diffeological spaces has its own canonical diffeological structure, making **Diffeol** Cartesian closed (Theorem **2.93**). It turns out that **Diffeol** is even locally Cartesian closed (Theorem **2.97**), and together with the construction of a weak subobject classifier in Section **2.5** we prove that **Diffeol** is a quasitopos (Theorem **2.118**). Finishing off this chapter, in Section **2.6** we discuss in detail a special class of smooth functions that we need to extend the notion of a submersion to the diffeological realm, and then finally give a short discussion of diffeological spaces as sheaves in Section **2.7**.

Diffeological Morita equivalence. In Chapter **III** we introduce the notion of a *diffeological groupoid* (Definition **3.16**), in preparation for the later discussion to come. Diffeological groupoids have already been studied in [**Igl85**; ***Diffeology***]. We give some basic examples that generalise from the theory of Lie groupoids, and also discuss some diffeological examples. The main definition in this chapter is of the 2-category **DiffeolGrpd** of diffeological groupoids, *smooth functors*, and *smooth natural transformations* (Definition **3.18**). This is not our working category, however, and the subsequent chapters are devoted to the construction and study of a bigger bicategory. In Section **3.3** we introduce the special class of *fibration groupoids* (Definition **3.41**), which have been used to study *diffeological fibre bundles* [***Diffeology***, Chapter 8] (Definition **3.42**). In Section **3.4** we propose a definition for *smooth linear representations* (Definition **3.51**) of diffeological groupoids, based on the notion of *diffeological vector pseudo-bundles* [**Per16**] (Definition **3.47**). As an example, we construct the *smooth left regular representation* (Example **3.52**), which is generally an infinite-dimensional representation.

Chapter **IV** contains the main new results of this thesis. Its first part is a proposal for a theory of *diffeological Morita equivalence*, in which we define, study, and give examples of: diffeological groupoids, -actions, -bundles, and -bibundles. The presentation of this material closely follows the theory of Lie groupoids and bibundles as already studied in the literature. Especially we like to recommend [**Blo08**; **dHo12**], and other references are [**Lan01a**; **Lan01b**; **Lan01c**; **Ler08**; **Li15**]. In Chapter **V** we also draw inspiration from, in particular, [**Moe02**; **MM03**; **MM05**], which focuses on the *calculus of fractions* approach to Morita equivalence of Lie groupoids. We discuss a calculus of fractions approach for diffeological groupoids in that chapter, while Chapter **IV** focuses on the use of bibundles.

In Section **4.1** we define and study *diffeological groupoid actions* (Definition **4.1**), and the corresponding category $\mathbf{Act}(G \rightrightarrows G_0)$ of smooth groupoid actions and equivariant smooth maps. This category can be thought of as a general setting of smooth groupoid representation theory. It is also here that we lay the groundwork for the *balanced tensor product* (Construction **4.12**). In Section **4.2** we introduce the new notion of *diffeological groupoid bundles* (Definition **4.14**). These are smooth groupoid actions, together with an invariant smooth map. It is here that the diffeological theory of groupoids starts to diverge conceptually from the Lie theory. This is mostly because the flexibility of diffeology circumvents all of the technical restrictions that appear in the theory Lie groupoids. In particular, in many instances we need special conditions to ensure that quotients and fibred products stay inside the category of smooth manifolds. Since **Diffeol** is closed under such constructions, these special conditions become redundant. This allows us to dissect the conceptual progression in the Lie groupoid theory, and adapt it here to give a more transparent presentation in the diffeological setting. Most importantly, the definition of a *principal* groupoid bundle splits into two separate components (Definitions **4.17** and **4.18**). Continuing into Section **4.2.2**, we prove that equivariant smooth maps between principal groupoid bundles have to be diffeomorphisms (Proposition **4.30**). We then arrive at Section **4.3**, where we finally define *diffeological bibundles* (Definition **4.31**). In Section **4.3.2** we show how the earlier introduced balanced tensor product allows us to transfer smooth groupoid actions along bibundles (functorially). Then in Section **4.3.3** we use this to define the composition of diffeological bibundles (Construction **4.48**), which gives the promised bicategory **DiffeolBiBund** of diffeological groupoids, bibundles, and smooth biequivariant maps (Theorem **4.51**). It is already worthwhile to mention that this bicategory has no analogue in the Lie theory, because there bibundle composition can only be defined for *left principal* bibundles. The sections Sections **4.3.4** and **4.3.5** are where we obtain our main new results. The first of these (Theorem **4.62**) is an analogue of the theorem about weak invertibility of left principal bibundles between Lie groupoids (see e.g. [**Blo08**, Section 3] or [**Lan01a**, Proposition 6.7]). But we go further: we prove that *any* diffeological bibundle is weakly invertible if and only if it is biprincipal (Theorem **4.69**). This fully justifies the bicategory **DiffeolBiBund** as being the correct setting for Morita equivalence of diffeological groupoids. Closing off Chapter **IV**, we discuss some applications of this framework in Section **4.4**. The first of these is a proof that Morita equivalent diffeological groupoids have equivalent action categories (Theorem **4.70**), which directly generalises [**Lan01a**, Theorem 6.6], in which a similar result for the Lie case is proved. Next, in Section **4.4.2** we prove that the property of a diffeological groupoid being *fibrating* is preserved under Morita equivalence. In Section **4.4.3** we pose the open question whether diffeological Morita equivalence between Lie groupoids reduces to the standard notion of Morita equivalence between Lie groupoids (Question **4.80**).

The contents of Chapter **V** consists mainly of an alternative development of the notion of Morita equivalence through the use of a *calculus of fractions*. We prove that this notion of Morita equivalence coincides with the one defined in terms of bibundles (Theorem **5.14**). As an application of this alternative framework, in Section **5.2** we prove that the orbit spaces of two Morita equivalent groupoids are diffeomorphic (Theorem **5.18**).

Lastly, in Chapter **VI** we study the *groupoid of germs* of a diffeological space, and use it to study the local structures of spaces. We use it to define a rudimentary notion of *atlas* (Definition **6.9**), and define diffeological *groupoids of transition functions* (Definition **6.11**). We prove that two diffeological spaces are diffeomorphic if and only if they admit atlases whose groupoid of transition functions are Morita equivalenct (Theorem **6.20**).

Future questions. We list here some open questions and ideas for future research:

1. The construction of a theory of bibundles for a more general framework of concrete sheaves ([**BH11**, Definition 19]), or even arbitrary sheaves, perhaps in relation to [**MZ15**]. A theory of principal bibundles seems to exist in a general setting for groupoids in ∞-toposes: [n**L18a**].
2. Finding an answer to the open Question **4.80** about *diffeological* Morita equivalence between *Lie* groupoids.
3. What is the precise relation between differentiable stacks and diffeological groupoids (cf. [**WW19**])? Using our notion of Morita equivalence, what types of objects are *"diffeological stacks"* (i.e., Morita equivalence classes of diffeological groupoids)?
4. Can the *Hausdorff Morita equivalence* for holonomy groupoids of singular foliations introduced in [**GZ19**] be understood as a Morita equivalence between diffeological groupoids?
5. The general construction of diffeological groupoid C*-algebras and the preservation of Morita equivalence. One step in this direction has already been taken in [**ASZ19**]. More generally: is there a precise relation between diffeology and noncommutative geometry? There is a small amount of research investigating this question: [**Ber16**; **IZL18**; **IZP20**]. There could be a tremendous benefit to link these theories more closely. On the one hand, we have the powerful framework of noncommutative geometry, which has deep implications for physics and mathematics, but is analysis-heavy. On the other hand we have diffeology, which is simple, light-weight and intuitive. A hybrid theory, combining the best of both worlds, could bring intuition to noncommutative geometry and analytic power to diffeology. We hope that the framework of diffeological Morita equivalence that we present here can help in building this link.
6. A theory of 'VpB-groupoids', generalising the theory of VB-groupoids to the diffeological setting. We use the abbreviation 'VpB' in reference to the theory of *vector pseudo-bundles* [**Per16**]. This could be used to study infinite-dimensional linear representations of groupoids. A linear representation of a Lie groupoid is defined in terms of the frame groupoid of a vector bundle. As long as the vector bundle is finite-dimensional, the frame groupoid is a Lie groupoid. Having introduced these notions in Section **3.4**, what could be their applications in the representation theory of groupoids (cf. e.g. [**Bos07**])? Using diffeology, the frame groupoid can be extended also to infinite-dimensional vector bundles, or even vector bundles with varying fibres. Since Hilbert spaces have a canonical diffeology (the *fine* diffeology), this can also be used to define unitary representations of groupoids. Choosing a notion of tangency on diffeological spaces, we should define a *tangent groupoid*, which would naturally form a VpB-groupoid over itself. From Lie groupoid theory it is known that the tangent groupoid is closely related to its Lie algebroid. This theory of VpB-groupoids could shine more light on the question of *"diffeological algebroids."* We briefly remark on this in Section **3.4.1**.
7. What is the physical interpretation of the notion of Morita equivalence for the diffeological groupoids used in general relativity [**BFW13**; **Gł19**]? What physical notion does the Morita equivalence class of the groupoid of Σ-evolutions represent?

Chapter II
Diffeology

Diffeology[19] was introduced by Souriau in [**Sou80**]. In that publication, Souriau describes *five* axioms for *"groupes différentiels,"* which we can now recognise as the definition of a *diffeological group* (cf. Definition **3.1**). It took a further four years for the definition of *"espaces différentiels"* to be distilled from these five axioms into the *three Axioms of Diffeology* we know today (Definition **2.2**). We recommend [**IZ13**] for a short first-person account of these developments.

A very extensive development of the elementary theory of diffeology can be found in the first two chapters of the *Diffeology* textbook by Iglesias-Zemmour. This special reference will be denoted by: [***Diffeology***]. A lot of the content in this section is based on that monograph. Some results appear to be new (although not profound), as they were tailor-made to develop some of the theory in later sections. For a thorough treatment on the categorical constructions of diffeological spaces we refer to the expositions in [**Gł19**, Section I.1] and the paper [**BH11**]. Another good elementary introduction to diffeology is in [**Vin08**], which resembles more closely the approach we take here.

Intuitively, diffeology is somewhere in the realm of differential topology and differential geometry. But it is not quite either of those: diffeology doesn't require topology (although it does generate one), and it is as much geometric as the theory of smooth manifolds. But clearly diffeology is differential *something*. We could call it differential *pre*-topology. Maybe we *should* call it differential *set* theory[20].

The basic goal of diffeology is to lift the differential topology of Euclidean domains to a larger class of objects. Recall that a *Euclidean domain* is an open subset of a Euclidean space $\mathbb{R}^n$, for some $n \in \mathbb{N}$. These, together with the next definition, form the building blocks of the theory:

Definition 2.1. Let X be a set. A *parametrisation* on X is a function $U \to X$ defined on a Euclidean domain. We denote by $\mathrm{Param}(X)$ the set of all parametrisations on X.

A diffeology on a set X determines which parametrisations $U \to X$ are 'smooth'. These smooth parametrisations are called *plots*. It does this in a way that translates the local behaviour of Euclidean domains onto X. In particular, we can describe the following intuitive understanding of a diffeology. First, every point in the set X must be in the image of some plot. Second, plots may be reparametrised by smooth functions between Euclidean domains. Here the notion of smooth function between Euclidean domains is just the usual one, as if they were smooth manifolds. Lastly, whether a parametrisation is smooth is determined by its local behaviour. That means if the domain of a parametrisation $U \to X$ has an open cover for which each of the restrictions are plots, then the entire parametrisation must be a plot. This reflects the fact that a function should be smooth if and only if it is locally smooth everywhere.

Definition 2.2 (Axioms of Diffeology)**.** Let X be a set. A *diffeology* on X is a collection of parametrisations $\mathcal{D}_X \subseteq \mathrm{Param}(X)$, containing what we call *plots*, satisfying the following three axioms:

1. *(Covering)* Every constant map $U \to X$ is a plot.
2. *(Smooth Compatibility)* For every plot $\alpha : U_\alpha \to X$ in $\mathcal{D}_X$ and every smooth function $h : V \to U_\alpha$ between Euclidean domains, we have that $\alpha \circ h \in \mathcal{D}_X$.
3. *(Locality)* If $\alpha : U_\alpha \to X$ is a parametrisation and $(U_i)_{i\in I}$ an open cover of U_α such that each restriction $\alpha|_{U_i}$ is a plot of X, then so is α.

[19] The etymology of the word is explained in the afterword to [***Diffeology***]. Souriau first used the term *"différentiel"*, as in 'differential' (from the Latin *differentia*, "difference"). Through a suggestion by Van Est, the name was later changed to *"difféologie,"* as in *"topologie"* ('topology', from the Ancient Greek *tópos*, "place," and *-(o)logy*, "study of"). Hence the term: diffeology.

[20] This would definitely be in line with the definition of a *smooth set*. Recall from Section **1.1.3** that smooth sets are the sheaves on the site of Euclidean domains. I think the terminology 'smooth set' is not quite appropriate, since sheaves do not in general have an underlying set. (But calling them *smooth spaces* could also be confusing.)

A set X, paired with a diffeology $(X, \mathcal{D}_X)$, is called a *diffeological space*. Note that the distinction between diffeological spaces and diffeologies is merely superficial, because, by the Axiom of Covering, the information of the set is entirely coded in the constant plots of its diffeology. Nevertheless, we find the viewpoint of diffeological *spaces* more appealing and intuitive, so we shall continue to view diffeologies as structures *on* sets[21].

The following equivalent characterisation of the Axiom of Covering explains its name.

Proposition 2.3. *If $\mathcal{D}_X$ is a diffeology, the Axiom of Covering is equivalent to $\bigcup_{\alpha \in \mathcal{D}_X} \operatorname{im}(\alpha) = X$.*

Proof. If the Axiom of Covering holds, then the values of the plots certainly cover X, because each point $x \in X$ is hit by any of its corresponding constant plots.

Conversely, suppose that $\bigcup_{\alpha \in \mathcal{D}_X} \operatorname{im}(\alpha) = X$, and let $\operatorname{const}_x : U \to X$ be a constant plot, say, taking the value $x \in X$. Since the images of the plots cover X, we can find an actual plot $\alpha : U_\alpha \to X$ such that $x \in \operatorname{im}(\alpha)$. There must therefore be at least one point $t_0 \in U_\alpha$ such that $\alpha(t_0) = x$. Now, the constant map $\operatorname{const}_{t_0} : U \to U_\alpha$ is certainly smooth between Euclidean domains, so by the Axiom of Smooth Compatibility $\operatorname{const}_x = \alpha \circ \operatorname{const}_{t_0}$ is a plot in $\mathcal{D}_X$. □

Definition 2.4. A function $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ between diffeological spaces is called *smooth* if for every plot $\alpha \in \mathcal{D}_X$, the composition with f also gives a plot: $f \circ \alpha \in \mathcal{D}_Y$. When the diffeologies on the sets X and Y are understood, we will just write $f : X \to Y$. Unlike in Section **1.1**, we will make no terminological distinction with the usual definition of smooth functions between manifolds, since we will see later that they are equivalent. Hence, from now on, *smooth* will be used synonymously with *diffeologically smooth*. A function $f : X \to Y$ is called a *diffeomorphism* if it is a smooth bijection whose inverse is also smooth.

Proposition 2.5. *The composition of two smooth maps between diffeological spaces is again smooth.*

Proof. Let $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ and $g : (Y, \mathcal{D}_Y) \to (Z, \mathcal{D}_Z)$ be two smooth maps. To show that $g \circ f$ is smooth, we need to show that $(g \circ f) \circ \alpha \in \mathcal{D}_Z$, for any $\alpha \in \mathcal{D}_X$. But, if α is a plot of X, then $g \circ \alpha \in \mathcal{D}_Y$, because g is a smooth map. And, since f is also smooth, we thus get $f \circ (g \circ \alpha) \in \mathcal{D}_Z$, which is just what we need up to associativity. □

It is also clear that the identity map $\operatorname{id}_X : (X, \mathcal{D}_X) \to (X, \mathcal{D}_X)$ is smooth (in fact, it is a diffeomorphism). The underlying structure of the category **Set** of sets and functions now transfers directly to diffeological spaces and smooth maps:

Definition 2.6. The category of diffeological spaces and smooth maps between them is denoted by **Diffeol**. The set of smooth maps $X \to Y$ is denoted by $C^\infty(X, Y)$, and the set of diffeomorphisms is denoted by $\operatorname{Diff}(X, Y)$. The set of diffeomorphisms $X \to X$ is denoted by just $\operatorname{Diff}(X)$, instead of $\operatorname{Diff}(X, X)$.

Before we dive further into the study of diffeology, we will demonstrate how this theory encapsulates that of smooth manifolds, in a fully faithful way. In this way we will show that our intuitions of smooth manifolds are genuinely allowed to transfer to that of diffeological spaces.

Example 2.7. Any Euclidean domain $U \in \mathbf{Eucl}$ has a *Euclidean diffeology* $\mathcal{D}_U$ (also called the *canonical-* or *standard diffeology*), which is just the collection of parametrisations that are smooth in the classical sense:

$$\mathcal{D}_U := \bigcup_{V \in \mathbf{Eucl}} C^\infty_{\mathbf{Mnfd}}(V, U),$$

where $C^\infty_{\mathbf{Mnfd}}(M, N)$ denotes the set of smooth functions between two manifolds[22]. It clear that this satisfies the axioms of Definition **2.2**. Note that, in particular, this gives us standard diffeologies $\mathcal{D}_{\mathbb{R}^n}$ for each Cartesian space $\mathbb{R}^n$.

[21] Although, when taken seriously, the idea that the underlying set is captured by the diffeology itself is what leads naturally to the sheaf-theoretic point of view.

[22] This notation is introduced only temporarily, as we will soon see (Proposition **2.10**) that it is a superficial distinction.

Proposition 2.8. *With respect to the Euclidean diffeologies, plots are exactly the smooth parametrisations.*

Proof. Let $\alpha : U \to X$ be a plot of a diffeological space $(X, \mathcal{D}_X)$, and endow U with its Euclidean diffeology $\mathcal{D}_U$. Then take a plot $h : V \to U$ in $\mathcal{D}_U$. The map $\alpha \circ h$ is then a plot of X by the Axiom of Smooth Compatibility, which proves that α is smooth.

Conversely, let $\alpha : U \to X$ be a parametrisation that is smooth with respect to the Euclidean diffeology $\mathcal{D}_U$. The identity map id_U is a plot in $\mathcal{D}_U$, so it follows immediately that $\alpha = \alpha \circ \mathrm{id}_U \in \mathcal{D}_X$. □

Once we know that the smoothness of maps between manifolds is determined by their local behaviour, Example **2.7** can be extended to all smooth manifolds. But this we do know, for instance through [**Lee13**, Corollary 2.8], which says that a compatible family of smooth maps on an open cover of a manifold glues together uniquely into a smooth map on the entire manifold, and every smooth map arises in this way. Given moreover the fact that the composition of smooth maps between manifolds is smooth, and that constant maps are smooth, the following definition gives a genuine diffeology for every smooth manifold:

Definition 2.9. Let M be a smooth manifold. The *manifold diffeology* $\mathcal{D}_M$ (also known as the *standard diffeology*) on M is just the collection of parametrisations that are smooth in the usual sense:

$$\mathcal{D}_M := \bigcup_{U \in \mathbf{Eucl}} C^\infty_{\mathbf{Mnfd}}(U, M).$$

Given the above remarks, we see that this satisfies all three axioms of Definition **2.2**. See also [***Diffeology***, Chapter 4] for an in-depth treatment of manifolds in the theory of diffeology. Note also that the Euclidean diffeology of Example **2.7** coincides with the manifold diffeology on Euclidean domains.

Proposition 2.10. *Let $f : M \to N$ be a function between smooth manifolds. Then f is smooth as a function between manifolds if and only if it is smooth with respect to the manifold diffeologies. In other words: $C^\infty(M, N) = C^\infty_{\mathbf{Mnfd}}(M, N)$.*

Proof. Suppose first that f is a smooth map between manifolds. Take a plot $\alpha \in \mathcal{D}_M$ in the manifold diffeology. That means that $\alpha : U_\alpha \to M$ is smooth as a function between manifolds. Therefore the composition $f \circ \alpha$ is also smooth between manifolds, which immediately gives $f \circ \alpha \in \mathcal{D}_N$ by the definition of the manifold diffeology on N.

Conversely, suppose that $f : (M, \mathcal{D}_M) \to (N, \mathcal{D}_N)$ is smooth as a map between diffeological spaces. Let $\mathcal{A} = (V_i \xrightarrow{\varphi_i} \mathbb{R}^m)_{i \in I}$ be an atlas of M, with $m = \dim(M)$. Given the differentiable structure on M, the charts $\varphi_i : V_i \to \mathrm{im}(\varphi_i) =: U_i$ become diffeomorphisms, where each U_i is an open subset of $\mathbb{R}^m$. By [**Lee13**, Corollary 2.8], f is smooth if and only if each $f|_{V_i}$ is smooth. In turn, since each φ_i is a diffeomorphism onto its image U_i, the restrictions of f are smooth if and only if

$$f \circ \varphi_i^{-1} = f|_{V_i} \circ \varphi_i^{-1} : U_i \longrightarrow N$$

are smooth. Since $\varphi_i^{-1} : U_i \to V_i \subseteq M$ are smooth in the manifold sense, they are in fact plots in the manifold diffeology $\mathcal{D}_M$. It follows by smoothness of $f : (M, \mathcal{D}_M) \to (N, \mathcal{D}_N)$ that $f \circ \varphi_i^{-1} \in \mathcal{D}_N$, for each $i \in I$. But, by definition of the manifold diffeology on N, this just means that each $f \circ \varphi_i^{-1}$ is smooth as a map on manifolds, and hence gives that each restriction $f|_{V_i}$, and hence f itself, must be smooth. □

This proposition now fully justifies the following theorem:

Theorem 2.11. *The inclusion functor* **Mnfd** $\hookrightarrow$ **Diffeol***, sending each smooth manifold to the diffeological space with its manifold diffeology, is fully faithful.*

The smooth manifolds are clearly a big class of examples, already making diffeology very rich. And as a contender for an extension of classical differential topology, Theorem **2.11** is essential. But the power of diffeology comes from its ability to reach beyond the classical realm. Many diffeological constructions in the rest of this chapter will actually do so: disjoint unions, quotients, fibred products, and subsets, just to name some elementary ones (all of which we discuss in Section **2.2**.) Before we go there, let us demonstrate some of the most basic examples.

Example 2.12. The easiest way to ensure that the Axioms of Diffeology are satisfied is to declare $\mathcal{D}_X = \mathrm{Param}(X)$. This makes the smooth structure on X particularly unstructured, since there is no distinction between plots and parametrisations. This diffeology is called the *coarse diffeology*, see Definition **2.23**.

Example 2.13. The one-point set $1 := \{*\}$ has a unique diffeology: $\mathcal{D}_1$, containing all parametrisations. To see that this is the only option, consider that the Axiom of Covering demands that there be at least one plot $\mathrm{const}_* : \mathbb{R}^0 \to 1$. But then any other parametrisation $\alpha \in \mathrm{Param}(1)$ can be written as $\alpha = \mathrm{const}_* \circ \mathrm{const}_0^\alpha$, where $\mathrm{const}_0^\alpha : \mathrm{dom}(\alpha) \to \mathbb{R}^0$ is the (unique) smooth constant map from the domain of α to the origin $\mathbb{R}^0 = \{0\}$. The Axiom of Smooth Compatibility forces α to be a plot, and hence $\mathcal{D}_1 = \mathrm{Param}(1)$. It is easy to see that any function $f : X \to 1$ defined on a diffeological space has to be smooth, and therefore $(1, \mathcal{D}_1)$ defines a terminal object in **Diffeol**.

The empty set $\varnothing$ also has a unique diffeology. Any set X admits a unique *empty parametrisation* $\varnothing \to X$, since the empty set is itself a Euclidean domain. Hence, there is a unique parametrisation $\varnothing \to \varnothing$ of the empty set, which is vacuously constant, so the Axiom of Covering declares that it must be a plot. The only admissible diffeology is thus $\mathcal{D}_\varnothing = \{\varnothing \to \varnothing\}$ (whereas the empty family $\varnothing \subsetneq \mathrm{Param}(\varnothing)$ violates the Axiom of Covering). Any function $f : \varnothing \to X$ defined onto a diffeological space is vacuously constant, and hence smooth. Therefore $(\varnothing, \mathcal{D}_\varnothing)$ defines an initial object in **Diffeol**. In general it will be safe to assume that all of our diffeological spaces are non-empty.

Example 2.14. There exists exactly two diffeologies on the two-point set $\Omega := \{0, 1\}$. One of them is the set of all parametrisations $\mathrm{Param}(\Omega)$. The other one is described as follows. Its plots are those parametrisations $\alpha : U_\alpha \to \Omega$ such that, for all $t \in U_\alpha$, there is an open neighbourhood $t \in V \subseteq U_\alpha$ such that $\alpha|_V = \mathrm{const}_0$ or $\alpha|_V = \mathrm{const}_1$. That is, it contains exactly those plots that are locally constant. This diffeology is called the *discrete diffeology*, see Definition **2.23**. We will discuss this object Ω more in Section **2.5**.

Example 2.15. Let (X, τ) be a topological space. It then makes sense to talk about *continuous* parametrisations $\alpha : U_\alpha \to X$, since every Euclidean domain has a standard topology, defined by the Euclidean metric on $\mathbb{R}^n$. The collection $\mathcal{D}_\tau$ of all continuous parametrisations, called the *continuous diffeology* [**Don84**, Section 2.8], forms a diffeology on X. It is easy to see that if $f : (X, \tau_X) \to (Y, \tau_Y)$ is a continuous map between topological spaces, then $f : (X, \mathcal{D}_{\tau_X}) \to (Y, \mathcal{D}_{\tau_Y})$ is a smooth map with respect to the continuous diffeologies. See [**CSW14**, Proposition 3.3] for more.

Example 2.16 (The circle)**.** The circle is arguably the simplest manifold that is not a Euclidean space. We describe here its diffeological structure ([***Diffeology***, Article 1.11]). As a set, we take the unit circle in the complex numbers:

$$S^1 := \{z \in \mathbb{C} : |z| = 1\} = \{e^{2\pi i x} : x \in \mathbb{R}\}.$$

We claim that the following characterisation defines a diffeology $\mathcal{D}_{S^1}$ on S^1:

> A parametrisation $\alpha : U_\alpha \to S^1$ is in $\mathcal{D}_{S^1}$ if and only if for every $t \in U_\alpha$ we can find an open neighbourhood $t \in V \subseteq U_\alpha$ and a smooth map $\theta : V \to \mathbb{R}$ such that $\alpha|_V(s) = e^{2\pi i\theta(s)}$.

Proof. We check that $\mathcal{D}_{S^1}$ satisfies the three Axioms of Diffeology. First let α be a constant parametrisation. Then there exists a real number $x \in \mathbb{R}$ such that $\alpha(t) = e^{2\pi i x}$ for all $t \in U_\alpha$. Clearly, setting $\theta = \mathrm{const}_x$ to be the constant plot of $\mathbb{R}$ then gives the desired equation. Hence $\mathcal{D}_{S^1}$ satisfies the Axiom of Locality.

For the Axiom of Smooth Compatibility, let $\alpha : U_\alpha \to S^1$ be an element of $\mathcal{D}_{S^1}$ and consider a smooth map $h : V \to U_\alpha$ between Euclidean domains. For any $t \in U_\alpha$ we can write $\alpha|_V(s) = e^{2\pi i\theta(s)}$ for some plot $\theta : W \to \mathbb{R}$. Then $\alpha \circ h|_{h^{-1}(W)}(s) = e^{2\pi i(\theta \circ h(s))}$. But the diffeology on $\mathbb{R}$ satisfies the Axiom of Smooth Compatibility, so $\theta \circ h|_{h^{-1}(W)}$ is a plot. But this means $\alpha \circ h$ satisfies exactly the defining characteristic of parametrisations in $\mathcal{D}_{S^1}$, so $\alpha \circ h \in \mathcal{D}_{S^1}$. This shows $\mathcal{D}_{S^1}$ satisfies the Axiom of Smooth Compatibility.

Lastly, for the Axiom of Locality, take a parametrisation $\alpha : U_\alpha \to S^1$ that admits an open cover $(V_i)_{i \in I}$ of U_α such that for each $i \in I$ we have $\alpha|_{V_i} \in \mathcal{D}_{S^1}$. Then for each $t \in U_\alpha$ there is an element V_i of the open cover such that $t \in V_i \subseteq U_\alpha$. In turn, since $\alpha|_{V_i} \in \mathcal{D}_{S^1}$, there is another open neighbourhood

$t \in W \subseteq V_i$ such that $\alpha|_W(s) = e^{2\pi i\theta(s)}$ for some smooth $\theta : W \to \mathbb{R}$. Since $t \in U_\alpha$ was arbitrary, this just shows that the entire parametrisation α is an element of $\mathcal{D}_{S^1}$, so the Axiom of Locality is satisfied. □

In Example **2.68** we will see that the circle is diffeomorphic to the quotient $\mathbb{R}/\mathbb{Z}$.

Example 2.17 (The wire diffeology)**.** The *wire diffeology* (called the *spaghetti diffeology* by Souriau) is an example of a diffeology on $\mathbb{R}^n$ that does not coincide with the Euclidean diffeology ([***Diffeology***, Article 1.10]). It is characterised by those parametrisations that locally factor through $\mathbb{R}$. This means that a parametrisation $\alpha : U_\alpha \to \mathbb{R}^n$ is a plot in the wire diffeology if and only if for every $t \in U_\alpha$ there exists an open neighbourhood $t \in V \subseteq U_\alpha$, a smooth map $h : V \to \mathbb{R}$, and a smooth map $f \in C^\infty(\mathbb{R}, \mathbb{R}^n)$ such that $\alpha|_V = f \circ h$. It is an easy exercise to check that this defines a collection of parametrisations that satisfies all three Axioms of Diffeology. That $\mathbb{R}^n$ with the wire diffeology is not diffeomorphic to the Euclidean $\mathbb{R}^n$ can be seen by noting that the identity map $\mathrm{id}_{\mathbb{R}^n}$ is not a plot in the wire diffeology (unless $n = 1$), while it is trivially a plot for the Euclidean diffeology. This example is a specific case of a diffeology *generated* by a predetermined family of parametrisations (Definition **2.26**), which we shall encounter in Section **2.1**.

Example 2.18 (The crosses)**.** Another important illustrative example is the space X that is the union of the coordinate axes in $\mathbb{R}^2$ ([**Vin08**, Example 3], [**CW14**, Example 3.19]). As a set we have

$$X = \{(x, y) \in \mathbb{R}^2 : xy = 0\}.$$

It turns out that there are two natural yet non-diffeomorphic diffeological structures on X. One comes from seeing X as a subset of $\mathbb{R}^2$ (Definition **2.51**), and the other from seeing X as a gluing of two real lines at the origin (which could be realised as a pushout). We can give precise descriptions of these diffeologies once we know about limits and colimits (Section **2.2**). Two conceptual descriptions are as follows.

The first, denoted $\mathcal{D}_{\mathrm{sub}}$, consists of all parametrisations $\alpha \in \mathrm{Param}(X)$ such that, when composed with the natural inclusion $i : X \hookrightarrow \mathbb{R}^2$, gives a genuine plot $i \circ \alpha \in \mathcal{D}_{\mathbb{R}^2}$. The other, denoted $\mathcal{D}_{\mathrm{line}}$, consists of parametrisations $\alpha \in \mathrm{Param}(X)$ that are locally contained entirely either in the x- or y-axis, and are there smooth in the usual sense. The identity map $\mathrm{id}_X : (X, \mathcal{D}_{\mathrm{line}}) \to (X, \mathcal{D}_{\mathrm{sub}})$ is then smooth, but not a diffeomorphism. To see this, define a smooth map

$$h : \mathbb{R} \longrightarrow \mathbb{R}; \qquad x \longmapsto \begin{cases} \exp(-1/x) & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

We then get a parametrisation $\alpha : x \mapsto (h(x), h(-x))$ of the cross X. Clearly $\alpha \in \mathcal{D}_{\mathrm{sub}}$, but $\alpha \notin \mathcal{D}_{\mathrm{line}}$, because there is no open neighbourhood around $0 \in \mathbb{R}$ such that α takes values exclusively in the x- or y-axis. In fact, $\alpha^{-1}(\{(0, 0)\}) = \{0\}$. In that sense $\mathcal{D}_{\mathrm{sub}}$ contains 'singular' curves, while $\mathcal{D}_{\mathrm{line}}$ does not.

Example 2.19 (Diffeological vector spaces)**.** As is well known, addition and multiplication in $\mathbb{R}$ are smooth. More generally, addition and scalar multiplication in the vector space $\mathbb{R}^n$ are smooth. This shows that Euclidean spaces are examples of *diffeological vector spaces*. More generally, consider a vector space V that is also equipped with a diffeology $\mathcal{D}_V$. In that case we can put a natural *product diffeology* on $V \times V$, which contains exactly those parametrisations $\alpha : U_\alpha \to V \times V$ such that both $\mathrm{pr}_1 \circ \alpha$ and $\mathrm{pr}_2 \circ \alpha$ are smooth. We discuss the product diffeology in detail in Section **2.2**. If the addition $+ : V \times V \to V$ and scalar multiplication $\mathbb{R} \times V \to V$ are smooth (where $\mathbb{R}$ has the Euclidean diffeology), then we say $(V, \mathcal{D}_V)$ is a *diffeological vector space*, and $\mathcal{D}_V$ is called a *vector space diffeology* for V. The diffeological vector spaces $\mathbb{R}^n$ are the special class of finite-dimensional *fine vector spaces*. These are the vector spaces equipped with the *finest* diffeology (Definition **2.22**) that turns them into a diffeological vector space. This diffeology always exists, and in finite dimensions, each fine vector space is isomorphic to some $\mathbb{R}^n$.

Example 2.20 (Fréchet diffeology)**.** Recall that a *Fréchet space* is a locally convex topological vector space that is complete with respect to a translation-invariant metric. There exists a notion of calculus for functions between Fréchet spaces, defined by the *Gateaux derivative*. If $F : U \to Y$ is a function defined on an open subset $U \subseteq X$ between Fréchet spaces, the derivative of F at $u \in U$ in the direction $v \in X$ is the function $\mathrm{d}_v F : U \to Y$ defined by

$$\mathrm{d}_v F(u) := \lim_{t \to 0} \frac{F(u + tv) - F(u)}{t},$$

whenever the limit exists. If it exists for all $u \in U$ and $v \in X$ and the induced map

$$\mathrm{d}F : U \times X \longrightarrow Y; \qquad (u, v) \longmapsto \mathrm{d}_v F(u)$$

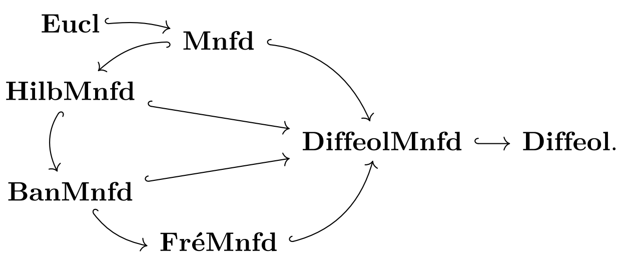
is continuous, we say that F is *continuously differentiable*, or a C^1*-function*. Inductively, we can define higher order derivatives and notions of C^k-functions, for natural numbers $k \in \mathbb{N}$. We say that F is *Fréchet smooth* if it is a C^k-function for all $k \in \mathbb{N}$. Euclidean spaces are naturally Fréchet spaces, so it makes sense to talk about the smoothness of parametrisations of Fréchet spaces in this sense. In this way we get a canonical *Fréchet diffeology* for any Fréchet space X, which is the collection of all Fréchet smooth parametrisations. (On Banach spaces this reduces to the *Banach diffeology* defined in [***Diffeology***, Exercise 72].) Even more generally, any Fréchet *manifold* gets a canonical Fréchet diffeology in the same way. It was proved in [**Los92**] (and noted in [**Los94**, Theorem 3.1.1]) that the inclusion functor

$$\mathbf{Fr\acute{e}Mnfd} \hookrightarrow \mathbf{Diffeol}$$

of Fréchet manifolds into the category of diffeological spaces is fully faithful. This generalises the result about the inclusion functor of finite-dimensional smooth manifolds in Theorem **2.11**.

This example generalises further in the diffeological setting:

Example 2.21 (Diffeological manifolds)**.** One other advantage of diffeology is that it is even less technical than the ordinary definition of a smooth manifold, yet subsumes the latter. In fact, we could study smooth manifolds entirely within the framework of diffeology without ever defining notions such as smooth atlases, differentiable structures or manifold diffeologies. A *diffeological manifold* is just a diffeological space X that is *locally diffeomorphic* (Section **2.4.1**) to a diffeological vector space E. A smooth manifold is just a diffeological space that is locally diffeomorphic to some $\mathbb{R}^n$. Diffeological manifolds are treated in detail in [***Diffeology***, Chapter 4], where they also prove that diffeological manifolds modelled on $E = \mathbb{R}^m$ correspond exactly to smooth m-dimensional manifolds. In this way Hilbert-, Banach-, and Fréchet manifolds get canonical diffeologies that make them into diffeological manifolds, and we get a diagram of inclusion functors:



Before we give more examples of diffeological spaces, we first discuss what types of constructions we can perform with them. This will allow us to create many new interesting examples.

2.1 Constructions with diffeologies

One of the main appeals of working with generalised smooth spaces (over smooth manifolds) is that they are closed under many categorical constructions, especially those that smooth manifolds are notoriously bad at handling. In the next section we will give an exposition on some of these constructions, such as products, pullbacks, and quotients. We do this here from the perspective of Definition **2.2**. In this way we can see, through very natural definitions, that **Diffeol** indeed handles these categorical constructions much better. The profound reason that all of this works is because diffeological spaces are (concrete) *sheaves*, a point of view that we will explain in Section **2.7**. The advantage of the more down to earth view of diffeological spaces in terms of plots is that one gets nearly all the advantages of working with sheaves, while being much more accessible. Besides, there are already excellent expositions exploiting the sheaf-theoretic point of view in [**BH11**] and [**Gł19**, Section I.1]. For those reasons, we choose here to give a detailed exposition of the various categorical constructions in **Diffeol** from an as elementary viewpoint as possible. This is also described in [**Vin08**, Section 1.3], which however delegates the proofs to another paper (by the same author), which we were unable to find. Here we redevelop those results independently, under guidance of that reference.

But before we can treat the constructions with diffeological spaces, we need to study constructions with diffeologies themselves. We will discover that the structure of diffeologies on one set is already quite rich. In fact, the collection of diffeologies on a set forms a *complete lattice*. The main contents of this section are also developed in [***Diffeology***, Chapter 1].

Let us draw a short analogy to topology. Suppose we have two topologies τ_1 and τ_2 on a set X. Then τ_1 is called *finer* than τ_2 if $\tau_2 \subseteq \tau_1$. This makes sense: τ_1 has more open sets, and can therefore possibly detect more local behaviour of the points of X, and hence could be called 'finer'. We also say τ_2 is *coarser* than τ_1. Now, this can be expressed in terms of the continuity of the identity function id_X: τ_1 is finer than τ_2 if and only if the map $\mathrm{id}_X : (X, \tau_1) \to (X, \tau_2)$ is continuous. We can mimic this definition in diffeology.

Definition 2.22. Consider two diffeologies $\mathcal{D}_1$ and $\mathcal{D}_2$ on a set X. We say $\mathcal{D}_1$ is *finer* than $\mathcal{D}_2$ if the identity map $\mathrm{id}_X : (X, \mathcal{D}_1) \to (X, \mathcal{D}_2)$ is smooth. We also say that $\mathcal{D}_2$ is *coarser* than $\mathcal{D}_1$.

Note that $\mathcal{D}_1$ is finer than $\mathcal{D}_2$ if and only if $\mathcal{D}_1 \subseteq \mathcal{D}_2$. The relation of *fineness* on diffeologies can be expressed as subset inclusions that are exactly opposite to the case of topology! At first glance it may appear counterintuitive to say a diffeology is finer if it contains *less* plots than another, but we can understand this as follows. A diffeology determines what functions into a set are smooth. If all of them are smooth (which is the case when the diffeology is just $\mathrm{Param}(X)$, see Definition **2.23**), then we may as well treat it like a set, because there is no distinction between smooth- and non-smooth functions. That is the least refined notion of smoothness on a set, the *coarsest* smooth structure. The *fewer* plots we allow, then, the *finer* the smooth structure becomes.

Definition 2.23. Let X be a fixed set. The set of all diffeologies on X is a partially ordered set, whose relation is that of fineness, defined just previously. We will see now that the partial order is bounded.

The *coarse diffeology* on X is just the collection of all parametrisations: $\mathcal{D}_X^\bullet := \mathrm{Param}(X)$. All three axioms of Definition **2.2** are trivially fulfilled. If $X^\bullet$ denotes the set X equipped with its coarse diffeology, then every function into a coarse space is smooth: $C^\infty(Y, X^\bullet) = \mathrm{Hom}_{\mathbf{Set}}(Y, X)$. The coarse diffeology $\mathcal{D}_X^\bullet$ is an upper bound for the partial order of diffeologies on X.

On the other hand, we can define a lower bound as follows. A parametrisation $\alpha : U_\alpha \to X$ is called *locally constant* if there exists an open cover $(U_i)_{i \in I}$ of U_α such that each $\alpha|_{U_i}$ is constant. The collection $\mathcal{D}_X^\circ$ of all locally constant parametrisations forms a diffeology, called the *discrete diffeology*. We prove that the discrete diffeology on X is indeed the finest. Take any other diffeology $\mathcal{D}_X$ on X, and pick a locally constant plot $\alpha \in \mathcal{D}_X^\circ$ as above. By the Axiom of Covering, each $\alpha|_{U_i}$ is an element of $\mathcal{D}_X$. But then it follows immediately by the Axiom of Locality that the entire plot α must be in $\mathcal{D}_X$, and hence $\mathcal{D}_X^\circ \subseteq \mathcal{D}_X$. If we denote the set X endowed with its discrete diffeology by X°, then we get $C^\infty(X^\circ, Y) = \mathrm{Hom}_{\mathbf{Set}}(X, Y)$.

Every diffeology $\mathcal{D}_X$ on a set X therefore lives somewhere in between the discrete- and coarse diffeologies:

$$\mathcal{D}_X^\circ \subseteq \mathcal{D}_X \subseteq \mathcal{D}_X^\bullet,$$

proving that the diffeologies on a fixed set, ordered by fineness, form a bounded partially ordered set. We will see now that it is in fact a bounded *complete lattice*, which means that all infima and suprema exist. The following proposition will help us prove this.

Proposition 2.24. *Consider a family of diffeologies $(\mathcal{D}_i)_{i \in I}$ on a fixed set X. Then $\mathcal{D} := \bigcap_{i \in I} \mathcal{D}_i$ is also a diffeology. In fact, it is the unique coarsest diffeology on X contained in each $\mathcal{D}_i$.*

Proof. We verify that the three Axioms of Diffeology are satisfied for $\mathcal{D}$.

The Axiom of Covering is clearly satisfied, for each $\mathcal{D}_i$ contains every constant parametrisation, and hence so must $\mathcal{D}$.

For the Axiom of Smooth Compatibility, let $\alpha : U_\alpha \to X$ be a plot in $\mathcal{D}$, and consider a smooth map $h : V \to U_\alpha$ between Euclidean domains. Smooth Compatibility holds in each $\mathcal{D}_i$, so we get that $\alpha \circ h \in \mathcal{D}_i$, for every $i \in I$. Hence $\alpha \circ h \in \mathcal{D}$.

Lastly, for the Axiom of Locality, pick a parametrisation $\alpha \in \mathrm{Param}(X)$, with an open cover $(U_j)_{j\in J}$ of its domain such that the restrictions $\alpha|_{U_j}$ are plots in $\mathcal{D}$. Then for each pair $(i, j) \in I \times J$ we have $\alpha|_{U_j} \in \mathcal{D}_i$, so by the Axiom of Locality for $\mathcal{D}_i$ we get that $\alpha \in \mathcal{D}_i$, and since i is arbitrary, we get $\alpha \in \mathcal{D}$.

That $\mathcal{D}$ is the unique coarsest diffeology on X contained in each $\mathcal{D}_i$ is clear from its construction. □

The result of this proposition will help us create many interesting diffeological constructions, because it allows us to define the infimum and supremum of any family of diffeologies.

Definition 2.25. Let X be a fixed set, and consider a family of diffeologies $\mathbf{D} = (\mathcal{D}_i)_{i\in I}$ on X. The *infimum* of $(\mathcal{D}_i)_{i\in I}$ is the diffeology obtained by taking the intersection:

$$\inf(\mathbf{D}) = \inf_{i\in I} \mathcal{D}_i := \bigcap_{i\in I} \mathcal{D}_i.$$

This is the coarsest diffeology that is contained in each $\mathcal{D}_i$.

To define the supremum, we introduce the collection of diffeologies that contains every element of the family: $\overline{\mathbf{D}} := \{\text{diffeology } \mathcal{D} : \forall i \in I : \mathcal{D}_i \subseteq \mathcal{D}\}$. Note that $\overline{\mathbf{D}}$ is not empty, because it contains the coarse diffeology $\mathcal{D}^{\bullet}_X$. We then define

$$\sup(\mathbf{D}) = \sup_{i\in I} \mathcal{D}_i := \inf(\overline{\mathbf{D}}).$$

This is the finest diffeology containing each $\mathcal{D}_i$.

We refer to [***Diffeology***, Article 1.25] for more discussion on these notions. We will use them to construct the diffeologies of the categorical constructions. Note that the existence of infima and suprema of diffeologies on X means that the partially ordered set of diffeologies, seen as a category, is complete and cocomplete (Section **2.2.6**). An important and useful tool in the description of diffeologies is the following.

Definition 2.26. Let $\mathcal{F} \subseteq \mathrm{Param}(X)$ be a family of parametrisations on a set X. There is a unique finest diffeology $\langle\mathcal{F}\rangle$ on X containing $\mathcal{F}$, called the *diffeology generated by* $\mathcal{F}$. It is defined as the infimum over the family of diffeologies that contain $\mathcal{F}$, i.e., the supremum over the single element family $\mathbf{D} = \{\mathcal{F}\}$. Given a diffeology $\mathcal{D}_X$ such that $\mathcal{D}_X = \langle\mathcal{F}\rangle$, we say that $\mathcal{F}$ is a *generating family* for $\mathcal{D}_X$. If, in addition, the images of the elements of $\mathcal{F}$ cover X, i.e. $\bigcup_{f\in\mathcal{F}} \mathrm{im}(f) = X$, we say that $\mathcal{F}$ is a *covering generating family.*

The plots of a diffeology generated by a family are characterised as follows ([***Diffeology***, Article 1.68]):

Proposition 2.27. *Let $\mathcal{F}$ be a family of parametrisations on a set X. In general, the plots of $\langle\mathcal{F}\rangle$ are characterised as follows:*

> *A parametrisation $\alpha : U_\alpha \to X$ is a plot in $\langle\mathcal{F}\rangle$ if and only if for every $t \in U_\alpha$ there exists an open neighbourhood $t \in V \subseteq U_\alpha$ such that either $\alpha|_V$ is constant, or is of the form $\alpha|_V = f \circ h$, where $f : W \to X$ is an element in $\mathcal{F}$, and $h : V \to W$ is a smooth map between Euclidean domains.*

If $\mathcal{F}$ is a covering *generating family, the condition of $\alpha|_V$ to be constant can be left out, and we get a simpler characterisation:*

> *A parametrisation $\alpha : U_\alpha \to X$ is a plot in $\langle\mathcal{F}\rangle$ if and only if for every $t \in U_\alpha$ there exists an open neighbourhood $t \in V \subseteq U_\alpha$ such that $\alpha|_V = f \circ h$, where $f : W \to X$ is an element in $\mathcal{F}$, and $h : V \to W$ is a smooth map between Euclidean domains.*

Proof. Let us denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the collections of parametrisations defined by these two characterisations. (So, $\mathcal{D}_1$ is the set of all parametrisations that are locally constant, or factor smoothly through an element of $\mathcal{F}$.) Starting with the first claim, it suffices to show that $\mathcal{D}_1$ is a diffeology that contains $\mathcal{F}$, and is contained in every other diffeology that contains $\mathcal{F}$. The very definition of $\mathcal{D}_1$ ensures that the Axioms of Covering and Locality hold. We therefore need only confirm the Axiom of Smooth Compatibility. For that, let $\alpha : U_\alpha \to X$ be a parametrisation in $\mathcal{D}_1$, and fix a smooth map $h : W \to U_\alpha$ between Euclidean domains. We need to show that $\alpha \circ h \in \mathcal{D}_1$. Each $t \in W$ gives a point $h(t) \in U_\alpha$. Since $\alpha \in \mathcal{D}_1$, we can find an open neighbourhood $h(t) \in V \subseteq U_\alpha$ such that either $\alpha|_V$ is constant, or $\alpha|_V = f \circ k$ as described. If $\alpha|_V$ is constant, then $h^{-1}(V)$ is an open neighbourhood of $t \in W$, and $(\alpha \circ h)|_{h^{-1}(V)} = \alpha|_V \circ h|_{h^{-1}(V)}$ is then also constant. Similarly, if $\alpha|_V = f \circ k$, we get $(\alpha \circ h)|_{h^{-1}(V)} = f \circ (k \circ h|_{h^{-1}(V)})$, which is exactly of the desired form. This proves that $\mathcal{D}_1$ is a diffeology on X.

The diffeologies $\mathcal{D}_1$ and $\mathcal{D}_2$ both contain $\mathcal{F}$, which we can see by setting $h = \mathrm{id}_W$. Suppose that $\mathcal{D}$ is another diffeology on X that contains $\mathcal{F}$. Let $\alpha : U_\alpha \to X$ be a plot in $\mathcal{D}_1$, and consider the restriction $\alpha|_V$ that is either constant or of the form $f \circ h$. If it is constant, then $\alpha|_V$ is clearly also in $\mathcal{D}$. And by the Axiom of Smooth Compatibility, together with the assumption that $\mathcal{D}$ contains $\mathcal{F}$, it follows that also $\alpha|_V = f \circ h \in \mathcal{D}$. In this way we can create an open cover $(V_t)_{t \in U_\alpha}$ of U_α such that each restriction $\alpha|_{V_t}$ is in $\mathcal{D}$. The Axiom of Locality then gives that the entire plot α must be in $\mathcal{D}$, and we conclude $\mathcal{D}_1 \subseteq \mathcal{D}$. This proves that $\mathcal{D}_1$ is a diffeology on X that contains $\mathcal{F}$, and is contained in any other diffeology that contains $\mathcal{F}$, and so $\mathcal{D}_1 = \langle \mathcal{F} \rangle$, as claimed.

The result for the second characterisation follows similarly. Almost the exact same argument proves that $\mathcal{D}_2$ is also a diffeology. Only, instead of using the local constancy of the plots, we need to use Proposition **2.3** together with the fact $\mathcal{F}$ covers X to show that $\mathcal{D}_2$ satisfies the Axiom of Covering. The rest of the argument is the same, which shows that if $\mathcal{F}$ is a covering family of parametrisations, then $\mathcal{D}_2 = \langle \mathcal{F} \rangle$. □

The definition of generating families gives a more intuitive definition of the supremum of a family of diffeological spaces:

Proposition 2.28. *Let $\mathbf{D} = (\mathcal{D}_i)_{i \in I}$ be a family of diffeologies on X. Then the supremum of $\mathbf{D}$ is the diffeology generated by the union of all its elements:*

$$\sup(\mathbf{D}) = \left\langle \bigcup_{i \in I} \mathcal{D}_i \right\rangle.$$

Proof. The supremum $\sup(\mathbf{D})$ is, by definition, the finest diffeology on X that contains each $\mathcal{D}_i$. But $\left\langle \bigcup_{i \in I} \mathcal{D}_i \right\rangle$ contains $\bigcup_{i \in I} \mathcal{D}_i$, so in particular each individual diffeology $\mathcal{D}_i$. Hence $\sup(\mathbf{D}) \subseteq \left\langle \bigcup_{i \in I} \mathcal{D}_i \right\rangle$.

Conversely, $\left\langle \bigcup_{i \in I} \mathcal{D}_i \right\rangle$ is the finest diffeology containing $\bigcup_{i \in I} \mathcal{D}_i$. But $\sup(\mathbf{D})$ contains each $\mathcal{D}_i$, so in particular their union. Hence $\left\langle \bigcup_{i \in I} \mathcal{D}_i \right\rangle \subseteq \sup(\mathbf{D})$, and the equality follows. □

Example 2.29. The empty family of parametrisations generates the discrete diffeology: $\langle \varnothing \rangle = \mathcal{D}^{\circ}_X$, and the collection of all parametrisations generates the coarse diffeology: $\langle \mathrm{Param}(X) \rangle = \mathcal{D}^{\bullet}_X$.

Example 2.30. Unsurprisingly, any diffeology $\mathcal{D}$ generates itself: $\mathcal{D} = \langle \mathcal{D} \rangle$.

Example 2.31. Any smooth manifold M admits an atlas $\mathscr{A}$, consisting of (depending on your conventions) a covering family of local homeomorphisms $\mathscr{A} = (\varphi_i : U_i \to M)_{i \in I}$, defined on Euclidean domains. With respect to the differentiable structure on M, these charts become smooth, so from the definition of the manifold diffeology $\mathcal{D}_M$ (Definition **2.9**), it follows that $\langle \mathscr{A} \rangle \subseteq \mathcal{D}_M$. But the elementary characterisation of smooth maps between manifolds is that their local coordinate expressions are smooth (cf. [**Lee13**, Proposition 2.5][23]). This means that if $\alpha \in \mathcal{D}_M$ is a plot $U_\alpha \to M$ in the manifold diffeology, which just means that α is smooth in the manifold sense, then for every point

[23]Note that in the standard definition of an atlas, charts map from the manifold to a Euclidean domain. Here it is more convenient to work with charts that map in the other direction, because they directly form a family of parametrisations, making it easier to think about the generated diffeology.

$t \in U_\alpha$ there exists a chart $h : V \to U_\alpha$ around t, and a chart $\varphi : W \to M$ containing $\alpha(t)$, such that $\varphi^{-1} \circ \alpha \circ h$ is smooth between the Euclidean domains $h^{-1}(\alpha^{-1}(\mathrm{im}(\varphi))) \to W$. Since h is a local diffeomorphism, we even get that $\varphi^{-1} \circ \alpha|_{\alpha^{-1}(\mathrm{im}(\varphi))}$ is smooth. Looking at Proposition **2.27**, we then see that $\alpha|_{\alpha^{-1}(\mathrm{im}(\varphi))} = \varphi \circ (\varphi^{-1} \circ \alpha|_{\alpha^{-1}(\mathrm{im}(\varphi))})$ is exactly of the form that plots in $\langle \mathscr{A} \rangle$ are supposed to have. This shows that we have the other inclusion $\mathcal{D}_M \subseteq \langle \mathscr{A} \rangle$ as well, and can thus conclude that any atlas of a smooth manifold generates the manifold diffeology:

$$\langle \mathscr{A} \rangle = \mathcal{D}_M.$$

This example shows that any given diffeology is generally not generated by a unique family of parametrisations. Namely, a smooth manifold generally admits many non-equal atlases $\mathscr{A}_1 \neq \mathscr{A}_2$ that describe the same differentiable structure, but their generated diffeologies are equal: $\langle \mathscr{A}_1 \rangle = \mathcal{D}_M = \langle \mathscr{A}_2 \rangle$.

Example 2.32. In Example **2.17** we found a diffeology on $\mathbb{R}^2$ that was not diffeomorphic to the Euclidean diffeology. This was the *wire diffeology*, whose plots all factor locally through curves in $\mathbb{R}$. With the language of generated diffeologies, we now see that the wire diffeology is just the one generated by the family of all smooth curves:

$$\mathcal{D}_{\text{wire}} = \left\langle C^\infty(\mathbb{R}, \mathbb{R}^2) \right\rangle.$$

One very important tool the construction of a generated diffeology provides is the following lemma: the smoothness of functions is completely determined by those plots that come from the generating family. This result will be used many times throughout our proofs. Remark that we use all three Axioms of Diffeology in this proof.

Lemma 2.33. *Let X be a diffeological space whose diffeology is generated by a family $\mathcal{F}$. Then a function $f : X \to Y$ between diffeological spaces is smooth if and only if for every $g \in \mathcal{F}$ we have $f \circ g \in \mathcal{D}_Y$.*

Proof. The "only if" direction is immediate, because $\mathcal{F} \subseteq \mathcal{D}_X$. For the other direction, suppose that f sends all generating parametrisations $g \in \mathcal{F}$ to plots $f \circ g \in \mathcal{D}_Y$. We need to show that f is smooth. Take then $\alpha \in \mathcal{D}_X$ to be an arbitrary plot, so that by Proposition **2.27** we know that locally $\alpha|_V = g \circ h$ for some $g \in \mathcal{F}$, or $\alpha|_V$ is constant. If $\alpha|_V$ is constant, clearly also $f \circ \alpha|_V$ will be, and hence in that case $f \circ \alpha|_V \in \mathcal{D}_Y$ by the Axiom of Covering. If on the other hand $\alpha|_V = g \circ h$, then the Axiom of Smooth Compatibility and our hypothesis gives $f \circ \alpha|_V = f \circ g \circ h \in \mathcal{D}_Y$. Hence in either case $f \circ \alpha|_V \in \mathcal{D}_Y$, and it follows by the Axiom of Locality that $f \circ \alpha \in \mathcal{D}_Y$, showing that f is smooth. □

Lemma 2.34. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two families of parametrisations on X, such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Generating diffeologies preserves this inclusion: $\langle \mathcal{F}_1 \rangle \subseteq \langle \mathcal{F}_2 \rangle$.*

Proof. Using Proposition **2.27**, a plot $\alpha \in \langle \mathcal{F}_1 \rangle$ is locally constant, or locally of the form $f \circ h$, where $f \in \mathcal{F}_1$, and hence $f \in \mathcal{F}_2$. In either case, α is locally in $\langle \mathcal{F}_2 \rangle$, and the result follows by the Axiom of Locality. □

Lemma 2.35. *Suppose that $\mathcal{D}_1 = \langle \mathcal{F}_1 \rangle$ and $\mathcal{D}_2 = \langle \mathcal{F}_2 \rangle$ are two generated diffeologies on X. Then*

$$\langle \mathcal{F}_1 \cup \mathcal{F}_2 \rangle = \langle \mathcal{D}_1 \cup \mathcal{D}_2 \rangle.$$

Proof. Since either diffeology $\mathcal{D}_1$ and $\mathcal{D}_2$ contains its generating family, the inclusion $\langle \mathcal{F}_1 \cup \mathcal{F}_2 \rangle \subseteq \langle \mathcal{D}_1 \cup \mathcal{D}_2 \rangle$ follows immediately from the previous Lemma **2.34**.

For the converse inclusion, suppose that $\alpha \in \langle \mathcal{D}_1 \cup \mathcal{D}_2 \rangle$ is a plot. Then, locally, we have $\alpha|_V = \beta \circ h$, where $\beta \in \mathcal{D}_1 \cup \mathcal{D}_2$. We can do this by the second characterisation in Proposition **2.27**. However, since in turn these diffeologies are generated by $\mathcal{F}_1$ and $\mathcal{F}_2$, we can further find a restriction such that either $\beta|_W$ is constant, or $\beta|_W = f \circ k$, where $f \in \mathcal{F}_1 \cup \mathcal{F}_2$. In that case, we get $\alpha|_{h^{-1}(W)} = \beta|_W \circ h|_{h^{-1}(W)} = f \circ k \circ h|_{h^{-1}(W)}$, which is exactly what the plots of $\langle \mathcal{F}_1 \cup \mathcal{F}_2 \rangle$ locally look like. □

Pulling back diffeologies. Given a function $f : X \to Y$ on sets, it will be of interest to know how a diffeology on either its domain or codomain interacts with f itself. We have already seen in Definition **2.23** that if its domain is discrete, or its codomain is coarse, then f will automatically be smooth. But what if this is not the case? In particular, we may ask: given a diffeology $\mathcal{D}_Y$ on its codomain, what is the coarsest diffeology on the domain X such that f is smooth?

Definition 2.36. Let $f : X \to Y$ be a function, and let $\mathcal{D}_Y$ be a diffeology on Y. Let $\mathbf{D}$ be the family of diffeologies on X such that f is smooth. The *pullback diffeology* is defined as $f^*(\mathcal{D}_Y) := \sup(\mathbf{D})$. We claim that this is the coarsest diffeology on X such that f is smooth, and that it is given by:

$$f^*(\mathcal{D}_Y) = \{\alpha \in \mathrm{Param}(X) : f \circ \alpha \in \mathcal{D}_Y\}.$$

Proof. Note that $\mathbf{D}$ is non-empty, because it contains $\mathcal{D}_X^\circ$. Let us denote the collection of parametrisations on the right hand side of the above equation by $\mathcal{D}$. It is easy to see that $\mathcal{D}$ is itself a diffeology on X that makes f smooth, so $\mathcal{D} \in \mathbf{D}$. It follows that $\mathcal{D} \subseteq f^*(\mathcal{D}_Y)$. At the same time, if $\mathcal{D}' \in \mathbf{D}$, then it follows immediately from the explicit definition of $\mathcal{D}$ that $\mathcal{D}' \subseteq \mathcal{D}$. This means that $\mathcal{D}$ is an upper bound for $\mathbf{D}$, and hence $f^*(\mathcal{D}_Y) \subseteq \mathcal{D}$, giving equality. □

Corollary 2.37. *A function $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ is smooth if and only if $\mathcal{D}_X \subseteq f^*(\mathcal{D}_Y)$.*

A special class of functions is the one consisting of those injective functions for which the diffeology on their domain is the coarsest making it smooth:

Definition 2.38. An injective function $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ between diffeological spaces is called an *induction* if $\mathcal{D}_X = f^*(\mathcal{D}_Y)$. Inductions are exactly the smooth injective functions that are diffeomorphisms onto their images (Proposition **2.54**).

[***Diffeology***, Article 1.31] gives a criterion for a smooth map to be an induction:

> An injective smooth map $f : X \to Y$ is an induction if and only if, for every plot $\alpha : U_\alpha \to Y$ taking values in $\mathrm{im}(f)$, the parametrisation $f^{-1} \circ \alpha$ is a plot for X.

Proof. We prove that the criterion is valid. Suppose we start with an induction $f : X \to Y$, and a plot $\alpha \in \mathcal{D}_Y$ taking values in the image $\mathrm{im}(f)$. Then $f^{-1} \circ \alpha$ is a plot for X if and only if $f^{-1} \circ \alpha \in f^*(\mathcal{D}_Y)$, which by the definition of the pullback diffeology is in turn equivalent to $f \circ f^{-1}\alpha = \alpha \in \mathcal{D}_Y$.

Conversely, suppose that f satisfies the property described in the criterion. Since f is smooth, we already know that $\mathcal{D}_X \subseteq f^*(\mathcal{D}_Y)$ (Corollary **2.37**). To prove the other inclusion, take a plot $\alpha \in f^*(\mathcal{D}_Y)$, which means that $f \circ \alpha \in \mathcal{D}_Y$. Obviously, $f \circ \alpha$ takes values in the image of f, so by hypothesis $f^{-1} \circ f \circ \alpha = \alpha \in \mathcal{D}_X$, proving the equality $\mathcal{D}_X = f^*(\mathcal{D}_Y)$. □

Lemma 2.39. *Let $f : Y \to Z$ and $g : X \to Y$ be two smooth functions. If $f \circ g$ is an induction, then g is an induction.*

Proof. We use the criterion of Definition **2.38** to prove this claim. First, from elementary set theory we know that g must be injective. We therefore consider a plot $\alpha : U_\alpha \to Y$ taking values in $\mathrm{im}(g)$. Then, since f is smooth, we get another plot $f \circ \alpha : U_\alpha \to Z$ taking values in $\mathrm{im}(f \circ g)$. Since the composition is an induction, it follows by the criterion that $(f \circ g)^{-1} \circ f \circ \alpha$ is a plot of X. We do not know whether the left inverse of f exists or not, but in any case, this function is equal to $g^{-1} \circ \alpha$. Again using the criterion, this shows that g is an induction. □

Pushing forward diffeologies. Of course, we also get a converse question: given a diffeology $\mathcal{D}_X$ on the domain of f, what is the finest diffeology on Y such that f is smooth?

Definition 2.40. Let $f : X \to Y$ be a function between sets, and let $\mathcal{D}_X$ be a diffeology on X. Let $\mathbf{D}$ be the family of diffeologies on Y such that f becomes smooth. The *pushforward diffeology* is defined as $f_*(\mathcal{D}_X) := \inf(\mathbf{D})$. Note that $\mathbf{D}$ is non-empty, because it contains the coarse diffeology $\mathcal{D}_Y^\bullet$.

It is characterised as follows:

A parametrisation $\alpha : U_\alpha \to Y$ is a plot in $f_*(\mathcal{D}_X)$ if and only if for every point $t \in U_\alpha$ there exists an open neighbourhood $t \in V \subseteq U_\alpha$ such that $\alpha|_V$ is either constant, or there exists a plot $\beta : V \to X$ in $\mathcal{D}_X$ such that $\alpha|_V = f \circ \beta$.

In other words, every plot in $f_*(\mathcal{D}_X)$ lifts locally through a plot in $\mathcal{D}_X$ along f. We claim that $f_*(\mathcal{D}_X)$ is the finest diffeology on Y such that f is smooth.

Proof. Let $\mathcal{D}$ denote the set of parametrisations described in the characterisation above, i.e., the ones that are locally constant or locally lift through plots of X along f. It is easy to check that $\mathcal{D}$ forms a diffeology on Y. We will show that it is simultaneously a lower bound and an element of $\mathbf{D}$. It is clear from the characterisation of $\mathcal{D}$ that f becomes smooth, because for every $\beta \in \mathcal{D}_X$, the composition $f \circ \beta$ even globally lifts along f. Hence $\mathcal{D} \in \mathbf{D}$. Now let $\mathcal{D}' \in \mathbf{D}$ be another diffeology on Y such that f becomes smooth. Every plot $\alpha \in \mathcal{D}$ then has the property that locally $\alpha|_V$ is constant, or $\alpha|_V = f \circ \beta$, for $\beta \in \mathcal{D}_X$. If $\alpha|_V$ is constant, then it is an element of $\mathcal{D}'$ by the Axiom of Covering. If $\alpha|_V = f \circ \beta$, then since f is smooth with respect to $\mathcal{D}'$, we also get $\alpha|_V \in \mathcal{D}'$. By the Axiom of Locality it then follows the entire plot α must be in $\mathcal{D}'$, showing that $\mathcal{D} \subseteq \mathcal{D}'$. It follows immediately that $\mathcal{D} = f_*(\mathcal{D}_X)$ is the finest diffeology on Y such that f is smooth. □

Corollary 2.41. *A function $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ is smooth if and only if $f_*(\mathcal{D}_X) \subseteq \mathcal{D}_Y$.*

The dual notion of an induction (Definition **2.38**) is the following. They are the class of surjective functions whose diffeology on the codomain is the finest making them smooth:

Definition 2.42. Let $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ be a function between diffeological spaces. We call f a *subduction* if it is surjective, and $f_*(\mathcal{D}_X) = \mathcal{D}_Y$. Note that f automatically becomes smooth by Corollary **2.41**.

Since subductions will play a very special rôle for us in Chapter **IV**, we will discuss them in more detail in Section **2.6**. They also play a key part in defining diffeologies on quotients of spaces, see Definition **2.64**.

Lemma 2.43. *Let $f : X \to Y$ be a function, and consider a diffeology $\mathcal{D}_X$ on the domain. Then the pushforward diffeology satisfies $f_*(\mathcal{D}_X) = \langle f \circ \mathcal{D}_X \rangle$, where $f \circ \mathcal{D}_X := \{f \circ \alpha : \alpha \in \mathcal{D}_X\}$.*

Proof. Clearly $\langle f \circ \mathcal{D}_X \rangle$ makes f smooth, so $f_*(\mathcal{D}_X) \subseteq \langle f \circ \mathcal{D}_X \rangle$. The other inclusion directly follows from the characterisation in Proposition **2.27** of plots in a generated diffeology, which agrees with the defining characterisation in Definition **2.40**. □

This lemma makes it easy to check smoothness of functions defined on the codomain of a subduction. Namely, if $f : X \to Y$ is a subduction, and $g : Y \to Z$ is a function, then we only need to check whether the parametrisations of the form $g \circ f \circ \alpha$ are plots of Z, for $\alpha \in \mathcal{D}_X$. This follows directly from Lemmas **2.33** and **2.43**. This will become useful later, when we study quotients.

The initial- and final diffeologies. In what follows we describe two constructions that generalise the pushforward- and pullback diffeologies to allow instead for diffeologies to be generated by *families* of smooth maps. These constructions are defined in [**Vin08**, Section 1.3.2], and are called the *strong-* and *weak diffeologies*. For their proofs they refer to an article (by the same author) that we could not find, so we reproduce them here, independently. Note that these two constructions are the same as the *initial-* and *final diffeologies*, respectively, mentioned (but not constructed) in [**CSW14**, Section 2]. We adopt their terminology.

Definition 2.44. Let $\mathcal{D}_X$ be a diffeology on a set X, and consider a family $\mathbf{D}$ of diffeologies on X. We say that $\mathbf{D}$ *covers* $\mathcal{D}_X$ if $\sup(\mathbf{D}) = \mathcal{D}_X$, and we say that $\mathbf{D}$ *cocovers* $\mathcal{D}_X$ if $\inf(\mathbf{D}) = \mathcal{D}_X$. In those cases, we say that $\mathbf{D}$ is a *cover* or *cocover* of $\mathcal{D}_X$, respectively.

The reason for the terminology of *covering* makes more sense in light of the notation of Proposition **2.28**, which says that $(\mathcal{D}_i)_{i \in I}$ covers $\mathcal{D}_X$ if and only if $\left\langle \bigcup_{i \in I} \mathcal{D}_i \right\rangle = \mathcal{D}_X$. The following proposition gives even more justification.

Proposition 2.45. *Let* $\mathbf{D} = (\mathcal{D}_i)_{i\in I}$ *be a cover for* $\mathcal{D}_X$*. Then* $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ *is smooth if and only if* $f : (X, \mathcal{D}_i) \to (Y, \mathcal{D}_Y)$ *is smooth for all* $i \in I$*.*

Similarly, if $\mathbf{D} = (\mathcal{D}_i)_{i\in I}$ *cocovers* $\mathcal{D}_X$*, then* $f : (Z, \mathcal{D}_Z) \to (X, \mathcal{D}_X)$ *is smooth if and only if for each* $i \in I$ *the map* $f : (Z, \mathcal{D}_Z) \to (X, \mathcal{D}_i)$ *is smooth.*

Proof. We prove the first assertion. Suppose that $\mathbf{D}$ covers $\mathcal{D}_X$. Note that $\sup(\mathbf{D})$ is the finest diffeology containing each $\mathcal{D}_i$, so in particular we have $\mathcal{D}_i \subseteq \mathcal{D}_X$. It follows immediately that if $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ is smooth, then each $f : (X, \mathcal{D}_i) \to (Y, \mathcal{D}_Y)$ is also smooth.

Conversely, suppose that $f : (X, \mathcal{D}_i) \to (Y, \mathcal{D}_Y)$ is smooth for every $i \in I$, and pick a plot $\alpha \in \mathcal{D}_X$. By Propositions **2.27** and **2.28** we know that we can find an open cover $(V_t)_{t\in U_\alpha}$ of U_α, together with plots $\beta_t \in \bigcup_{i\in I} \mathcal{D}_i$ and smooth maps h_t, such that $\alpha|_{V_t} = \beta_t \circ h_t$. Now each β_t is contained in some $\mathcal{D}_i$, for which f is smooth, so $f \circ \beta_t \in \mathcal{D}_Y$. It follows from the Axiom of Smooth Compatibility for $\mathcal{D}_Y$ that $f \circ \alpha|_{V_t} = f \circ \beta_t \circ h_t \in \mathcal{D}_Y$. The Axiom of Locality hence gives that $f \circ \alpha \in \mathcal{D}_Y$, which proves $f : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ is smooth.

The claim about cocovers is much easier, because if $\mathbf{D}$ is a cocover for $\mathcal{D}_X$, then we know $\mathcal{D}_X$ is just the intersection $\bigcap_{i\in I} \mathcal{D}_i$. The result then follows quite straightforwardly from the fact that $f \circ \alpha \in \mathcal{D}_X$ if and only if $f \circ \alpha \in \mathcal{D}_i$, for every $i \in I$. □

Lemma 2.46. *Let* $\mathbf{D} = (\mathcal{D}_i)_{i\in I}$ *be a family of diffeologies on the diffeological space* $(X, \mathcal{D}_X)$*. We then have the following characterisations of covers and cocovers:*

1. *The family* $\mathbf{D}$ *covers* $\mathcal{D}_X$ *if and only if for every diffeological space* $(Y, \mathcal{D}_Y)$ *we have*

$$C^\infty(X, Y) = \bigcap_{i\in I} C^\infty(X_i, Y),$$

where X_i *denotes the set* X *equipped with the diffeology* $\mathcal{D}_i$*.*

2. *The family* $\mathbf{D}$ *cocovers* $\mathcal{D}_X$ *if and only if for every diffeological space* $(Z, \mathcal{D}_Z)$ *we have*

$$C^\infty(Z, X) = \bigcap_{i\in I} C^\infty(Z, X_i).$$

Proof. Let us prove the first point. If $\mathbf{D}$ covers $\mathcal{D}_X$, the desired equation follows immediately from Proposition **2.45**. We therefore focus on the converse implication. Suppose that the displayed equation holds; we need to show $\mathcal{D}_X = \sup(\mathbf{D})$. In particular, we can set $Y = X$, as diffeological spaces. Then $\mathrm{id}_X \in C^\infty(X, X)$, so it follows that for all $i \in I$ the identity map $\mathrm{id}_X : X_i \to X$ is smooth, which just means that $\mathcal{D}_i \subseteq \mathcal{D}_X$. But $\sup(\mathbf{D})$ is the finest diffeology containing each $\mathcal{D}_i$, so it follows $\sup(\mathbf{D}) \subseteq \mathcal{D}_X$. For the other inclusion, note that each identity map $\mathrm{id}_X : (X, \mathcal{D}_i) \to (X, \sup(\mathbf{D}))$ is smooth because $\sup(\mathbf{D})$ contains each $\mathcal{D}_i$. But then the displayed equation then gives that $\mathrm{id}_X : (X, \mathcal{D}_X) \to (X, \sup(\mathbf{D}))$ is smooth, which gives the other inclusion: $\mathcal{D}_X \subseteq \sup(\mathbf{D})$. This proves the first claim.

The proof of the second claim is quite similar. Again, if $\mathbf{D}$ cocovers $\mathcal{D}_X$, the desired equation follows from the second part of Proposition **2.45**. Suppose therefore that the second displayed equation in the lemma holds; we need to show $\mathcal{D}_X = \inf(\mathbf{D})$. Setting $Z = X$ as diffeological spaces give that for every $i \in I$ the identity map $\mathrm{id}_X : X \to X_i$ is smooth, which gives $\mathcal{D} \subseteq \mathcal{D}_i$. But $\inf(\mathbf{D})$ is the coarsest diffeology contained in each $\mathcal{D}_i$, so $\mathcal{D}_X \subseteq \inf(\mathbf{D})$. For the other inclusion, note that $\inf(\mathbf{D})$ is contained in each $\mathcal{D}_i$, so that each identity map $\mathrm{id}_X : (X, \inf(\mathbf{D})) \to (X, \mathcal{D}_i)$ is smooth, which by the displayed equation gives that $\inf(\mathbf{D}) \subseteq \mathcal{D}_X$. Hence $\mathbf{D}$ is a cocover for $\mathcal{D}_X$, and the second claim is proven. □

Theorem 2.47. *Consider a family of functions* $(f_i : X_i \to A)_{i\in I}$*, defined on a family of diffeological spaces* $(X_i, \mathcal{D}_i)_{i\in I}$*. Then the following three conditions are equivalent for a diffeology* $\mathcal{D}_A$ *on* A*:*

1. *The diffeology* $\mathcal{D}_A$ *is the finest diffeology on* A *such that each map* f_i *is smooth.*

2. *The family* $((f_i)_*(\mathcal{D}_i))_{i\in I}$ *of pushforward diffeologies covers* $\mathcal{D}_A$*.*

3. *A function* $g : A \to B$ *is smooth if and only if each* $g \circ f_i$ *is smooth.*

Proof. *(1 ⇒ 2)* Each pushforward diffeology $(f_i)_*(\mathcal{D}_i)$ makes the map f_i smooth, so $\sup_{i\in I}(f_i)_*(\mathcal{D}_i)$ is a diffeology on A that makes every map f_i smooth. But $\mathcal{D}_A$ is the finest such diffeology, so $\mathcal{D}_A \subseteq \sup_{i\in I}(f_i)_*(\mathcal{D}_i)$. For the other inclusion, note that all plots in $(f_i)_*(\mathcal{D}_i)$ are locally constant, or of the form $f_i \circ \beta$, where $\beta \in \mathcal{D}_i$ (recall Definition **2.40**). But $f_i : X_i \to B$ is smooth, so it follows $(f_i)_*(\mathcal{D}_i) \subseteq \mathcal{D}_A$ for every $i \in I$. This means that $\mathcal{D}_A$ contains the union $\bigcup_{i\in I}(f_i)_*(\mathcal{D}_i)$, but from Proposition **2.28** we know that $\sup_{i\in I}(f_i)_*(\mathcal{D}_i)$ is the finest diffeology that contains that union, so we obtain the other inclusion: $\sup_{i\in I}(f_i)_*(\mathcal{D}_i) \subseteq \mathcal{D}_A$.

(2 ⇒ 3) For this, we use the first part of Lemma **2.46**, and the characterisation of the pushforward diffeology in Lemma **2.43**. A function $g : A \to B$ is smooth if and only if $g : A_i \to B$ is smooth for every $i \in I$, where A_i is the set A endowed with the pushforward diffeology $(f_i)_*(\mathcal{D}_i) = \langle f_i \circ \mathcal{D}_i \rangle$. Then Lemma **2.33** says that g_i is smooth if and only if for every $\beta \in \mathcal{D}_i$ we have $g \circ f_i \circ \beta \in \mathcal{D}_B$, which is just equivalent to $g \circ f_i$ being smooth.

(3 ⇒ 1) Lastly, suppose that any function $g : A \to B$ is smooth if and only if each $g \circ f_i$ is smooth. Setting $g = \mathrm{id}_A$ and $\mathcal{D}_B = \mathcal{D}_A$ gives that every f_i is smooth with respect to $\mathcal{D}_A$. If $\mathcal{D}'$ is another diffeology on A that makes every f_i smooth, then $\mathrm{id}_A : (A, \mathcal{D}_A) \to (A, \mathcal{D}')$ is smooth, which gives $\mathcal{D}_A \subseteq \mathcal{D}'$. □

Theorem 2.48. *Consider a family of functions $(g_i : B \to X_i)_{i\in I}$, defined into a family of diffeological spaces $(X_i, \mathcal{D}_i)_{i\in I}$. The following three conditions for a diffeology $\mathcal{D}_B$ on B are equivalent:*

1. *The diffeology $\mathcal{D}_B$ is the coarsest diffeology on B such that all the maps g_i become smooth.*
2. *The family $(g_i^*(\mathcal{D}_i))_{i\in I}$ of pullback diffeologies cocovers $\mathcal{D}_B$.*
3. *A function $f : A \to B$ is smooth if and only if each $g_i \circ f$ is smooth.*

Proof. The proof of this is, to a large extent, analogous to (and simpler than) that of Theorem **2.47**. We shall leave the details to the reader.

(1 ⇒ 2) The intersection $\inf_{i\in I} g_i^*(\mathcal{D}_i)$ makes every map g_i smooth, so if $\mathcal{D}_B$ is the coarsest such diffeology, then $\inf_{i\in I} g_i^*(\mathcal{D}_i) \subseteq \mathcal{D}_B$. Conversely, because each pullback $g_i^*(\mathcal{D}_i)$ is the coarsest diffeology such that g_i is smooth, we get $\mathcal{D}_B \subseteq \inf_{i\in I} g_i^*(\mathcal{D}_i)$.

(2 ⇒ 3) This follows from the second part of Lemma **2.46**.

(3 ⇒ 1) The identity map id_B is smooth, and hence every g_i is smooth with respect to $\mathcal{D}_B$. If $\mathcal{D}'$ is another diffeology on B making each g_i smooth, then $\mathrm{id}_B : (B, \mathcal{D}') \to (B, \mathcal{D}_B)$ is smooth, which proves that $\mathcal{D}' \subseteq \mathcal{D}_B$. □

Definition 2.49. Given a family of functions $(f_i : X_i \to A)_{i\in I}$, defined on a family of diffeological spaces $(X_i, \mathcal{D}_i)_{i\in I}$, we call the diffeology $\sup_{i\in I}(f_i)_*(\mathcal{D}_i)$ satisfying the equivalent conditions of Theorem **2.47** the *final diffeology on A generated by $(f_i)_{i\in I}$*.

Given a family of functions $(g_i : B \to X_i)_{i\in I}$, defined on a family of diffeological spaces $(X_i, \mathcal{D}_i)_{i\in I}$, we call the diffeology $\inf_{i\in I} g_i^*(\mathcal{D}_i)$ satisfying the equivalent conditions of Theorem **2.48** the *initial diffeology on B generated by $(g_i)_{i\in I}$*.

Example 2.50. A diffeology is itself the final diffeology induced by its plots with their Euclidean diffeologies. Let $\mathcal{D}_X$ be a diffeology on X. Then $(\alpha_*(\mathcal{D}_{\mathrm{dom}(\alpha)}))_{\alpha\in\mathcal{D}_X}$ covers $\mathcal{D}_X$, that is:

$$\mathcal{D}_X = \sup_{\alpha\in\mathcal{D}_X} \alpha_*(\mathcal{D}_{\mathrm{dom}(\alpha)}) = \left\langle \bigcup_{\alpha\in\mathcal{D}_X} \alpha_*(\mathcal{D}_{\mathrm{dom}(\alpha)}) \right\rangle.$$

Proof. We already know that plots are smooth (Proposition **2.8**), which by Theorem **2.47**(*1*) immediately gives $\sup_{\alpha\in\mathcal{D}_X} \alpha_*(\mathcal{D}_{\mathrm{dom}(\alpha)}) \subseteq \mathcal{D}_X$. The other inclusion is trivial, because if $\alpha \in \mathcal{D}_X$, then α lifts globally along itself, so $\alpha \in \alpha_*(\mathcal{D}_{\mathrm{dom}(\alpha)})$. □

In the next section we will use the initial- and final diffeologies to describe basic categorical constructions in **Diffeol**. In Section **2.2.6** we will use these to construct arbitrary limits and colimits.

2.2 Constructions with diffeological spaces

In this section we describe how to perform elementary categorical (or set-theoretical[24]) constructions in the category of diffeological spaces. Some of the material here is also discussed in [**BH11**, Section 3] and [**Gł19**, Section I.1]. We focus on: subsets, products, pullbacks, quotients, and coproducts. It should be noted that **Diffeol** is in fact complete and cocomplete, something which will follow very directly from the sheaf analysis in Section **2.7**. It is also from that analysis that we can explain why the underlying set structure of the diffeological spaces transforms in the ways we expect: namely because the limits and colimits of (pre)sheaves can be calculated point-wise. This implies that the forgetful functor **Diffeol** → **Set** preserves limits and colimits. (We prove this directly in Proposition **2.69**.) The underlying set of the product of diffeological spaces then *has* to be the product of sets, and the underlying set of the disjoint union of diffeological spaces *has* to be the disjoint union of sets, etc. The general idea is then as follows: each of the underlying set-theoretic constructions is accompanied by certain canonical maps defined to or from the original diffeological spaces. The point of the new diffeology is then that these canonical maps become smooth in the *"nicest"* way possible. The main tools for this will be the pushforward- and pullback diffeologies, defined in Section **2.1**.

2.2.1 Subsets

In classical differential topology, there are several conditions to ensure that a subset of a smooth manifold is itself a smooth manifold (sometimes in a unique way), see for example [**Lee13**, Chapter 5]. In a diffeological space, any old subset gets a canonical diffeology from the ambient one:

Definition 2.51. Let X be a fixed, ambient diffeological space. The *subset diffeology*[25], defined on a subset $A \subseteq X$, is the collection $\mathcal{D}_{A\subseteq X}$ of plots in X that take values in A. That means that, if $i_A : A \hookrightarrow X$ is the inclusion map, then

$$\mathcal{D}_{A\subseteq X} = \{\alpha \in \mathrm{Param}(A) : i_A \circ \alpha \in \mathcal{D}_X\}.$$

To ease the notation, we will usually just denote the subset diffeology of A by $\mathcal{D}_A$, since it will be clear from the context which ambient space it inherits the smooth structure from. Note that if A is endowed with its subset diffeology, the inclusion map $i_A : A \hookrightarrow X$ becomes smooth. In fact, the subset diffeology is exactly the pullback diffeology $i_A^*(\mathcal{D}_X)$ from Definition **2.36**. Therefore the subset diffeology is exactly the diffeology that makes the inclusion map into an induction (Definition **2.38**).

Example 2.52. The subset diffeology of an open subset in Cartesian space is just the Euclidean diffeology. An open subset of a smooth manifold gets a unique differentiable structure. The subset diffeology is exactly the manifold diffeology it gets from the ambient manifold.

Example 2.53. The diffeology $\mathcal{D}_{\mathrm{sub}}$ of the cross in $\mathbb{R}^2$ defined in Example **2.18** is just the subset diffeology inherited from $\mathbb{R}^2$.

The definition of the subset diffeology ensures that the inclusion map defines a diffeomorphism onto its image.

Proposition 2.54. *Let $i : A \to X$ be an induction (Definition **2.38**), and endow $\mathrm{im}(i)$ with the subset diffeology in X. Then $A \cong \mathrm{im}(i)$.*

Proof. Clearly $i : A \to \mathrm{im}(i)$ is a bijection, so it has a set-theoretic inverse $p : \mathrm{im}(i) \to A$, sending $i(a) \mapsto a$. We claim that p is smooth. To see that, let $\alpha \in \mathcal{D}_{\mathrm{im}(i)}$ be a plot of the subset diffeology of $\mathrm{im}(i)$. Hence α is just a plot in $\mathcal{D}_X$, taking values in the image of i. We need to show that $p\circ\alpha : U_\alpha \to A$ is a plot. But i is an induction, so the plots of A are exactly those such that, when composed with i, are plots in of X. This holds for $p \circ \alpha$ because p is the inverse of i, so that we get $i \circ p \circ \alpha = \alpha \in \mathcal{D}_X$. This proves that p is smooth, and hence we get a diffeomorphism $A \cong \mathrm{im}(i)$. □

[24]Since the forgetful functor U : **Diffeol** → **Set** preserves limits and colimits (Proposition **2.69**), the terms 'categorical-' and 'set-theoretical' constructions really are almost synonymous here. See also the discussion in Section **2.2.6**.

[25]Some authors call this the sub*space* diffeology. Since the term 'space' may imply some extra conditions (think of linear subspace, embedded submanifolds, etc.), we prefer the term sub*set* diffeology.

Proposition 2.55. *Let $f : X \to Y$ be a smooth function between diffeological spaces. For any subset $A \subseteq X$, when endowed with the subset diffeology, the restriction $f|_A : A \to Y$ is smooth.*

Proof. The restriction $f|_A$ is smooth because for every $\alpha \in \mathcal{D}_{A\subseteq X}$, $f|_A \circ \alpha = f \circ \alpha \in \mathcal{D}_Y$. □

2.2.2 Coproducts

In the theory of smooth manifolds, we can only take disjoint unions of manifolds as long as they have the same dimension. That means $\mathbb{R} \sqcup \mathbb{R}^2$ is not even a manifold, classically speaking. Diffeology deals easily with such objects, because it is built on bare sets, and does not discriminate based on dimension. Recall the definition of the disjoint union of sets. If $(X_i)_{i\in I}$ is a family of sets, then the disjoint union is defined as

$$\coprod_{i\in I} X_i := \{(i,x) : i \in I, x \in X_i\}\,.$$

Each component X_i is naturally injected into the disjoint union through the function $\iota_j : X_j \to \coprod_{i\in I} X_i$ defined by $x \mapsto (j,x)$. If each X_i is accompanied by a diffeology $\mathcal{D}_i$, it is our task to construct a good diffeology on $\coprod_{i\in I} X_i$. The most obvious candidate is the final diffeology induced by the family $(\iota_i)_{i\in I}$.

Definition 2.56. Let $(X_i, \mathcal{D}_i)_{i\in I}$ be a family of diffeological spaces. The *coproduct diffeology* on the disjoint union $\coprod_{i\in I} X_i$ is the final diffeology induced by the family of canonical inclusion maps $(\iota_i)_{i\in I}$.

Whenever we encounter a disjoint union of diffeological spaces, we will assume that it is endowed with the coproduct diffeology, unless mentioned otherwise, the main exception being Definition **2.95**. Unpacking the definition of the final diffeology, we see that a parametrisation α of $\coprod_{i\in I} X_i$ is a plot if and only if it is locally of the form $\iota_i \circ \beta$, where $\beta \in \mathcal{D}_i$. As we would expect, each component X_i of the family sits diffeomorphically in the disjoint union. We can demonstrate this by showing that each canonical inclusion ι_i is an induction. The property that $X_i \cong \mathrm{im}(\iota_i)$ then follows from Proposition **2.54**.

Proposition 2.57. *The canonical injections $\iota_i : X_i \to \coprod_{i\in I} X_i$ are inductions.*

Proof. Let $\mathcal{D}$ denote the coproduct diffeology on $\coprod_{i\in I} X_i$. The pullback diffeology $\iota_i^*(\mathcal{D})$ contains exactly the parametrisations $\alpha \in \mathrm{Param}(X_i)$ such that $\iota_i \circ \alpha \in \mathcal{D}$. Propositions **2.27** and **2.28** then tell us that $\iota_i \circ \alpha$ is locally of the form $\iota_j \circ \beta$, for some $j \in I$ and $\beta \in \mathcal{D}_j$. But for this to hold it has to be the case that $i = j$, and hence $\beta \in \mathcal{D}_i$. Since the inclusion ι_i is injective, it follows that α itself has to be locally, and hence globally, in $\mathcal{D}_i$. This shows that $\iota_i^*(\mathcal{D}) = \mathcal{D}_i$, proving that ι_i is an induction. □

A special case is the disjoint union of just two diffeological spaces. Following Proposition **2.28** and Lemmas **2.35** and **2.43**, we get that the coproduct diffeology on $X \sqcup Y$ is

$$\mathcal{D}_{X\sqcup Y} = \langle \iota_X \circ \mathcal{D}_X \cup \iota_Y \circ \mathcal{D}_Y \rangle\,.$$

This means that the plots of $X \sqcup Y$ are just the ones that are locally either plots of X or Y.

An interesting application of the coproduct diffeology is the following.

Proposition 2.58. *Let $\mathcal{F}$ be a covering generating family for a diffeology $\mathcal{D}_X$ on X. Then the map*

$$\mathrm{ev} : \coprod_{f\in\mathcal{F}} \mathrm{dom}(f) \longrightarrow X; \qquad (f,t) \longmapsto f(t)$$

*is a subduction, and in particular, borrowing Proposition **2.66** below, we get a diffeomorphism*

$$X \cong \coprod_{f\in\mathcal{F}} \mathrm{dom}(f) \Big/ \mathrm{ev}\,.$$

Proof. First we need to prove that ev is smooth with respect to the coproduct diffeology induced by the Euclidean diffeologies on the domains $\mathrm{dom}(f)$ of each generating plot $f \in \mathcal{F}$. These plots $\alpha : U_\alpha \to \coprod_{f\in\mathcal{F}} \mathrm{dom}(f)$ are locally of the form $\alpha|_V = \iota_f \circ h$, where $h : V \to \mathrm{dom}(f)$ is a plot in the Euclidean diffeology. Then $\mathrm{ev} \circ \alpha|_V = f \circ h$, and since $\mathcal{F}$ generates the diffeology $\mathcal{D}_X$ on X, this expression is smooth. It follows by the Axiom of Locality for $\mathcal{D}_X$ that ev is smooth.

To prove subductiveness, fix a plot $\alpha : U_\alpha \to X$ in $\mathcal{D}_X$, and a point $t \in U_\alpha$. Since the family $\mathcal{F}$ is covering and generates $\mathcal{D}_X$, the second characterisation in Proposition **2.27** allows us to find an open neighbourhood $t \in V \subseteq U_\alpha$, together with some generating plot $f \in \mathcal{F}$ and a smooth function $h : V \to \mathrm{dom}(f)$ such that $\alpha|_V = f \circ h$. This data gives a plot $\iota_f \circ h$ of $\coprod_{f\in\mathcal{F}} \mathrm{dom}(f)$, and it is easy to check that $\mathrm{ev} \circ \iota_f \circ h = f \circ h = \alpha|_V$. This proves that ev is a subduction. □

The interpretation of this result is that any diffeological space can always be obtained as the quotient (see Definition **2.64** below) of some disjoint union of Euclidean domains. Note that this does *not* mean that any diffeological space is always the quotient of a manifold, since this disjoint union can certainly be very big or change dimension.

2.2.3 Products

Forming the product of spaces is one of the few categorical constructions that the theory of manifolds handles fairly well. The main exceptions to this rule are infinite products, which usually are not finite-dimensional. We shall now construct the product of an arbitrary family of diffeological spaces $(X_i, \mathcal{D}_i)_{i\in I}$. We start by sketching the construction for an arbitrary indexing family, and then move on to the special case of binary products, which are the only ones that we will really need.

First we recall the set-theoretic definition of a product. If $(X_i)_{i\in I}$ is a family of sets, then we have a projection $\mathrm{pr}_1 : \coprod_{i\in I} X_i \to I$, sending each point $(i, x) \mapsto i$ to its corresponding index. Elements of the product of $(X_i)_{i\in I}$ are exactly the sections of this projection:

$$\prod_{i\in I} X_i := \left\{ I \xrightarrow{s} \coprod_{i\in I} X_i : \mathrm{pr}_1 \circ s = \mathrm{id}_I \right\}.$$

We think of elements of $\prod_{i\in I} X_i$ as families $(x_i)_{i\in I}$ of points $x_i \in X_i$, which determines (and is determined by) the section $s : i \mapsto (i, x_i)$. The canonical family of maps belonging to a product are the projections: $(x_i)_{i\in I} \mapsto x_j$, which formally corresponds to the function

$$\mathrm{pr}_j : \prod_{i\in I} X_i \longrightarrow X_j; \qquad s \longmapsto \mathrm{pr}_2 \circ s(j).$$

Definition 2.59. Let $(X_i, \mathcal{D}_i)_{i\in I}$ be a family of diffeological spaces. The *product diffeology* on $\prod_{i\in I} X_i$ is the initial diffeology induced by the family of canonical projections $(\mathrm{pr}_i)_{i\in I}$.

Set-theoretically, a function $f : A \to \prod_{i\in I} X_i$ into the product can be decomposed into components $f = (f_i)_{i\in I}$, where $f_i = \mathrm{pr}_i \circ f : A \to X_i$. Theorem **2.48**(*3*) then says that f is smooth if and only if each f_i is smooth.

Just like the coproduct diffeology makes the canonical inclusions inductions, the projection maps are subductions:

Proposition 2.60. *The canonical projections* $\mathrm{pr}_j : \prod_{i\in I} X_i \to X_j$ *are subductions.*

Proof. We borrow the result Proposition **2.120** from Section **2.6** below. Since the projections are smooth surjections, it suffices to check the second condition in that proposition. Given a plot $\alpha : U_\alpha \to X_j$, we define a plot Ω of $\prod_{i\in I} X_i$ as follows. The idea is that Ω moves like α in the jth component, and stays constant everywhere else. Using the Axiom of Choice, for every $i \in I \setminus \{j\}$ we can pick $x_i \in X_i$. For each $t \in U_\alpha$, $\Omega(t)$ is the function

$$\Omega(t) : I \longrightarrow \coprod_{i\in I} X_i; \qquad \Omega(t)(i) := \begin{cases} (i, x_i) & \text{if } i \neq j, \\ (j, \alpha(t)) & \text{if } i = j. \end{cases}$$

It is then obvious that $\Omega : U_\alpha \to \prod_{i\in I} X_i$ is a parametrisation of the product, and that it satisfies $\mathrm{pr}_j \circ \Omega = \alpha$. It therefore suffices to show that Ω is a plot. For this we use Theorem **2.48**(*3*). The ith component of Ω is just the constant map const_{x_i}, which is smooth, and the jth component is α, which is also smooth, proving that Ω is smooth. □

The construction of binary products will be very important for us. If $X, Y \in$ **Diffeol**, how are the plots of $X \times Y$ characterised? By definition, the product diffeology is the initial diffeology generated by the projection maps:

$$\mathcal{D}_{X\times Y} = \mathrm{pr}_1^*(\mathcal{D}_X) \cap \mathrm{pr}_2^*(\mathcal{D}_Y).$$

Since the plots of $X \times Y$ are exactly the smooth maps $U \to X \times Y$, Theorem **2.48** shows that $\alpha \in \mathcal{D}_{X\times Y}$ if and only if $\mathrm{pr}_1 \circ \alpha \in \mathcal{D}_X$ and $\mathrm{pr}_2 \circ \alpha \in \mathcal{D}_Y$. This means that a parametrisation $\alpha = (\alpha_1, \alpha_2)$ is a plot if and only if each of its components α_1 and α_2 are plots. This is, in practise, often an easy condition to check (or ensure). Another characterisation of the product diffeology is the following:

Lemma 2.61. *Let X and Y be two diffeological spaces whose diffeologies are generated by two families of parametrisations $\mathcal{F}_X$ and $\mathcal{F}_Y$, respectively. Then the product diffeology on $X \times Y$ is generated by $\mathcal{F}_X \times \mathcal{F}_Y := \{\alpha \times \beta : \alpha \in \mathcal{F}_X, \beta \in \mathcal{F}_Y\}$.*

Proof. The proof of this lemma relies on the trick that for a parametrisation $\alpha : U_\alpha \to X \times Y$, we have $(\alpha_1, \alpha_2) = (\alpha_1 \times \alpha_2) \circ \Delta_{U_\alpha}$, where $\Delta_{U_\alpha} : U_\alpha \to U_\alpha \times U_\alpha$ is the diagonal map $t \mapsto (t, t)$. Note that Δ_{U_α} is smooth, because it is just the map whose components are the identity maps on U_α. Hence we have already shown that $\mathcal{D}_{X\times Y} \subseteq \langle \mathcal{F}_X \times \mathcal{F}_Y \rangle$, because every plot of $X \times Y$ globally factors through an element of $\mathcal{F}_X \times \mathcal{F}_Y$.

For the other inclusion it suffices to show that each $\alpha_1 \times \alpha_2$ is a plot. But this follows in turn from another trick, namely that $\mathrm{pr}_i \circ (\alpha_1 \times \alpha_2) = \alpha_i \circ \mathrm{pr}_i$, where the projection on the right hand side is that of the product $U_{\alpha_1} \times U_{\alpha_2}$, which is clearly smooth. Thus both projections of $\alpha_1 \times \alpha_2$ are smooth, so it must be a plot of $X \times Y$. □

Note that in particular a diffeology generates itself, so that the product diffeology is generated by the product of the diffeologies: $\mathcal{D}_{X\times Y} = \langle \mathcal{D}_X \times \mathcal{D}_Y \rangle$. This observation, together with Lemma **2.33**, will be used countless times throughout this thesis.

Example 2.62. The product diffeology on $\mathbb{R} \times \mathbb{R}$ agrees with the Euclidean diffeology on $\mathbb{R}^2$.

2.2.4 Pullbacks

Consider two functions $f : X \to Z$ and $g : Y \to Z$, with the same codomain. We know that the pullback of f and g in **Set** is (up to unique bijection):

$$X \times_Z^{f,g} Y = \{(x, y) \in X \times Y : f(x) = g(y)\}.$$

This is just a subset of the product $X \times Y$, which, if X and Y have diffeologies, already gets the product diffeology. We therefore define:

Definition 2.63. If $f : X \to Z$ and $g : Y \to Z$ are two smooth maps between diffeological spaces, the *pullback diffeology* on $X \times_Z^{f,g} Y$ is the subset diffeology it gets from the product diffeology on $X \times Y$.

The characterisation of the plots of the pullback diffeology is quite straightforward, and again something that is easy to check in practise. Namely, by definition of the subset diffeology, $\alpha : U_\alpha \to X \times_Z^{f,g} Y$ is a plot if and only if it is a plot of $X \times Y$ that takes values in the pullback. Hence this means the components α_1 and α_2 must be plots of X and Y, respectively, satisfying $f \circ \alpha_1 = g \circ \alpha_2$:

$$\mathcal{D}_{X\times_Z^{f,g} Y} = \{\alpha \in \mathcal{D}_{X\times Y} : f \circ \alpha_1 = g \circ \alpha_2\}.$$

2.2.5 Quotients

The last construction we discuss is that of *quotients*, which provide one of the other main advantages of diffeological spaces over manifolds. In general, if we have an equivalence relation $\sim$ on a smooth manifold M, the quotient set $M/\sim$ does *not* have a differentiable structure such that the projection map $M \to M/\sim$ is a submersion (or even smooth). The quotient behaves nicely as a manifold if and only if the *Godement criterion* is satisfied ([**Ser65**, Theorem 2, p. 92]):

1. The set $R = \{(x, y) \in M \times M : x \sim y\}$ is an embedded submanifold of $M \times M$.

2. The projection map $\mathrm{pr}_2|_R : R \to M$ is a submersion.

This gives in particular the famous fact that the quotient M/G of a smooth Lie group action $G \curvearrowright M$ exists as a manifold if and only if the action is free and proper ([**AM78**, Proposition 4.1.23]). In diffeology, however, *every* quotient $X/\sim$ of a diffeological space has a natural diffeology:

Definition 2.64. Consider a diffeological space X, with diffeology $\mathcal{D}_X$. Let $\sim$ be an equivalence relation on the set X. We denote the equivalence classes under this relation by $[x] := \{y \in X : x \sim y\}$, and the *quotient* $X/\sim$ is naturally the collection of all equivalence classes. In other words, it is the image of the *canonical projection map* $p : X \to X/\sim : x \mapsto [x]$, also known as the *quotient map.* The *quotient diffeology* on $X/\sim$ is the pushforward diffeology $p_*(\mathcal{D}_X)$ of $\mathcal{D}_X$ along the canonical projection map. This is the finest diffeology on the quotient that makes the projection map smooth. In fact, p becomes a subduction.

The way in which diffeology handles quotients is in fact what distinguishes it from other approaches to generalised smooth structures. Lots of interesting examples come from two types of constructions: fibres of surjections, or quotients of group actions.

Example 2.65. Let $\pi : X \to B$ be a surjection between sets. This defines an equivalence relation on X, where elements are equivalent if and only if they inhabit the same π-fibre. The equivalence classes are therefore $[x]_\pi = \{y \in X : \pi(x) = \pi(y)\} = \pi^{-1}(\{x\})$. We denote the quotient of X by this equivalence relation by X/π. If X has a diffeology, then X/π gets the *quotient diffeology* as the pushforward of $\mathcal{D}_X$ along the projection map $X \to X/\pi : x \mapsto [x]_\pi$.

Proposition 2.66. *If $\pi : X \to B$ is a subduction, there is a diffeomorphism $B \cong X/\pi$.*

Proof. Note, first of all, that there is a bijection $\Phi : X/\pi \to B$ which sends each fibre $[x]_\pi$ to its base point $\pi(x)$. Writing the projection of the quotient as $p : X \to X/\pi$ we have $\pi = \Phi \circ p$.

Suppose now that π is a subduction, so that both the diffeology $\mathcal{D}_B$ on B, and the diffeology on X/π are determined by a pushforward of the diffeology $\mathcal{D}_X$ on X. Lemma **2.121** and Proposition **2.123** below immediately give that Φ is a diffeomorphism, but we prove this here explicitly. First we show that Φ is a subduction. For that, take a plot $\alpha : U_\alpha \to B$, and pick a point $t \in U_\alpha$. The map π is a subduction, so we can find a plot $\beta : V \to X$ defined on an open neighbourhood $t \in V \subseteq U_\alpha$, and providing a local lift $\alpha|_V = \pi \circ \beta$. If we now substitute $\pi = \Phi \circ p$, we see that the plot $p \circ \beta : V \to X/\pi$ defines a local lift for α along Φ. This shows that Φ is a subduction. But Φ is also a bijection, so it has a set-theoretic inverse Φ^{-1}. If we can show this is smooth, we are done. For that, let $\alpha : U_\alpha \to B$ be a plot, and find a local lift $\alpha|_V = \Phi \circ \beta$. We can do this because Φ is a subduction. But then: $\Phi^{-1} \circ \alpha|_V = \beta \in \mathcal{D}_{X/\pi}$. It follows by the Axiom of Locality of the quotient diffeology that $\Phi^{-1} \circ \alpha$ is a plot, and hence that Φ^{-1} is smooth. This gives the desired diffeomorphism. □

The other common way of obtaining quotients is through the orbits of group actions. Quotients M/G of Lie groups acting smoothly on a manifold M sometimes admit a differentiable structure, but to define the diffeological quotient the action does not even need to be smooth. We discuss smooth group actions in Section **3.1.2**.

Example 2.67. Let X be a diffeological space, and consider a group action $G \curvearrowright X$, denoted

$$G \times X \longrightarrow X; \qquad (g, x) \longmapsto gx.$$

We denote the *orbits* of this action by $\mathrm{Orb}_G(x) := \{gx : g \in G\}$. We denote the *orbit space* by X/G, which is defined as the set of all orbits:

$$X/G := \{\mathrm{Orb}_G(x) : x \in X\}.$$

The *quotient diffeology* on X/G is the pushforward of $\mathcal{D}_X$ along the projection map $X \to X/G : x \mapsto \mathrm{Orb}_G(x)$. This is the same diffeology as defined by the equivalence relation $\sim_G$ on X, which identifies two points if and only if they are in the same orbit.

A special case of this is the quotient G/H of a group by some subgroup $H \subseteq G$. The cosets of this subgroup are just the orbits of the right multiplication of H on G. If G carries a diffeology, then G/H gets a quotient diffeology by the equivalence relation where two group elements are identified if and only if they are in the same coset. We study groups with a diffeological structure further in Section **3.1**. Here we assume nothing about the compatibility between the group structure of G and its diffeology.

Example 2.68. Recall from Example **2.16** that we had a diffeology on the circle S^1. This diffeology agrees with its standard manifold structure. We exhibit here two equivalent ways to realise the circle as a diffeological quotient.

The first of these characterisations follows trivially, given the diffeological structure exhibited in Example **2.16**. Namely, the smooth map $\exp : \mathbb{R} \to S^1$ given by $x \mapsto e^{2\pi i x}$ is a subduction. Therefore, by Proposition **2.66** we immediately get $S^1 \cong \mathbb{R}/\exp$.

On the other hand, the additive group of integers $\mathbb{Z}$ acts by translation on the real numbers $\mathbb{R}$. For the sake of simplicity, we assume that this action is scaled by a factor of 2π, so that the action is given by $\mathbb{Z} \times \mathbb{R} \to \mathbb{R} : (n, x) \mapsto x + 2\pi n$. We denote this quotient by $\mathbb{R}/\mathbb{Z}$, and its equivalence classes are $[x]_\mathbb{Z} = \{x + 2\pi n : n \in \mathbb{Z}\} \subseteq \mathbb{R}$. It is easy to see that, with our conventions, the orbits of this action are exactly the fibres $[x]_{\exp}$ of the exponential map. We therefore get a commutative square

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\mathrm{id}_\mathbb{R}} & \mathbb{R} \\ \downarrow & & \downarrow \\ \mathbb{R}/\mathbb{Z} & \xrightarrow[\Phi]{} & \mathbb{R}/\exp, \end{array}$$

where the vertical arrows are the quotient maps, and $\Phi : [x]_\mathbb{Z} \mapsto [x]_{\exp}$. It is easy to see that Φ is a diffeomorphism (this follows also from Lemma **2.121** below). We therefore have diffeomorphisms

$$S^1 \cong \mathbb{R}/\mathbb{Z} \cong \mathbb{R}/\exp.$$

The circle S^1 is an example of a diffeological quotient that happens to be a manifold. [***Diffeology***, Article 4.6] gives a general recipe of necessary and sufficient conditions for the quotient $X/\sim$ of any diffeological space to be a smooth manifold.

2.2.6 Limits and colimits of diffeological spaces

In this section we calculate the diffeologies of arbitrary limits and colimits in **Diffeol**. Again, some of these results already appear in [**Vin08**, Section 1.3.2], whose proofs we were unable to locate. Hence we reproduce them here independently. They are also sketched in [**CSW14**, Theorem 2.5]. We will see that (co)completeness of **Diffeol** follows entirely from (co)completeness of **Set**, together with the (co)completeness of the partial orders of diffeologies on a set. Compare also to [**BH11**, Sections 5.1, 5.3], where (co)limits are calculated using sheaves.

Proposition 2.69. *The forgetful functor U : **Diffeol** $\to$ **Set** preserves limits and colimits.*

Proof. From Definition **2.23** we get two functors, $(-)^\bullet$: **Set** $\to$ **Diffeol**, sending each set to its coarse diffeological space, and $(-)^\circ$: **Set** $\to$ **Diffeol**, sending each set to its corresponding discrete diffeological space. Then, as we have already seen, since every map defined on a discrete space is smooth, and every map defined into a coarse space is smooth, we get natural bijections:

$$C^\infty(X^\circ, Y) \cong \mathrm{Hom}_{\mathbf{Set}}(X, U(Y)) \qquad \text{and} \qquad \mathrm{Hom}_{\mathbf{Set}}(U(X), Y) \cong C^\infty(X, Y^\bullet),$$

so we have a chain of adjunctions $(-)^\circ \dashv U \dashv (-)^\bullet$ that proves the claim[26]:

$$\mathbf{Diffeol} \underset{(-)^\bullet}{\overset{(-)^\circ}{\underset{\perp}{\overset{\perp}{\leftleftarrows U \rightarrow}}}} \mathbf{Set}. \qquad \square$$

This shows that the underlying set of the (co)limit of a diagram of diffeological spaces (up to a unique bijection) has to be the (co)limit of the underlying set of each space in the diagram. Since we know **Set** is complete and cocomplete, to construct (co)limits in **Diffeol** is therefore equivalent to putting diffeologies on the corresponding (co)limits in **Set**.

The following two theorems provide these diffeologies, and prove that the category **Diffeol** of diffeological spaces and smooth maps is complete and cocomplete. Together, they encapsulate all of the constructions that we have discussed in Section 2.2. (See [Vin08, Theorems 1.3.17, 1.3.18].)

Theorem 2.70. *Let $F : \mathbf{I} \to \mathbf{Diffeol}$ be a small diagram of diffeological spaces, and denote $Fi = (X_i, \mathcal{D}_i)$ for each $i \in \mathbf{I}$. Then $U \circ F : \mathbf{I} \to \mathbf{Set}$ has a limit, which we denote by $X := \lim_{i\in\mathbf{I}} X_i$, with accompanying cone $(X \xrightarrow{\mu_i} X_i)_{i\in\mathbf{I}}$. Then the limit of F exists, and is given by the cone:*

$$\left(\left(\lim_{i\in\mathbf{I}} X_i, \inf_{i\in\mathbf{I}} \mu_i^*(\mathcal{D}_i)\right) \xrightarrow{\mu_i} (X_i, \mathcal{D}_i)\right)_{i\in\mathbf{I}} \qquad \textit{so that} \qquad \lim_{i\in\mathbf{I}}(X_i, \mathcal{D}_i) \cong \left(\lim_{i\in\mathbf{I}} X_i, \inf_{i\in\mathbf{I}} \mu_i^*(\mathcal{D}_i)\right).$$

Proof. Note that $U \circ F : \mathbf{I} \to \mathbf{Set}$ has a limit, because **Set** is complete (see e.g. [Mac71, Theorem V.1.1]). We denote the limiting cone of this limit by $\mu = (X \xrightarrow{\mu_i} X_i)_{i\in\mathbf{I}}$, as in the theorem statement. Thus we have a (natural) family of functions defined into a family of diffeological spaces, so on the limit $X = \lim_{i\in\mathbf{I}} X_i$ we put the initial diffeology $\mathcal{D}_X := \inf_{i\in\mathbf{I}} \mu_i^*(\mathcal{D}_i)$ (Definition 2.49). By Theorem 2.48(*1*), each function μ_i then becomes smooth. Hence $\mu : \Delta_{(X,\mathcal{D}_X)} \to F$ is a cone for F, and not just for $U \circ F$. We prove that it is also limiting. For that, let ν be another cone for F, consisting of smooth functions $\nu_i : (Y, \mathcal{D}_Y) \to (X_i, \mathcal{D}_i)$. Then $U \circ \nu = (U\nu_i : Y \to X_i)_{i\in\mathbf{I}}$ is a cone for the limit X in **Set**, so by the universal property there exists a unique function $\Omega : Y \to X$ such that for all $i \in \mathbf{I}$ we have $\mu_i \circ \Omega = \nu_i$. However, each ν_i is smooth, so by Theorem 2.48(*3*) it follows that $\Omega : (Y, \mathcal{D}_Y) \to (X, \mathcal{D}_X)$ must be smooth as well. So not only does Ω form the unique morphism of cones from $U \circ \nu$ to $\mu : \Delta_X \to U \circ F$, but also from ν to $\mu : \Delta_{(X,\mathcal{D}_X)} \to F$. This proves that μ is a limiting cone for F. Therefore, F has a limit, and since limits are unique up to isomorphism, it is of the form as claimed in the theorem. $\square$

Theorem 2.71. *Let $F : \mathbf{I} \to \mathbf{Diffeol}$ be a small diagram of diffeological spaces, and write $Fi = (X_i, \mathcal{D}_i)$ for each $i \in \mathbf{I}$. Then $U \circ F : \mathbf{I} \to \mathbf{Set}$ has a colimit, which we denote by $X := \operatorname{colim}_{i\in\mathbf{I}} X_i$, with accompanying cocone $(X_i \xrightarrow{\mu_i} X)_{i\in\mathbf{I}}$. Then the colimit of F exists, and is given by the cocone:*

$$\left((X_i, \mathcal{D}_i) \xrightarrow{\mu_i} \left(\operatorname*{colim}_{i\in\mathbf{I}} X_i, \sup_{i\in\mathbf{I}} (\mu_i)_*(\mathcal{D}_i)\right)\right)_{i\in\mathbf{I}} \quad \textit{so that} \quad \operatorname*{colim}_{i\in\mathbf{I}}(X_i, \mathcal{D}_i) \cong \left(\operatorname*{colim}_{i\in\mathbf{I}} X_i, \sup_{i\in\mathbf{I}} (\mu_i)_*(\mathcal{D}_i)\right).$$

Proof. This proof will obviously be analogous to that of Theorem 2.70. The functor $U \circ F : \mathbf{I} \to \mathbf{Set}$ has a colimit, because **Set** is cocomplete (solve e.g. [Mac71, Exercise V.1.8]). We denote the colimiting cocone by $\mu = (X_i \xrightarrow{\mu_i} X)_{i\in\mathbf{I}}$. On $X = \operatorname{colim}_{i\in\mathbf{I}} X_i$ we put the final diffeology $\mathcal{D}_X := \sup_{i\in\mathbf{I}} (\mu_i)_*(\mathcal{D}_i)$. Theorem 2.47(*1*) then ensures each μ_i becomes smooth, and hence $\mu : F \to \Delta_{(X,\mathcal{D}_X)}$ is a cocone for F. To prove that it is colimiting, take another cocone $\nu : F \to \Delta_{(Y,\mathcal{D}_Y)}$ on F. Applying the forgetful functor U to ν, the universal property of the colimit X gives a unique function $\Omega : X \to Y$ satisfying $\Omega \circ \mu_i = \nu_i$, for all $i \in \mathbf{I}$. Theorem 2.47(*3*) then makes $\Omega : (X, \mathcal{D}_X) \to (Y, \mathcal{D}_Y)$ smooth, and we may conclude that μ is a colimiting cocone for F. By construction of the cocone μ, it follows that the colimit of F is indeed of the desired form. $\square$

Theorem 2.72. *The category* **Diffeol** *of diffeological spaces is complete and cocomplete.*

[26]Coincidentally, this adjoint triple, which we could write as Disc $\dashv U \dashv$ coDisc, defines the fundamental cohesive structure of **SmoothSet** [*n*L19c].

Infima and suprema as limits and colimits. For purely aesthetic reasons, we can force our perspective such that limits in **Diffeol** can be fully expressed as limits over other categories (as opposed to using the infimum and supremum). We have seen in Definition **2.25** that any non-empty family of diffeologies has an infimum and a supremum. This can be explained as the (co)completeness of the partial order of diffeologies on a set. For any given fixed set X, let $\mathbf{Diffeol}(X)$ denote the set of all diffeologies on X:

$$\mathbf{Diffeol}(X) := \{\mathcal{D} \subseteq \mathrm{Param}(X) : \mathcal{D} \text{ is a diffeology}\}.$$

Together with the relation of fineness (Definition **2.22**), this becomes a partially ordered set. We view it as a category, which has arrows

$$\mathrm{Hom}_{\mathbf{Diffeol}(X)}(\mathcal{D}_1, \mathcal{D}_2) = \begin{cases} \{(\mathcal{D}_1, \mathcal{D}_2)\} & \text{if } \mathcal{D}_1 \subseteq \mathcal{D}_2, \\ \varnothing & \text{otherwise.} \end{cases}$$

A diagram $F : \mathbf{I} \to \mathbf{Diffeol}(X)$ in this category is just a family of diffeologies $(\mathcal{D}_i)_{i\in\mathbf{I}}$, where $\mathcal{D}_i = Fi$. If there exists an arrow $i \to j$ in $\mathbf{I}$, this just means that $\mathcal{D}_i \subseteq \mathcal{D}_j$. A cone for F is then just a diffeology on X that is contained in every member of this family. It is then easy to see that the limit $\lim F$ is just the coarsest diffeology on X that is contained in each $\mathcal{D}_i$. In other words, $\lim F$ has to be the infimum over $(Fi)_{i\in\mathbf{I}}$. Similarly, we find that the colimit over F has to be the supremum over $(Fi)_{i\in\mathbf{I}}$. We therefore have:

Theorem 2.73. *For any set X, the category $\mathbf{Diffeol}(X)$ of diffeologies on X is complete and cocomplete, and for any small diagram $F : \mathbf{I} \to \mathbf{Diffeol}(X)$ we have*

$$\lim F = \inf_{i\in\mathbf{I}} Fi, \qquad \textit{and} \qquad \operatorname{colim} F = \sup_{i\in\mathbf{I}} Fi.$$

This viewpoint leads to the following two results (whose easy proofs we leave to the reader).

Proposition 2.74. *Let $f : X \to Y$ be a function. Then the functors $f_* : \mathbf{Diffeol}(X) \to \mathbf{Diffeol}(Y)$ and $f^* : \mathbf{Diffeol}(Y) \to \mathbf{Diffeol}(X)$ form an adjunction: $f_* \dashv f^*$.*

Corollary 2.75. *Let $(\mathcal{D}_i)_{i\in I}$ be a family of diffeologies on X. Consider one function $f : A \to X$ into X, and another function $g : X \to B$ defined on X. Then:*

$$f^*\left(\inf_{i\in I} \mathcal{D}_i\right) = \inf_{i\in I} f^*(\mathcal{D}_i), \qquad \textit{and} \qquad g_*\left(\sup_{i\in I} \mathcal{D}_i\right) = \sup_{i\in I} g_*(\mathcal{D}_i).$$

2.3 The irrational torus

The way diffeology is able to capture the smooth structure of quotients is one of its key strengths. Here we describe in detail one of the most illuminating examples of this fact: *the irrational torus.* This section is based on Exercises 4 and 31 in [***Diffeology***], and [**IZ17**]. The irrational torus was originally studied within diffeology in [**DI83**]. The noncommutative geometry approach is summarised in [**CM08**, Section 6], to which we here make a light comparison.

Definition 2.76. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational number. We consider the reals $\mathbb{R}$ as an additive group, with its Euclidean diffeology. We can see this as a diffeological group (Definition **3.1**), since the diffeology comes from the Lie group structure of $\mathbb{R}$. It has an additive subgroup

$$\mathbb{Z} + \theta\mathbb{Z} = \{n + \theta m : n, m \in \mathbb{Z}\},$$

acting naturally on $\mathbb{R}$. Two real numbers $x, y \in \mathbb{R}$ are therefore in the same equivalence class if and only if $x = y + n + \theta m$, for some $n, m \in \mathbb{Z}$. With the quotient diffeology (Example **2.67**), the *irrational torus* is defined as the diffeological space

$$T_\theta := \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z}).$$

We can also realise the irrational torus as the leaf space of a *Kronecker foliation*. We explain this in the following construction:

Construction 2.77. The *2-torus* is the product $S^1 \times S^1$ of two circles, equipped with the product diffeology. This is just the diffeology that it gets from the usual manifold structure. Extending the argument in Example **2.68** shows that there is a diffeomorphism $S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$. This diffeomorphism comes from the subduction

$$F : \mathbb{R}^2 \longrightarrow S^1 \times S^1; \qquad (x, y) \longmapsto (e^{2\pi i x}, e^{2\pi i y}).$$

With this identification of the 2-torus as a quotient of $\mathbb{R}^2$, we can project curves in the plane onto $S^1 \times S^1$. Of interest here will be the straight lines through the origin. Let us denote the line with slope θ by

$$\ell_\theta := \{(x, \theta x) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2.$$

Under the subduction $F : \mathbb{R}^2 \to S^1 \times S^1$ we then get a subgroup

$$\Delta_\theta := F(\ell_\theta) = \{(e^{2\pi i x}, e^{2\pi i \theta x}) : x \in \mathbb{R}\} \subseteq S^1 \times S^1.$$

This is known as a *linear flow* on the 2-torus. To characterise it, we consider the smooth map $f_\theta : \mathbb{R} \to S^1 \times S^1$, which defines Δ_θ as the image of the infinite curve $f_\theta : x \mapsto (e^{2\pi i x}, e^{2\pi i \theta x})$. Suppose first that the slope is rational: $\theta = m/n$, for some $m, n \in \mathbb{Z}$. Then the function $f_{m/n}$ becomes periodic: $f_{m/n}(x) = f_{m/n}(x + n)$. This induces a diffeomorphism $\mathbb{R}/\mathbb{Z} \cong \Delta_{m/n}$, so that $\Delta_{m/n}$ is just a circle. The idea is that, after a sufficient number of revolutions, the curve $f_{m/n}$ arrives back at its starting point $f_{m/n}(0)$. This can be seen in the following illustrations:

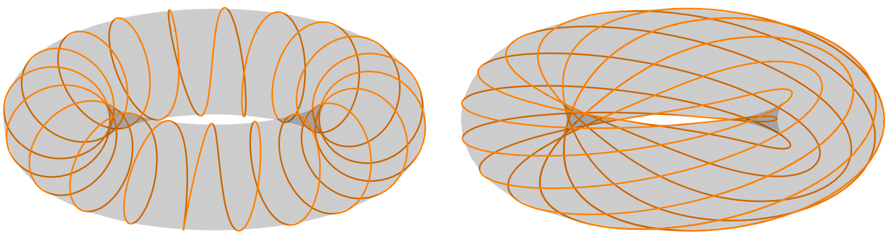


Figure: illustrations of Δ_{20} and $\Delta_{9/10}$, respectively[27].

The orange lines represent the orbits of single points in $S^1 \times S^1$ that $\Delta_{m/n}$ sweeps out through its action. Equivalently, $\Delta_{m/n}$ *is* the orbit of the identity element $(1, 1) \in S^1 \times S^1$. It is then easy to imagine that $(S^1 \times S^1)/\Delta_{m/n} \cong S^1$ (but this will also follow from Proposition **2.79**). In the language of foliation theory, this quotient is known as the *leaf space*.

The case we are interested in is when the slope is irrational: $\theta \in \mathbb{R} \setminus \mathbb{Q}$. When θ is irrational, the curve $f_\theta : \mathbb{R} \to S^1 \times S^1$ is injective. To see this, suppose for the sake of contradiction that $x, y \in \mathbb{R}$ are two real numbers such that $x \neq y$ and $f_\theta(x) = f_\theta(y)$. From the latter equation it follows that there exists integers $n, m \in \mathbb{Z}$ such that $x = y + n$ and $\theta x = \theta y + m$. Since $x \neq y$ it follows that $n \neq 0$, and we get $\theta = m/n$, contradicting our assumption that θ is irrational. Whereas for rational slope the curve $f_{m/n}$ defines a closed path in $S^1 \times S^1$, the fact that θ is irrational means that the resulting curve f_θ on $S^1 \times S^1$ never intersects itself. This can be (approximately) pictured as follows:

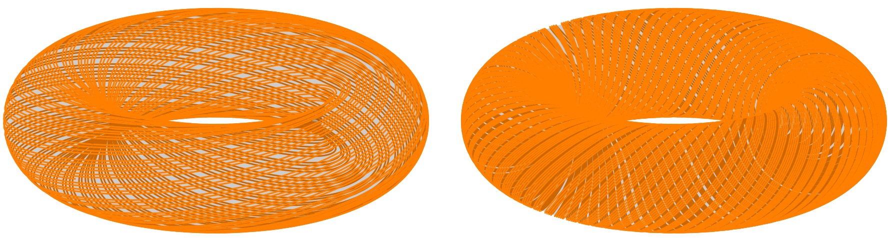


Figure: illustrations of $\Delta_{1/\sqrt{\pi}}$ and $\Delta_{\sqrt{20}}$, respectively.

[27] Programmed using the Sketch software, based on the example code [**Jak12**].

In fact, f_θ induces a diffeomorphism $\mathbb{R} \cong \Delta_\theta$, which we can prove by showing that f_θ is an induction. For that, take a plot $\alpha : U_\alpha \to S^1 \times S^1$ taking values in $\mathrm{im}(f_\theta) = \Delta_\theta$. From Example **2.16** we know that plots of the circle factor locally through the exponential map, so we can find a plot $\beta : V \to \mathbb{R}^2$ such that $\alpha|_V = F \circ \beta$. This is of the form $\alpha|_V(t) = (e^{2\pi i \beta_1(t)}, e^{2\pi i \beta_2(t)})$, and it follows that $f_\theta^{-1} \circ \alpha|_V = \beta_1 \in \mathcal{D}_\mathbb{R}$. The map $f_\theta^{-1} : \Delta_\theta \to \mathbb{R}$ is therefore smooth, and the diffeomorphism $\mathbb{R} \cong \Delta_\theta$ follows. This embedding of the real line into the 2-torus is known as the *irrational winding*. The famous *Kronecker Density Theorem* (see e.g. [**BS06**]), which states that the set $\{e^{2\pi i\theta n} : n \in \mathbb{Z}\}$ is dense in S^1, shows that the irrational winding Δ_θ lies densely in the 2-torus $S^1 \times S^1$. We have seen above that for a rational winding $(S^1 \times S^1)/\Delta_{m/n} \cong S^1$, which has both a nice topological- and smooth structure (it is a smooth manifold). But what about the quotient $(S^1 \times S^1)/\Delta_\theta$ of an irrational winding? Recall the following proposition (cf. [**IZ17**, p.4]):

Proposition 2.78. *Let G be a topological group, and consider a dense subgroup $H \subseteq G$. Then the quotient G/H has no non-trivial open sets.*

Proof. That H is dense just means that the unit coset $1_G H = H \subseteq G$ is dense, and since the left multiplication by an arbitrary group element $g \in G$ defines a diffeomorphism $G \to G$, it follows that the coset $gH \subseteq G$ is also dense.

We need to show that the only non-empty open subset of the quotient G/H is the entire space itself. Suppose therefore that $U \subseteq G/H$ is a non-empty open subset. By definition of the quotient topology this means that the preimage $\pi^{-1}(U)$ of U along the projection map $\pi : G \to G/H$ is a non-empty open subset of the original group G. Since the coset gH is dense, there is a non-empty intersection $\pi^{-1}(U) \cap gH \neq \varnothing$. We can therefore find an element $h \in H$ in the subgroup such that $gh \in \pi^{-1}(U)$, which gives $\pi(gh) \in U$. However π is constant on the cosets, so that in fact every element $h \in H$ then satisfies $\pi(gh) \in U$, so that $gH \subseteq \pi^{-1}(U)$. Since H is a subgroup, we note that $g \in gH$. And since the element $g \in G$ was arbitrary, it follows $G \subseteq \pi^{-1}(U)$, which gives that $U = G/H$. ☐

The quotient $(S^1 \times S^1)/\Delta_\theta$ therefore loses all topological information[28], and it is therefore certainly no longer a smooth manifold. But, as a diffeological space, it is perfectly acceptable, and even *non-trivial*. This is something that we will demonstrate below. First we will prove that $(S^1 \times S^1)/\Delta_\theta$ is the irrational torus.

Proposition 2.79. *For any real number $\theta \in \mathbb{R}$, there is a diffeomorphism*

$$\mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z}) =: T_\theta \cong (S^1 \times S^1)/\Delta_\theta.$$

Proof. Let us denote the equivalence classes in the quotient $(S^1 \times S^1)/\Delta_\theta$ by $[e^{2\pi ix}, e^{2\pi iy}]$. We then have identities of the form $[e^{2\pi ix}, e^{2\pi iy}] = [1, e^{2\pi i(y-\theta x)}]$. With this in mind, we define the smooth map $\varphi : \mathbb{R} \to S^1 \times S^1$ by $x \mapsto (1, e^{2\pi ix})$. Now, for two equivalence classes $[x] = [y] \in \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z})$ in the (irrational) torus, we have a relation $x = y + n + \theta m$ between the representatives, where $n, m \in \mathbb{Z}$. In that case, a simple calculation shows that

$$[\varphi(x)] = [1, e^{2\pi ix}] = [1, e^{2\pi i(y+n+\theta m)}] = [e^{-2\pi im}, e^{2\pi iy}] = [\varphi(y)].$$

We therefore find that φ induces a well-defined function

$$\Phi : \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z}) \longrightarrow (S^1 \times S^1)/\Delta_\theta; \qquad [x] \longmapsto [1, e^{2\pi ix}]$$

at the level of quotients, fitting into the commutative diagram

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\varphi} & S^1 \times S^1 \\ \downarrow & & \downarrow \\ \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z}) & \xrightarrow[\Phi]{} & (S^1 \times S^1)/\Delta_\theta. \end{array}$$

[28] So far we have not discussed the relation between diffeology and topology. Diffeology purposefully does not rely on topology for its definition, but every diffeology defines a canonical topology on its underlying set, called the *D-topology*. We discuss this briefly in Section **2.4.1**. It can be shown that the D-topology of a diffeological quotient agrees with the quotient topology.

Here the vertical maps are the canonical projections. By Lemma **2.122** it follows immediately that Φ is smooth. We will now construct a smooth inverse for Φ, exhibiting the desired diffeomorphism. We propose

$$\Psi : (S^1 \times S^1)/\Delta_\theta \longrightarrow \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z}); \qquad [e^{2\pi i x}, e^{2\pi i y}] \longmapsto [y - \theta x].$$

To show that Ψ is a well-defined function, take $[e^{2\pi i x}, e^{2\pi i y}] = [e^{2\pi i a}, e^{2\pi i b}]$. Then there exists a real number $z \in \mathbb{R}$ such that $e^{2\pi i x} = e^{2\pi i (a+z)}$ and $e^{2\pi i y} = e^{2\pi i (b+\theta z)}$, so that in turn we can find integers $n, m \in \mathbb{Z}$ giving $x = a + z + n$ and $y = b + \theta z + m$. Well-definedness then follows:

$$\Psi([e^{2\pi i x}, e^{2\pi i y}]) = [y - \theta x] = [(b - \theta a) + (m - \theta n)] = [b - \theta a] = \Psi([e^{2\pi i a}, e^{2\pi i b}]).$$

To prove that Ψ is smooth, recall that the subduction $F : \mathbb{R}^2 \to S^1 \times S^1$ from the start of Construction **2.77** defines a diffeomorphism $\Gamma : \mathbb{R}^2/\mathbb{Z}^2 \to S^1 \times S^1$. We further get a subduction $\rho : \mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto y - \theta x$. Together, we get a commutative diagram

$$\begin{array}{ccc}
\mathbb{R}^2/\mathbb{Z}^2 & \longleftarrow & \mathbb{R}^2 \\
\downarrow{\scriptstyle \Gamma} & & \downarrow{\scriptstyle \rho} \\
S^1 \times S^1 & & \mathbb{R} \\
\downarrow & & \downarrow \\
(S^1 \times S^1)/\Delta_\theta & \xrightarrow[\Psi]{} & \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z}).
\end{array}$$

Here, each unlabelled arrow is a quotient map, so that every arrow in this diagram besides Ψ is a subduction. Then by Lemma **2.122** it follows that Ψ is smooth (and in fact also a subduction). It is easy to see that Ψ and Φ are mutual inverses, so the desired diffeomorphism has been constructed. □

We have thus realised the irrational torus T_θ as the orbit space of the subgroup $\Delta_\theta \subseteq S^1 \times S^1$ in the 2-torus. It is worth noting that if θ is rational it follows that $\mathbb{R}/\mathbb{Z} \cong \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z})$. The special case where the slope is zero gives $\Delta_0 = S^1 \times \{1\}$, and the orbits are the horizontal sections of $S^1 \times S^1$. In the rest of this section we shall investigate the diffeological structure of the irrational tori. First we note that their real-valued functions are trivial, since the trivial topology of T_θ allows for no others:

Proposition 2.80. *Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be irrational. Then $C^\infty(T_\theta)$ contains only the constant functions.*

Proof. Let $f \in C^\infty(T_\theta)$ be a smooth real-valued function on an irrational torus. Denote the quotient map by $\pi_\theta : \mathbb{R} \to \mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z}) = T_\theta$. This is a plot of the irrational torus, so $f \circ \pi_\theta \in C^\infty(\mathbb{R}, \mathbb{R})$ is smooth. But, note that for any $x \in \mathbb{R}$ and $n, m \in \mathbb{Z}$ we then have $f \circ \pi_\theta(x) = f \circ \pi_\theta(x + n + \theta m)$. This shows that $f \circ \pi_\theta : \mathbb{R} \to \mathbb{R}$ must be constant on the subset $\mathbb{Z} + \theta\mathbb{Z} \subseteq \mathbb{R}$. But this is a dense subset of a Hausdorff space, which therefore completely determines the function $f \circ \pi_\theta$ as a constant by continuity. Since π_θ is a surjection it follows that f itself must also be constant. □

This result motivates the noncommutative geometry approach to the irrational tori. The usual philosophy of the Gelfand duality would have us study the commutative algebra of real-valued functions $C^\infty(T_\theta)$ to get information about T_θ itself. But since the algebra $C^\infty(T_\theta)$ is so trivial, it serves as a poor algebraic model. The noncommutative geometry solution is to consider instead the *irrational rotation algebra* (cf. [**CM08**, Section 6])[29]. This is the (C*-)algebra generated by two elements u and v, subject to the commutation relation:

$$vu = e^{2\pi i \theta} uv.$$

[29]The noncommutative geometry approach can be motivated by using groupoid C*-algebras. As we have remarked in Chapter **I**, Lie groupoids are used to model singular quotient spaces. The appropriate model for the irrational torus is the *action groupoid* (cf. Example **3.28**) $\mathbb{Z} \ltimes_\theta S^1 \rightrightarrows S^1$ corresponding to the action of $\mathbb{Z}$ on S^1 described by $n \cdot e^{2\pi i x} := e^{2\pi i (x + \theta n)}$, whose quotient is yet a third description of the irrational torus. In general, if G is a Lie group acting smoothly on a manifold M, there is an isomorphism of C*-algebras $C^*(G \ltimes M) \cong G \ltimes C_0(M)$, where on the right hand side we have a *crossed product C*-algebra*. In the case that the action is free and proper, so that the quotient M/G has a canonical manifold structure, these C*-algebras are Morita equivalent (in the sense of Rieffel) to the commutative coordinate algebra $C_0(M/G)$. In the case the quotient is not a manifold, the noncommutative geometry approach is to study instead the Morita equivalent groupoid C*-algebra $C^*(G \ltimes M)$. For the irrational torus, the groupoid C*-algebra $C^*(\mathbb{Z} \ltimes_\theta S^1) \cong \mathbb{Z} \ltimes_\theta C(S^1)$ is exactly the irrational rotation algebra.

The corresponding noncommutative space is called the *noncommutative torus*, and has served as an important and illuminating example in that theory, being classified first by Rieffel in [**Rie81**]. The next result shows that the diffeological approach to the irrational torus returns an equivalent characterisation:

Theorem 2.81 ([**DI83**])**.** *Two irrational tori T_θ and T_ϱ are diffeomorphic if and only if*

$$\textit{there exists} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \quad \textit{such that} \quad \theta = \frac{a+\varrho b}{c+\varrho d}.$$

We recall that $\mathrm{GL}(2,\mathbb{Z})$ is the group of invertible 2×2 matrices with integer coefficients[30]. Compare this theorem to the Morita equivalence of the irrational rotation C*-algebras, which was obtained only a few years earlier in noncommutative geometry [**Rie81**, Theorem 4], [**Con80**]. The result in fact translates verbatim if we replace "irrational" by "noncommutative," and "diffeomorphic" by "Morita equivalent." This result was the first hint that diffeology could be an alternative for noncommutative geometry. The rest of this section will be dedicated to proving Theorem **2.81**, for which we first have a lemma:

Lemma 2.82. *Let $\theta, \varrho \in \mathbb{R}\setminus\mathbb{Q}$ be two irrational numbers, and consider a smooth map $f : T_\theta \to T_\varrho$ between irrational tori. Then there exists a smooth* affine *map $F : \mathbb{R}\to\mathbb{R}$ such that $\pi_\varrho \circ F = f\circ \pi_\theta$.*

Proof. (This proof is based on the solution of [***Diffeology***, Exercise 4 (3)], found at the end of the book.) The canonical projection π_θ is a plot, so that $f\circ\pi_\theta : \mathbb{R}\to T_\varrho$ is again a plot. Since π_ϱ is a subduction, for every $x_0 \in \mathbb{R}$ we can find an open neighbourhood $x_0 \in V \subseteq \mathbb{R}$ and a smooth map $F : V \to \mathbb{R}$ such that $\pi_\varrho \circ F = f\circ\pi_\theta|_V$. Inside of the open neighbourhood we can find an open interval $I \subseteq V$, centered around $x_0 \in V$. For any $x\in I$ and $n, m\in\mathbb{Z}$ such that $x+n+\theta m\in I$ is an element in this interval, we can then write

$$\pi_\varrho \circ F|_I(x) = f\circ\pi_\theta|_I(x) = f\circ\pi_\theta|_I(x+n+\theta m) = \pi_\varrho\circ F|_I(x+n+\theta m),$$

which means that there exists two integers $l, k\in\mathbb{Z}$ such that

$$F(x+n+\theta m) = F(x) + l + \varrho k. \qquad (\heartsuit)$$

In fact, since ϱ is irrational, these integers are determined uniquely by $x\in I$ and $n,m\in\mathbb{Z}$. Possibly by shrinking I, we can pick a non-empty open interval $U\subseteq\mathbb{R}$ centered around the origin, such that if $x\in I$ and $n+\theta m\in U$ then $x+n+\theta m\in V$. This new interval is the domain for $n+\theta m$ for which expressions of the form $F(x+n+\theta m)$ are still well-defined. We can moreover ensure that $I\subseteq x_0+U$, perhaps by shrinking I further. Consider then the smooth function

$$Q : I \longrightarrow \mathbb{R}; \qquad x\longmapsto F(x+n+\theta m) - F(x),$$

fixing the integers $n,m\in\mathbb{Z}$ as above so that $n+\theta m\in U$. As a topological space, $\mathbb{Z}+\theta\mathbb{Z}$ is totally disconnected, which means that its connected components are singletons. The map Q is continuous, so by equation ($\heartsuit$) it maps the connected interval I to a connected subset $Q(I)\subseteq\mathbb{Z}+\theta\mathbb{Z}$, which therefore has to be a singleton, and shows that Q is in fact constant. The derivative with respect to the variable x is then zero, so we get that $F'(x) = F'(x+n+\theta m)$ for all $x\in I$ and $n+\theta m\in U$, where F' denotes the derivative of F. Since θ is irrational, the intersection $(\mathbb{Z}+\theta\mathbb{Z})\cap U$ is dense in U, so that for every $x\in I$ we can find a sequence $(n_i,m_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}\times\mathbb{Z}$ such that $x = \lim_{i\in\mathbb{N}}(x_0+n_i+\theta m_i)$. By continuity of the derivative F' we thus get

$$F'(x) = \lim_{i\in\mathbb{N}} F'(x_0+n_i+\theta m_i) = \lim_{i\in\mathbb{N}} F'(x_0) = F'(x_0),$$

so that not only Q is constant, but the derivative F' is also constant on I. Let us denote that slope of F then by $F'(x_0) = \lambda\in\mathbb{R}$, so that F can be written as

$$F|_I(x) = \lambda x + \mu, \qquad (\spadesuit)$$

[30]Note that, since $\mathbb{Z}$ is not a field but merely a ring, a matrix is in $\mathrm{GL}(2,\mathbb{Z})$ if and only if its determinant is invertible in $\mathbb{Z}$, i.e., if and only if its determinant is ± 1.

for some $\mu \in \mathbb{R}$ satisfying $f \circ \pi_\theta(0) = \pi_\varrho(\mu)$. We have therefore constructed an affine map $F|_I$ such that $\pi_\varrho \circ F|_I = f \circ \pi_\theta|_I$. Since $\mathbb{Z}+\theta\mathbb{Z}$ is even dense in the interval I, we have $\pi_\theta(I) = T_\theta$, so that this already completely determines the function f. But $F|_I$ can be extended to the entire real line. To show this we need an intermediate technical result. If we substitute equation (♠) into (♥) we get that for all $x \in I$ and $n + \theta m \in U$, we have $\lambda(n + \theta m) = l + \varrho k$, which shows:

$$\forall n + \theta m \in U : \lambda(n + \theta m) \in \mathbb{Z} + \varrho\mathbb{Z}.$$

We claim that this in fact holds for all $n+\theta m \in \mathbb{Z}+\theta\mathbb{Z}$. To show this, let us take explicitly $U = (-a, a)$, where $a \notin \mathbb{Z} + \theta\mathbb{Z}$. If the original boundaries of U happened to be in $\mathbb{Z} + \theta\mathbb{Z}$, we can pick $a \in \mathbb{R}$ by slightly shrinking U. To prove the claim we take an element $x \in \mathbb{Z} + \theta\mathbb{Z}$ outside of the interval U, say $x > a$. By the Archimedean property of the reals we can find a natural number $N \in \mathbb{N}$ such that $x/N \in (0, a)$. The intersection $(\mathbb{Z} + \theta\mathbb{Z}) \cap (0, a)$ is dense in the interval $(0, a)$, so the number x/N is arbitrarily close to an element in $\mathbb{Z} + \theta\mathbb{Z}$. Explicitly, this means that for every $\varepsilon > 0$ we can find $y \in \mathbb{Z} + \theta\mathbb{Z}$ such that $0 < x/N - y < \varepsilon$. If not already small enough, we can pick $\varepsilon < a/N$, so that we even get $x - Ny \in (0, a)$ and $y \in (0, a)$. We therefore get an element $x - Ny \in (\mathbb{Z} + \theta\mathbb{Z}) \cap U$, for which we then know that $\lambda(x - Ny) \in \mathbb{Z} + \varrho\mathbb{Z}$. However y was an element of $(\mathbb{Z}+\theta\mathbb{Z}) \cap U$ already, so that we also have $\lambda Ny \in \mathbb{Z} + \varrho\mathbb{Z}$. It must therefore follow that $\lambda x \in \mathbb{Z} + \varrho\mathbb{Z}$. It is clear that the argument also works for $x < -a$, so that the claim follows:

$$\forall n + \theta m \in \mathbb{Z} + \theta\mathbb{Z} : \lambda(n + \theta m) \in \mathbb{Z} + \varrho\mathbb{Z}.$$

We can now extend $F|_I$ to $F : \mathbb{R} \to \mathbb{R}$ by defining $x \mapsto \lambda x + \mu$. All that is left to show is that the equation $\pi_\varrho \circ F = f \circ \pi_\theta$ still holds. First, since the intersection $(\mathbb{Z} + \theta\mathbb{Z}) \cap I$ is dense in the interval I, it follows that $\pi_\theta(I) = T_\theta$. This means that for any $x \in \mathbb{R}$ we can find $y \in I$ such that $\pi_\theta(x) = \pi_\theta(y)$, or in other words: $y = x + n + \theta m$, for some $n, m \in \mathbb{Z}$. Equation (♠) and our starting assumption that $\pi_\varrho \circ F|_I = f \circ \pi_\theta|_I$ then give:

$$f \circ \pi_\theta(x) = f \circ \pi_\theta|_I(y) = \pi_\varrho \circ F|_I(y) = \pi_\varrho(\lambda x + \underbrace{\lambda(n + \theta m)}_{\in \mathbb{Z}+\varrho\mathbb{Z}} + \mu).$$

By the claim we have proved above, it follows that $\lambda(n + \theta m) \in \mathbb{Z} + \varrho\mathbb{Z}$, making that term vanish, and we get:

$$f \circ \pi_\theta(x) = \pi_\varrho(\lambda x + \mu) = \pi_\varrho \circ F(x),$$

at last proving the lemma. □

The claim that we have proved in this argument also implies a relation between the coefficients θ and ϱ. Recall that the claim says that if $n + \theta m \in \mathbb{Z} + \theta\mathbb{Z}$, then $\lambda(n + \theta m) \in \mathbb{Z} + \varrho\mathbb{Z}$. In particular, we may set $n + \theta m = \theta$ or $n + \theta m = 1$, so that there are integers $a, b, c, d \in \mathbb{Z}$ such that

$$\lambda\theta = a + \varrho b, \qquad \text{and} \qquad \lambda = c + \varrho d.$$

This will help us prove the next result.

Corollary 2.83. *There exists non-constant smooth functions in $C^\infty(T_\theta, T_\varrho)$ between irrational tori if and only if there are integers $a, b, c, d \in \mathbb{Z}$ such that $\theta = (a + \varrho b)/(c + \varrho d)$.*

Proof. Start with the case that there exists a non-constant smooth function $f \in C^\infty(T_\theta, T_\varrho)$, and let $F : x \mapsto \lambda x+\mu$ be the underlying affine map obtained from Lemma 2.82. The fact that f is not constant implies that $\lambda \neq 0$, because otherwise $f \circ \pi_\theta(x) = \pi_\varrho(\mu)$ for all $x \in \mathbb{R}$. From the remarks preceding this corollary, we can find integers $a, b, c, d \in \mathbb{Z}$ such that $\lambda\theta = a + \varrho b$ and $\lambda = c + \varrho d$, where now in addition we know that $c + \varrho d \neq 0$. This allows us to perform a division, from which it easily follows that:

$$\theta = \frac{a + \varrho b}{c + \varrho d}.$$

Suppose now that $\theta = (a + \varrho b)/(c + \varrho d)$. We then get a smooth function $f \in C^\infty(T_\theta, T_\varrho)$ defined by the affine map $F : x \mapsto \lambda x + \mu$, where $\lambda = c + \varrho d$, and $\mu \in \mathbb{R}$ is arbitrary. Then $\lambda \neq 0$, because otherwise the coefficient θ would not be well-defined. Pick a real number $z \in \mathbb{R} \setminus (\mathbb{Z} + \varrho\mathbb{Z})$, so that in particular $z \neq 0$ as well. If f were to be constant, we would have $f \circ \pi_\theta(0) = f \circ \pi_\theta(z/\lambda)$. Writing this out in terms of F, this would mean the existence of integers $l, k \in \mathbb{Z}$ such that $z = l + \varrho k$, contradicting the assumption that $z \notin \mathbb{Z} + \varrho\mathbb{Z}$. The function f is therefore not constant. ☐

Corollary **2.83** now allows us to show that, even though the topology on the irrational torus T_θ is trivial, its canonical quotient diffeology is not! To see this, suppose otherwise. If T_θ is discrete, then any function $f : T_\theta \to T_\varrho$ is smooth, including any non-constant ones. But we can then just choose ϱ arbitrarily to contradict the result of Corollary **2.83**. If instead T_ϱ is coarse, so that again any function $f : T_\theta \to T_\varrho$ is smooth, we can perform a similar trick by picking θ to obtain a contradiction. We therefore have *strict* inclusions of diffeologies:

$$\mathcal{D}^\circ_{T_\theta} \subsetneq \mathcal{D}_{T_\theta} \subsetneq \mathcal{D}^\bullet_{T_\theta}.$$

This is one of the aspects that so distinguishes diffeology from the other set-based smooth theories such as those of Sikorski- and Frölicher spaces, whose descriptions of the irrational tori are necessarily trivial for topological reasons (Proposition **2.80**).

We now prove the classification of irrational tori:

Proof of Theorem ***2.81***. Let us start by giving a characterisation of the injective and surjective smooth maps $f \in C^\infty(T_\theta, T_\varrho)$. Given such an f, let $F : x \mapsto \lambda x + \mu$ be the underlying affine map from Lemma **2.82**. We claim that f is surjective if and only if $\lambda \neq 0$. If f is surjective then it cannot be constant, so λ cannot be zero. Conversely, if $\lambda \neq 0$, any point $\pi_\varrho(z) \in T_\varrho$ can be obtained as $f \circ \pi_\theta((z - \mu)/\lambda) = \pi_\varrho(z)$.

The characterisation of injectivity is slightly more involved. Assume that $\lambda \neq 0$. We claim that f is injective if and only if $\frac{1}{\lambda}(\mathbb{Z} + \varrho\mathbb{Z}) \subseteq \mathbb{Z} + \theta\mathbb{Z}$. Suppose that f is injective, and take $x \in \mathbb{R}$ such that $\lambda x \in \mathbb{Z} + \varrho\mathbb{Z}$. Then $f \circ \pi_\theta(x) = \pi_\varrho(\mu) = f \circ \pi_\theta(0)$, so the injectivity gives that $\pi_\theta(x) = \pi_\theta(0)$, which just means that $x \in \mathbb{Z} + \theta\mathbb{Z}$. Hence we have proved that if $\lambda x \in \mathbb{Z} + \varrho\mathbb{Z}$ then $x \in \mathbb{Z} + \theta\mathbb{Z}$, which (when $\lambda \neq 0$) gives the desired inclusion.

Conversely, suppose that $\frac{1}{\lambda}(\mathbb{Z} + \varrho\mathbb{Z}) \subseteq \mathbb{Z} + \theta\mathbb{Z}$, and take $x, y \in \mathbb{R}$ such that $f \circ \pi_\theta(x) = f \circ \pi_\theta(y)$. In that case we have $\lambda(x - y) \in \mathbb{Z} + \varrho\mathbb{Z}$, so that by the assumption we get $x - y \in \mathbb{Z} + \theta\mathbb{Z}$, which just translates to $\pi_\theta(x) = \pi_\theta(y)$, proving that f is injective.

We can now prove the claim made in Theorem **2.81**. Suppose that there exists a diffeomorphism $f : T_\theta \to T_\varrho$. Then $f \in C^\infty(T_\theta, T_\varrho)$ is a bijection, so that $\lambda \neq 0$ and $\frac{1}{\lambda}(\mathbb{Z} + \varrho\mathbb{Z}) \subseteq \mathbb{Z} + \theta\mathbb{Z}$. We view $\mathbb{Z} + \theta\mathbb{Z}$ and $\mathbb{Z} + \varrho\mathbb{Z}$ as $\mathbb{Z}$-modules, defined by the bases $\{1_\theta, \theta\}$ and $\{1_\varrho, \varrho\}$, respectively. They are both isomorphic to $\mathbb{Z} \times \mathbb{Z}$ as $\mathbb{Z}$-modules. The constant λ defines a $\mathbb{Z}$-module map, defined on the basis as follows:

$$L_\lambda : \mathbb{Z} + \theta\mathbb{Z} \longrightarrow \mathbb{Z} + \varrho\mathbb{Z}; \qquad L_\lambda(1_\theta) := c1_\varrho + d\varrho, \quad \text{and} \quad L_\lambda(\theta) := a1_\varrho + b\varrho.$$

This reflects the relations $\lambda\theta = a + \varrho b$ and $\lambda = c + \varrho d$ obtained above. The fact that f is bijective, and hence $\lambda \neq 0$ and $\frac{1}{\lambda}(\mathbb{Z} + \varrho\mathbb{Z}) \subseteq \mathbb{Z} + \theta\mathbb{Z}$, means that the map L_λ has a well-defined inverse $L_\lambda^{-1} : \mathbb{Z} + \varrho\mathbb{Z} \to \mathbb{Z} + \theta\mathbb{Z}$, and is hence a $\mathbb{Z}$-module isomorphism. Hence, in this basis, its matrix is invertible and given by:

$$[L_\lambda] = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}).$$

Up to the order of the basis, this proves one direction of the desired claim.

Conversely, if the matrix $[L_\lambda] \in \mathrm{GL}(2, \mathbb{Z})$ exists and $\theta = (a + \varrho b)/(c + \varrho d)$, then we get a smooth map $f \in C^\infty(T_\theta, T_\varrho)$ defined by the affine curve $F : x \mapsto \lambda x$ for $\lambda = c + \varrho d$. Since $[L_\lambda]$ is invertible it follows by the above characterisation that f is a bijection. Its inverse is obtained by looking at the smooth function defined by the inverse matrix $[L_\theta]^{-1}$, and the result follows. ☐

In recent work [**IZP20**] the philosophy of orbifolds has been extended so that it can treat objects such as irrational tori. The authors study the groupoid C*-algebra of the locally compact Hausdorff groupoid associated to the atlas of a diffeological quasifold. They prove that different atlases give Morita equivalent groupoids (in the original sense of [**MRW87**]), and hence that the resulting C*-algebras must be Morita equivalent. In this way, the noncommutative geometric result of Rieffel [**Rie81**, Theorem 4] follows as a corollary of Theorem **2.81** ([**DI83**]). The construction of these groupoids fits into the larger picture we sketch in Chapter **VI**.

2.4 Functional diffeologies and local constructions

We have seen in Theorem **2.72** that all small limits and colimits in **Diffeol** exist. This gives us various interesting constructions on diffeological spaces to work with, many of which do not exist in **Mnfd** (as we have already seen in Section **2.2**). There are some other categorical properties that make **Diffeol** particularly nice, but that are not captured by the (co)completeness. One of those properties is that **Diffeol** is *Cartesian closed* (Definition **A.5**). In essence, this means that the function space $C^\infty(X, Y)$ has its own natural diffeology. This set is, in almost all cases, *not* a finite-dimensional smooth manifold (although it can, in some cases, be treated as an infinite-dimensional one, see [**KM97**, Chapter VI].) This is one of the other key advantages of diffeological spaces over smooth manifolds, and is a big part of the motivation for generalised smooth spaces that we discussed in Section **1.1**. In this section we shall show how to define a natural diffeology on spaces of smooth functions, and prove that **Diffeol** is Cartesian closed.

Definition 2.84. Let X and Y be diffeological spaces. The *evaluation map* is the function

$$\mathrm{ev} : C^\infty(X, Y) \times X \longrightarrow Y; \qquad (f, x) \longmapsto f(x).$$

A diffeology on $C^\infty(X, Y)$ making the evaluation map smooth is called a *functional diffeology.*

It is not immediate that a functional diffeology has to be unique, however one always exists, because the discrete diffeology on $C^\infty(X, Y)$ makes the evaluation map smooth. To see this, let

$$(\Omega, \alpha) : U \to C^\infty(X, Y) \times X$$

be a plot, i.e., $\alpha \in \mathcal{D}_X$, and Ω is locally constant. Find an open neighbourhood $V \subseteq U$ on which Ω is constant, say equal to some smooth function $f \in C^\infty(X, Y)$. Then $\mathrm{ev} \circ (\Omega, \alpha)|_V = f \circ \alpha|_V$, which is a plot of Y by the very definition of smoothness of f. It follows by the Axiom of Locality that ev is smooth with respect to the discrete diffeology on $C^\infty(X, Y)$. The discrete diffeology is not known to be very interesting, so we now exhibit a diffeology on $C^\infty(X, Y)$ that makes the evaluation map smooth in the nicest way possible. For that, we first have the following lemma, characterising when evaluating a family of smooth maps is itself smooth.

Lemma 2.85. *Let $\Omega : U_\Omega \to C^\infty(X, Y)$ be a parametrisation (where the function space does not necessarily have a diffeology). Then the map $\mathrm{ev} \circ (\Omega \times \mathrm{id}_X) : U_\Omega \times X \to Y$ is smooth if and only if for every plot $\alpha \in \mathcal{D}_X$ the map $\mathrm{ev} \circ (\Omega \times \alpha) : U_\Omega \times U_\alpha \to Y$ is smooth.*

Proof. If $\mathrm{ev} \circ (\Omega \times \mathrm{id}_X)$ is smooth and $\alpha \in \mathcal{D}_X$ is a plot, then

$$\mathrm{ev} \circ (\Omega \times \alpha) = \mathrm{ev} \circ (\Omega \times \mathrm{id}_X) \circ (\mathrm{id}_{U_\Omega} \times \alpha)$$

is the composition of smooth maps, and hence itself smooth. Conversely, suppose that $\mathrm{ev} \circ (\Omega \times \alpha)$ is smooth, for each $\alpha \in \mathcal{D}_X$. By Lemmas **2.33** and **2.61** it suffices to check that $\mathrm{ev} \circ (\Omega \times \mathrm{id}_X) \circ (\beta \times \alpha)$ is smooth, for every $\alpha, \beta \in \mathcal{D}_X$. This again follows by decomposing the composition in a smart way:

$$\mathrm{ev} \circ (\Omega \times \mathrm{id}_X) \circ (\beta \times \alpha) = \mathrm{ev} \circ (\Omega \times \mathrm{id}_X) \circ (\mathrm{id}_{U_\beta} \times \alpha) \circ (\beta \times \mathrm{id}_{U_\alpha}) = \mathrm{ev} \circ (\Omega \times \alpha) \circ (\beta \times \mathrm{id}_{U_\alpha}).$$

By assumption, the right hand side is the composition of smooth maps, and the result follows. □

A lot of the other proofs in this section will include compositional acrobatics of this sort. We find this slightly more illuminating than the presentation in [***Diffeology***, pp. 34–40], where instead of $\mathrm{ev} \circ (\Omega \times \alpha)$ the author writes $\Omega(\alpha(t))$. In our form it is more apparent that certain expressions are just the composition of other smooth functions.

We will now prove that the equivalent conditions in Lemma **2.85** define a diffeology on $C^\infty(X, Y)$, making the evaluation map smooth.

Lemma 2.86. *Consider two diffeological spaces X and Y. The collection $\mathcal{D}$, consisting of all parametrisations $\Omega : U_\Omega \to C^\infty(X, Y)$ such that $\mathrm{ev} \circ (\Omega \times \mathrm{id}_X)$ is smooth, defines a functional diffeology on $C^\infty(X, Y)$.*

Proof. For the Axiom of Covering, let $\Omega = \text{const}_f : U_\Omega \to C^\infty(X, Y)$ be a constant parametrisation, with $f \in C^\infty(X, Y)$. Then

$$\text{ev} \circ (\Omega \times \text{id}_X)(t, x) = \text{ev}(\Omega(t), x) = \text{ev}(f, x) = f(x),$$

so that $\text{ev} \circ (\Omega \times \text{id}_X) = f \circ \text{pr}_2$, where $\text{pr}_2 : U_\Omega \times X \to X$ is the projection onto X. This composition is clearly smooth, since f is smooth.

For the Axiom of Smooth Compatibility, take $\Omega \in \mathfrak{D}$, together with a smooth map $h : V \to U_\Omega$. Then $\text{ev} \circ (\Omega \times \text{id}_X)$ is smooth, so that the composition

$$\text{ev} \circ ((\Omega \circ h) \times \text{id}_X) = \text{ev} \circ (\Omega \times \text{id}_X) \circ (h \times \text{id}_X)$$

is smooth, and hence $\Omega \circ h \in \mathfrak{D}$.

Lastly, for the Axiom of Locality, we will need Lemma **2.85**. Suppose $\Omega : U_\Omega \to C^\infty(X, Y)$ is a parametrisation that is locally in $\mathfrak{D}$. This means that, for every $t \in U_\Omega$, we can find an open neighbourhood $t \in V \subseteq U_\Omega$, such that $\Omega|_V \in \mathfrak{D}$. In other words, we can find an open cover $(V_t)_{t \in U_\Omega}$ of U_Ω such that each restriction $\Omega|_{V_t}$ is in $\mathfrak{D}$. By Lemma **2.85** this means that for every $\alpha \in \mathfrak{D}_X$, the map

$$\text{ev} \circ (\Omega|_{V_t} \times \alpha) = \text{ev} \circ (\Omega \times \alpha)|_{V_t \times U_\alpha}$$

is smooth. Now $(V_t \times U_\alpha)_{t \in U_\Omega}$ is an open cover of $U_\Omega \times U_\alpha$ on which the restrictions of $\text{ev} \circ (\Omega \times \alpha)$ are smooth, so by the Axiom of Locality of the diffeology of Y, we obtain $\text{ev} \circ (\Omega \times \alpha) \in \mathfrak{D}_Y$. Since α was arbitrary, it follows that $\Omega \in \mathfrak{D}$.

To finish the proof, we need to show that, if $C^\infty(X, Y)$ is endowed with the diffeology $\mathfrak{D}$, then the evaluation map becomes smooth. But this follows very easily from the characterisation of $\mathfrak{D}$. Namely, by Lemmas **2.33** and **2.61**, the evaluation map is smooth if for every $\Omega \in \mathfrak{D}$ and $\alpha \in \mathfrak{D}_X$ we have $\text{ev} \circ (\Omega \times \alpha) \in \mathfrak{D}_Y$. But this is exactly the defining characteristic of the plots in $\mathfrak{D}$, by Lemma **2.85**. □

Definition 2.87. Let X and Y be two diffeological spaces. The functional diffeology $\mathfrak{D}$ on $C^\infty(X, Y)$ from the previous Lemma **2.86** is called the *standard functional diffeology*. It is the coarsest diffeology on $C^\infty(X, Y)$ making the evaluation map smooth. Indeed, if $\mathfrak{D}'$ was another functional diffeology, and $\Omega \in \mathfrak{D}'$, it follows directly that $\text{ev} \circ (\Omega \times \text{id}_X)$ is smooth, because it is a composition of smooth maps, and hence $\Omega \in \mathfrak{D}$.

Whenever we encounter a functional space $C^\infty(X, Y)$, we will assume that it is endowed with the standard functional diffeology. While the standard functional diffeology may not be the *only* interesting diffeology on $C^\infty(X, Y)$, as we will find, it makes for some particularly nice categorical properties. It also makes the composition of smooth maps smooth:

Proposition 2.88. *Let X, Y, and Z be diffeological spaces. Then the composition map is smooth:*

$$\text{comp} : C^\infty(Y, Z) \times C^\infty(X, Y) \longrightarrow C^\infty(X, Z); \qquad (f, g) \longmapsto f \circ g.$$

Proof. We use the sufficient criteria from Lemmas **2.33** and **2.61**. Let $\Omega : U_\Omega \to C^\infty(Y, Z)$ and $\Psi : U_\Psi \to C^\infty(X, Y)$ be two plots in the standard functional diffeologies. We need to show that $\text{comp} \circ (\Omega \times \Psi)$ is a plot of $C^\infty(X, Z)$. By Lemma **2.85**, this means $\text{ev} \circ ([\text{comp} \circ (\Omega \times \Psi)] \times \text{id}_X)$ should be smooth. When evaluated at $(t, s, x) \in U_\Omega \times U_\Psi \times X$, this expression gives $\Omega(t) \circ \Psi(s)(x)$, which can be rewritten:

$$\begin{aligned} \Omega(t) \circ \Psi(s)(x) &= \text{ev}\,(\Omega(t), \Psi(s)(x)) \\ &= \text{ev} \circ (\Omega \times \text{id}_Y)\,(t, \text{ev} \circ (\Psi \times \text{id}_X)(s, x)) \\ &= \text{ev} \circ (\Omega \times \text{id}_Y) \circ (\text{id}_{U_\Omega} \times [\text{ev} \circ (\Psi \times \text{id}_X)])\,(t, s, x). \end{aligned}$$

Now, the right hand side of this equation is a composition of the smooth maps $\text{ev} \circ (\Omega \times \text{id}_Y)$ and $\text{ev} \circ (\Psi \times \text{id}_X)$, which proves that $\text{comp} \circ (\Omega \times \Psi)$ is a plot for $C^\infty(X, Z)$ in the standard functional diffeology, and the result follows. □

Example 2.89. The *pullback map*

$$f^* : C^\infty(Y) \longrightarrow C^\infty(X); \qquad g \longmapsto g \circ f$$

of any smooth function $f : X \to Y$ is smooth with respect to the standard functional diffeologies. (See [**Vin08**, Proposition 2.4.2].) This follows because we have a similar computation as in Proposition **2.88** above. Note that the rôle of $\mathbb{R}$ is not special here, so we can replace the spaces of real-valued functions by $C^\infty(X, Z)$ and $C^\infty(Y, Z)$ and the claim still holds.

Example 2.90. Let R be an equivalence relation on a diffeological space X, and endow the quotient X/R with the quotient diffeology, as usual. If we denote by $C^\infty(X, Y)^R$ the subset of $C^\infty(X, Y)$ of those smooth functions that are constant on equivalence classes of R, and equip it with the subset diffeology it gets from the standard functional diffeology, then there is a diffeomorphism:

$$C^\infty(X/R, Y) \cong C^\infty(X, Y)^R.$$

Proof. We shall construct a diffeomorphism between the two spaces. If we denote by $\pi : X \to X/R$ the canonical quotient map, we can define a function

$$\Phi : C^\infty(X/R, Y) \longrightarrow C^\infty(X, Y)^R; \qquad f \longmapsto f \circ \pi.$$

It is easy to see from Proposition **2.88** that Φ is smooth. If, on the other hand, we start with a smooth function $g : X \to Y$ that is constant on the equivalence classes of R, we get a unique function $\overline{g} : X/R \to Y$ defined by $\overline{g}(\pi(x)) := g(x)$. As we will formally prove in Lemma **2.122**, the map $\overline{g}$ is in fact smooth. The essence of this argument is to use Lemmas **2.33** and **2.43**: any plot of X/R is locally of the form $\pi \circ \beta$, where $\beta \in \mathcal{D}_X$, and since g is smooth we therefore find that the expression $\overline{g} \circ \pi \circ \beta = g \circ \beta$ is smooth. We thus get a function

$$\Psi : C^\infty(X, Y)^R \longrightarrow C^\infty(X/R, Y); \qquad g \longmapsto \overline{g},$$

which we claim defines a smooth inverse for Φ. That it is an inverse is obvious, so we are left to check smoothness. For that, we take a plot $\Omega : U_\Omega \to C^\infty(X, Y)^R$ in the standard functional diffeology, taking values in the R-invariant smooth functions. We need to show that $\Psi \circ \Omega$ is a plot for $C^\infty(X/R, Y)$. By Lemma **2.85** it then suffices to show that $\mathrm{ev} \circ (\overline{\Omega} \times \alpha)$ is smooth, where we denote $\overline{\Omega} := \Psi \circ \Omega$, and $\alpha \in \mathcal{D}_{X/R}$ is an arbitrary plot of the quotient. Since π is a subduction, we know that α is locally of the form $\alpha|_V = \pi \circ \beta$, where $\beta : V \to X$ is a plot in $\mathcal{D}_X$. For $t \in U_\Omega$ and $s \in V$, we then find:

$$\mathrm{ev} \circ (\overline{\Omega} \times \alpha|_V)(t, s) = \overline{\Omega(t)}\,(\pi(\beta(s))) = \Omega(t)(\beta(s)) = \mathrm{ev} \circ (\Omega \times \beta)(t, s).$$

Since Ω is a plot, another application of Lemma **2.85** shows that the expression on the right hand side is smooth. The Axiom of Locality thus shows that $\mathrm{ev} \circ (\overline{\Omega} \times \alpha)$ is smooth, and we are done. □

Example 2.91 (A familiar infinite-dimensional vector space)**.** Consider an arbitrary diffeological space X, and consider its space $C^\infty(X)$ of real-valued smooth functions. The vector space structure of $\mathbb{R}$ then transfers to $C^\infty(X)$ as usual, by defining a point-wise addition $+ : C^\infty(X) \times C^\infty(X) \to C^\infty(X)$ and scalar multiplication $\mathbb{R} \times C^\infty(X) \to C^\infty(X)$. We claim that, with the standard functional diffeology, this turns $C^\infty(X)$ into a diffeological vector space. (Actually, this works for any space $C^\infty(X, V)$, where V is a diffeological vector space.)

Proof. We show that the addition $+ : C^\infty(X) \times C^\infty(X) \to C^\infty(X)$ is smooth. Take two plots Ω and Ψ in the standard functional diffeology on $C^\infty(X)$, defined on a common Euclidean domain U. This means that $\mathrm{ev} \circ (\Omega \times \mathrm{id}_X) : U \times X \to \mathbb{R}$ is smooth, and similarly for Ψ. We need to show that $+ \circ (\Omega, \Psi)$ is again a plot in the standard functional diffeology. For this, it is easy to calculate:

$$\mathrm{ev} \circ ([+ \circ (\Omega, \Psi)] \times \mathrm{id}_X)\,(t, x) = \Omega(t)(x) + \Psi(t)(x) = +_{\mathbb{R}} \circ (\mathrm{ev} \circ (\Omega \times \mathrm{id}_X), \mathrm{ev} \circ (\Psi \times \mathrm{id}_X))\,(t, x),$$

where $+_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the smooth addition map for $\mathbb{R}$. The right hand side is clearly smooth, since Ω and Ψ are standard functional plots, so this proves that the point-wise addition $+$ has to be smooth. That the point-wise scalar multiplication is smooth is proved analogously. □

The space $\mathrm{Diff}(X, Y)$ of diffeomorphisms between two spaces is clearly a subset of $C^\infty(X, Y)$. In particular, we are interested in $\mathrm{Diff}(X)$, because it forms a group. Clearly the group operation in $\mathrm{Diff}(X)$ is smooth by Proposition **2.88**. But it turns out that the subset diffeology on $\mathrm{Diff}(X)$ does not ensure that the *inversion* is smooth. For that we need to construct a refinement of the standard functional diffeology. We discuss this in Section **3.1**.

Cartesian closedness. Since a functional space $C^\infty(Y, Z)$ is itself a diffeological space, it makes sense to talk about smooth functions on or into such a space. This means that we get an *internal hom-functor:*

$$C^\infty(X, -) : \mathbf{Diffeol} \longrightarrow \mathbf{Diffeol}; \quad \left(Y \xrightarrow{f} Z\right) \longmapsto \left(C^\infty(X, Y) \xrightarrow{f \circ -} C^\infty(X, Z)\right).$$

In turn, the space of smooth functions $C^\infty(X, C^\infty(Y, Z))$ is also a diffeological space. The following proposition characterises this space (using implicitly the Cartesian closedness of **Set**), and will help us prove that **Diffeol** is Cartesian closed (recall Definition **A.5**).

Proposition 2.92. *Let X, Y, and Z be diffeological spaces. Then there is a diffeomorphism*

$$C^\infty(X, C^\infty(Y, Z)) \xrightarrow{\ \Phi\ } C^\infty(X \times Y, Z).$$

Proof. What we need to show, essentially, is that rewriting a pair of parentheses is a smooth operation. Given $F \in C^\infty(X, C^\infty(Y, Z))$, we define

$$\Phi(F) = f : X \times Y \longrightarrow Z; \qquad (x, y) \longmapsto f(x, y) := F(x)(y).$$

First, to show that Φ is even well-defined, we need to show that $f : X \times Y \to Z$ is smooth. But if $\alpha \in \mathcal{D}_X$ and $\beta \in \mathcal{D}_Y$ then $f \circ (\alpha \times \beta) = \mathrm{ev} \circ ([F \circ \alpha] \times \beta)$ is clearly smooth, which is sufficient by Lemmas **2.33** and **2.61**. It is then clear that Φ is a well-defined bijective function. Now, to prove that Φ itself is smooth, let $\Omega : U_\Omega \to C^\infty(X, C^\infty(Y, Z))$ be a plot. According to Lemma **2.86**, we need to show that $\mathrm{ev} \circ ([\Phi \circ \Omega] \times \mathrm{id}_{X \times Y})$ is smooth. If we evaluate this at a point $(t, x, y) \in U_\Omega \times X \times Y$, we get the expression $\Phi(\Omega(t))(x, y) = (\Omega(t)(x))(y)$. This can be written as:

$$\begin{aligned}(\Omega(t)(x))(y) &= \mathrm{ev}(\Omega(t)(x), y)\\ &= \mathrm{ev}\left([\mathrm{ev} \circ (\Omega \times \mathrm{id}_X)](t, x), y\right)\\ &= \mathrm{ev} \circ \left([\mathrm{ev} \circ (\Omega \times \mathrm{id}_X)] \times \mathrm{id}_X\right)(t, x, y),\end{aligned}$$

which is smooth because Ω is a plot. This proves that Φ is smooth.

To finish the proof, we are left to show that the inverse $\Phi^{-1} : f \mapsto F$ is smooth. Note that this is also well defined. To prove smoothness, fix a plot $\Omega : U_\Omega \to C^\infty(X \times Y, Z)$. By the equivalent condition in Lemma **2.85**, we need to show that for every $\alpha \in \mathcal{D}_X$ the map $\mathrm{ev} \circ ([\Phi^{-1} \circ \Omega] \times \alpha)$ is smooth. In turn, by the definition of the standard functional diffeology on $C^\infty(Y, Z)$, this means that

$$\mathrm{ev} \circ \left(\left[\mathrm{ev} \circ \left([\Phi^{-1} \circ \Omega] \times \alpha\right)\right] \times \mathrm{id}_Y\right) : U_\Omega \times U_\alpha \times Y \longrightarrow Z$$

has to be smooth. To make this legible, let us evaluate it at a point (t, s, y), which gives the expression $\Omega(t)(\alpha(s), y)$. This can be expanded as

$$\begin{aligned}\Omega(t)(\alpha(s), y) &= \mathrm{ev}\left(\Omega(t), (\alpha(s), y)\right)\\ &= \mathrm{ev} \circ \left(\Omega \times (\alpha \times \mathrm{id}_Y)\right)(t, s, y)\\ &= \mathrm{ev} \circ \left(\Omega \times \mathrm{id}_{X \times Y}\right) \circ \left(\mathrm{id}_{U_\Omega} \times (\alpha \times \mathrm{id}_Y)\right)(t, s, y).\end{aligned}$$

Since Ω is a plot of $C^\infty(X \times Y, Z)$, the term $\mathrm{ev} \circ (\Omega \times \mathrm{id}_{X \times Y})$ is smooth, and since α is smooth, it follows that the entire expression must be smooth. This proves that Φ^{-1} is smooth, and hence that the map Φ exhibits the desired diffeomorphism. $\square$

It is not much more work to then prove our goal:

Theorem 2.93. *The category* **Diffeol** *of diffeological spaces and smooth maps is Cartesian closed.*

Proof. Consider a fixed diffeological space $Y \in \mathbf{Diffeol}$, together with its induced product functor $- \times Y : \mathbf{Diffeol} \to \mathbf{Diffeol}$, which sends a smooth map $f : X \to Z$ to $f \times \mathrm{id}_Y$. There should then be an adjunction

$$- \times Y \dashv C^\infty(Y, -).$$

But this is literally the bijection constructed in Proposition **2.92**, which we know from the Cartesian closedness of **Set** is natural. ☐

Local Cartesian closedness. The category of diffeological spaces enjoys an even stronger categorical property: that of being *localy Cartesian closed* (Definition **A.7**). To prove that **Diffeol** is locally Cartesian closed we need to show, first, that every slice category **Diffeol**$/B$ has finite products, and further, that these finite products admit a right adjoint. It is easy to see that products in slice categories **Diffeol**$/B$ are just the pullbacks in **Diffeol**, which we know to exist. Therefore, to study local Cartesian closedness of **Diffeol**, we need to study function spaces of pullbacks. These are also called *parametrised mapping spaces* in [**BH11**, p. 15]. We elaborate here on the constructions discussed in that paper, and prove that the diffeology they define is actually a generalisation of the standard functional diffeology.

To start, fix two smooth maps $p_X : X \to B$ and $p_Y : Y \to B$, playing the rôle of objects in the slice category **Diffeol**$/B$. We shall denote the fibres of such maps by $X_b := p_X^{-1}(\{b\})$, for $b \in B$. These fibres naturally get a subset diffeology from X and Y, and so it makes sense to talk about the function spaces $C^\infty(X_b, Y_b)$. The *parametrised mapping space (over B)* is then defined as the collection of function spaces between each of the fibres:

$$C^\infty_B(X, Y) := \coprod_{b \in B} C^\infty(X_b, Y_b).$$

Since each function space $C^\infty(X_b, Y_b)$ between one of the fibres carries the standard functional diffeology, we could endow the parametrised mapping space $C^\infty_B(X, Y)$ with the coproduct diffeology. This diffeology is, however, too fine, because it does not allow for smooth variation between the fibres. We therefore now give a natural extension of the standard functional diffeology to parametrised mapping spaces. This diffeology is also sketched in [**BH11**, p. 16]. Again, this diffeology will be constructed to make certain structure maps smooth. In the first place, the first projection

$$\mathrm{pr}_1 : C^\infty_B(X, Y) \longrightarrow B; \qquad (b, f) \longmapsto b$$

should be smooth. This ensures that a family of functions can vary smoothly across fibres. And secondly, just as for any functional diffeology, the evaluation map should be smooth:

$$\mathrm{ev}_B : C^\infty_B(X, Y) \times_B^{\mathrm{pr}_1, p_X} X \longrightarrow Y; \qquad ((b, f), x) \longmapsto f(x).$$

The following is then a direct generalisation of Lemma **2.86**. Note that for a parametrisation Ω of $C^\infty_B(X, Y)$, the expression $\mathrm{ev}_B \circ (\Omega \times \mathrm{id}_X)$ only makes sense when restricted to the domain $U_\Omega \times_B^{\mathrm{pr}_1 \Omega, p_X} X$.

Lemma 2.94. *The collection of parametrisations* $\Omega : U_\Omega \to C^\infty_B(X, Y)$ *such that* $\mathrm{ev}_B \circ (\Omega \times \mathrm{id}_X)|_{U_\Omega \times_B X}$ *and* $\mathrm{pr}_1 \circ \Omega$ *are smooth defines a diffeology on* $C^\infty_B(X, Y)$ *such that* pr_1 *and* ev_B *are smooth.*

Proof. Let us denote by $\mathcal{D}$ the collection of parametrisations that satisfy these two conditions. Take $\Omega \in \mathcal{D}$ to be a constant parametrisation, taking values as $\Omega(t) = (b, f)$, for some $b \in B$ and $f \in C^\infty(X_b, Y_b)$. Then $\mathrm{pr}_1 \circ \Omega = \mathrm{const}_b$, which is smooth by the Axiom of Covering for the diffeology on B. Similarly, when restricted to the right domain, we get $\mathrm{ev}_B \circ (\Omega \times \mathrm{id}_X) = \mathrm{ev} \circ (\mathrm{const}_f \times \mathrm{id}_X)$, which is smooth by the Axiom of Covering for the standard functional diffeology on $C^\infty(X_b, Y_b)$. The proof that the Axioms of Smooth Compatibility and Locality hold for $\mathcal{D}$ can be copied almost directly from the proof of Lemma **2.86**, which we leave to the reader. Hence $\mathcal{D}$ defines a diffeology on $C^\infty_B(X, Y)$.

That pr_1 is smooth with respect to $\mathcal{D}$ follows directly from its characterisation. We are therefore left to check smoothness of ev_B. For that, let $\Omega \in \mathcal{D}$ be a plot of $C^\infty_B(X, Y)$ and $\alpha \in \mathcal{D}_X$ be a plot of X, both defined on the same Euclidean domain U, satisfying $\mathrm{pr}_1 \circ \Omega = p_X \circ \alpha$. Then we can write $\mathrm{ev}_B \circ (\Omega, \alpha) = \mathrm{ev}_B \circ (\Omega \times \mathrm{id}_X) \circ (\mathrm{id}_U, \alpha)$. However, the first term (id_U, α) takes values in $U \times_B^{\mathrm{pr}_1 \Omega, p_X} X$, so it follows $\mathrm{ev}_B \circ (\Omega \times \mathrm{id}_X)$ is defined on the right domain, on which it is smooth. The result follows. ☐

Definition 2.95. The diffeology on $C^\infty_B(X,Y)$ defined by the previous Lemma **2.94** is called the *standard parametrised functional diffeology*. It is in fact the coarsest diffeology on $C^\infty_B(X,Y)$ such that both pr_1 and ev_B are smooth.

Note that the standard parametrised functional diffeology contains the coproduct diffeology. Namely, if $\Omega : U_\Omega \to C^\infty_B(X,Y)$ is a plot in the coproduct diffeology on the disjoint union, then Ω is locally of the form $\Omega|_V = \iota_b \circ \Psi$, where $\Psi : V \to C^\infty(X_b, Y_b)$ is a plot in the standard functional diffeology, and ι_b is the natural inclusion. Clearly then $\mathrm{pr}_1 \circ \Omega|_V = \mathrm{const}_b$, which is smooth. Moreover, the domain for $\mathrm{ev}_B \circ (\Omega|_V \times \mathrm{id}_X)$ just becomes $V \times X_b$, on which it is equal to $\mathrm{ev} \circ (\Psi \times \mathrm{id}_X)$. This is also clearly smooth. Hence from the Axiom of Locality it follows that Ω is an element of the standard parametrised functional diffeology as well. The main difference is that in the coproduct diffeology, the plots are not allowed to transfer between fibres. In this sense the coproduct diffeology describes a functional diffeology that is discretely parametrised, and if B is discrete it is easy to see that they coincide.

Proposition 2.96. *For $X \xrightarrow{p_X} B$, $Y \xrightarrow{p_Y} B$, and $Z \xrightarrow{p_Z} B$ three smooth maps, there is a natural diffeomorphism*

$$C^\infty_B(X, C^\infty_B(Y,Z)) \xrightarrow{\Phi} C^\infty_B(X \times_B Y, Z).$$

Proof. This is a straightforward generalisation of Proposition **2.92**. □

We now strengthen Theorem **2.93**:

Theorem 2.97. *The category* **Diffeol** *of diffeological spaces is locally Cartesian closed.*

Note that since **Diffeol** has a terminal object, which is just the singleton set $1 = \{*\}$ with its unique diffeology (Example **2.13**), Theorem **2.93** is recovered from the Cartesian closedness of **Diffeol**$/1$.

2.4.1 Locally smooth maps and the D-topology

Since diffeology deals so well with quotients, it may happen that diffeological spaces have what are called *singularities* in classical differential topology. Orbifolds are one type of space that are a good example of this phenomena. For such spaces it may be interesting to look the smooth structure only on a local level, instead of a global level. In this section, based on [***Diffeology***, Chapter 2], we develop the tools to do this. This will eventually lead to a notion of an *atlas* on a diffeological space Section **6.1.1**. Note however that any such atlas concept has been absent at this point in the development of the diffeology, as it can be developed fully without such a notion.

The original definition of *local smoothness* is [***Diffeology***, Article 2.1], which reads as follows[31].

Definition 2.98. Let X and Y be diffeological spaces. A function $f : X \supseteq A \to Y$, defined on a subset $A \subseteq X$, is called *locally smooth* if for every plot $\alpha \in \mathcal{D}_X$, the composition $f \circ \alpha|_{\alpha^{-1}(A)} \in \mathcal{D}_Y$. We denote such a map purposefully by $f : X \supseteq A \to Y$, because the definition involves *all* plots of X, and not just those taking values in the subset A. It is therefore good to remember X in the notation.

A map $f : X \to Y$ is called *locally smooth at* $x \in X$ if there exists a subset $x \in A \subseteq X$ such that $f|_A : X \supseteq A \to Y$ is locally smooth.

There is a subtle implicit condition in this definition. For $f : X \supseteq A \to Y$ to be locally smooth, each $f \circ \alpha|_{\alpha^{-1}(A)}$ has to be a plot. But plots are special kinds of parametrisations, which are defined only on Euclidean domains. For that composition to be a plot, then, the preimage $\alpha^{-1}(A)$ has to be an *open* subset of $\mathrm{dom}(\alpha)$, for every $\alpha \in \mathcal{D}_X$. Therefore the domains of locally smooth maps have to be a special kind of subset. They are exactly the open subsets in a special topology:

Definition 2.99. Let $(X, \mathcal{D}_X)$ be a diffeological space. The *D-topology* on X is the topology whose open sets are exactly those subsets $A \subseteq X$ such that $\alpha^{-1}(A)$ is open for all $\alpha \in \mathcal{D}_X$. This is the *final topology* on X induced by the family of plots $(\alpha : U_\alpha \to X)_{\alpha \in \mathcal{D}_X}$. (See any textbook on topology for more on the definition of the final topology, e.g. [**Müg20**, Section 6.1].) The open subsets in the D-topology are called *D-open*.

[31] Beware that *local* smoothness has nothing to do with *local* Cartesian closedness.

We will not be studying the D-topology *per se* in this thesis. We refer to [***Diffeology***, Chapter 2] for some more elementary results, and the paper [**CSW14**] for a more in-depth discussion on D-topologies. It is interesting to remark that, with respect to the D-topologies: smooth functions are continuous, and diffeomorphisms are homeomorphisms ([***Diffeology***, Article 2.9]).

Proposition 2.100. *Smooth maps are D-continuous.*

Proof. Let $f : X \to Y$ be a smooth map between diffeological spaces, and let $A \subseteq Y$ be a D-open subset. We need to show that the preimage $f^{-1}(A)$ is D-open in X. For that, let $\alpha \in \mathcal{D}_X$ be a plot. Then $\alpha^{-1}(f^{-1}(A)) = (f \circ \alpha)^{-1}(A)$ is D-open, because f is smooth. □

Example 2.101. The D-topology given to a manifold by its manifold diffeology is just its own topology. (See [**CSW14**, Example 3.2].) The standard Euclidean topology on a Euclidean domain therefore also agrees with the D-topology it gets from its Euclidean diffeology.

We only collect one lemma, which we will need later to study locally smooth maps.

Lemma 2.102. *Let $\mathcal{D}_X$ be a diffeology on some set X that is generated by a family of parametrisations $\mathcal{F}$. Then $A \subseteq X$ is D-open if and only if for every $f \in \mathcal{F}$ the preimage $f^{-1}(A)$ is open.*

Proof. The parametrisations $f \in \mathcal{F}$ are plots in $\mathcal{D}_X$, so the "only if" implication follows immediately from the definition of the D-topology. We therefore prove the converse. Suppose that $A \subseteq X$ is a subset such that for all $f \in \mathcal{F}$ we have $f^{-1}(A)$ is open. We need to show that this is the case instead for arbitrary $\alpha \in \mathcal{D}_X$. But since $\mathcal{F}$ generates $\mathcal{D}_X$, we can use Proposition **2.27** to deduce that $\alpha : U_\alpha \to X$ is locally constant or factors through an element of $\mathcal{F}$. This means that we can find two families $(V_i)_{i \in I}$ and $(W_j)_{j \in J}$ of open subsets of the domain U_α such that: the union of these families covers U_α, $\alpha|_{V_i} = f_i \circ h_i$ for some $f_i \in \mathcal{F}$ and h_i smooth, and $\alpha|_{W_j} = \text{const}_{x_j}$ for some $x_j \in X$. Now the preimages of A under each of these restrictions of α is open: first we have that $(\alpha|_{V_i})^{-1}(A) = h_i^{-1}(f_i^{-1}(A))$ is open because h_i is a smooth map between Euclidean domains, and $f_i^{-1}(A)$ is open by assumption. On the other hand $(\alpha|_{W_j})^{-1}(A) = \text{const}_{x_j}^{-1}(A)$ is empty if $x_j \notin A$, and equal to the open subset W_j if otherwise, both of which are open. The preimage $\alpha^{-1}(A)$ can therefore be written as a union of open sets:

$$\alpha^{-1}(A) = \bigcup_{i \in I} (\alpha|_{V_i}^{-1})(A) \cup \bigcup_{j \in J} (\alpha|_{W_j})^{-1}(A) = \bigcup_{i \in I} h_i^{-1}(f_i^{-1}(A)) \cup \bigcup_{j \in J} \text{const}_{x_j}^{-1}(A).$$

Since the plot α was arbitrary, this shows that $A \subseteq X$ is D-open. □

Back to locally smooth maps. The following proposition shows us that the locally smooth maps are just those that are smooth on D-open subsets. We prefer this characterisation to the original definition Definition **2.98** ([***Diffeology***, Article 2.1]), because it allows us to talk about locally smooth maps using the original definition of smooth maps in Definition **2.4**.

Proposition 2.103. *Let X and Y be two diffeological spaces, and consider a subset $A \subseteq X$. Then $f : X \supseteq A \to Y$ is local smooth if and only if A is D-open and $f : A \to Y$ is smooth with respect to the subset diffeology on A.*

Proof. Suppose $f : X \supseteq A \to Y$ is local smooth. The definition of the D-topology on X then ensures A is D-open. Let $\alpha \in \mathcal{D}_A$ be a plot of X taking values in A. Then $\alpha^{-1}(A) = \text{dom}(\alpha)$, and since $f : X \supseteq A \to Y$ is local smooth we then get $f \circ \alpha|_{\alpha^{-1}(A)} = f \circ \alpha \in \mathcal{D}_Y$, which just means that $f : A \to Y$ is smooth. Conversely, suppose that A is D-open and $f : A \to Y$ is smooth. Let $\alpha \in \mathcal{D}_X$ be an arbitrary plot (not necessarily taking values in A). Then $\alpha^{-1}(A)$ is an open subset of $\text{dom}(\alpha)$, making $\alpha|_{\alpha^{-1}(A)}$ into a plot of A in the subset diffeology. But f is smooth with respect to that diffeology, so $f \circ \alpha|_{\alpha^{-1}(A)} \in \mathcal{D}_Y$, as required by Definition **2.98**. □

We find this point of view much nicer, in which a function $f : A \to Y$ is locally smooth if we can just check that it is smooth and it its domain is D-open. Smoothness has only to be checked for the plots of A, and not for the plots of the entire ambient space $X \supseteq A$. From now on, if we say $f : A \to Y$ is locally smooth, we mean that it is smooth as a function $A \to Y$ and that A is D-open. A function

$f : X \to Y$ is locally smooth at $x \in X$ if and only if there exists a D-open neighbourhood $x \in A \subseteq X$ such that $f|_A : A \to Y$ is smooth. This gives a nice result, which is analogous to [**Lee13**, Corollary 2.8] in the case of manifolds.

Proposition 2.104. *A map $f : X \to Y$ is smooth if and only if it is locally smooth at every point $x \in X$.*

Proof. If f is smooth, then the desired D-open neighbourhood of $x \in X$ is just X itself. Conversely, suppose that f is locally smooth at every point. Then we can find a D-open cover $(A_x)_{x\in X}$, where $x \in A_x \subseteq X$, such that $f|_{A_x}$ is smooth. Take then a plot $\alpha : U_\alpha \to X$ in $\mathcal{D}_X$. We need to show that $f \circ \alpha \in \mathcal{D}_Y$. Each $t \in U_\alpha$ gives a point $\alpha(t) \in X$, which is covered by a D-open neighbourhood $A_{\alpha(t)}$. The preimage $\alpha^{-1}(A_{\alpha(t)})$ is then an open neighbourhood of $t \in U_\alpha$, and the family $(\alpha^{-1}(A_{\alpha(t)}))_{t\in U_\alpha}$ forms an open cover of U_α. If we restrict the plot α to elements of this open cover, we get plots of X taking values in the respective $A_{\alpha(t)}$, which f sends to plots in $\mathcal{D}_Y$. In other words, since $f|_{A_{\alpha(t)}}$ is smooth, we get $f \circ \alpha|_{\alpha^{-1}(A_{\alpha(t)})} = f|_{A_{\alpha(t)}} \circ \alpha|_{\alpha^{-1}(A_{\alpha(t)})} \in \mathcal{D}_Y$, for every $t \in U_\alpha$. Hence by the Axiom of Locality for $\mathcal{D}_Y$, it follows that $f \circ \alpha$ is a plot, and hence that $f : X \to Y$ is smooth. □

Lemma 2.105. *Let $f : X \supseteq A \to Y$ and $g : Y \supseteq B \to Z$ be two locally smooth maps. Then the map $g \circ f : X \supseteq f^{-1}(B) \to Z$ is also locally smooth.*

Proof. In light of Proposition **2.103**, we know $f : A \to Y$ and $g : B \to Z$ are two smooth maps defined on D-open domains. By Proposition **2.55** it follows $g \circ f|_{f^{-1}(B)}$ is smooth, so we are left to show that $f^{-1}(B)$ is D-open. So let $\alpha \in \mathcal{D}_X$ be a plot of X. Then $\alpha^{-1}(f^{-1}(B)) = (f \circ \alpha|_{\alpha^{-1}(A)})^{-1}(B)$. But since $f : A \to Y$ is smooth, $f \circ \alpha|_{\alpha^{-1}(A)} \in \mathcal{D}_Y$, and since B is D-open, it follows that $\alpha^{-1}(f^{-1}(B))$ is open. Since α was arbitrary, the claim follows. □

The rest of this section is based on [**IZL18**, Section 2], filling in some details on their proofs. We shall lay the groundwork here for the construction of an interesting diffeological groupoid in Section **6.1**.

The space of all local smooth maps is denoted $C^\infty_{\text{loc}}(X, Y)$. If τ_X denotes the D-topology of a diffeological space $(X, \mathcal{D}_X)$, then this space is explicitly given by

$$C^\infty_{\text{loc}}(X, Y) := \{f \in C^\infty(A, Y) : A \in \tau_X\}.$$

If the D-topology on X is rich, this is clearly a much bigger space than $C^\infty(X, Y)$. Just like the standard functional diffeology on $C^\infty(X, Y)$, which is the unique coarsest diffeology making the evaluation map smooth, there is a *standard local functional diffeology* on the bigger space $C^\infty_{\text{loc}}(X, Y)$, which is the unique coarsest diffeology making the evaluation map local smooth on a natural domain. This natural domain is just the pairs (f, x) for which $x \in \text{dom}(f)$:

$$\mathcal{E}_{X,Y} := \coprod_{f\in C^\infty_{\text{loc}}(X,Y)} \text{dom}(f) = \{(f, x) \in C^\infty_{\text{loc}}(X, Y) \times X : x \in \text{dom}(f)\}.$$

Here '$\mathcal{E}$' stands for *evaluable*. In analogy to the construction of the standard functional diffeology on $C^\infty(X, Y)$ (Lemma **2.85**), we shall construct a coarsest diffeology on $C^\infty_{\text{loc}}(X, Y)$ such that $\mathcal{E}_{X,Y}$ is D-open and $\text{ev} : \mathcal{E}_{X,Y} \to Y$ is smooth. It is clear that the discrete diffeology on $C^\infty_{\text{loc}}(X, Y)$ makes $\text{ev} : \mathcal{E}_{X,Y} \to Y$ locally smooth, so there are diffeologies that satisfy this. We want to adapt the characterisation of the plots of the standard functional diffeology on $C^\infty(X, Y)$, as described in Lemma **2.86**. However, if $\Omega : U_\Omega \to C^\infty_{\text{loc}}(X, Y)$ is a parametrisation, the expression $\text{ev} \circ (\Omega \times \text{id}_X)$ may not be defined everywhere, because there could exist a pair $(t, x) \in U_\Omega \times X$ such that $x \notin \text{dom}(\Omega(t))$. To remedy this, instead of demanding full-on smoothness of $\text{ev} \circ (\Omega \times \text{id}_X)$, we will only demand local smoothness. The maximal domain that we can allow for this is the following:

$$\mathcal{U}_\Omega := (\Omega \times \text{id}_X)^{-1}(\mathcal{E}_{X,Y}) = \{(t, x) \in U_\Omega \times X : x \in \text{dom}(\Omega(t))\}.$$

We can then say something about the local smoothness of $\text{ev} \circ (\Omega \times \text{id}_X)|_{\mathcal{U}_\Omega}$, and we will now show that the collection of parametrisations Ω that make this expressions locally smooth defines a diffeology:

Lemma 2.106. *The collection of parametrisations* $\Omega : U_\Omega \to C^\infty_{\text{loc}}(X,Y)$*, making* $\text{ev} \circ (\Omega \times \text{id}_X)|_{\mathcal{U}_\Omega}$ *locally smooth, defines a diffeology on* $C^\infty_{\text{loc}}(X,Y)$*.*

Proof. This is a generalisation of the proof of Lemma **2.86**, but the idea remains the same. In addition we need to prove that each of the domains $\mathcal{U}_\Omega$ are D-open. Denote the collection of parametrisations as in the claim by $\mathcal{D}$.

Let us start with the Axiom of Covering. For that, let $\Omega : U_\Omega \to C^\infty_{\text{loc}}(X,Y)$ be a constant parametrisation, taking values on a locally smooth map $f \in C^\infty_{\text{loc}}(X,Y)$. Then $\mathcal{U}_\Omega = U_\Omega \times \text{dom}(f)$, which is clearly D-open. Then $\text{ev} \circ (\Omega \times \text{id}_X)|_{\mathcal{U}_\Omega} = f \circ \text{pr}_2|_{\mathcal{U}_\Omega}$ is smooth by Proposition **2.55**. This proves the collection $\mathcal{D}$ contains all constant parametrisations.

For the Axiom of Smooth Compatibility, let $\Omega : U_\Omega \to C^\infty_{\text{loc}}(X,Y)$ be a parametrisation in $\mathcal{D}$, and take a smooth function $h : V \to U_\Omega$. Then $h \times \text{id}_X$ is smooth and D-continuous, and an easy calculation shows that $\mathcal{U}_{\Omega\circ h} = (h \times \text{id}_X)^{-1}(\mathcal{U}_\Omega)$, which is therefore D-open because $\mathcal{U}_\Omega$ is D-open. It follows then that

$$\text{ev} \circ ((\Omega \circ h) \times \text{id}_X)\,|_{\mathcal{U}_{\Omega\circ h}} = \text{ev} \circ (\Omega \times \text{id}_X)|_{\mathcal{U}_\Omega} \circ (h \times \text{id}_X)|_{\mathcal{U}_{\Omega\circ h}}$$

is smooth, because $\text{ev} \circ (\Omega \times \text{id}_X)|_{\mathcal{U}_\Omega}$ is smooth (and Proposition **2.55**).

Lastly, for the Axiom of Locality, let $\Omega : U_\Omega \to C^\infty_{\text{loc}}(X,Y)$ be a parametrisation together with an open cover $(V_i)_{i\in I}$ of U_Ω, such that each restriction $\Omega|_{V_i}$ is an element of $\mathcal{D}$. Then we know that each $\mathcal{U}_{\Omega|_{V_i}} = \mathcal{U}_\Omega \cap (V_i \times X)$ is D-open. Since $(V_i)_{i\in I}$ covers U_Ω, we get that

$$\bigcup_{i\in I} \mathcal{U}_{\Omega|_{V_i}} = \bigcup_{i\in I} \mathcal{U}_\Omega \cap (V_i \times X) = \mathcal{U}_\Omega \cap (U_\Omega \times X) = \mathcal{U}_\Omega$$

forms a D-open cover, which proves that $\mathcal{U}_\Omega$ itself also has to be D-open. Lastly, because each term $\text{ev} \circ (\Omega|_{V_i} \times \text{id}_X)|_{\mathcal{U}_{\Omega|_{V_i}}}$ is smooth, it follows that $\text{ev} \circ (\Omega \times \text{id}_X)|_{\mathcal{U}_\Omega}$ restricts to smooth maps on the D-open cover $(\mathcal{U}_{\Omega|_{V_i}})_{i\in I}$, so by the Axiom of Locality for $\mathcal{D}_Y$ and Proposition **2.104** it follows that Ω satisfies the defining condition of $\mathcal{D}$. This proves that $\mathcal{D}$ satisfies all three Axioms of Diffeology. □

We equip $C^\infty_{\text{loc}}(X,Y)$ with this diffeology, and subsequently $\mathcal{E}_{X,Y}$ with the induced subset diffeology. Note that, for similar reasons we touched upon in our discussion on the standard parametrised functional diffeology on parametrised mapping spaces, the diffeology on $\mathcal{E}_{X,Y}$ is not simply the coproduct diffeology.

Proposition 2.107. *The evaluation map* $\text{ev} : \mathcal{E}_{X,Y} \to Y$ *is locally smooth with respect to the diffeology in Lemma* ***2.106****.*

Proof. First we need to prove that $\mathcal{E}_{X,Y}$ is D-open in $C^\infty_{\text{loc}}(X,Y) \times X$, when $C^\infty_{\text{loc}}(X,Y)$ is endowed with this diffeology. By Lemmas **2.61** and **2.102** it suffices to check for preimages by plots of the form $\Omega \times \alpha$, where $\alpha \in \mathcal{D}_X$ and $\Omega : U_\Omega \to C^\infty_{\text{loc}}(X,Y)$ is a plot as in Lemma **2.106**. We can then write

$$(\Omega \times \alpha)^{-1}(\mathcal{E}_{X,Y}) = (\text{id}_{U_\Omega} \times \alpha)^{-1}\left((\Omega \times \text{id}_X)^{-1}(\mathcal{E}_{X,Y})\right) = (\text{id}_{U_\Omega} \times \alpha)^{-1}(\mathcal{U}_\Omega).$$

But $\mathcal{U}_\Omega$ is D-open since Ω is a plot, and $\text{id}_{U_\Omega} \times \alpha$ is D-continuous, so the right hand side of this equation is open. Hence $\mathcal{E}_{X,Y}$ is D-open.

That leaves us to show that $\text{ev} : \mathcal{E}_{X,Y} \to Y$ is smooth with respect to this diffeology. By Lemmas **2.33** and **2.61** it suffices to check smoothness for plots of the form $\Omega \times \alpha$ as above taking values in $\mathcal{E}_{X,Y}$, meaning that for every $(t,s) \in U_\Omega \times U_\alpha$ we have $\alpha(s) \in \text{dom}(\Omega(t))$. But, in that case, $U_\Omega \times \text{im}(\alpha) \subseteq \mathcal{U}_\Omega$, so that

$$\text{ev} \circ (\Omega \times \alpha) = \text{ev} \circ (\Omega \times \text{id}_X)|_{\mathcal{U}_\Omega} \circ (\text{id}_{U_\Omega} \times \alpha),$$

in which every term is smooth, so the result follows. □

Definition 2.108. The diffeology on $C^\infty_{\text{loc}}(X,Y)$ from Lemma **2.106** is called the *standard local functional diffeology*. Just like the standard functional diffeology on $C^\infty(X,Y)$, it is easy to see that this diffeology is the coarsest one on $C^\infty_{\text{loc}}(X,Y)$ such that the evaluation map is locally smooth on $\mathcal{E}_{X,Y}$. From now, we will assume that $C^\infty_{\text{loc}}(X,Y)$ is always equipped with the standard local functional diffeology.

Since the space of globally smooth functions $C^\infty(X,Y)$ sits naturally in the locally smooth ones, we can ask what diffeology $C^\infty(X,Y)$ inherits from the standard local functional diffeology. It turns out that it is just the standard functional diffeology:

Proposition 2.109. *The subset diffeology of $C^\infty(X,Y) \subseteq C^\infty_{\mathrm{loc}}(X,Y)$ coincides with the standard functional diffeology of Definition 2.87.*

Proof. This is easy to see, once we observe that for any plot $\Omega : U_\Omega \to C^\infty_{\mathrm{loc}}(X,Y)$ taking values in $C^\infty(X,Y)$ we have $\mathcal{U}_\Omega = U_\Omega \times X$. □

We also have a result generalising Proposition **2.88** to the local setting.

Proposition 2.110. *Let X, Y, and Z be diffeological spaces. The local composition map is then smooth:*

$$\mathrm{comp}_{\mathrm{loc}} : C^\infty_{\mathrm{loc}}(Y,Z) \times C^\infty_{\mathrm{loc}}(X,Y) \longrightarrow C^\infty_{\mathrm{loc}}(X,Z); \qquad (f,g) \longmapsto (f\circ g)|_{g^{-1}(\mathrm{dom}(f))}.$$

Proof. Note first that $\mathrm{comp}_{\mathrm{loc}}$ is well-defined by Lemma **2.105**. To prove that $\mathrm{comp}_{\mathrm{loc}}$ is smooth, take two plots $\Omega : U_\Omega \to C^\infty_{\mathrm{loc}}(Y,Z)$ and $\Psi : U_\Psi \to C^\infty_{\mathrm{loc}}(X,Y)$. As we know, by Lemmas **2.33** and **2.61** it suffices to prove that $\mathrm{comp}_{\mathrm{loc}} \circ (\Omega\times\Psi)$ is a plot of $C^\infty_{\mathrm{loc}}(X,Z)$. To shorten notation, let us denote $\mathcal{U} := \mathcal{U}_{\mathrm{comp}_{\mathrm{loc}}\circ(\Omega\times\Psi)}$, which is the set of points $(t,s,x) \in U_\Omega \times U_\Psi \times X$ satisfying $x \in \Psi(s)^{-1}(\mathrm{dom}(\Omega(t)))$. We need to prove that $\mathcal{U}$ is D-open. To see this, note that $(t,s,x)\in\mathcal{U}$ if and only if $\Psi(s)(x) \in \mathrm{dom}(\Omega(t))$. We can rewrite $\Psi(s)(x) = \mathrm{ev}\circ(\Psi\times\mathrm{id}_X)(s,x)$. If we discard the term U_Ω through the smooth projection $\hat{\mathrm{pr}} : U_\Omega \times U_\Psi \times X \to U_\Psi \times X$, which maps $(t,s,x) \mapsto (s,x)$, then we get a smooth map:

$$\left(\mathrm{pr}_{U_\Omega}, \mathrm{ev}\circ(\Psi\times\mathrm{id}_X)|_{\mathcal{U}_\Psi}\circ\hat{\mathrm{pr}}\right) : U_\Omega\times U_\Psi\times X \longrightarrow U_\Omega\times Y; \qquad (t,s,x)\longmapsto (t,\Psi(s)(x)).$$

The codomain of this map contains $\mathcal{U}_\Omega$. It is straightforward to see that the preimage of $\mathcal{U}_\Omega$ under this map is $\mathcal{U}$, which proves that it is D-open. That leaves us to show that $\mathrm{ev}\circ([\mathrm{comp}_{\mathrm{loc}}\circ(\Omega\times\Psi)]\times\mathrm{id}_X)|_{\mathcal{U}}$ is smooth. This transpires much the same way as in the proof of Proposition **2.88**. From the displayed equation in that proof, we get a similar equation for each $(t,s,x)\in\mathcal{U}$:

$$\mathrm{ev}\circ([\mathrm{comp}_{\mathrm{loc}}\circ(\Omega\times\Psi)]\times\mathrm{id}_X)|_{\mathcal{U}}(t,s,x) = \mathrm{ev}\circ(\Omega\times\mathrm{id}_Y)\circ(\mathrm{id}_{U_\Omega}\times[\mathrm{ev}\circ(\Psi\times\mathrm{id}_X)])\,(t,s,x).$$

We know something about the smoothness of the terms $\mathrm{ev}\circ(\Omega\times\mathrm{id}_Y)|_{\mathcal{U}_\Omega}$ and $\mathrm{ev}\circ(\Psi\times\mathrm{id}_X)|_{\mathcal{U}_\Psi}$, since Ω and Ψ are plots. Therefore, if we can show that those terms in the right hand side of the above equation take values in $\mathcal{U}_\Omega$ and $\mathcal{U}_\Psi$, we are done. But this follows because $(t,s,x)\in\mathcal{U}$, which implies $x\in\Psi(s)^{-1}(\mathrm{dom}(\Omega(t)))$, which immediately gives $(s,x)\in\mathcal{U}_\Psi$ and $(t,\Psi(s)(x))\in\mathcal{U}_\Omega$. This shows that the left hand side of the above equation is smooth, and hence $\mathrm{ev}\circ([\mathrm{comp}_{\mathrm{loc}}\circ(\Omega\times\Psi)]\times\mathrm{id}_X)|_{\mathcal{U}}$ satisfies the conditions to be a plot of $C^\infty_{\mathrm{loc}}(X,Z)$, which proves that $\mathrm{comp}_{\mathrm{loc}}$ is smooth. □

All of this technology about locally smooth maps will be put to good use in Section **6.1**.

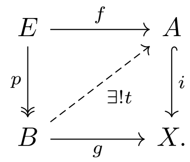
2.5 A weak subobject classifier for diffeological spaces

So far we have seen that **Diffeol** is (co)complete, Cartesian closed, and even locally Cartesian closed. To finish off our discussion of the categorical aspects of diffeology, we will here provide the proverbial icing on the diffeological cake by showing that **Diffeol** has a *weak subobject classifier* (Definition **2.114**), and is hence a *quasitopos* (Definition **2.117**). Much of this section is based on the sheaf-theoretic results found throughout [**BH11**], but we have translated them here into the language of plots.

Proposition 2.111. *The epimorphisms and monomorphisms in* **Diffeol** *are just the surjective and injective smooth functions, respectively.*

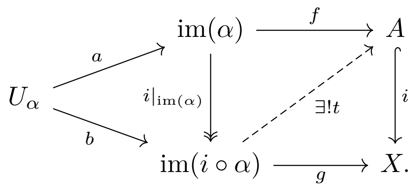
Proof. This follows in one direction because the forgetful functor U : **Diffeol** $\to$ **Set** preserves limits and colimits (Proposition **2.69**). The other direction follows because U is faithful, and hence reflects mono- and epimorphisms. □

Definition 2.112. A monomorphism $i : A \hookrightarrow X$ in a category $\mathbf{C}$ is called *strong* if for every epimorphism $p : E \twoheadrightarrow B$ and two arrows $f : E \to A$, $g : B \to X$ making a commutative square, there is a unique arrow fitting into the following commutative diagram:



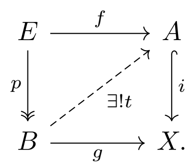
Proposition 2.113. *The strong monomorphisms in* **Diffeol** *are exactly the inductions.*

Proof. Let $i : A \hookrightarrow X$ be a strong monomorphism in **Diffeol**. From the characterisation of monomorphisms between diffeological spaces in Proposition **2.111** we already know that i is a smooth injection. In light of the fact that this map is smooth if and only if $\mathcal{D}_A \subseteq i^*(\mathcal{D}_X)$ (Corollary **2.37**), we are left to show $i^*(\mathcal{D}_X) \subseteq \mathcal{D}_A$. For that, let $\alpha : U_\alpha \to A$ be a *parametrisation* of A such that $i \circ \alpha \in \mathcal{D}_X$. We need to show that $\alpha \in \mathcal{D}_A$. For this we shall use the universal defining property of strong monomorphisms. Let $f : \mathrm{im}(\alpha) \hookrightarrow A$ and $g : \mathrm{im}(i \circ \alpha) \hookrightarrow X$ be the respective smooth inclusion maps, where each of the domains has the subset diffeology. We then get a commuting diagram:



In this diagram, $a : U_\alpha \to \mathrm{im}(\alpha)$ is the unique function so that $f \circ a = \alpha$, and $b : U_\alpha \to \mathrm{im}(i \circ \alpha)$ is the unique arrow such that $g \circ b = i \circ \alpha$. Then, save for a (the corestriction of α), we know that everything is smooth. In particular, b is smooth because $\mathrm{im}(i \circ \alpha)$ has the subset diffeology. From the commutativity of the diagram it follows that $i \circ \alpha = i \circ t \circ b$, and since i is injective this gives $\alpha = t \circ b$. Hence α is the composition of two smooth maps, and must therefore be a plot in A. This shows that $\mathcal{D}_A = i^*(\mathcal{D}_X)$, and therefore the map i is an induction.

For the converse implication, suppose now that we have an induction $i : A \hookrightarrow X$. Start with the data of Definition **2.112**, i.e., we have a smooth surjection (Proposition **2.111**) $p : E \twoheadrightarrow B$ and two smooth maps $f : E \to A$ and $g : B \to X$ such that $i \circ f = g \circ p$. We will construct the unique arrow $t : B \to A$ making the following diagram commute:



To construct this smooth map, recall that by Proposition **2.54** we get a diffeomorphism $i : A \to \mathrm{im}(i)$. Moreover, since the outer square commutes and p is surjective, we have $\mathrm{im}(g) = \mathrm{im}(g \circ p) = \mathrm{im}(i \circ f) \subseteq \mathrm{im}(i)$. In particular, the induction then restricts to a diffeomorphism $i|_{\mathrm{im}(f)} : \mathrm{im}(f) \to \mathrm{im}(i \circ f) = \mathrm{im}(g)$, which allows us to define $t := (i|_{\mathrm{im}(f)})^{-1} \circ g : B \to A$. This is clearly the unique smooth map that we are after, proving that i is a strong monomorphism. □

Definition 2.114. Let $\mathbf{C}$ be a category with finite limits, whose terminal object we denote by $1_{\mathbf{C}}$. For any object $C \in \mathrm{ob}(\mathbf{C})$ we then have a unique arrow $t_C : C \to 1_{\mathbf{C}}$. A *weak subobject classifier* is an object $\Omega \in \mathrm{ob}(\mathbf{C})$ together with an arrow called $\mathrm{true} : 1_{\mathbf{C}} \to \Omega$, so that for every strong monomorphism $i : C \hookrightarrow D$ in $\mathbf{C}$ there is a pullback diagram:

$$\begin{array}{ccc} C & \xrightarrow{t_C} & 1_{\mathbf{C}} \\ {\scriptstyle i}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{true}} \\ D & \underset{\exists! \chi_i}{\dashrightarrow} & \Omega. \end{array}$$

We call these subobject classifiers *weak* because they only classify the subobjects of *strong* monomorphisms, whereas a genuine subobject classifier classifies every monomorphism. The arrow $\chi_i : D \to \Omega$ is called the *characteristic function* of $i : C \hookrightarrow D$. In the category of sets, these are actually the characteristic functions of subsets, so that $\Omega = \{0, 1\}$ and $\mathrm{true} = \mathrm{const}_1$.

Construction 2.115. We construct a weak subobject classifier for the category **Diffeol** of diffeological spaces. This construction just adopts the construction of the subobject classifier for **Set** into the diffeological setting (cf. [**Mac71**, Section IV.9]). Recall that the terminal object in **Diffeol** is just the one-point space $1 = \{*\}$, which has a unique diffeology (the discrete one, which happens to align with the coarse one). As a set, we then define $\Omega = \{0, 1\}$. This set only has two diffeologies, and we choose $\mathcal{D}_\Omega := \mathrm{Param}(\Omega)$. The elements of Ω are usually interpreted as bivalent truth-values. This also explains the notation of the function $\mathrm{true} : 1 \to \Omega$, which is then just the constant map $* \mapsto 1$ taking values on the positive truth-value. Since Ω carries the coarse diffeology, this map is smooth. Moreover, for any induction $i : A \hookrightarrow X$ we define the characteristic function

$$\chi_i : X \longrightarrow \Omega; \qquad x \longmapsto \begin{cases} 1 & \text{if } x \in \mathrm{im}(i), \\ 0 & \text{if } x \notin \mathrm{im}(i). \end{cases}$$

Theorem 2.116. *The arrow* $\mathrm{true} : 1 \to \Omega$ *and the characteristic functions from Construction* ***2.115*** *form a weak subobject classifier for* **Diffeol**.

Proof. We just need to prove that if $i : A \hookrightarrow X$ is an induction (Proposition **2.113**), then we have a pullback:

$$\begin{array}{ccc} A & \xrightarrow{t_A} & 1 \\ {\scriptstyle i}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{true}} \\ X & \underset{\exists! \chi_i}{\dashrightarrow} & \Omega. \end{array}$$

But for that it suffices to show there exists a diffeomorphism $A \cong X \times_\Omega 1$ compatible with the projections of $X \times_\Omega 1$ onto its components. The fibred product is just $X \times_\Omega 1 = \{(x, *) : \chi_i(x) = 1\} = \mathrm{im}(i) \times 1$. Since 1 is the terminal object, the projection therefore induces a diffeomorphism $X \times_\Omega 1 \cong \mathrm{im}(i)$. However, we know that inductions induce diffeomorphisms onto their images (Proposition **2.54**), which immediately gives $A \cong \mathrm{im}(i) \cong X \times_\Omega 1$. More precisely, this diffeomorphism is induced by the map $\Phi : A \to X \times_\Omega 1$ defined as $a \mapsto (i(a), *)$. It is easy to check that $\mathrm{pr}_1 \circ \Phi = i$ and $\mathrm{pr}_2 \circ \Phi = t_A$, which proves that the diagram is a pullback. □

It is from this proof that we can also see why **Diffeol** might not be a genuine topos: because the smooth injections $i : A \hookrightarrow X$ in general do not induce diffeomorphisms $A \cong \mathrm{im}(i)$.

Definition 2.117. A category **C** is called a *quasitopos* if it has all finite limits and finite colimits, is locally Cartesian closed, and admits a weak subobject classifier. (See also [**BH11**; *n***L18b**].)

Just as Grothendieck toposes (sheaves on a site) form an important class of examples of a topos, the special class of *concrete sheaves* on a *concrete site* form an important example of quasitoposes. For the general study of these types of categories we refer to [**Dub79**; **GL12**]. [**BH11**] contains an account of the quasitoposes of concrete sheaves, including case-studies on diffeological- and Chen spaces.

Theorem 2.118. *The category* **Diffeol** *of diffeological spaces and smooth maps is a quasitopos.*

Proof. This is the culmination of Theorems **2.72**, **2.97** and **2.116**. □

According to [**GL12**, p.2], the main difference between a topos and a quasitopos is that the latter does not have to be *balanced*. This means that, in a quasitopos, there may exist morphisms that are epic and monic, but not invertible. On the other hand, every topos is balanced (and in fact, a balanced quasitopos is a topos). This is clear in the case of **Set**, where every injective surjection is a bijection. In **Diffeol**, since the epi- and monomorphisms are just the surjective- and injective functions respectively (Proposition **2.111**), it is easy to find a bijective smooth function that is not a diffeomorphism (for example $\mathbb{R} \to \mathbb{R} : x \mapsto x^3$). We do not know an interpretation or the significance of this fact.

2.6 Subductions

The concept of a *subduction* will be very important in Chapter **IV**, because it will replace that of a submersion. There is no unambiguous notion of submersion between arbitrary diffeological spaces as there is for smooth manifolds, in terms of differentials. Luckily, one can show that *local* subductions agree precisely with submersions between Euclidean domains (Proposition **2.128**). So it would be acceptable to take local subductions as the replacement for the concept of submersion. However, as we will see, it turns out that everything works for subductions as well, although this is a strictly *weaker* notion than of a local subduction[32].

Recall Definition **2.42**:

Definition 2.119. A map $f : X \to Y$ between a diffeological spaces is called a *subduction* if f is surjective and $f_*(\mathcal{D}_X) = \mathcal{D}_Y$. Note that subductions are automatically smooth. The subductions are exactly the surjective functions such that the diffeology of its codomain is the finest diffeology making it smooth.

In practice, it is cumbersome to continually calculate pushforwards of diffeologies to verify that a map is a subduction. We therefore need a more hands-on characterisation. This is provided by the following proposition, which will almost entirely replace Definition **2.119** throughout our proofs.

Proposition 2.120. *Let $f : X \to Y$ be a function between diffeological spaces. Then f is a subduction if and only if the following two conditions are both satisfied:*

1. *The map f is smooth.*
2. *For every plot $\alpha : U_\alpha \to Y$ and every point $t \in U_\alpha$, there exists an open neighbourhood $t \in V \subseteq U_\alpha$ and a plot $\beta : V \to X$ such that $\alpha|_V = f \circ \beta$. (In other words, the plots of Y lift locally along f.)*

Proof. First let f be a subduction. The first condition is then immediately satisfied. Moreover, since f is surjective, the family of parametrisations $f \circ \mathcal{D}_X$ on Y is covering, and so by Lemma **2.43** and the second characterisation in Proposition **2.27** we find the second condition fulfilled.

Conversely, suppose that the two listed conditions are satisfied for some arbitrary function $f : X \to Y$. That f is smooth gives the inclusion $f_*(\mathcal{D}_X) \subseteq \mathcal{D}_Y$, as per Corollary **2.41**. The second condition shows that every plot in $\mathcal{D}_Y$ is locally in $f \circ \mathcal{D}_X$, so the Axiom of Locality gives the other inclusion. Lastly, to prove that f is surjective, take $y \in Y$ and consider the constant plot $\mathrm{const}_y : \mathbb{R} \to Y$. Even this must lift through f, so locally $\mathrm{const}_y|_V = f \circ \beta$. It follows $y = f(x)$ for $x = \beta(0) \in X$. ☐

Note that this proposition is a slight improvement on [***Diffeology***, Article 1.48], where the author in addition assumes in the first condition that f has to be a surjection. As we see here, this is redundant, as it is already ensured by the second condition. A big upside to Proposition **2.120** is that most of the maps we *want* to be subductions are already smooth (surjections), so that we only have to verify the second condition.

Lemma 2.121. *Let $f : Y \to Z$ and $g : X \to Y$ be two smooth maps between diffeological spaces. If $f \circ g$ is a subduction, then so is f.*

[32] The example of a subduction that is not a local subduction in [***Diffeology***, Exercise 61] is based on a non-manifold. We will show in Example **2.129** that these notions do not even collapse on Euclidean domains.

Proof. Assuming the hypothesis of the claim, and taking a plot $\alpha : U_\alpha \to Z$, then for every $t \in U_\alpha$ we can find an open neighbourhood $t \in V \subseteq U_\alpha$ and a plot $\beta : V \to X$ such that $\alpha|_V = (f \circ g) \circ \beta$. Using associativity then shows that $g \circ \beta \in \mathfrak{D}_Y$ is the desired local lift of α along f. Also, it is elementary that f is a surjection, so that by Proposition **2.120** the result follows. ☐

A direct and very useful corollary of this is that any map that admits a global smooth section has to be a subduction.

The following generalises a property of submersions in the case of manifolds (see e.g. [**Lee13**, Theorem 4.29]) to the diffeological world:

Lemma 2.122. *Let $\pi : X \to B$ be a subduction, and $f : B \to Y$ a function. Then f is smooth if and only if $f \circ \pi$ is smooth. Moreover, f is a subduction if and only if $f \circ \pi$ is a subduction.*

Proof. The first claim follows directly from Theorem **2.47**(*3*), or, alternatively, from Lemmas **2.33** and **2.43**. But, to see the underlying mechanism at work, we give an explicit proof here as well. Our proof looks different, but is equivalent, to the one in [***Diffeology***, Article 1.51]. Let's start with the first claim. If f is smooth, it follows immediately that $f \circ \pi$ is smooth. Conversely, suppose that $f \circ \pi$ is smooth, and fix a plot $\alpha : U_\alpha \to X$. Since π is a subduction we can find an open cover $(V_t)_{t \in U_\alpha}$ of U_α together with plots $\beta_t : V_t \to X$ such that $\alpha|_{V_t} = \pi \circ \beta_t$. It follows that each restriction $f \circ \alpha|_{V_t} = f \circ \pi \circ \beta$ is a plot, and hence by the Axiom of Locality it follows $f \circ \alpha \in \mathfrak{D}_Y$.

For the claim about subductions, again it is trivial if f is a subduction (with the above). The converse follows from Lemma **2.121**. ☐

Proposition 2.123. *Any injective subduction is a diffeomorphism.*

Lemma 2.124. *Let $f : X \to Z$ be a subduction, and $g : Y \to Z$ be a smooth map. Then the projection onto the second component restricted to the fibred product $\mathrm{pr}_2|_{X \times^{f,g}_Z Y} : X \times^{f,g}_Z Y \to Y$ is also a subduction. In other words, in* **Diffeol**, *subductions are preserved under pullback.*

Proof. Consider a plot $\alpha : U_\alpha \to Y$, giving another plot $g \circ \alpha \in \mathfrak{D}_Z$. Since f is a subduction, for every $t \in U_\alpha$ we can find a plot $\beta : V \to X$ on an open subset $t \in V \subseteq U_\alpha$ such that $g \circ \alpha|_V = f \circ \beta$. Now $(\beta, \alpha|_V) : V \to X \times_Z Y$ is a plot that satisfies $\mathrm{pr}_2|_{X \times_Z Y} \circ (\beta, \alpha|_V) = \alpha|_V$, showing that the restricted projection is a subduction. ☐

We need to know how subductions interact with fibred products.

Lemma 2.125. *Let $f : X \to Y$ and $g : V \to W$ be two subductions, and consider two commutative diagrams of smooth maps between diffeological spaces:*

Then the map

$$(f \times g)|_{X \times_A V} : X \times^{r,l}_A V \longrightarrow Y \times^{R,L}_A W; \qquad (x, v) \longmapsto (f(x), g(v))$$

is also a subduction.

Proof. Clearly $f \times g$ is smooth, so we are left to show that it is surjective and that the second condition in Proposition **2.120** is fulfilled. For surjectivity, take $(y, w) \in Y \times^{R,L}_A W$. Since f and g themselves are surjective, we can find $x \in X$ and $v \in V$ such that $f(x) = y$ and $g(v) = w$. But then

$$r(x) = R \circ f(x) = R(y) = L(w) = l(v),$$

so we get an element $(x, v) \in X \times^{r,l}_A V$ such that $(f \times g)(x, v) = (y, w)$.

Now take a plot $(\alpha, \beta) : U \to Y \times^{R,L}_A W$, i.e., we have two plots $\alpha \in \mathfrak{D}_Y$ and $\beta \in \mathfrak{D}_W$ such that $R \circ \alpha = L \circ \beta$. Fix a point $t \in U$ in the domain. Then since f and g are subductive we can find two plots

$\overline{\alpha} : U_\alpha \to X$ and $\overline{\beta} : U_\beta \to V$, defined on open neighbourhoods of $t \in U$, such that $\alpha|_{U_\alpha} = f \circ \overline{\alpha}$ and $\beta|_{U_\beta} = g \circ \overline{\beta}$. Now $(\overline{\alpha}|_{U_\alpha \cap U_\beta}, \overline{\beta}|_{U_\alpha \cap U_\beta}) : U_\alpha \cap U_\beta \to X \times V$ takes values in the fibred product because

$$r \circ \overline{\alpha}|_{U_\beta} = R \circ f \circ \overline{\alpha}|_{U_\beta} = R \circ \alpha|_{U_\alpha \cap U_\beta} = L \circ \beta|_{U_\alpha \cap U_\beta} = l \circ \overline{\beta}|_{U_\alpha},$$

and moreover we have that $(f \times g) \circ (\overline{\alpha}|_{U_\alpha \cap U_\beta}, \overline{\beta}|_{U_\alpha \cap U_\beta}) = (\alpha, \beta)|_{U_\alpha \cap U_\beta}$, proving the claim. □

Corollary 2.126. *If $f : X \to Y$ and $g : V \to W$ are two subductions, then so is their Cartesian product $f \times g : X \times V \to Y \times W$.*

Proof. This follows immediately from Lemma **2.125** by setting A to be the singleton space $\{*\}$. □

2.6.1 Local subductions and submersions

We have seen in Lemma **2.121** that subductiveness is a relatively weak condition, since any smooth map that admits a global smooth section is a subduction. Here we introduce a refinement of this notion, which is called *local subductiveness*. The jump from subductions to local subductions is tantamount to the jump from global section-admitting maps to submersions. We will make this precise in Proposition **2.128**. The idea is as follows: if $\pi : X \to B$ is a global section-admitting smooth map (and hence a subduction), then it is possible to map any point $b \in B$ to a point $\sigma(b) \in X$, in such a way that $\pi \circ \sigma(b) = b$. In a sense, this means that it is possible to pick for each point b in the base B a corresponding point $\sigma(b)$ in the π-fibre of that point. But this does not imply that every point in the total space X can be reached in such a fashion. In other words, there may be points $x \in X$ through which no section runs. A *local subduction*, then, is a map that *does* allow (local) sections through each point of its domain. Let us make this precise now:

Definition 2.127. A smooth surjection $f : X \to Y$ is called a *local subduction* if for every pointed plot $\alpha : (U_\alpha, 0) \to (Y, f(x))$, there exists another pointed plot $\beta : (V, 0) \to (X, x)$, defined on an open neighbourhood $0 \in V \subseteq U_\alpha$, such that $\alpha|_V = f \circ \beta$.

Compare this to the characterisation of a subduction in Proposition **2.120**. Clearly every local subduction is also a subduction. But for subductions, the plot β does not have to hit the point x at all.

We expand on the remarks in [***Diffeology***, Article 2.16]:

Proposition 2.128. *The local subductions between manifolds are exactly the surjective submersions.*

Proof. Since smooth manifolds are locally diffeomorphic to Euclidean domains, it suffices to prove the claim for the latter. Suppose first that $\pi : U \to V$ is a local subduction at $x \in U$ between two Euclidean domains, as usual equipped with their Euclidean diffeologies. We need to show that the differential $\mathrm{d}_x\pi : T_xU \to T_{\pi(x)}V$ is surjective. For that, let $v \in T_{\pi(x)}V$ be a tangent vector that is the velocity $v = \mathrm{d}_0\gamma(\partial_t)$ of some smooth curve $\gamma : (\mathbb{R}, 0) \to (V, \pi(x))$. Note that we can interpret γ as a pointed plot of V. Therefore, by local subductiveness, we can find a plot $\beta : (I, 0) \to (U, x)$ defined on an open interval $0 \in I \subseteq \mathbb{R}$ such that $\gamma|_I = \pi \circ \beta$. It then follows by the chain rule that

$$v = \mathrm{d}_0\gamma(\partial_t) = \mathrm{d}_0(\gamma|_I)(\partial_t) = \mathrm{d}_0(\pi \circ \beta)(\partial_t) = \mathrm{d}_x\pi \circ \mathrm{d}_0\beta(\partial_t),$$

which is clearly in the image of $\mathrm{d}_x\pi$. The argument does not depend on the point $x \in U$, so π must be a submersion.

The converse follows from the *rank theorem*, see e.g. [**Lee13**, Theorem 4.12]. This says that constant rank maps, in particular submersions, look locally like projections. For us, if $\pi : U \to V$ is a submersion and $x \in U$, there exist charts $(\overline{U}, \varphi)$ and $(\overline{V}, \psi)$ around x and $\pi(x)$ respectively, such that $\pi(\overline{U}) \subseteq \overline{V}$, and the coordinate representation of π in these charts is of the form

$$\psi \circ \pi \circ \varphi^{-1}(x_1, \ldots, x_n, x_{n+1}, \ldots, x_m) = (x_1, \ldots, x_n),$$

where $m = \dim(U)$ and $n = \dim(V)$. Now take a pointed plot $\alpha : (U_\alpha, 0) \to (V, \pi(x))$. If we project the plot along ψ, it is clear from the previous equation how α lifts along π, at least locally. Namely, we can define $\beta : (W, 0) \to (U, x)$ in such a way that $\varphi \circ \beta = (\psi \circ \alpha, \mathrm{const})$, where $\mathrm{const} : W \to \mathbb{R}^{m-n}$ is some constant function. Locally, π then just projects β onto the $\psi \circ \alpha$ component, and lifting back to V along ψ^{-1} then shows that $\pi \circ \beta = \alpha|_W$. □

Since the concept of a submersion is so central to the definition of a Lie groupoid, this proposition provides a key part in building the bridge to the diffeological theory of bibundles we describe in Chapter **IV**. (Although we show that the theory generalises further to where we can even work with plain subductions.)

Example 2.129. Proposition **2.128** proves that local subductions between Euclidean domains are exactly the submersions. From Lemma **2.121** it follows that if a smooth map has a global section, then it must be a subduction. This clearly is not sufficient for local subductiveness, and it is therefore easy to find examples of subductions that are not local subductions, even on manifolds. Consider the map $\pi : \mathbb{R}^2 \to \mathbb{R}$, defined by the product $(x, y) \mapsto xy$. This is clearly smooth, when $\mathbb{R}$ is endowed with the Euclidean diffeology. The map $x \mapsto (1, x)$ gives a global section of π, so it must be a subduction. However, by [**Lee13**, Problem 4-8], π is not a submersion at the origin (its derivatives are zero there), and hence not a local subduction.

Note that this distinguishes submersions from subductions in the sense that having a global smooth section implies that a map is a subduction, whereas it does *not* imply it is a submersion.

Proposition 2.130. *The composition of local subductions is a local subduction.*

Proof. Clearly if $f : Y \to Z$ and $g : X \to Y$ are local subductions, then $f \circ g$ is a smooth surjection. Consider then a pointed plot $\alpha : (U_\alpha, 0) \to (Z, f \circ g(x))$. Since f is a local subduction, there exists a pointed plot $\beta : (V, 0) \to (Y, g(x))$, defined on an open subset $0 \in V \subseteq U_\alpha$, such that $\alpha|_V = f \circ \beta$. In turn, since g is a local subduction, there exists a pointed plot $\gamma : (W, 0) \to (X, x)$, again defined on an open neighbourhood $0 \in W \subseteq V$, such that $\beta|_W = g \circ \gamma$. It is then clear that γ defines the desired lift of $f \circ g$, since $\alpha|_W = f \circ \beta|_W = f \circ g \circ \gamma$. □

Lemma 2.131. *Let $f : Y \to Z$ and $g : X \to Y$ be two smooth functions. If $f \circ g$ is a local subduction and g is a surjection, then f is a local subduction.*

Proof. If $f \circ g$ is a local subduction, it follows that f has to be a smooth surjection. Then consider a pointed plot $\alpha : (U_\alpha, 0) \to (Z, f(y))$. Since g is a surjection, we can choose $y = g(x)$, for some $x \in X$. In that case, since $f \circ g$ is a local subduction we can find a pointed plot $\beta : (V, 0) \to (X, x)$, defined on an open neighbourhood $0 \in V \subseteq U_\alpha$, such that $\alpha|_V = f \circ g \circ \beta$. But then $g \circ \beta : (V, 0) \to (Y, g(x))$ is the desired lift of f. □

The result of Lemma **2.122** extends to local subductions as well:

Lemma 2.132. *Let $f : B \to Y$ be a smooth map, and $\pi : X \to B$ a local subduction. Then f is a local subduction if and only if $f \circ \pi$ is a local subduction.*

Proof. This follows directly from Proposition **2.130** and Lemma **2.131**. □

2.7 Diffeological spaces as concrete sheaves

The many wonderful properties of the category of diffeological spaces (by which we mean Theorems **2.72** and **2.118**) can be explained by their realisation as the *concrete sheaves on* **Eucl**. One of the first places where this idea is made explicit is the paper [**BH11**]. But, for another great and detailed discussion of this perspective, we refer to [**Gł19**, Section I.1]. This section is fully based on these two references, and we leave the details to them.

This viewpoint of diffeological spaces is already motivated by some remarks we made in Definition **2.2**: that the underlying set of a diffeological space is captured by the diffeology itself. If we denote the category of sheaves on the category of Euclidean domains (with its natural Grothendieck topology) by Sh(**Eucl**), then we can naturally associate the following sheaf to any diffeological space:

Definition 2.133. For $X \in \mathbf{Diffeol}$ any diffeological space, we define $\overline{X} \in \mathrm{Sh}(\mathbf{Eucl})$ as the presheaf

$$\overline{X} := C^\infty(-, X) : \mathbf{Eucl}^{\mathrm{op}} \longrightarrow \mathbf{Set}; \qquad \overline{X}(U) := C^\infty(U, X).$$

The underlying set X is encoded in the set $\overline{X}(1) = C^\infty(1, X)$, where $1 = \{0\} = \mathbb{R}^0$ is the zero-dimensional Euclidean domain. Each point $x \in X$ corresponds to the unique smooth function $1 \to X$ given by $0 \mapsto x$. Generalising this observation gives rise to the idea of concrete sheaves. A sheaf $F : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ on a *concrete site* $\mathbf{C}$ is called *concrete* if for every object $D \in \mathrm{ob}(\mathbf{C})$ there is a bijection

$$F(D) \cong \mathrm{Hom}_{\mathbf{Set}}\left(\mathrm{Hom}_{\mathbf{C}}(1, D), F(1)\right),$$

where $1 \in \mathrm{ob}(\mathbf{C})$ is a terminal object. Here $\mathrm{Hom}_{\mathbf{C}}(1, D)$ stands in analogy to $C^\infty(1, X)$, and therefore represents something like the underlying set of points of the object $D \in \mathrm{ob}(\mathbf{C})$. The same holds for $X_F := F(1)$, which represents the underlying space of the sheaf F. This bijection, and hence the concreteness of F, therefore means that $F(D)$, which is supposed to represent the allowed morphisms $D \to X_F$, is in bijective correspondence with the functions $\mathrm{Hom}_{\mathbf{C}}(1, D) \to X_F$. Concreteness can therefore broadly be interpreted as "admitting an underlying set." See [**BH11**, Definitions 18-19] for details. We denote the category of concrete sheaves on a concrete site $\mathbf{C}$ and natural transformations between them by $\mathrm{cSh}(\mathbf{C})$.

Given this observation, we see that the sheaf $\overline{X} \in \mathrm{Sh}(\mathbf{Eucl})$ defined by a diffeological space $X \in \mathbf{Diffeol}$ actually satisfies this concreteness condition. We therefore get a functor

$$\mathbf{Diffeol} \longrightarrow \mathrm{cSh}(\mathbf{Eucl}); \qquad X \longmapsto \overline{X}.$$

This functor induces an equivalence of categories (cf. [**GI19**, Lemma I.1.16]):

Theorem 2.134 ([**BH11**, Proposition 24])**.** *The category* **Diffeol** *of diffeological spaces is equivalent to the category of concrete sheaves on the category* **Eucl** *of Euclidean domains:*

$$\mathbf{Diffeol} \simeq \mathrm{cSh}(\mathbf{Eucl}).$$

We further have a characterisation of the concrete sheaves on **Eucl** as the concrete sheaves on **Mnfd**:

Proposition 2.135 ([**WW19**, Lemma 2.9])**.** *The categories of concrete sheaves on* **Eucl** *and on* **Mnfd** *are equivalent:* $\mathrm{cSh}(\mathbf{Mnfd}) \simeq \mathrm{cSh}(\mathbf{Eucl})$.

Given this proposition, it follows that the three Axioms of Diffeology in Definition **2.2** could also have been given using smooth manifolds, instead of Euclidean domains:

Corollary 2.136 ([**WW19**, Corollary 2.10])**.** *The category* **Diffeol** *of diffeological spaces is equivalent to the category of concrete sheaves on the category* **Mnfd** *of smooth manifolds:*

$$\mathbf{Diffeol} \simeq \mathrm{cSh}(\mathbf{Mnfd}).$$

[**BH11**, Theorem 52] proves that any category of the form $\mathrm{cSh}(\mathbf{C})$ is a quasitopos, and given the equivalence $\mathbf{Diffeol} \simeq \mathrm{cSh}(\mathbf{Eucl})$ our Theorem **2.118** follows as a special case.

Chapter III

Diffeological groupoids

3.1 Diffeological groups

Historically, diffeology started as a theory of groups. Souriau first introduced “differentiable groups” [**Sou80**] to study the smooth structure of infinite-dimensional groups of symplectomorphisms. As [**IZ17**] recalls, it took some years to extract from these “differentiable groups” the concept of a “differentiable space,” and the first formal definition of *diffeology* was given in [**Sou84**]. Diffeological groups are discussed in the textbook [***Diffeology***, Chapter 7], but it also returns in other literature on diffeology, such as [**Hec95**; **HMV02**; **Les03**; **CW17b**]. Before we move to groupoids, let us therefore first briefly discuss that what gave birth to the theory in the first place:

Definition 3.1. A *diffeological group* is a group G with a diffeological structure such that the multiplication map $\mathrm{m}_G : G \times G \to G$ and inversion $\mathrm{inv}_G : G \to G$ are smooth. These are exactly the groups internal to the category **Diffeol**. A diffeology $\mathcal{D}_G$ on a group G that makes its multiplication and inversion smooth is called a *group diffeology*.

The morphisms between diffeological groups are what one would expect: a *smooth (group) homomorphism* $f : G \to H$ between two diffeological groups is a group homomorphism that is smooth with respect to the underlying diffeologies. The category of diffeological groups and smooth group homomorphisms is denoted **DiffeolGrp**. The isomorphisms in this category are just the arrows that are also diffeomorphisms.

Any given group G may have multiple diffeologies that turn it into a diffeological group, and generally there are at least two of them:

Example 3.2 (Coarse- and discrete groups)**.** Both the coarse- and discrete diffeologies on a group G define diffeological groups. For the coarse diffeology this is easy, since both multiplication and inversion are then functions into a coarse space, and hence are smooth. If G has the discrete diffeology, it follows immediately that the inversion is smooth. For the smoothness of multiplication we need to make the small mental jump that the product of two discrete spaces is again discrete (use e.g. Lemma **2.61**).

A group structure alone is therefore not enough to define a canonical diffeology (putting them in contrast to vector spaces, which *do* have the canonical *fine* diffeology). Despite this non-uniqueness, there is often a natural diffeology compatible with the group structure. For instance, it may occur (see Section **3.1.1**) that G already gets a diffeology from someplace else, in which case there may be a natural refinement or coarsening to turn it into a diffeological group. Of course, another class of examples where the diffeology is canonical is the following one:

Example 3.3. Every Lie group with its manifold diffeology is a diffeological group.

Many constructions of Lie groups extend and generalise in the diffeological setting.

Example 3.4 (Subgroups)**.** Let $H \subseteq G$ be a subgroup, and suppose that G has a group diffeology $\mathcal{D}_G$. Then H, endowed with its subset diffeology, is also a diffeological group. This follows in essence from Proposition **2.55**; since the inversion and multiplication of H are just that of G, restricted appropriately, they remain smooth with respect to the subset diffeology. To be precise, the inclusion function $i : H \hookrightarrow G$ is an induction (hence smooth), and if we denote the multiplication and inversion maps of G and H by m_G, inv_G, m_H and inv_H, respectively, then we get $\mathrm{m}_H = \mathrm{m}_G \circ (i \times i)$ and $\mathrm{inv}_H = \mathrm{inv}_G \circ i$, which shows that they must be smooth.

Contrast this to a theorem of É. Cartan (first proved by von Neumann for matrix groups), which says that any *closed* subgroup of a Lie group is naturally a Lie group itself. The induction $i : H \hookrightarrow G$ here plays the rôle of the embedding.

Example 3.5 (Products)**.** If G and H are diffeological groups, then the product diffeology on the direct product $G \times H$ is a group diffeology.

Example 3.6 (Normal subgroups)**.** Suppose that $N \lhd G$ is a normal subgroup, meaning that for every group element $g \in G$ and $n \in N$ we have $gng^{-1} \in N$. The *quotient group* is the group G/N, consisting of the *cosets* $gN := \{gn : n \in N\}$, with multiplication $(gN)(hN) := (gh)N$ and inversion $(gN)^{-1} := g^{-1}N$. We denote the canonical projection homomorphism by $\pi : G \to G/N$.

Suppose now that G is furnished with a group diffeology $\mathcal{D}_G$. The projection map π induces the *quotient group diffeology* on G/N, which is just $\mathcal{D}_{G/N} = \pi_*(\mathcal{D}_G)$. We claim that, with this diffeology, G/N is also diffeological group.

Proof. Let us show that the multiplication $\mathrm{m}_{G/N} : G/N \times G/N \to G/N$ is smooth. Keeping Lemmas **2.33** and **2.43** in mind, take two plots of the form $\pi \circ \alpha, \pi \circ \beta \in \mathcal{D}_{G/N}$, where $\alpha, \beta \in \mathcal{D}_G$. It is then easy to calculate

$$\mathrm{m}_{G/N} \circ [(\pi \circ \alpha) \times (\pi \circ \beta)]\,(t,s) = (\alpha(t)N)(\beta(s)N) = (\alpha(t)\beta(s)N) = \pi \circ \mathrm{m}_G \circ (\alpha \times \beta)(t,s),$$

which is indeed smooth. The smoothness of the inversion map is just as simple, since we have a similar expression: $\mathrm{inv}_{G/N} \circ \pi \circ \alpha = \pi \circ \mathrm{inv}_G \circ \alpha$. $\square$

3.1.1 Diffeomorphism groups

The space $\mathrm{Diff}(X)$ of diffeomorphisms on X gets the subset diffeology from the standard functional diffeology on $C^\infty(X,X)$. By the above Proposition **2.88**, the composition of diffeomorphisms therefore also becomes smooth. Since $\mathrm{Diff}(X)$ naturally has the structure of a group, we may ask if the inversion map $\mathrm{inv} : \mathrm{Diff}(X) \to \mathrm{Diff}(X)$ is also smooth with respect to this subset diffeology, which would make it into a diffeological group. For smooth manifolds, this seems to be ensured by the *inverse function theorem* ([**Lee13**, Theorem 4.5], cf. [***Diffeology***, Article 1.61]). It is not known if this is the case for arbitrary diffeological spaces, so we need to refine the diffeology on $\mathrm{Diff}(X)$ to make the inversion map smooth.

Definition 3.7. Let X be a diffeological space, and let $\mathcal{D}$ be the subset diffeology on $\mathrm{Diff}(X)$ inherited from the standard functional diffeology $\mathcal{D}$ on $C^\infty(X,X)$. The *standard diffeomorphism diffeology* on $\mathrm{Diff}(X)$ is the coarsest diffeology such that the evaluation and inversion maps are smooth. Concretely, it is the intersection $\mathrm{inv}^*(\mathcal{D}) \cap \mathcal{D}$. This means that a parametrisation $\Omega : U_\Omega \to \mathrm{Diff}(X)$ is a plot in the standard diffeomorphism diffeology if and only if Ω and $\Omega^{-1} := \mathrm{inv} \circ \Omega$ are both plots in $\mathcal{D}$ (i.e., in the standard functional diffeology). As remarked above, the inverse function theorem ensures that the standard diffeomorphism diffeology agrees with the standard functional diffeology whenever X is a manifold.

Example 3.8. For any diffeological vector space E we get the *general linear group* $\mathrm{GL}(E)$, containing all linear diffeomorphisms on E. As such, $\mathrm{GL}(E)$ is a subgroup of $\mathrm{Diff}(E)$, and so inherits the subset diffeology from the standard diffeomorphism diffeology, making it into a diffeological group.

If E is a fine finite-dimensional space, then $\mathrm{GL}(E)$ is isomorphic (as a diffeological group) to the Lie group $\mathrm{GL}(n;\mathbb{R})$, where $n = \dim(E)$. When E is infinite-dimensional, its general linear group is no longer a Lie group. In particular we can consider a Hilbert space $\mathscr{H}$, with its fine vector space diffeology (where now $\mathbb{C}$ is equipped with its standard diffeology). The group $\mathrm{U}(\mathscr{H})$ of unitary operators on $\mathscr{H}$ also becomes a diffeological group with respect to the subset diffeology it gets from $\mathrm{GL}(\mathscr{H})$. When $\mathscr{H}$ is finite-dimensional we reobtain the classical Lie groups $\mathrm{U}(n)$. We are not aware of any research on (unitary or infinite-dimensional) representation theory of diffeological groups. In particular, in functional analysis the unitary group is often treated from a topological viewpoint, in which it is important to distinguish between several non-equivalent topologies on $\mathrm{U}(\mathscr{H})$ (such as the norm-, strong-, or weak operator topologies). We do not know about the D-topological properties of the diffeology on $\mathrm{U}(\mathscr{H})$, if it can be refined in useful ways, or what its relation is to the functional analytic approaches.

Example 3.9. The following is an example suggested in [**Hec95**, Example 4.2]. We recall the definition of an *infinite general linear group* (also known as the *stable general linear group*, and which should not be confused with a general linear group of an infinite-dimensional vector space). For simplicity, we use the base field $\mathbb{R}$, with its standard Euclidean diffeology, making it into a *diffeological field.* We denote

$$\mathrm{GL}(n;\mathbb{R}) := \mathrm{GL}(\mathbb{R}^n),$$

where on the right hand side we have the diffeological general linear group as in in the previous example. The idea is that, for each $n \in \mathbb{N}$, we have an inclusion homomorphism, represented by the following function on matrices:

$$I_n : \mathrm{GL}(n;\mathbb{R}) \longrightarrow \mathrm{GL}(n+1;\mathbb{R}); \qquad M \longmapsto \mathrm{diag}(M,1) := \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix},$$

mapping each invertible $n \times n$ matrix M to the $(n+1) \times (n+1)$ matrix containing M in the upper left corner, filled out with zeroes on the bottom row and rightmost column, and containing 1 on the final entry. It can be shown that each I_n is smooth with respect to the standard diffeomorphism diffeologies on $\mathrm{Diff}(\mathbb{R}^n)$ and $\mathrm{Diff}(\mathbb{R}^{n+1})$, and hence we get an infinite tower of inclusions in **DiffeolGrp**:

$$\mathrm{GL}(1;\mathbb{R}) \hookrightarrow \mathrm{GL}(2;\mathbb{R}) \hookrightarrow \mathrm{GL}(3;\mathbb{R}) \hookrightarrow \mathrm{GL}(4;\mathbb{R}) \hookrightarrow \mathrm{GL}(5;\mathbb{R}) \hookrightarrow \cdots.$$

The *infinite general linear group* is defined, as a group, as the colimit of this diagram:

$$\mathrm{GL}(\infty;\mathbb{R}) := \operatorname*{colim}_{n\in\mathbb{N}} \mathrm{GL}(n;\mathbb{R}).$$

This colimit is the union over each $\mathrm{GL}(n;\mathbb{R})$ (which is the colimit of the underlying sets), where we identify each group with its image under the inclusion I_n. One may think of this as the space of infinite square invertible matrices with only finitely many non-zero entries, and only finitely many entries on the diagonal not equal to 1. The plots in the colimit diffeology on $\mathrm{GL}(\infty;\mathbb{R})$ are all locally of the form $I_n^\infty \circ \beta$, where β is a plot of some $\mathrm{GL}(n;\mathbb{R})$, and $I_n^\infty : \mathrm{GL}(n;\mathbb{R}) \hookrightarrow \mathrm{GL}(\infty;\mathbb{R})$ is the smooth inclusion. From this, and since each $\mathrm{GL}(n;\mathbb{R})$ is a diffeological group, it follows that $\mathrm{GL}(\infty;\mathbb{R})$ with the colimit diffeology is also a diffeological group.

The above example shows that a specific colimit of groups, when endowed with its corresponding colimit diffeology, again forms a diffeological group. In general, we conjecture that all (co)limits of groups, when endowed with the (co)limit diffeologies constructed in Section **2.2.6**, form the (co)limits in **DiffeolGrp**. We suspect that the argument could be made analogously to the proof that the category of topological groups is (co)complete. We will not attempt a proof here, so we leave it as an exercise for the reader.

3.1.2 Smooth group actions

The notion of a *smooth action* of a Lie groupoid G on a smooth manifold M is an important notion throughout Lie theory, or indeed differential geometry as a whole. It extends to diffeology in a straightforward fashion:

Definition 3.10. Given a set X, recall that an *action* $G \curvearrowright X$ of G on X is a function $G \times X \to X$, usually written as $(g,x) \mapsto g \cdot x = gx$, such that the identity element of G acts trivially: $1_G \cdot x = x$, and the action is associative in the following sense: $g \cdot (h \cdot x) = (gh) \cdot x$. If G is a diffeological group, and X itself has a diffeological structure, we say that the action $G \curvearrowright X$ is *smooth* if the action map $\mu : G \times X \to X$ is smooth.

Example 3.11. Let X be a diffeological space, and let $\mathrm{Diff}(X)$ be its diffeomorphism group, equipped with the standard diffeomorphism diffeology as in Definition **3.7**. The evaluation map

$$\mathrm{ev} : \mathrm{Diff}(X) \times X \longrightarrow X; \qquad (\varphi, x) \longmapsto \varphi(x)$$

then defines a smooth action on X, since it is smooth with respect to the standard functional diffeology.

Example 3.12. We have already seen in Example **2.67** that the quotient space X/G of *any* action $G \curvearrowright X$ (regardless of smoothness) inherits a natural diffeology. Of course, this still works when the action is smooth, and this is in fact necessary to define *action groupoids*, see Example **3.28**.

3.2 Diffeological groupoids

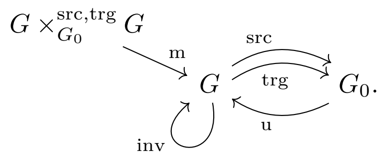
Groupoids, as their name would suggest, generalise the notion of a group. Simultaneously, they generalise the notions of a space, actions, and equivalence relations. If the theory of groups is the theory of symmetry, groupoids provide a framework for *Unifying Internal and External Symmetry* [**Wei96**]. One way to think about groupoids is as a collection of objects, each with its own group of symmetries, and a way to determine how these symmetries vary along the objects. We recall Appendix **A** for our basics on category theory. In the most formal sense:

Definition 3.13. A *groupoid* is a small category[33] in which every arrow has an inverse.

If we unpack this definition, we will find a more convenient point of view. Concretely, a *groupoid* consists of two sets: G_0 and G, together with five *structure maps*. A groupoid will be denoted $G \rightrightarrows G_0$, or just G, which is meant to stand for '*groupoid*'. Here G_0 is the set of objects of the category, and G is the set of arrows. The two arrows in the notation $G \rightrightarrows G_0$ represent the source and target maps $\mathrm{src}, \mathrm{trg} : G \to G_0$, sending each arrow $g : x \to y$ in G to its domain $\mathrm{src}(g) = x$ and codomain $\mathrm{trg}(g) = y$, respectively. Besides these two, the three other structure maps are as follows: first, since every arrow in G has an inverse, we have the *inversion map* $\mathrm{inv} : G \to G$, mapping $g \mapsto g^{-1}$. We also have a map $\mathrm{u} : G_0 \to G$, sending each object in the groupoid to its identity arrow: $x \mapsto \mathrm{id}_x$. The fifth and final structure map is the *composition*. This function is defined on the pairs (g, h) of composable arrows in G. Specifically, this set is the fibred product

$$G \times_{G_0}^{\mathrm{src},\mathrm{trg}} G = \{(g, h) \in G \times G : \mathrm{src}(g) = \mathrm{trg}(h)\}.$$

The composition map is then denoted $\mathrm{m} : G \times_{G_0}^{\mathrm{src},\mathrm{trg}} G \to G$, sending each pair $(g, h) \mapsto g \circ h$. The structure maps of a groupoid fully capture its properties as a category, and the entire situation can be depicted as:



In particular, besides the defining properties of a category, these structure maps satisfy:

1. For any pair of composable arrows $g, h \in G$, we have $\mathrm{src}(g \circ h) = \mathrm{src}(h)$, and $\mathrm{trg}(g \circ h) = \mathrm{trg}(g)$.

2. We have $\mathrm{src} \circ \mathrm{u} = \mathrm{trg} \circ \mathrm{u} = \mathrm{id}_{G_0}$.

3. The inversion map is involutive: $\mathrm{inv} \circ \mathrm{inv} = \mathrm{id}_G$.

4. And we further have $\mathrm{src} \circ \mathrm{inv} = \mathrm{trg}$ and $\mathrm{trg} \circ \mathrm{inv} = \mathrm{src}$.

Example 3.14. Something special happens when we look at groupoids $G \rightrightarrows G_0$ whose space of objects is just the singleton set $G_0 = \{*\}$. First of all, every pair of arrows in G becomes composable, since there is no room for differing sources and targets. In fact the base space G_0 completely disappears from the structure maps. What we are left with is simply a *group*. The arrows of the groupoid are the group elements, composition of arrows is group multiplication, and inversion of arrows is inversion of group elements. In this sense, a groupoid is a *"many-object group."* From the symmetry point of view, a group captures the automorphisms of a single object, and a groupoid captures the (possibly varying) automorphisms of a family of objects.

As already mentioned in Chapter **I**, it was one of the insights of Charles Ehresmann that traditional mathematical structures can be *internalised* into other categories [**Ehr59**] (see also [**Pra07**]). Examples of this are natural numbers objects, monoids, groups, and even categories themselves. This is because the structure of a category, for example, can be captured entirely by the commutativity of diagrams

[33] Recall that a small category is one where both the collection of objects and the collection of arrows are sets.

of its structure maps. And so it is for groupoids. The *internalisation* of a groupoid into a category $\mathbf{C}$ means that G_0 and G are objects in $\mathbf{C}$, and each of the structure maps are arrows in $\mathbf{C}$ whose relations are captured by suitable commutative diagrams. Essentially, it means that the above diagram depicting each of the structure maps takes place entirely in the category $\mathbf{C}$. If we pick $\mathbf{C} = \mathbf{Mnfd}$, this leads us to the definition of a Lie groupoid:

Definition 3.15. A *Lie groupoid* is a groupoid $G \rightrightarrows G_0$, such that G_0 and G have the structure of a smooth manifold, making each of the five structure maps smooth, and making the source and target maps into submersions.

For extensive developments on the theory of Lie groupoids we refer to [**Lan98**; **MM03**; **Mac05**], and the lecture notes [**Mei17**]. See also [**Lan06**] for a survey of how Lie groupoids are used in noncommutative geometry. The requirements that the source and target maps are submersions merely ensures that the fibred product $G \times_{G_0}^{\mathrm{src},\mathrm{trg}} G$, representing the space of composable arrows, is itself a manifold. If this were not the case, it would not make much sense to speak about the smoothness of composition. This means that Lie groupoids are not in the most precise sense *internal* to $\mathbf{Mnfd}$. Diffeological groupoids, on the other hand, are immune to this problem, since the space of composable arrows can always be furnished with the pullback diffeology. Diffeological groupoids are therefore truly the internal groupoids in $\mathbf{Diffeol}$:

Definition 3.16. A *diffeological groupoid* is a groupoid internal to the category of diffeological spaces.

Concretely, this means that it is a groupoid $G \rightrightarrows G_0$ such that the object space G_0 and arrow space G are endowed with diffeologies that make all of the structure maps smooth. The theory of diffeological groupoids started nearly simultaneously with diffeology itself [**Igl85**]. This was not only because diffeology started as a theory of groups, but also because diffeological groupoids capture the theory of diffeological fibre bundles. The textbook account of diffeological groupoids is in [***Diffeology***, Chapter 8].

Despite not being submersions, as we had to assume in the case of Lie groupoids, the algebraic relations between the structure morphisms of a groupoid will ensure that the source and target maps are not just ordinary smooth maps:

Proposition 3.17. *The source and target maps of a diffeological groupoid are subductions.*

Proof. The smooth map $G_0 \to G$ sending each object to its identity arrow is a global section of the source map, and hence by Lemma **2.121** the source map must be a subduction. Since the inversion map is a diffeomorphism, it follows that the target map is a subduction as well. □

The same argument (but with Lemma **2.39**) also shows that the identity section $\mathrm{u} : G_0 \to G$ has to be an induction. In this way, the diffeologies of the object- and arrow spaces of a diffeological groupoid $G \rightrightarrows G_0$ are tightly linked: we have $\mathcal{D}_{G_0} = \mathrm{src}_*(\mathcal{D}_G)$ and $\mathcal{D}_{G_0} = \mathrm{u}^*(\mathcal{D}_G)$. We will use this in Example **3.25**.

Definition 3.18. The morphisms of diffeological groupoids are exactly the *smooth functors* $\phi : G \to H$. Here, a functor ϕ is smooth if its two underlying maps $\phi_0 : G_0 \to H_0$ and $\phi_1 : G \to H$ are smooth. To ease notation, we will denote the function between the arrow spaces just by $\phi_1 = \phi$. To keep our notation light on parentheses, we might also write $\phi_0 x$ and ϕg to mean $\phi_0(x)$ and $\phi(g)$, for $x \in G_0$ and $g \in G$, respectively.

A natural transformation $T : \phi \to \psi$ between two smooth functors $\phi, \psi : G \to H$ is called *smooth* if the underlying map $G_0 \to H : x \mapsto T_x$ is smooth.

Together, we get a strict *2-category* **DiffeolGrpd** consisting of diffeological groupoids, smooth functors, and smooth natural transformations. (Cf. Appendix **A.2** for the definition of a 2-category.)

As diffeology fully subsumes smooth manifolds, so do diffeological groupoids capture Lie groupoids. We therefore get a breadth of examples that we can generalise to our setting. To get a grip on Definition **3.16**, let us discuss some of them here. The simplest possible groupoid is the following:

Example 3.19 (Unit groupoids)**.** Any diffeological space X can be seen as a diffeological groupoid $X \rightrightarrows X$, by setting all of the structure maps to be the identity function id_X on X. The composition in $X \rightrightarrows X$ is then defined on the *diagonal* $\Delta_X = \{(x,x) : x \in X\}$, and maps $\mathrm{m} : (x,x) \mapsto x$. The interpretation of this is that $X \rightrightarrows X$ represents the category whose space of objects is X, and whose only arrows are the identity arrows id_x at each point $x \in X$. This groupoid is called the *unit groupoid*.

Example 3.20 (Pair groupoids)**.** Again start with a diffeological space X. We define the *pair groupoid* $X \times X \rightrightarrows X$ as follows. The idea is in a sense opposite to that of the unit groupoid. There, we have allowed the bare minimum of categorical structure on a base space X. Here, we choose to have every two points X connected by a unique arrow. Between every two points $x, y \in X$ there exists exactly one arrow, which we denote by the pair $(y,x) \in X \times X$. In the tradition of the usual categorical notation, we define $\mathrm{src}(y,x) = x$ and $\mathrm{trg}(y,x) = y$, so that we can write the composition as $(z,y) \circ (y,x) := (z,x)$. This clearly defines a diffeological groupoid $X \times X \rightrightarrows X$.

Example 3.21 (Diffeological groups)**.** Following the idea of Example **3.14**, we see that any diffeological group G can be seen as a diffeological groupoid $G \rightrightarrows \{*\}$, also denoted $G \rightrightarrows 1$.

Example 3.22 (Subgroupoids)**.** Consider a set-theoretic groupoid $G \rightrightarrows G_0$. A *subgroupoid* is a subset $H \subseteq G$ of arrows in G that is closed under the composition and inversion of G. If H_0 is the set of sources of arrows in H, this gives a groupoid $H \rightrightarrows H_0$. If $G \rightrightarrows G_0$ is a diffeological groupoid, then a subgroupoid $H \rightrightarrows H_0$ gets a diffeological structure by putting the subset diffeologies on $H_0 \subseteq G_0$ and $H \subseteq G$. With these diffeologies, it is an easy exercise to see that $H \rightrightarrows H_0$ becomes a diffeological groupoid. We may refer to this diffeology on $H \rightrightarrows H_0$ as the *subgroupoid diffeology* induced by $G \rightrightarrows G_0$.

Example 3.23. A natural way to obtain subgroupoids of $G \rightrightarrows G_0$ is to take a subset $A \subseteq G_0$ of the base space. We then get a groupoid $G|_A \rightrightarrows A$ consisting exactly of the arrows in G whose sources and targets are elements in A:

$$G|_A := \mathrm{src}^{-1}(A) \cap \mathrm{trg}^{-1}(A).$$

This type of subgroupoid is special, because the inclusion functor is fully faithful. With the subgroupoid diffeology, we might call $G|_A \rightrightarrows A$ the *restriction* of G to A, or the *full subgroupoid* generated by A.

Every diffeological groupoid carries a natural family of diffeological groups within it:

Definition 3.24. Let $G \rightrightarrows G_0$ be a diffeological groupoid. The *isotropy group* (or *automorphism group*) at $x \in G_0$ is the collection G_x consisting of all arrows in G from and to x:

$$G_x := \mathrm{Hom}_G(x,x) = \mathrm{src}^{-1}(\{x\}) \cap \mathrm{trg}^{-1}(\{x\}).$$

Clearly the isotropy groups define subgroupoids of $G \rightrightarrows G_0$, and with the subgroupoid diffeology from Example **3.22** they become diffeological groups. An isotropy group G_x can be thought of as the smooth internal symmetry group of the object $x \in G_0$.

Example 3.25 (Discrete- and course groupoids)**.** Consider a set-theoretic groupoid $G \rightrightarrows G_0$, without any diffeological (or other) structure. What happens if we apply the distinguished coarse- and discrete diffeologies (Definition **2.23**) to this groupoid? There are several possibilities to explore.

A natural one is to put the discrete diffeology on the space of arrows G, since then the source, target, inversion, and composition maps are already smooth. What happens to the diffeology on G_0? We have seen that this diffeology is completely determined as the pullback $\mathrm{u}^*(\mathcal{D}^\circ_G)$. Therefore, a parametrisation $\alpha : U_\alpha \to G_0$ is a plot if and only if $\mathrm{u} \circ \alpha$ is locally constant, meaning that we can find an open neighbourhood $V \subseteq U_\alpha$ around each point such that $\mathrm{u} \circ \alpha|_V(t) = \mathrm{id}_x$, for some $x \in G_0$. But the identity section is injective, as the source map provides a left inverse, so we obtain $\alpha|_V(t) = \mathrm{src}(\mathrm{id}_x) = x$, for all $t \in V$. This proves that the diffeology on the object space has to be discrete, as well. More generally, it can be shown that the pullback of the discrete diffeology along an injective function is also discrete. The entire groupoid therefore carries the discrete diffeology, from which it follows that also the unit section is smooth, so that we get the *discrete diffeological groupoid*[34] $G^\circ \rightrightarrows G^\circ_0$.

[34] In the literature the term 'discrete groupoid' is also sometimes used to refer to the unit groupoids from Example **3.19**.

Another possibility is to put not the discrete, but the coarse diffeology on G. in that case, it is very easy to see that the diffeology on the object space G_0 also has to be coarse. Namely, $\mathrm{u}^*(\mathcal{D}^\bullet_G)$ is the coarsest diffeology on G_0 such that u is smooth. But, with respect to the coarse diffeology on G, any diffeology on G_0 would make u smooth, and so we get $\mathcal{D}_{G_0} = \mathrm{u}^*(\mathcal{D}^\bullet_G) = \mathcal{D}^\bullet_{G_0}$. It is clear that, also with these diffeologies, each of the structure maps is smooth, and we get the *coarse diffeological groupoid* $G^\bullet \rightrightarrows G_0^\bullet$.

Definition 3.26. Let $G \rightrightarrows G_0$ be a diffeological groupoid. We can define an equivalence relation on the object space by saying that two points $x, y \in G_0$ are equivalent if and only if they are connected by an arrow $g : x \to y$ in G. The equivalence classes are called *orbits*:

$$\mathrm{Orb}_G(x) := \{y \in G_0 : \exists g \in G : \mathrm{src}(g) = x, \mathrm{trg}(g) = y\} = \mathrm{trg}(\mathrm{src}^{-1}(\{x\})).$$

The *orbit space* (also known as the *coarse moduli space*) of the groupoid is the space G_0/G consisting of these orbits. We furnish the orbit space with the quotient diffeology, so that $\mathrm{Orb}_G : G_0 \to G_0/G$ is a subduction.

Example 3.27. Let X be a diffeological space, and let R be an equivalence relation on X. We define the *relation groupoid* $X \times_R X \rightrightarrows X$ as follows. The space of arrows consists of exactly those pairs $(x, y) \in X \times X$ such that xRy:

$$X \times_R X := \{(x, y) \in X \times X : xRy\}.$$

This gets a diffeological groupoid structure, being a subgroupoid of the pair groupoid $X \times X \rightrightarrows X$.

The orbit space $X/(X \times_R X)$ of $X \times_R X \rightrightarrows X$ is just the quotient X/R. We see that the unit groupoids and pair groupoids from Examples **3.19** and **3.20** are special cases of relation groupoids. In particular, the orbit space of the unit groupoid is $X/X = X$, and the orbit space of the pair groupoid is $X/(X \times X) = \{*\}$.

Another special case is the equivalence relation induced by a smooth surjection $\pi : X \to B$. Then the fibred product $X \times_B X$ gets the structure of a diffeological groupoid as a subgroupoid of the pair groupoid $X \times X \rightrightarrows X$. In this groupoid, every two points in the same π-fibre are connected by a unique arrow, but the points in different fibres are disjoint. Hence the orbit space is $X/(X \times_B X) = X/\pi$. Hence, if π is a subduction, the orbit space is diffeomorphic to the base B.

Example 3.28 (Action groupoids)**.** Any smooth group action (Section **3.1.2**) has a corresponding diffeological groupoid, denoted $G \ltimes X \rightrightarrows X$, that captures its structure. The idea is that the arrows in this groupoid represent the mappings $(g, x) \mapsto gx$. Specifically, the space of arrows in this groupoid is $G \times X$, and the source and target maps are determined by the action:

$$\mathrm{src}, \mathrm{trg} : G \ltimes X \longrightarrow X; \qquad \mathrm{src}(g, x) := x, \qquad \mathrm{trg}(g, x) := gx.$$

The set of arrows in $G \ltimes X$ between two points $x, y \in X$ therefore represents the group elements $g \in G$ such that $gx = y$. The composition in $G \ltimes X$ is just the multiplication of the group:

$$(h, gx) \circ (g, x) := (hg, x),$$

and the inversion can be obtained as $(g, x)^{-1} = (g^{-1}, gx)$. Each of the structure maps of $G \ltimes X$ is built up out of the smooth action map, the smooth multiplication map, and the smooth inversion map of the group (and some projections), so it follows that $G \ltimes X \rightrightarrows X$ defines a diffeological groupoid. It is called the *action groupoid*. In the case of a right action $X \curvearrowleft H$, we define its action groupoid analogously, and denote it by $X \rtimes H \rightrightarrows X$.

The isotropy group $(G \ltimes X)_x$ of a point $x \in X$ represents the group elements $g \in G$ that act trivially: $gx = x$. It is in fact the stabilizer group of the point x. This shows that the group action $G \curvearrowright X$ is *free* if and only if each isotropy group in $G \ltimes X$ is trivial.

The orbits of an action groupoid $G \ltimes X \rightrightarrows X$ are simply just the orbits of the group action in X. Hence, the orbit space $X/(G \ltimes X)$ is just the quotient X/G of the underlying group action.

Example 3.29. As a particular example of an action groupoid, consider a diffeological space X, and its space of diffeomorphisms $\mathrm{Diff}(X)$, equipped with the standard diffeomorphism diffeology (Definition 3.7). By definition, its evaluation map is smooth, and therefore defines a smooth action $\mathrm{Diff}(X) \curvearrowright X$ by

$$\mathrm{ev} : \mathrm{Diff}(X) \times X \longrightarrow X; \qquad (\varphi, x) \longmapsto \varphi(x).$$

Hence we get a diffeological groupoid $\mathrm{Diff}(X) \ltimes X \rightrightarrows X$. The isotropy group of a point $x \in X$ is just the group of diffeomorphisms that have x as a fixed point.

Example 3.30. Let $G \rightrightarrows G_0$ be a diffeological groupoid, and for any object $x \in G_0$ denote by G_x its isotropy group. We can then consider the subgroupoid of G that only consists of elements in isotropy groups:

$$I_G := \bigcup_{x \in G_0} G_x \subseteq G.$$

With the subgroupoid diffeology, this becomes a diffeological groupoid $I_G \rightrightarrows G_0$ called the *isotropy groupoid*. This has been studied in [**Bos07**, Example 2.1.9] in the context of Lie groupoids. Note that if $G \rightrightarrows G_0$ is a Lie groupoid, then generally I_G is not a submanifold of G, so the isotropy groupoid may no longer be a Lie groupoid.

Example 3.31. The *thin fundamental groupoid* (or *path groupoid*) $\Pi^{\mathrm{thin}}(M)$ of any smooth manifold M is a diffeological groupoid [**CLW16**, Proposition A.25].

Example 3.32. The *groupoid of* Σ*-evolutions* is a diffeological groupoid [**Gł19**, Section II.2.2].

3.3 Diffeological fibrations

3.3.1 The structure groupoid of a smooth surjection

The following describes a generalisation of the idea of a *frame bundle* (or *general linear groupoid*) of a smooth vector bundle. This is a groupoid that describes the linear isomorphisms between the fibres of the bundle. In the case that the vector bundle has finite-dimensional fibres, this is in fact a Lie groupoid. In the case that the fibres are infinite-dimensional, the frame groupoid is no longer strictly a Lie groupoid. Here we will generalise the idea of a frame groupoid to an arbitrary smooth surjection $\pi : X \to B$. This is the concept that lies at the heart of the theory of diffeological fibre bundles [***Diffeology***, Chapter 8]. The question of frame bundles in diffeology will be discussed briefly in Section 3.4.

Definition 3.33. Consider a smooth surjection $\pi : X \to B$, which we think of as template for a bundle. For any given base point $b \in B$, we denote the π-fibre by $X_b := \pi^{-1}(\{b\})$. The *structure groupoid* $\mathbf{G}(\pi) \rightrightarrows B$ of π is defined as follows. The space $\mathbf{G}(\pi)$ of arrows is the collection of all diffeomorphisms between the fibres of π:

$$\mathbf{G}(\pi) := \bigcup_{a,b \in B} \mathrm{Diff}(X_a, X_b),$$

where each fibre $X_b \subseteq X$ is furnished with the subset diffeology. We define the source and target maps by projecting a diffeomorphism $f : X_a \to X_b$ to $\mathrm{src}(f) := a$ and $\mathrm{trg}(f) := b$. Composition in $\mathbf{G}(\pi)$ is just the composition of the diffeomorphisms between the fibres, as is the inversion.

Besides the groupoid structure maps, the structure groupoid $\mathbf{G}(\pi) \rightrightarrows B$ also carries an evaluation map. As in Section 2.4.1, we have the maximal domain of evaluable pairs, which here can be written as a fibred product:

$$\mathcal{E}_X^{\mathrm{src}} := \mathbf{G}(\pi) \times_B^{\mathrm{src},\pi} X = \coprod_{f \in \mathbf{G}(\pi)} \mathrm{dom}(f).$$

The evaluation map is then defined as usual:

$$\mathrm{ev} : \mathcal{E}_X^{\mathrm{src}} \longrightarrow X; \qquad (f, x) \longmapsto f(x).$$

We want to construct a canonical groupoid diffeology on $\mathbf{G}(\pi) \rightrightarrows B$ such that the evaluation map becomes smooth. This is done in [***Diffeology***, Article 8.7], which we describe here below. Since the proofs of the results in this section are so similar to the ones we have encountered in Section **2.4**, we will skip over some of them here.

Construction 3.34. To define a diffeology on $\mathbf{G}(\pi)$ we mimic the construction of the standard functional diffeology as described in Section **2.4**. Given a parametrisation $\Omega : U_\Omega \to \mathbf{G}(\pi)$, we introduce the following notation:

$$\mathcal{E}^{\mathrm{src}}_{\Omega} := U_\Omega \times_B^{\mathrm{src}\circ\Omega,\pi} X \qquad \text{and} \qquad \mathcal{E}^{\mathrm{trg}}_{\Omega} := U_\Omega \times_B^{\mathrm{trg}\circ\Omega,\pi} X.$$

These spaces contain, respectively, the pairs $(t, x) \in U_\Omega \times X$ such that $x \in \mathrm{dom}(\Omega(t))$ and $x \in \mathrm{im}(\Omega(t))$. On $\mathcal{E}^{\mathrm{src}}_{\Omega}$ it makes sense to define expressions such as $\mathrm{ev} \circ (\Omega \times \mathrm{id}_X)$, while on $\mathcal{E}^{\mathrm{trg}}_{\Omega}$ we can define $\mathrm{ev} \circ (\Omega^{-1} \times \mathrm{id}_X)$, where $\Omega^{-1} := \mathrm{inv} \circ \Omega$ is the point-wise inverse. These domains fit into commutative diagrams of the sorts

$$\begin{array}{ccccc} \mathcal{E}^{\mathrm{src}}_{\Omega} & \xrightarrow{\Omega\times\mathrm{id}_X} & \mathcal{E}^{\mathrm{src}}_{X} & \xrightarrow{\mathrm{pr}_2} & X \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \pi} \\ U_\Omega & \xrightarrow[\Omega]{} & \mathbf{G}(\pi) & \xrightarrow[\mathrm{src}]{} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} \mathcal{E}^{\mathrm{trg}}_{\Omega} & \xrightarrow{\Omega^{-1}\times\mathrm{id}_X} & \mathcal{E}^{\mathrm{src}}_{X} & \xrightarrow{\mathrm{ev}} & X \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \pi} \\ U_\Omega & \xrightarrow[\Omega^{-1}]{} & \mathbf{G}(\pi) & \xrightarrow[\mathrm{trg}]{} & B. \end{array}$$

We define a collection $\mathcal{D}_\pi$ of parametrisations on $\mathbf{G}(\pi)$ through the following two conditions. We say a parametrisation $\Omega : U_\Omega \to \mathbf{G}(\pi)$ is in $\mathcal{D}_\pi$ if and only if the following two conditions are satisfied:

1. The map $(\mathrm{src}, \mathrm{trg}) \circ \Omega : U_\Omega \to B \times B$ is smooth.
2. The maps $\mathrm{ev} \circ (\Omega \times \mathrm{id}_X) : \mathcal{E}^{\mathrm{src}}_{\Omega} \to X$ and $\mathrm{ev} \circ (\Omega^{-1} \times \mathrm{id}_X) : \mathcal{E}^{\mathrm{trg}}_{\Omega} \to X$ are smooth.

Lemma 3.35. *The collection $\mathcal{D}_\pi$ defined in Construction 3.34 is a diffeology on $\mathbf{G}(\pi)$.*

Proof. Note that we can treat both conditions separately, since $\mathcal{D}_\pi$ is just the intersection of the parametrisations that satisfy either one. The first condition is just the defining characterisation of the pullback diffeology along the map $(\mathrm{src}, \mathrm{trg}) : \mathbf{G}(\pi) \to B \times B$, of which it is known that it defines a diffeology. It hence suffices to show that the second condition defines a diffeology. This follows from an argument that is closely analogous to Lemma **2.94**. ☐

Lemma 3.36. *The diffeology $\mathcal{D}_\pi$ on $\mathbf{G}(\pi)$ makes the evaluation map smooth.*

Proof. This follows immediately from the decomposition $\mathrm{ev} \circ (\Omega \times \alpha) = \mathrm{ev} \circ (\Omega \times \mathrm{id}_X) \circ (\mathrm{id}_{U_\Omega} \times \alpha)$ and Lemmas **2.33** and **2.61**. ☐

Lemma 3.37. *The diffeology $\mathcal{D}_\pi$ is a groupoid diffeology for $\mathbf{G}(\pi) \rightrightarrows B$.*

Proof. We need to show that each of the structure maps of $\mathbf{G}(\pi) \rightrightarrows B$ become smooth when $\mathbb{G}(\pi)$ is equipped with the diffeology $\mathcal{D}_\pi$. The first condition in Construction **3.34** directly ensures that the source and target maps are smooth. That inversion is smooth follows immediately from the second condition. We are therefore to show that the composition and unit maps are smooth.That composition is smooth follows since it is just the composition of smooth maps, which we saw was smooth in Proposition **2.88**. Lastly, let $\alpha : U_\alpha \to B$ be a plot of the base space. We need to show that $\mathrm{u} \circ \alpha : U_\alpha \to \mathbf{G}(\pi)$ is a plot in $\mathcal{D}_\pi$. First, note that $\mathrm{src} \circ \mathrm{u} \circ \alpha = \alpha$ and $\mathrm{trg} \circ \mathrm{u} \circ \alpha = \alpha$, so that the first condition is fulfilled. For the second condition, it is easy to see that $\mathrm{ev} \circ ([\mathrm{u} \circ \alpha] \times \mathrm{id}_X) = \mathrm{pr}_2$, which is also smooth. Every structure map of $\mathbf{G}(\pi) \rightrightarrows B$ is thus smooth, which was to be shown. ☐

Definition 3.38. The diffeology $\mathcal{D}_\pi$ on $\mathbf{G}(\pi)$ is called the *structure groupoid diffeology*. We always assume that the structure groupoid of a smooth surjection $\pi : X \to B$ is equipped with this diffeology. The structure groupoid diffeology is the coarsest groupoid diffeology on $\mathbf{G}(\pi) \rightrightarrows B$ such that the evaluation map is smooth.

This definition is the key ingredient of the theory of diffeological fibrations [***Diffeology***, Chapter 8].

Example 3.39. Suppose that $\pi : X \to B$ is a bijection. Then each fibre X_b is just a singleton $\pi^{-1}(\{b\}) = \{x_b\}$, and $\mathrm{Diff}(X_a, X_b)$ contains the unique function $f : x_a \mapsto x_b$. There is then a diffeomorphism $\mathbf{G}(\pi) \cong B \times B$, and the structure groupoid reduces to the pair groupoid of B (Example **3.20**).

Example 3.40. Consider a fixed diffeological space F, and set $\pi = \mathrm{pr}_1 : B \times F \to B$. The fibres of this surjection are just (diffeomorphic to) F, so the arrows in the structure groupoid $\mathbf{G}(\mathrm{pr}_1) \rightrightarrows B$ are just diffeomorphisms of F.

3.3.2 Diffeological fibre bundles

The theory of diffeological fibre bundles was first developed in the thesis [**Igl85**], motivated by the example of the irrational torus (Section **2.3**). A review of the theory of diffeological fibre bundles is [***Diffeology***, Chapter 8].

Definition 3.41. Let $G \rightrightarrows G_0$ be a diffeological groupoid. The map $(\mathrm{src}, \mathrm{trg}) : G \to G_0 \times G_0$ is called the *characteristic map*. A diffeological groupoid $G \rightrightarrows G_0$ is called *fibrating* (or a *fibration groupoid*) if its characteristic map is a subduction.

Even though the source and target maps themselves are subductions (Proposition **3.17**), this does not imply that the characteristic map is a subduction. In fact, the characteristic map of any fibrating diffeological groupoid is necessarily surjective, so that the groupoid has only one orbit. It is not hard to define a diffeological groupoid that has multiple orbits. The central definition of diffeological bundle theory is then the following:

Definition 3.42. A smooth surjection $\pi : X \to B$ is called a *diffeological fibre bundle* (or *diffeological fibration*) if its structure groupoid $\mathbf{G}(\pi) \rightrightarrows B$ is fibrating.

It is remarkable that the whole notion of diffeological fibre bundles (and hence also of usual smooth fibre bundles) is captured in such a succinct definition. It says nothing explicitly about local triviality or the relation of the fibres. Remark to that extent that diffeological fibre bundles are indeed in general *not locally trivial.* This is actually a welcome feature of the theory, because it means that we can treat examples like the irrational torus T_θ as nicely behaved bundles, even though their topology is trivial so notions of local triviality are hopeless. On the point about fibres, we can still see that all fibres of a diffeological fibration have to be diffeomorphic. If $\pi : X \to B$ is a diffeological fibre bundle, then the characteristic map of its structure groupoid $\mathbf{G}(\pi) \rightrightarrows B$ is subductive, and hence surjective. In particular this means that there exists an arrow in $\mathbf{G}(\pi)$ between any two points $a, b \in B$, which exhibits the diffeomorphisms between each of the fibres.

Despite the fact that local triviality plays no explicit rôle in the definition of a diffeological fibre bundle, we can nevertheless characterise them by a weaker notion of local triviality. This is sometimes more practical, and usually more intuitive. For that we introduce the following hierarchy of notions:

Definition 3.43. Let F be a fixed diffeological space, referred to as the *fibre.* The *trivial diffeological fibre bundle* with fibre F is the projection $\mathrm{pr}_1 : B \times F \to B$. It is clear from Example **3.40** that this is a fibration. We say a smooth surjection $\pi : X \to B$ is *trivial of type* F if there exists a diffeomorphism $\Phi : X \to B \times F$ such that the following triangle commutes:

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & B \times F \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\mathrm{pr}_1} & \\ & B. & \end{array}$$

A smooth surjection $\pi : X \to B$ is called *locally trivial of type* F if there exists a D-open cover $(U_i)_{i \in I}$ of the base space B, such that each restriction $\pi|_{\pi^{-1}(U_i)} : \pi^{-1}(U_i) \to U_i$ is trivial of type F.

Finally, a smooth surjection $\pi : X \to B$ is called *locally trivial of type F along the plots* if for every plot $\alpha : U_\alpha \to B$ of the base space, the pullback projection $\alpha^* X := U_\alpha \times_B^{\alpha,\pi} X \to U_\alpha$ is locally trivial of type F.

Theorem 3.44 ([***Diffeology***, Article 8.9])**.** *A smooth surjection $\pi : X \to B$ is a diffeological fibre bundle if and only if there exists a diffeological space F such that π is locally trivial of type F along the plots.*

Any locally trivial bundle is locally trivial along the plots, but the converse is not true. This shows that diffeological fibre bundles are more general than locally trivial bundles. Even though diffeological fibre bundles are not locally trivial, Theorem **3.44** allows us to prove that they are still stronger than 'submersions':

Proposition 3.45. *Every diffeological fibration is a local subduction.*

Proof. Suppose that $\pi : E \to B$ is a diffeological fibration, of fibre type $F \in \mathbf{Diffeol}$. Pick an arbitrary pointed plot $\alpha : (U_\alpha, 0) \to (B, \pi(x))$ of the base space. The pullback $\alpha^*E \to U$ is then locally trivial, so we can find an open neighbourhood $0 \in V \subseteq U_\alpha$ and a diffeomorphism $\Phi : \alpha^*E|_V \to V \times F$ such that $\mathrm{pr}_1 \circ \Phi = \mathrm{pr}_1|_{V \times_B E}$, where we note that $\alpha^*E|_V = V \times_B^{\alpha|_V, \pi} E$. Clearly we have $(0, x) \in \alpha^*E|_V$, so we get a special element $\Phi_2(0, x) =: \xi \in F$ in the fibre. If Ψ is the inverse for Φ, we get a plot

$$\beta : V \xrightarrow{(\mathrm{id}_V, \mathrm{const}_\xi)} V \times F \xrightarrow{\Psi} \alpha^*E|_V \xrightarrow{\mathrm{pr}_2|_{V \times_B E}} X$$

of X. In the way that $\xi \in F$ is chosen, we get that $\beta(0) = x \in E$. By definition of the pullback $\alpha^*E|_V$ we see that $\pi \circ \mathrm{pr}_2|_{V \times_B E} = \alpha|_V \circ \mathrm{pr}_1|_{V \times_B E}$, from which it follows that $\alpha|_V = \pi \circ \beta$. □

3.4 A definition of smooth linear representations for groupoids

In this section we propose a definition of smooth linear representations for diffeological groupoids. We do not know if and how this theory can be applied. One possible use could be for the description of infinite-dimensional representations of groupoids (cf. [**Bos07**]), since diffeology allows for frame groupoids of infinite-dimensional vector bundles. Recall first the definition of a linear representation of a Lie group G. If V is a (usually complex) finite-dimensional vector space, then its general linear group $\mathrm{GL}(V)$ is a Lie group. A linear representation then associates to every element $g \in G$ a linear isomorphism $\rho(g) : V \to V$, forming the "linear representation" of the group element g on V. A *linear representation* of the entire group G is then a smooth group homomorphism $\rho : G \to \mathrm{GL}(V)$.

This definition has a natural extension for Lie groupoids, which can be explained as follows. Thinking of a groupoid as an 'interlinked' family of groups, we see that a linear representation of a Lie groupoid $G \rightrightarrows G_0$ should, for every object $x \in G_0$ in the base, determine a linear representation ρ of the isotropy group G_x, on some vector space $E_{\rho_0(x)}$. To preserve the smooth behaviour of the groupoid, these vector spaces should be the fibres of some vector bundle $E \to B$, and $\rho_0 : G_0 \to B$ should be some smooth function that associates to each $x \in G_0$ such a vector space in a smooth way. For every arrow $g \in G$ in the groupoid, $\rho(g)$ should then define a linear diffeomorphism between $E_{\rho_0(\mathrm{src}(g))}$ and $E_{\rho_0(\mathrm{trg}(g))}$. To respect the 'interlinkedness' of the groupoid, this assignment ρ should be functorial, and the target of this functor is the *frame groupoid* $\mathrm{GL}(E) \rightrightarrows B$. The arrows in this Lie groupoid are exactly the linear isomorphisms between the fibres of $E \to B$. With this, we can state the definition of a linear representation of a Lie groupoid[35]:

Definition 3.46. Let $G \rightrightarrows G_0$ be a Lie groupoid, and let $\pi : E \to B$ be a smooth finite-dimensional vector bundle (in the usual sense of smooth manifolds). A *smooth linear representation* of G over E is a smooth functor $\rho : G \to \mathrm{GL}(E)$.

If $\pi : E \to B$ is no longer finite-dimensional, for example as is the case with some bundles of Hilbert spaces, then $\mathrm{GL}(E) \rightrightarrows B$ is no longer a Lie groupoid, and it is not straightforward what it means for ρ to be smooth. Diffeology can provide answers to these problems.

To start, we need to clarify what we mean by a diffeological vector bundle. Given the discussion in Section **3.3.2**, it is possible to define such a notion by taking a diffeological fibre bundle $\pi : E \to B$ whose

[35] We use the notation $E \to B$ for a bundle with some form of linear structure, while we write $X \to B$ for subductions or arbitrary diffeological fibre bundles.

fibres have a linear structure, and whose local trivialisations along the plots are linear diffeomorphisms. In this definition, if B is a smooth manifold and the fibre type F is some $\mathbb{R}^n$, it can be shown that this definition returns the usual notion of smooth finite-dimensional vector bundles on manifolds. We might call such objects *diffeological vector bundles*. More recently, an alternative and more general approach to the definition of diffeological vector bundles has been proposed [**Per16**]. This alternative definition, which we state below, can mainly be motivated from examples of *diffeological tangent bundles*. There are examples of diffeological spaces X where the *internal tangent bundle* $\check{T}X \to X$ is not a diffeological vector bundle in the above sense[36]. One main obstruction is that the fibres of $\check{T}X \to X$ can *vary* along the base. An example of this is the cross (Example **2.18**), where the internal tangent space at the origin is two-dimensional, but is one-dimensional everywhere else [**CW14**, Section 3.3]. This contradicts the fact that the fibre type of a diffeological fibre bundle is unique, and also appears to contradict Proposition **3.45**.

The following is a definition for this more general class of *pseudo-bundles*, which is able to handle vector bundles of varying dimension. We adopt the terminology proposed by [**Per16**]. The definition below is equivalent to the one for *regular vector bundles* in [**Vin08**], and to the one for *vector spaces over difeological spaces* in [**CW14**, Definition 4.5].

Definition 3.47. A *diffeological vector pseudo-bundle* is a smooth map $\pi : E \to B$, where every π-fibre has the structure of a vector space, such that:

1. The *addition map* $E \times_B^{\pi,\pi} E \to E : (v, w) \mapsto v + w$ is smooth.
2. The *scalar multiplication map* $\mathbb{R} \times E \to E : (\lambda, u) \mapsto \lambda u$ is smooth.
3. The *zero section* $0_E : B \to E$ is smooth.

We note that the existence of a smooth zero section $\pi \circ 0_E = \mathrm{id}_B$ ensures through Lemma **2.121** that the projection π is a subduction, and hence that $B \cong E/\pi$. If $\pi : E \to B$ is a vector pseudo-bundle, each π-fibre $E_b := \pi^{-1}(\{b\})$ gets the structure of a diffeological vector space (Example **2.19**) from the subset diffeology. It therefore makes sense to talk about smooth linear maps between the fibres of π. The motivating example is the following:

Example 3.48. The internal tangent bundle $\check{T}X \to X$ of any diffeological space is a vector pseudo-bundle. The diffeology on $\check{T}X$ is in fact tailored to guarantee this, see [**CW14**, Section 4].

We introduce the following interesting class of (generally infinite-dimensional) vector pseudo-bundles:

Construction 3.49. Let $\pi : X \to B$ be a smooth surjection, not necessarily carrying any linear structure. We construct a vector pseudo-bundle $\pi_B : C_B^\infty(X) \to B$, where each fibre is just the vector space $C^\infty(X_b)$ of smooth real-valued functions on the π-fibre X_b. We have seen in Example **2.91** that, with the standard functional diffeology, these are diffeological vector spaces.

For the construction of $C_B^\infty(X)$, we can use the terminology of parametrised mapping spaces in Section **2.4**. Namely, if we consider the smooth projection map $\mathbb{R} \times B \to B$, then we get a parametrised mapping space

$$C_B^\infty(X) := C_B^\infty(X, \mathbb{R} \times B) = \coprod_{b \in B} C^\infty(X_b),$$

where in the last equation we have identified $\mathbb{R} \times \{b\} \cong \mathbb{R}$ and hence $C^\infty(X_b, \mathbb{R} \times \{b\}) \cong C^\infty(X_b)$. We equip $C_B^\infty(X)$ with the standard parametrised functional diffeology from Definition **2.95**, so that the projection $\mathrm{pr}_1 : C_B^\infty(X) \to B$ and the evaluation map ev_B are smooth. Note that, as a consequence, the subset diffeology of each subspace $C^\infty(X_b) \subseteq C_B^\infty(X)$ coincides with the standard functional diffeology, turning them into diffeological vector spaces. We write $\pi_B := \mathrm{pr}_1$, and claim that

$$\pi_B : C_B^\infty(X) \longrightarrow B$$

[36] We refer to [**CW14**] for a detailed construction of the internal tangent bundle. Note that we use the notation '$\check{T}$', with a hat pointing *inwards*, while [**CW14**] use the notation 'T^{dvs}'.

is a diffeological vector pseudo-bundle, where the addition and scalar multiplication are induced fibre-wise by the maps

$$+_b : C^\infty(X_b) \times C^\infty(X_b) \longrightarrow C^\infty(X_b),$$
$$\mathrm{m}_b : \mathbb{R} \times C^\infty(X_b) \longrightarrow C^\infty(X_b),$$

respectively, and the zero section is composed of the zero functions $0_b = \mathrm{const}_0 : X_b \to \mathbb{R}$. Concretely, this means:

$$\begin{aligned} &+_B : C^\infty_B(X) \times_B^{\pi_B,\pi_B} C^\infty_B(X) \longrightarrow C^\infty_B(X); && ((b,f),(b,g)) \longmapsto (b, +_b(f,g)), \\ &\mathrm{m}_B : \mathbb{R} \times C^\infty_B(X) \longrightarrow C^\infty_B(X); && (\lambda,(b,f)) \longmapsto (b, \mathrm{m}_b(\lambda,f)), \\ &0_B : B \longrightarrow C^\infty_B(X); && b \longmapsto (b, 0_b). \end{aligned}$$

We need to show that $\pi_B : C^\infty_B(X) \to B$ satisfies all three conditions of Definition **3.47**, which just means that the addition, scalar multiplication, and the zero section should be smooth. We check these in turn.

To show that the fibre-wise addition is smooth, take two plots Ω and Ψ of the standard parametrised functional diffeology, defined on the same Euclidean domain U. Recall that the plots of this diffeology are characterised in Lemma **2.94**. First, it is easy to check from the definition that

$$\mathrm{pr}_1 \circ +_B \circ (\Omega, \Psi) = \mathrm{pr}_1 \circ \Omega = \mathrm{pr}_1 \circ \Psi,$$

and the right-hand side of this equation is smooth. This also shows that the domain on which the function $\mathrm{ev}_B \circ ([+_B \circ (\Omega,\Psi)] \times \mathrm{id}_X)$ is defined is $U \times_B^{\mathrm{pr}_1\Omega,\pi} X$. We can then calculate for an element (t,x) in this domain:

$$\begin{aligned} \mathrm{ev}_B \circ ([+_B \circ (\Omega,\Psi)] \times \mathrm{id}_X)(t,x) &= \mathrm{ev} \circ \left(+_{\pi(x)}(\Omega(t),\Psi(t)), x\right) \\ &= \Omega(t)(x) +_{\mathbb{R}} \Psi(t)(x) \\ &= +_{\mathbb{R}} \circ (\mathrm{ev}_B \circ (\Omega \times \mathrm{id}_X), \mathrm{ev}_B \circ (\Psi \times \mathrm{id}_X))(t,x). \end{aligned}$$

On the right hand side everything is smooth, since both Ω and Ψ are plots, and the addition $+_{\mathbb{R}}$ on $\mathbb{R}$ is smooth. This proves that the fibre-wise addition for $C^\infty_B(X)$ is smooth.

Second, we prove that scalar multiplication is smooth, which follows from a similar calculation. Let us take a plot $\alpha : U \to \mathbb{R}$, and the same plot $\Omega : U \to C^\infty_B(X)$ from before. Then it follows from the definition of m_B that

$$\mathrm{pr}_1 \circ \mathrm{m}_B \circ (\alpha, \Omega) = \mathrm{pr}_1 \circ \Omega,$$

which is smooth because Ω is a plot. Next, for any $(t,x) \in U \times_B^{\mathrm{pr}_1\Omega,\pi} X$ we find

$$\begin{aligned} \mathrm{ev}_B \circ ([\mathrm{m}_B \circ (\alpha,\Omega)] \times \mathrm{id}_X)(t,x) &= \mathrm{ev} \circ \left(\mathrm{m}_{\pi(x)}(\alpha(t),\Omega(t)), x\right) \\ &= \alpha(t) \cdot \Omega(t)(x) \\ &= \mathrm{m}_{\mathbb{R}}(\alpha(t), \Omega(t)(x)) \\ &= \mathrm{m}_{\mathbb{R}} \circ (\alpha \circ \mathrm{pr}_U, \mathrm{ev}_B \circ (\Omega \times \mathrm{id}_X))(t,x). \end{aligned}$$

Again, the right hand side is smooth, and we can conclude that the scalar multiplication map m_B of $C^\infty_X(X)$ is smooth.

Lastly, we need to check that the zero section $0_B : B \to C^\infty_B(X)$ is smooth. It is clear that $\mathrm{pr}_1 \circ 0_B = \mathrm{id}_B$, and for a plot $\alpha : U_\alpha \to B$ we calculate for $(t,x) \in U_\alpha \times_B^{\alpha,\pi} X$:

$$\mathrm{ev}_B \circ ([0_B \circ \alpha] \times \mathrm{id}_X)(t,x) = \mathrm{ev}_B(0_{\alpha(t)}, x) = 0 = \mathrm{const}_0(t,x),$$

which is evidently smooth as a function of (t,x).

In conclusion, for any smooth surjection $\pi : X \to B$ we have a diffeological vector pseudo-bundle $\pi_B : C^\infty_B(X) \to B$, and the fibres are exactly the diffeological vector spaces $C^\infty(X_b)$ of real-valued smooth functions on the π-fibres (with the standard functional diffeology).

Since the fibres of a diffeological vector pseudo-bundle are diffeological vector spaces, it makes sense to consider linear diffeomorphisms between the fibres. This allows us to make the following definition, generalising the traditional notion of a frame groupoid from the Lie groupoid theory:

Definition 3.50. Let $\pi : E \to B$ be a diffeological vector pseudo-bundle. We define the *frame groupoid* (also known as a *general linear groupoid*) $\mathrm{GL}(E) \rightrightarrows B$ of π as the subgroupoid of the structure groupoid $\mathbf{G}(\pi) \rightrightarrows B$ (Definition **3.33**) containing the linear diffeomorphisms between the fibres of π. With the subgroupoid diffeology, this turns $\mathrm{GL}(E) \rightrightarrows B$ into a diffeological groupoid such that the evaluation map is smooth:

$$\mathrm{ev} : \mathrm{GL}(E) \times_B^{\mathrm{src},\pi} E \longrightarrow E; \qquad (\varphi, e) \longmapsto \varphi(e).$$

We can now extend Definition **3.46** to the diffeological world:

Definition 3.51. A *smooth linear representation* of a diffeological groupoid $G \rightrightarrows G_0$ over a vector pseudo-bundle $\pi : E \to B$ is a smooth functor $\rho : G \to \mathrm{GL}(E)$. Unpacking this definition, we see that a representation induces a smooth groupoid action $G \curvearrowright^{\pi} E$ (Definition **4.1**), where the action is defined by the evaluation of the representation: $g \cdot e := \rho(g)(e)$. In particular, the maps $g \cdot - = \rho(g)$ are linear.

Using the construction of the infinite-dimensional pseudo-bundle in Construction **3.49**, we can construct a smooth left regular representation:

Example 3.52 (Smooth left regular representation)**.** For any diffeological groupoid $G \rightrightarrows G_0$, we shall define its *left regular representation* $\rho_L : G \to \mathrm{GL}(C^\infty_{G_0}(G))$. To do this, we first need to exhibit the vector pseudo-bundle $\Pi : C^\infty_{G_0}(G) \to G_0$. This is just the vector pseudo-bundle of real-valued smooth functions defined in Construction **3.49**, defined for the map $\pi = \mathrm{trg} : G \to G_0$. The fibres of the target map are denoted $G^x := \mathrm{trg}^{-1}(\{x\})$, for an object $x \in G_0$. The fibres of the pseudo-bundle $\Pi : C^\infty_{G_0}(G) \to G_0$ are then the spaces $C^\infty(G^x)$ of smooth real-valued functions on the space of arrows whose target is x. To define the smooth functor

$$\rho_L : (G \rightrightarrows G_0) \longrightarrow (\mathrm{GL}(C^\infty_{G_0}(G)) \rightrightarrows G_0),$$

we set the base arrow as $(\rho_L)_0 := \mathrm{id}_{G_0}$. Then, for any arrow $g \in G$, we are supposed to define a linear diffeomorphism $\rho_L(g) : C^\infty(G^{\mathrm{src}(g)}) \to C^\infty(G^{\mathrm{trg}(g)})$. This is done as follows. A function $f \in C^\infty(G^{\mathrm{src}(g)})$ is defined on the space of all arrows in G whose target is $\mathrm{src}(g)$. We then see that $f(g^{-1} \circ h)$ is a function that is defined for all arrows $h \in G$ such that $\mathrm{trg}(h) = \mathrm{src}(g^{-1}) = \mathrm{trg}(g)$, as desired. We can therefore define a linear map:

$$\rho_L(g) : C^\infty(G^{\mathrm{src}(g)}) \longrightarrow C^\infty(G^{\mathrm{trg}(g)}); \qquad f \longmapsto f(g^{-1} \circ -).$$

Using arguments similar to that in the proof of Construction **3.49**, we find that $\rho_L(g)$ is smooth, since the groupoid inversion and composition are smooth. An elementary calculation shows that this assignment is functorial: $\rho_L(g \circ h) = \rho_L(g) \circ \rho_L(h)$, $\rho_L(\mathrm{id}_x) = \mathrm{id}_{C^\infty(G^x)}$, which also immediately gives that each $\rho_L(g)$ is a linear diffeomorphism. Together, this proves that $\rho_L(g)$ is an arrow in the frame groupoid $\mathrm{GL}(C^\infty_{G_0}(G))$, and we have defined a genuine functor $\rho_L : G \to \mathrm{GL}(C^\infty_{G_0}(G))$. We are therefore left to show that ρ_L itself is smooth. Recall from Construction **3.34** the two defining conditions for plots on a structure groupoid. For an arbitrary plot $\alpha : U_\alpha \to G$ of the arrow space, we then need to show that

1. The map $(\mathrm{src}_{\mathrm{GL}}, \mathrm{trg}_{\mathrm{GL}}) \circ \rho_L \circ \alpha : U_\alpha \to G_0 \times G_0$ is smooth.
2. The map
$$\mathrm{ev} \circ ([\rho_L \circ \alpha] \times \mathrm{id}_{C^\infty_{G_0}(G)}) : U_\alpha \times_{G_0}^{\mathrm{src}_G \alpha, \Pi} C^\infty_{G_0}(G) \longrightarrow C^\infty_{G_0}(G),$$
and a similar expression for the point-wise inverse $\rho_L^{-1} \circ \alpha$, are smooth.

Here we use the notation $\mathrm{src}_G, \mathrm{trg}_G : G \to G_0$ and $\mathrm{src}_{\mathrm{GL}}, \mathrm{trg}_{\mathrm{GL}} : \mathrm{GL}(C^\infty_{G_0}(G)) \to G_0$ to distinguish between the source and target maps of the two groupoids.

Proof. (1). We have already seen that ρ_L is functorial, so that it intertwines the source and target maps of the two groupoids. This gives:

$$\mathrm{src}_{\mathrm{GL}} \circ \rho_L \circ \alpha = (\rho_L)_0 \circ \mathrm{src}_G \circ \alpha = \mathrm{id}_{G_0} \circ \mathrm{src}_G \circ \alpha = \mathrm{src}_G \circ \alpha,$$

which is smooth since α is a plot. Together with a similar computation for the target map $\mathrm{trg}_{\mathrm{GL}}$, we get that $(\mathrm{src}_{\mathrm{GL}}, \mathrm{trg}_{\mathrm{GL}}) \circ \rho_L \circ \alpha = (\mathrm{src}_G, \mathrm{trg}_G) \circ \alpha$, which is smooth.

(2). There are two steps in proving that $\mathrm{ev} \circ ([\rho_L \circ \alpha] \times \mathrm{id}_{C^\infty_{G_0}(G)})$ is smooth. First, we rewrite the expression into a simpler form. Then, we need to show that it defines a smooth function on $C^\infty_{G_0}(G)$, which has yet a different functional diffeology. For this, we need to take a plot $h : V \to U_\alpha$ and a plot $\Omega : V \to C^\infty_{G_0}(G)$ such that $\mathrm{src}_G \circ \alpha \circ h = \Pi \circ \Omega$, and then we need to prove that $\mathrm{ev} \circ ([\rho_L \circ \alpha] \times \mathrm{id}_{C^\infty_{G_0}(G)}) \circ (h, \Omega)$ satisfies the conditions of the standard parametrised functional diffeology in Definition **2.95**.

To start, take a pair $(t, (x, f)) \in U_\alpha \times^{\mathrm{src}_G \alpha, \Pi}_{G_0} C^\infty_{G_0}(G)$, meaning that $\mathrm{src}_G(\alpha(t)) = x$, and $f \in C^\infty(G^x)$. Let us abbreviate $\Phi_\alpha := \mathrm{ev} \circ ([\rho_L \circ \alpha] \times \mathrm{id}_{C^\infty_{G_0}(G)})$. We then calculate:

$$\begin{aligned} \Phi_\alpha(t, (x, f)) &= \mathrm{ev} \circ \left([\rho_L \circ \alpha] \times \mathrm{id}_{C^\infty_{G_0}(G)} \right) (t, (x, f)) \\ &= \mathrm{ev}\left(\rho_L(\alpha(t)), (x, f)\right) \qquad (\spadesuit) \\ &= \left(\mathrm{trg}_G(\alpha(t)), \rho_L(\alpha(t))(f)\right) \in C^\infty_{G_0}(G), \end{aligned}$$

where now $\rho_L(\alpha(t)) : C^\infty(G^x) \to C^\infty(G^{\mathrm{trg}_G(\alpha(t))})$. Next, with the two plots h and Ω as above, we need to show that $\Phi_\alpha \circ (h, \Omega)$ satisfies the two conditions of a plot in the standard parametrised functional diffeology. For that, looking at equation ($\spadesuit$), we find

$$\mathrm{pr}_1 \circ \Phi_\alpha \circ (h, \Omega) = \mathrm{trg}_G \circ \alpha \circ h,$$

which is evidently smooth. Second, we need to verify that

$$\mathrm{ev} \circ ([\Phi_\alpha \circ (h, \Omega)] \times \mathrm{id}_G) : V \times^{\mathrm{trg}_G \alpha h, \mathrm{trg}_G}_{G_0} G \longrightarrow \mathbb{R}$$

is smooth. For that, let $(s, g) \in V \times G$ be an element in the domain of this map, meaning that $\mathrm{trg}_G(\alpha(h(s))) = \mathrm{trg}_G(g)$. If we write $\Omega = (\Omega_1, \Omega_2)$, where $\Omega_1 : V \to G_0$ and $\Omega_2 : V \to \bigcup_{x \in G_0} C^\infty(G^x)$, then we see that $\rho_L(\alpha(h(s))) : C^\infty(G^{\Omega_1(s)}) \to C^\infty(G^{\mathrm{trg}_G(g)})$, so that the expression $\rho_L(\alpha(h(s)))(\Omega_2(s))$ is a well-defined element in $C^\infty(G^{\mathrm{trg}_G(g)})$. Again using equation ($\spadesuit$), we calculate:

$$\begin{aligned} \mathrm{ev} \circ ([\Phi_\alpha \circ (h, \Omega)] \times \mathrm{id}_G)(s, g) &= \mathrm{ev}\left(\Phi_\alpha(h(s), \Omega(s)), g\right) \\ &= \mathrm{ev}\left(\left(\mathrm{trg}_G(\alpha(h(s))), \rho_L(\alpha(h(s)))(\Omega_2(s))\right), g\right) \\ &= \rho_L\left(\alpha(h(s))\right)\left(\Omega_2(s)\right)(g) \\ &= \Omega_2(s)\left(\alpha(h(s))^{-1} \circ g\right) \\ &= \mathrm{ev} \circ (\Omega \times \mathrm{id}_G)(s, \alpha(h(s))^{-1} \circ g). \end{aligned}$$

On the right hand side we have the expression $\mathrm{ev} \circ (\Omega \times \mathrm{id}_G)$, which is guaranteed to be smooth since Ω is a plot in the standard parametrised functional diffeology on $C^\infty_{G_0}(G)$. The argument is clearly also a smooth expression in terms of (s, g), since both α and h are smooth, and the groupoid operations of G are smooth. This proves that $\Phi_\alpha \circ (h, \Omega)$ is a plot for $C^\infty_{G_0}(G)$. We can repeat these calculations for the point-wise inverse $\Phi_\alpha^{-1} = \rho_L^{-1} \circ \alpha$, since we already know that each map $\rho_L(\alpha(t))$ is a linear diffeomorphism, and moreover that the point-wise inverse Ω^{-1} is also smooth. We can therefore conclude that Φ_α is a plot for the structure groupoid diffeology on $\mathrm{GL}(C^\infty_{G_0}(G))$, and since throughout our calculations α was an arbitrary plot of G, we have proved that $\rho_L : G \to \mathrm{GL}(C^\infty_{G_0})$ is a smooth functor. □

3.4.1 A remark on 'VpB-groupoids'

Recall the definition of a *VB-groupoid* for Lie groupoids from [**Mac05**], or [**GA10**, Section 3]:

Definition 3.53. Let $G \rightrightarrows G_0$ and $\Gamma \rightrightarrows E$ be two Lie groupoids. A smooth functor $\pi : \Gamma \to G$ is called a *VB-groupoid* if the two smooth functions $\pi_1 : \Gamma \to G$ and $\pi_0 : E \to G_0$ between arrow- and base spaces are smooth vector bundles, and such that the source and target maps define smooth vector bundle morphisms:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\mathrm{src}_\Gamma} & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ G & \xrightarrow[\mathrm{src}_G]{} & G_0, \end{array} \qquad \begin{array}{ccc} \Gamma & \xrightarrow{\mathrm{trg}_\Gamma} & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ G & \xrightarrow[\mathrm{trg}_G]{} & G_0. \end{array}$$

Superimposing these two diagrams, a VB-groupoid is often depicted as:

$$\begin{array}{ccc} \Gamma & \rightrightarrows & E \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ G & \rightrightarrows & G_0. \end{array}$$

A VB-groupoid is a vector bundle object in the category of Lie groupoids, or equivalently, a groupoid object in the category of smooth vector bundles. This explains the terminology. We can extend this definition almost *verbatim* to the diffeological setting, using vector-pseudo bundles (Definition **3.47**) instead:

Definition 3.54. Let $G \rightrightarrows G_0$ and $\Gamma \rightrightarrows E$ be two diffeological groupoids. A smooth functor $\pi : \Gamma \to G$ is called a *VpB-groupoid* if the two smooth functions $\pi_1 : \Gamma \to G$ and $\pi_0 : E \to G_0$ are vector pseudo-bundles, such that the source and target maps define smooth vector pseudo-bundle morphisms. We use the term 'VpB-groupoid' to reflect the fact that we are using vector *pseudo*-bundles.

An important example of a VB-groupoid in the Lie groupoid case is the *tangent groupoid*, see e.g. [**Mei17**, Section 4.4]:

Example 3.55. Let $G \rightrightarrows G_0$ be a Lie groupoid, and consider the tangent bundles $\pi_G : TG \to G$ and $\pi_{G_0} : TG_0 \to G_0$ (in the usual sense). Differentiating the structure maps of $G \rightrightarrows G_0$ then gives rise to a Lie groupoid $TG \rightrightarrows TG_0$, with source and target maps $\mathrm{dsrc}, \mathrm{dtrg} : TG \to TG_0$, unit map $\mathrm{du} : TG_0 \to TG$, inverse $\mathrm{dinv} : TG \to TG$, and composition

$$\mathrm{dm} : T(G \times_{G_0}^{\mathrm{src},\mathrm{trg}} G) \cong TG \times_{TG_0}^{\mathrm{dsrc},\mathrm{dtrg}} TG \longrightarrow TG.$$

The groupoid $TG \rightrightarrows TG_0$ is called the *tangent groupoid* associated to $G \rightrightarrows G_0$. The canonical projection maps π_G and π_{G_0} induce a functor $\pi : TG \to G$, which forms the *tangent VB-groupoid*:

$$\begin{array}{ccc} TG & \rightrightarrows & TG_0 \\ {\scriptstyle \pi_G}\downarrow & & \downarrow{\scriptstyle \pi_{G_0}} \\ G & \rightrightarrows & G_0. \end{array}$$

A VB-groupoid over a Lie groupoid $G \rightrightarrows G_0$ can be seen as a geometric model for a "representation up to homotopy" [**GA10**], in which case the tangent groupoid $TG \rightrightarrows TG_0$ corresponds to the *adjoint representation*. Note that the adjoint representation of a groupoid is not canonically defined, since the expression ghg^{-1} only makes sense when $\mathrm{src}(g) = \mathrm{trg}(h) = \mathrm{src}(h)$. Under this correspondence, the tangent groupoid is a canonical geometric model of the adjoint representation (up to homotopy). The tangent VB-groupoid moreover encodes the Lie algebroid of $G \rightrightarrows G_0$, which can be obtained as the kernel of the source map src_Γ over the base G_0 (cf. [**dHo12**, Example 3.3.4]).

By drawing an analogy to this situation to diffeology, we can try to define a notion of *tangent VpB-groupoid* for a diffeological groupoid $G \rightrightarrows G_0$, serving as a model for the adjoint representation, and hence for a *"diffeological algebroid."* The most immediate problem that arises is that, in general, the internal tangent bundle of the fibred product $G \times_{G_0}^{\mathrm{src},\mathrm{trg}} G$ does not seem to decompose into a fibred product of internal tangent bundles, as in the case for smooth manifolds above. This relates to the fact that internal tangent bundles exhibit pathological behaviour around certain subsets, where the tangent space of a subset can have higher dimension than that of the ambient space. It is not clear if these problems can be resolved, and so, if a general internal tangent groupoid can be constructed. If such a thing exists, it should be an example of a VpB-groupoid.

Chapter IV

Diffeological bibundles

In this chapter we aim to generalise the theory of Morita equivalence of Lie groupoids to the diffeological setting. For this we need to define and study the diffeological analogues of: Lie groupoid actions, principal groupoid bundles, and principal groupoid bibundles. The theory appears almost identical to the Lie case, save for the fact that we need to replace the word 'submersion' by a suitable diffeological concept. We choose here for *subductions*, motivated below. Having shown that subductions behave sufficiently like submersions—and Section **2.6** served that purpose for us—all of the proofs for Lie groupoids can be copied almost *verbatim* to the diffeological case. But due to the flexibility of diffeological spaces many of these proofs can be dissected and pulled apart, which make our lemmas much shorter, and some wouldn't even make sense in the Lie case. An upside to this—besides generalising to diffeology—is that we obtain a clearer view of which of the technical conditions are exactly necessary along each step of the way, and it provides an account of the theory with proofs that contain almost every conceivably relevant detail. We hope that, if not the generalisation to diffeology, this is something that will be a nice complement to the existing literature.

The main results of this chapter can be summarised as follows. We define notions of groupoid bundles and -bibundles for diffeological groupoids. This includes a generalisation of the notion of a *principal groupoid bundle*. We generalise the *Hilsum-Skandalis tensor product* (what we call the *balanced tensor product*) of two bibundles, which forms the basis of the composition of bibundles in a bicategory **DiffeolBiBund**. This is a generalisation of the bicategory $\mathbf{LieGrpd}_{\mathrm{LP}}$ of Lie groupoids and left principal bibundles (and biequivariant maps). The construction of $\mathbf{LieGrpd}_{\mathrm{LP}}$ is discussed in detail in [**Blo08**]. Note that this Lie category is as general as it could get: there is no larger bicategory of arbitrary bibundles, since the construction of the Hilsum-Skandalis tensor product rests crucially on the left principality of the Lie bibundles. For diffeological bibundles the Hilsum-Skandalis tensor product can be defined in a more general sense (fundamentally due to the fact that pullbacks and quotients have natural diffeological structures), which is why we get a bicategory of *all* diffeological bibundles (as opposed to just left principal ones). Next, we generalise the result that two Lie groupoids are Morita equivalent if and only if they are equivalent (i.e. weakly isomorphic) in the bicategory $\mathbf{LieGrpd}_{\mathrm{LP}}$. This result holds *verbatim* for diffeological groupoids in the bicategory $\mathbf{DiffeolBiBund}_{\mathrm{LP}}$ of left principal diffeological bibundles. Our main theorem (Theorem **4.69**) generalises this further to the larger bicategory: a diffeological bibundle is biprincipal if and only if it is weakly invertible in **DiffeolBiBund**.

Lie groupoids provide geometric models for differentiable stacks, where two Lie groupoids that are Morita equivalent represent the same stack. In this sense we could say a diffeological groupoid is a model for a *"diffeological stack,"* which lives somewhere in between arbitrary stacks on manifolds and differentiable stacks (represented by Lie groupoids). See also [**WW19**]. As far as we know, the theory of diffeological groupoid bundles and bibundles has not appeared elsewhere. Diffeology is used in the theory of Lie groupoid bibundles and differentiable stacks in [**RV16**; **Wat17**], but they do not treat the general theory as we do here. The only place we could find that treats diffeological groupoid actions is [**KWW19**]. Seth Wolbert (one of the authors of that paper) has told the present author through private email that he is researching diffeological stacks, in which we suspect the theory of diffeological bibundles might make an appearance.

We should also mention the work [**MZ15**], where the authors treat groupoid bibundles internal to a category with a Grothendieck pretopology. We recommend this for a general discussion of groupoid bibundles, of which the authors provide a most complete and self-contained account. We do not know to what extent one of our intermediary results (Theorem **4.62**) follows from their [**MZ15**, Theorem 7.23]. They do treat various types of manifolds in [**MZ15**, Section 9.3], where they use a Grothendieck pretopology defined by submersions, and it takes some work to show that all constructions work in that setting. However, their bibundles are all *right principal*. This means that one of our main results (Theorem **4.69**) lies genuinely outside of the scope of their paper. We suspect that some of the other work in this thesis could partly be realised from their theory if we define a Grothendieck pretopology on **Diffeol** by the *subductions*, cf. [**Wal09**, Proposition A.2.3] and [**Car13**]. We would have to show that this pretopology satisfies all of their assumptions. We also do not know if Theorem **4.69** is unique to diffeology, or if the results of [**MZ15**] could be generalised to a bicategory of bibundles with no further assumptions on principality. For groupoids on manifolds this will not work (for reasons mentioned above), but we suspect it will work for a bicategory of topological groupoid and continuous bibundles.

More non-diffeological literature on groupoid bibundles can be found in: [**HS87**; **Mrc96**; **Mrč99**; **Lan01a**; **Moe02**; **MM03**; **Li15**]. For Lie groupoids, [**Blo08**] provides a complete and succinct account.

4.1 Diffeological groupoid actions

The fundamental definition for the upcoming theory is that of a *groupoid action.* These generalise group actions, and have already been studied extensively for the case of Lie groupoids. See e.g. [**Moe02**; **MM03**; **MM05**; **Mac05**; **Bos07**]. Also in [**Li15**, Section 1.4] for a 2-categorical context, and in [**Blo08**; **dHo12**] and [**Car11**, Section I.2.5] in the context of differentiable stacks.

The idea of a groupoid action is a natural generalisation of a group action. To motivate the definition, consider a groupoid $G \rightrightarrows G_0$, and consider its *left composition* (or *left multiplication*) L_g by an arrow $g \in G$. This is supposed to be an action of the groupoid on itself, sending an arrow $h \in G$ to $L_g(h) := g \circ h$. However, this is only defined if $\mathrm{src}(g) = \mathrm{trg}(h)$. If this holds, then the action 'moves' the target of h to the target of g:

$$\mathrm{trg}(L_g(h)) = \mathrm{trg}(g \circ h) = \mathrm{trg}(g).$$

In this sense, we see that the action of the arrow $g \in G$ is a way to move objects that are 'situated' around its *source* to an object that is situated around its *target.* The action of a groupoid $G \rightrightarrows G_0$ on an arbitrary space X should embody this idea. This can be accomplished by equipping the space X with an arrow $l_X : X \to G_0$, so that it makes sense to talk about what it means to be 'situated' at a source or target of an arrow $g \in G$. To be precise, this means that the action $g \cdot x$ is defined whenever $l_X(x) = \mathrm{src}(g)$. The point $g \cdot x \in X$ is subsequently situated around the target of g, as is represented by the equation $l_X(g \cdot x) = \mathrm{trg}(g)$. The formal definition of a groupoid action then reads as follows:

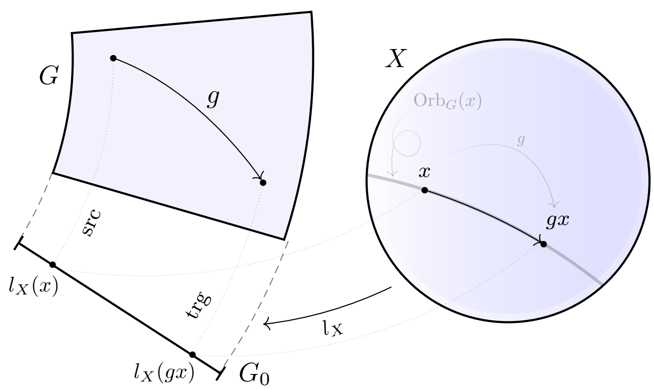
Definition 4.1. Fix a diffeological space X and a diffeological groupoid $G \rightrightarrows G_0$. A *smooth left G-action* on X along a smooth map $l_X : X \to G_0$ is a smooth map

$$G \times_{G_0}^{\mathrm{src}, l_X} X \longrightarrow X; \qquad (g, x) \longmapsto g \cdot x,$$

satisfying the following three conditions:

1. For any arrow $g \in G$ and point $x \in X$ satisfying $l_X(x) = \mathrm{src}(g)$, we have $l_X(g \cdot x) = \mathrm{trg}(g)$.
2. The identity arrows act trivially, meaning that $\mathrm{id}_{l_X(x)} \cdot x = x$ for every $x \in X$.
3. We have $h \cdot (g \cdot x) = (h \circ g) \cdot x$ whenever defined, i.e., when $l_X(x) = \mathrm{src}(g)$ and $\mathrm{src}(h) = \mathrm{trg}(g)$.

The smooth map $l_X : X \to G_0$ is called the *left moment map* (or sometimes the *anchor* by other authors). We denote such an action by $G \curvearrowright^{l_X} X$, or by $(G \rightrightarrows G_0) \curvearrowright^{l_X} X$ if we want to emphasise that it is a group*oid* action. To avoid spacious notation, we shall almost always use the more compact notation $(g, x) \mapsto gx$ instead of $g \cdot x$, except when it is absolutely necessary to distinguish between different types of actions. The following is a simplified illustration of how one can visualise a groupoid action:



A *smooth right G-action* on X along a smooth map $r_X : X \to G_0$ is a smooth map

$$X \times_{G_0}^{r_X,\mathrm{trg}} G \longrightarrow X; \qquad (x, g) \longmapsto x \cdot g,$$

satisfying $r_X(x \cdot g) = \mathrm{src}(g)$, $x \cdot \mathrm{id}_{r_X(x)} = x$, and $(x \cdot g) \cdot h = x \cdot (g \circ h)$ whenever defined. The smooth map $r_X : X \to G_0$ is called the *right moment map* (of the given action). We denote such an action by $X\ {}^{r_X}\!\curvearrowleft G$, and the same notational remarks from above apply.

Example 4.2. Let $G \rightrightarrows G_0$ be a diffeological groupoid. Then G acts smoothly on itself from the left along the moment map $l_G = \mathrm{trg} : G \to G_0$ by left translation: $g \cdot h := g \circ h$. Smoothness of this action is inherited from smoothness of composition in G.

Many notions from the actions of groups transfer to the actions of groupoids:

Definition 4.3. Consider a smooth left action $G \curvearrowright^{l_X} X$. Fix a point $x \in X$. The *orbit* of x is the set

$$\mathrm{Orb}_G(x) := \{gx : g \in \mathrm{src}^{-1}(l_X(x))\}.$$

The *orbit space*, or *quotient*, is the set

$$X/G := \{\mathrm{Orb}_G(x) : x \in X\},$$

with the unique diffeology so that the map $X \to X/G : x \mapsto \mathrm{Orb}_G(x)$ is a subduction, i.e., just the pushforward diffeology, or equivalently the quotient diffeology from Definition **2.64** with respect to the equivalence relation defined by being elements of the same orbit.

Example 4.4. Not only does a groupoid $G \rightrightarrows G_0$ act on its own *arrow* space by composition, it also acts on its own *base* space G_0. Here, the moment map is the identity map id_{G_0}, and the action $G \curvearrowright^{\mathrm{id}_{G_0}} G_0$ is defined by moving the source of an arrow to its target: $g \cdot \mathrm{src}(g) := \mathrm{trg}(g)$. The quotient of this groupoid action is exactly the orbit space G_0/G from Definition **3.26**.

Example 4.5. If $G \rightrightarrows G_0$ is a (diffeological) group, then we reobtain the definition of a (smooth) group action, since there is no choice in the moment map, and $G \times_{G_0}^{\mathrm{src},l_X} X$ just becomes $G \times X$.

Example 4.6. On the topic of group actions, we have seen that any group action defines an action groupoid (Example **3.28**). This construction can be generalised for groupoid actions $G \curvearrowright^{l_X} X$. The arrow space is defined as $G \ltimes X := G \times_{G_0}^{\mathrm{src},l_X} X$, on which we define the source and targets:

$$G \ltimes X \longrightarrow X; \qquad \mathrm{src}(g, x) := x, \quad \mathrm{trg}(g, x) := gx.$$

The composition $(h, y) \circ (g, x)$ can only be defined when $y = gx$, and in that case we set

$$(h, gx) \circ (g, x) := (h \circ g, x).$$

The resulting groupoid $G \ltimes X \rightrightarrows X$ is also called the *action groupoid*.

Example 4.7. Any diffeological space A can be seen as a groupoid in two ways: as a unit groupoid $A \rightrightarrows A$ or as a pair groupoid $A \times A \rightrightarrows A$. Any action $A \curvearrowright^{l_X} X$ of a unit groupoid has to be trivial, since it only contains identity arrows, and hence $X/A = X$.

The action of a pair groupoid $(A \times A) \curvearrowright^{l_X} X$ is defined on $(a, b) \cdot x$ only if $b = l_X(x)$. Such an action is therefore equivalent to a smooth map $\Phi : A \times X \to X$ defined by the action $(a, x) \mapsto (a, l_X(x)) \cdot x$, and which satisfies $l_X \circ \Phi = \mathrm{pr}_1$.

From now on, all diffeological groupoid actions will be assumed to be smooth. It will also be convenient to assume that whenever we encounter a term like gx, that it is actually well-defined, meaning $l_X(x) = \mathrm{src}(g)$. We do this for example in Definition **4.13** below. We may get away with that most of the time, but there will arise situations where it will be necessary to check that this assumption is valid. In that case we will point it out and prove that everything is allowed.

The category of G-actions will be important for us later. We want to define a category $\mathbf{Act}(G \rightrightarrows G_0)$ whose objects are smooth left G-actions $G \curvearrowright X$ on diffeological spaces. For that, we need to introduce an appropriate notion of morphism between such objects.

Definition 4.8. Consider two smooth groupoid actions $G \overset{l_X}{\curvearrowright} X$ and $G \overset{l_Y}{\curvearrowright} Y$. A smooth map $\varphi : X \to Y$ is called *G-equivariant* if it intertwines the moment maps: $l_X = l_Y \circ \varphi$, and commutes with the actions whenever defined: $\varphi(gx) = g\varphi(x)$. To be precise, this means that the following two diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ & {\scriptstyle l_X}\searrow \quad \swarrow{\scriptstyle l_Y} & \\ & G_0 & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} l_X^{-1}(\mathrm{src}(g)) & \xrightarrow{\varphi} & l_Y^{-1}(\mathrm{src}(g)) \\ {\scriptstyle g\cdot -}\downarrow & & \downarrow{\scriptstyle g\cdot -} \\ l_X^{-1}(\mathrm{trg}(g)) & \xrightarrow[\varphi]{} & l_Y^{-1}(\mathrm{trg}(g)). \end{array}$$

Proposition 4.9. *Suppose we have three smooth groupoid actions: $G \overset{l_X}{\curvearrowright} X$, $G \overset{l_Y}{\curvearrowright} Y$ and $G \overset{l_Z}{\curvearrowright} Z$, together with two G-equivariant smooth maps $\varphi : X \to Y$ and $\psi : Y \to Z$. Then $\psi \circ \varphi$ is also G-equivariant.*

Proof. This follows from a simple diagram chase. Let us sketch the result. By associativity we have $l_X = l_Y \circ \varphi = (_Z \circ \psi) \circ \varphi = l_Z \circ (\psi \circ \varphi)$. Moreover, if $g \in G$ and $x \in X$ are chosen appropriately, then we get $\psi \circ \varphi(gx) = \psi(g\varphi(x)) = g\psi(\varphi(x)) = g(\psi \circ \varphi(x))$. $\square$

Definition 4.10. Let $G \rightrightarrows G_0$ be a diffeological groupoid. The *(left) action category* $\mathbf{Act}(G \rightrightarrows G_0)$ of $G \rightrightarrows G_0$ consists of smooth groupoid actions $G \overset{l_X}{\curvearrowright} X$ as objects, and G-equivariant smooth maps as morphisms. Note that all identity maps are equivariant under any groupoid action. From Proposition **4.9** and associativity of **Diffeol** itself, we infer that $\mathbf{Act}(G \rightrightarrows G_0)$ is a category.

Example 4.11. A groupoid action of the form $(G \ltimes A) \overset{l_X}{\curvearrowright} X$ is just an action of the groupoid G on X, together with a G-equivariant map $l_X : X \to A$. The action on X is defined along the composition of $l_A \circ l_X : X \to A \to G_0$, and reads

$$G \overset{l_A \circ l_X}{\curvearrowright} X; \qquad g \cdot x := (g, l_X(x)) \cdot x.$$

On the right-hand side we have $(g, l_X(x)) \in G \ltimes A$, and the action is that of $G \ltimes A$ on X. The fact that l_X is G-equivariant follows from the first defining condition in Definition **4.1**.

The following is a construction that will become crucial to much of our later discussion. This is the main point at which submersions play a crucial rôle in the Lie groupoid category. There, it is necessary to have a (left) principal bibundle between Lie groupoids, but here we can introduce the construction even without knowing what a groupoid (bi)bundle is, and just work with smooth actions.

Construction 4.12 (Balanced tensor product)**.** Consider a smooth right action $X \overset{r_X}{\curvearrowleft} H$ and a smooth left action $H \overset{l_Y}{\curvearrowright} Y$. The *balanced tensor product* of X and Y over H is the diffeological space $X \otimes_H Y$ constructed as follows. On the fibred product $X \times_{H_0}^{r_X, l_Y} Y$ we define the following smooth right H-action along the moment map $R := r_X \circ \mathrm{pr}_1|_{X \times_{H_0} Y} = l_Y \circ \mathrm{pr}_2|_{X \times_{H_0} Y}$, called the *diagonal H-action*:

$$\left(X \times_{H_0}^{r_X, l_Y} Y\right) \times_{H_0}^{R,\mathrm{trg}} H \longrightarrow X \times_{H_0}^{r_X, l_Y} Y; \qquad ((x, y), h) \longmapsto (x \cdot h, h^{-1} \cdot y).$$

The balanced tensor product is defined as the quotient space of this smooth action:

$$X \otimes_H Y := \left(X \times_{H_0}^{r_X, l_Y} Y\right)/H.$$

We denote the orbit (recall Definition **4.3**) of a pair (x, y) of points in the balanced tensor product by $x \otimes y$. Hence, whenever we write something of the form $x \otimes y \in X \otimes_H Y$, we will assume that this is well defined, i.e. $r_X(x) = l_Y(y)$. The fact that we quotient out by this H-action gives the useful identity

$$xh \otimes h = x \otimes hy,$$

from which the name 'balanced tensor product' derives. In the literature, this is sometimes referred to as the *Hilsum-Skandalis tensor product.*

4.2 Diffeological groupoid bundles

This section introduces the theory of diffeological groupoid bundles. It draws heavy inspiration from the theory of diffeological principal *(group)* bundles as developed in [***Diffeology***, Chapter 8] and also studied in [**Wal09**; **CW17b**]. But mostly it is an extension of the theory of Lie groupoid bundles, as in e.g. [**MM05**; **Ler08**; **Blo08**; **dHo12**]. As far as we know, diffeological groupoid bundles have not appeared elsewhere. But our theory is not too novel, as the constructions are virtually copies of the ones for Lie groupoid bundles. Nevertheless, some notions are different. Mainly, in a Lie groupoid bundle

$$(G \rightrightarrows G_0) \curvearrowright X \xrightarrow{\pi} B,$$

the map π is always assumed to be a *submersion*. But the notion of submersiveness is not generally defined on diffeological spaces. An important note is that in the Lie case this assumption is absolutely necessary in all of the constructions, since they ensure the existence of certain fibred products: for instance, to say that the bundle is *principal*, the fibred product $X \times_B^{\pi;\pi} X$ needs to exist. In the diffeological category this assumption becomes redundant (at least for those purposes), and it appears that the constructions can be carried out more generally. Still, it seems reasonable to ask for π to have some nice properties, and the most obvious choice is to be made between subductiveness and *local* subductiveness. One may opt for the choice of local subductions, since these are the direct diffeological generalisation of submersions (Proposition **2.128**). However, it turns out that subductions behave sufficiently like submersions that they make the whole theory work, as will be demonstrated in the rest of this section. And in fact, there is a good reason not to choose local subductions. Given the theory of diffeological *pseudo*-bundles [**Per16**], which is now starting to look like a good contender for general diffeological bundle theory, there are many interesting examples of *"bundle-like"* objects that are not precisely local subductions[37] (and in particular are not diffeological fibre bundles). To be able to incorporate such objects into our theory of groupoid bundles, the appropriate choice seems to be to use subductions. We remark more on this choice in Section **4.4.3**. On the other hand, it seems that we cannot get away with bundles that are weaker than subductions, since (through Lemma **2.122**) subductiveness is used crucially to ensure smoothness of certain constructions that underlie our main theorems about Morita equivalence.

Our theory of diffeological groupoid bundles starts with the following simple definition:

Definition 4.13. Let $f : X \to Y$ be a smooth function between diffeological spaces, and suppose we have a smooth groupoid action $G \curvearrowright X$ on the domain of f. We say f is *G-invariant* if

$$f(gx) = f(x)$$

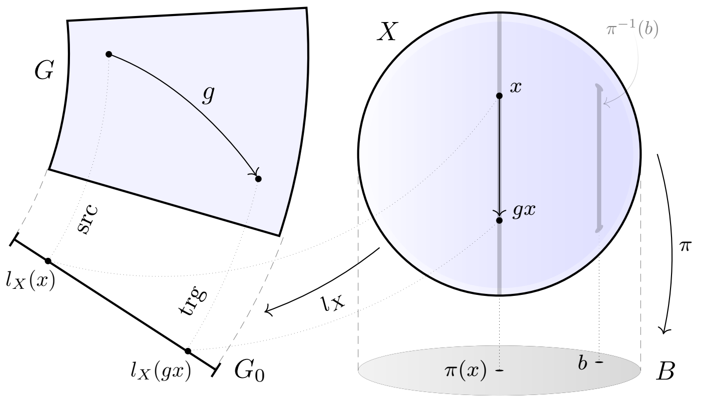
for all $g \in G$ and $x \in X$, whenever defined.

Definition 4.14. Fix a diffeological groupoid $G \rightrightarrows G_0$. A *left G-bundle* is a smooth left G-action $(G \rightrightarrows G_0) \curvearrowright^{l_X} X$ together with a G-invariant smooth map $\pi : X \to B$ between diffeological spaces. Diagrammatically such a bundle may be depicted as:

$$\begin{array}{ccc} G & \curvearrowright & X \\ \downarrow\downarrow & \swarrow_{l_X} & \downarrow \pi \\ G_0 & & B. \end{array}$$

[37] If anything, a tangent bundle should be an archetypal example of a "bundle-like" object. However, the internal tangent bundle is not generally a diffeological fibre bundle. This is most clearly demonstrated by various spaces that exhibit tangent bundles whose fibres have varying dimension, such as the two-dimensional coordinate axes Example **2.18**. This violates the fact that diffeological fibre bundles have diffeomorphic fibres at each point. By an example that Dan Christensen has communicated to the present author through email, it does not even generally hold that internal tangent bundles are local subductions, contradicting Proposition **3.45**. Pseudo-bundles seem to be a more natural concept if we want to incorporate examples such as the internal tangent bundle.

In-line, we also use the notation $(G \rightrightarrows G_0) \curvearrowright^{l_X} X \xrightarrow{\pi} B$, or $G \curvearrowright X \to B$. *Right G-bundles* are defined similarly, and written in the form $B \leftarrow X \curvearrowleft G$. The following illustrates a (left) G-bundle. Note that the G-orbits in X are contained in the π-fibres, since π is G-invariant.



Given the discussion above, the word 'bundle' is meant loosely. Indeed, we do not even assume that π is surjective, so that some π-fibres may be empty. Despite this we can still treat them like fibre bundles, and give the following definition:

Definition 4.15. Consider two diffeological groupoid bundles $G \curvearrowright^{l_X} X \xrightarrow{\pi_X} A$ and $G \curvearrowright^{l_Y} Y \xrightarrow{\pi_Y} B$. A *$G$-bundle morphism* consists of a pair of smooth maps (φ, a), where $\varphi : X \to Y$ is G-equivariant and the following square commutes:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ A & \xrightarrow[a]{} & B. \end{array}$$

For our purposes it will be useful to consider a restriction of Definition 4.15 to morphisms of bundles over the same base space, where the map between the bases is the identity. Whenever we encounter a G-bundle morphism of bundles over the same base, we assume the following definition:

Definition 4.16. Let $G \curvearrowright^{l_X} X \xrightarrow{\pi_X} B$ and $G \curvearrowright^{l_Y} Y \xrightarrow{\pi_Y} B$ be two left G-bundles over the same base space B. A *G-bundle morphism* is a smooth G-equivariant map $\varphi : X \to Y$ that preserves the fibres: $\pi_X = \pi_Y \circ \varphi$. We depict this situation as:

$$\begin{array}{ccc} G \curvearrowright^{l_X} X & & \\ {\scriptstyle \varphi}\downarrow & \searrow^{\pi_X} & \\ & & B. \\ & \nearrow_{\pi_Y} & \\ G \curvearrowright^{l_Y} Y & & \end{array}$$

Recalling the remarks made at the beginning of this section, in the Lie groupoid case it is important for subsequent constructions that the projection π of a bundle $G \curvearrowright X \xrightarrow{\pi} B$ is always a submersion. Since it seems in the diffeological category this is unnecessary, at least to replicate the constructions, we dissect the definition of a principal Lie groupoid bundle and separate it into its two underlying components. These are the groupoid action being *free and transitive on the π-fibres*, and the map π being a *submersion*. By separating these two conditions it becomes clearer where exactly they are necessary in our arguments, and it makes our proofs much neater by breaking them down into smaller chunks. So, instead of defining diffeological principal bundles in one go, we instead have the following two definitions. The first one speaks for itself as a substitute for the condition that the projection is a submersion:

Definition 4.17. A diffeological groupoid bundle $(G \rightrightarrows G_0) \curvearrowright X \xrightarrow{\pi} B$ is called *subductive* if the map $\pi : X \to B$ is a subduction. The definitions obviously extend to right bundles as well.

The second definition involves freeness and transitivity of the groupoid action:

Definition 4.18. A diffeological groupoid bundle $(G \rightrightarrows G_0) \curvearrowright^{l_X} X \xrightarrow{\pi} B$ is called *pre-principal* if the action map $G \times_{G_0}^{\mathrm{src}, l_X} X \to X \times_B^{\pi,\pi} X$ mapping $(g, x) \mapsto (gx, x)$ is a diffeomorphism.

Note that if the action map is a diffeomorphism, it follows that the action is free and transitive on the π-fibres. This is in fact guaranteed simply by bijectivity. Suppose that $gx = x$, so that the action mapping sends $(g, x) \mapsto (x, x)$. However, $(\mathrm{id}_{l_X(x)}, x)$ also gets sent to (x, x), so by injectivity $g = \mathrm{id}_{l_X(x)}$, proving the action is free. Surjectivity of the action map means that there exists a group element $g \in G$ for every pair of points $x, y \in X$ with $\pi(x) = \pi(y)$, such that $gx = y$, which is just transitivity on each π-fibre. We do not know if freeness and transitivity on the fibres are sufficient conditions for pre-principality. [**Blo08**, Page 13] and [**dHo12**, Section 3.6] claim that they are sufficient conditions for the action map to be a diffeomorphism in the case of a Lie groupoid bundle where the projection is a submersion. In their proofs they therefore use freeness and transitivity on the fibres as sufficient conditions, which simplifies them in the sense that they don't have to prove that the inverse of the action map is smooth. Since we do not know if freeness and fibre transitivity is sufficient in the diffeological setting, a small contribution of this thesis is therefore that we provide meticulous proofs that could also be applied in the classical theory. This makes some of them cumbersome (e.g. Propositions **4.54** and **4.55**), but fool-proof.

Definition 4.19. A diffeological groupoid bundle that is subductive and pre-principal is called *principal*.

This definition would serve as the most obvious analogue of principal Lie groupoid bundles, compare for instance to [**dHo12**, Section 3.6] and [**Blo08**, Definition 2.10]. The principality of a groupoid bundle $G \curvearrowright^{l_X} X \xrightarrow{\pi} B$ ensures that there is a special relation between the base B, the orbits $\mathrm{Orb}_G(x)$, and the π-fibres. First, note that, even for non-principal groupoid bundles, since π is G-invariant, the orbits $\mathrm{Orb}_G(x)$ are always contained in the π-fibre that x inhabits. The question is to what extent the orbit is embedded into the π-fibre. When the bundle is pre-principal, the action map is a diffeomorphism, and induces diffeomorphisms $\mathrm{Orb}_G(x) \cong X_{\pi(x)}$, where $X_{\pi(x)} := \pi^{-1}(\{\pi(x)\})$ is the π-fibre in X containing $x \in X$, and both sets are endowed with the subset diffeology. Subsequently, we get a diffeomorphism $X/G \cong X/\pi$ between quotient spaces. This generalises the concept of a principal (Lie) group bundle in the sense that every fibre is now not diffeomorphic to the group, but to the orbits. Of course, in the case of a free and transitive group action, the group may be identified with any given orbit. That the bundle $G \curvearrowright^{l_X} X \xrightarrow{\pi} B$ is subductive further gives a diffeomorphism:

$$B \cong X/\pi \cong X/G.$$

Example 4.20. We have seen in Example **4.2** that any groupoid $G \rightrightarrows G_0$ acts on its own arrow space along the target map by left translation. The source map is invariant under this action, so we get a diffeological groupoid bundle: $G \curvearrowright^{\mathrm{trg}} G \xrightarrow{\mathrm{src}} G_0$. By Proposition **3.17** it follows immediately that this bundle is subductive. It is actually principal, since the map $(g, h) \mapsto (g \circ h, h)$ is a diffeomorphism.

Example 4.21. The orbit is a principal bundle with the isotropy group, cf. [**CM18**, Proposition 2.4].

To make the connection to diffeological principal group bundles [***Diffeology***, Article 8.11], we make the following observation:

Proposition 4.22. *A diffeological groupoid bundle $G \curvearrowright^{l_X} X \xrightarrow{\pi} B$ is pre-principal if and only if G acts transitively on the π-fibres and the map $G \times^{\mathrm{src}, l_X}_{G_0} X \to X \times X : (g, x) \mapsto (gx, x)$ is an induction.*

Proof. This follows because surjective inductions are diffeomorphisms. □

We do not know to what extent the theory of principal diffeological *group* bundles [***Diffeology***] can be meaningfully extended to groupoid bundles. Principal group bundles are special cases of diffeological fibrations, whose fibre type is unique. That means that each of the fibres of a diffeological fibration have to be diffeomorphic. If the projection π of a principal groupoid bundle $G \curvearrowright X \xrightarrow{\pi} B$ is a diffeological fibration, then each orbit, and hence each isotropy group, has to be isomorphic. The proposition makes the following example clear:

Example 4.23. Let G be a diffeological group. A principal group*oid* bundle $(G \rightrightarrows \{*\}) \curvearrowright X \to B$ is the same as a *principal fibre bundle* in the sense of [***Diffeology***, Article 8.11]. In turn, this means that if G is a Lie group, then such a groupoid bundle is nothing but an ordinary principal G-bundle in the usual sense of the term.

Example 4.24. We here demonstrate that not every pre-principal bundle has to be principal. This is because the pre-principality poses no restrictions on the behaviour of the base space of the projection $\pi : X \to B$, but only on its fibres. Take $G \curvearrowright^{l_X} X \xrightarrow{\pi} B$ to be a principal G-bundle. Then π is a subduction, so in particular a surjection. We can destroy the subductiveness by enlarging the base space in a pathological manner. Define a new diffeological space $B_\infty := B \sqcup \{\infty\}$, and a smooth map $\pi_\infty : X \to B_\infty$ by $\pi_\infty(x) := \pi(x)$. Then π_∞ is clearly still G-invariant, but no subduction. We have $X \times_{B_\infty}^{\pi_\infty, \pi_\infty} X = X \times_B^{\pi,\pi} X$, so that the bundle $G \curvearrowright^{l_X} X \xrightarrow{\pi_\infty} B_\infty$ has the exact same action map as the original bundle, and is therefore pre-principal, but *not* principal.

This pathological counterexample merely aims to illustrate the independence of the Definitions **4.17** and **4.18**. We do this with the purpose of distinguishing pre-principal groupoid bundles from diffeological fibrations and principal group bundles as defined in [***Diffeology***, Article 8.11]. In particular, this shows that pre-principality is not a sufficient condition to ensure the projection π is a diffeological fibration.

4.2.1 The division map of a pre-principal bundle

In this section we construct the *division map* of any pre-principal groupoid bundle. This material is similar to [**Blo08**, Section 3.1]. For the duration of this section, fix a pre-principal diffeological groupoid bundle $(G \rightrightarrows G_0) \curvearrowright^{l_X} X \xrightarrow{\pi} B$. If we denote the smooth action by $\mu_G : G \times_{G_0}^{\mathrm{src}, l_X} X \to X$, then the action map can be written as

$$A_G := \left(\mu_G, \mathrm{pr}_2|_{G \times_{G_0} X}\right) : G \times_{G_0}^{\mathrm{src}, l_X} X \longrightarrow X \times_B^{\pi,\pi} X,$$

where the second component is just the projection map $\mathrm{pr}_2 : G \times X \to X$ restricted to the correct domain[38]. We will now describe a smooth map $\langle \cdot, \cdot \rangle_G : X \times_B^{\pi,\pi} X \to G$ such that

$$A_G^{-1} = \left(\langle \cdot, \cdot \rangle_G, \mathrm{pr}_2|_{X \times_B X}\right).$$

Definition 4.25. Let $G \curvearrowright^{l_X} X \xrightarrow{\pi} B$ be a pre-principal G-bundle. Then the *division map* associated to this bundle is the smooth map[39]

$$\langle \cdot, \cdot \rangle_G : X \times_B^{\pi,\pi} X \xrightarrow{A_G^{-1}} G \times_{G_0}^{\mathrm{src}, l_X} X \xrightarrow{\mathrm{pr}_1|_{G \times_{G_0} X}} G.$$

Note that, as we have remarked above, if the action map A_G is a bijection, then the action $G \curvearrowright X$ is free and transitive on the π-fibres. These two (set-theoretic) conditions already ensure that for any two points $x, y \in X$ in the same π-fibre there exists a unique arrow $g \in G$ such that $gy = x$. We denote this arrow by $\langle x, y \rangle_G$. The fact that the bundle is pre-principal ensures that the association $(x, y) \mapsto \langle x, y \rangle_G$ is actually smooth. In other words, $\langle x, y \rangle_G$ is the unique arrow in G that sends y to $\langle x, y \rangle_G y = x$. For this reason it is called the division map, because we may think of $\langle x, y \rangle_G$ as a fraction of x and y. We summarise some algebraic properties of the division map that will be used in our proofs throughout later sections.

Proposition 4.26. *Let $G \curvearrowright^{l_X} X \xrightarrow{\pi} B$ be a pre-principal G-bundle. Its division map $\langle \cdot, \cdot \rangle_G$ satisfies the following properties:*

1. *The source and targets are $\mathrm{src}(\langle x_1, x_2 \rangle_G) = l_X(x_2)$ and $\mathrm{trg}(\langle x_1, x_2 \rangle_G) = l_X(x_1)$.*
2. *The inverses are given by $\langle x_1, x_2 \rangle_G^{-1} = \langle x_2, x_1 \rangle_G$.*
3. *For every $x \in X$ we have $\langle x, x \rangle_G = \mathrm{id}_{l_X(x)}$.*
4. *Whenever well-defined, we have $\langle g x_1, x_2 \rangle_G = g \circ \langle x_1, x_2 \rangle_G$.*

[38]We are purposefully stringent in this regard, because whereas pr_2 is always a subduction, this is not true for the restriction. For this we need Lemma **2.124**. Keeping this in mind, in some places we may write $f : A \to Y$ instead of $f|_A : A \to Y$, to lighten up the notation. Lemma **2.124** will become especially important in this chapter.

[39]The notational resemblance to an inner-product is not accidental. The division map plays a very similar rôle to the inner product of a Hilbert C*-module. For more on this analogy, see [**Blo08**, Section 3].

The division map plays a crucial rôle in several of the upcoming constructions. To demonstrate its usefulness, we will use it in the next section to prove that G-bundle morphisms on principal G-bundles are invertible. They also respect the division maps on two pre-principal bundles:

Proposition 4.27. *Let $\varphi : X \to Y$ be a bundle morphism between two pre-principal G-bundles*

$$G \overset{l_X}{\curvearrowright} X \xrightarrow{\pi_X} B \qquad \text{and} \qquad G \overset{l_Y}{\curvearrowright} Y \xrightarrow{\pi_Y} B.$$

Denoting the division maps of these bundles respectively by $\langle \cdot, \cdot \rangle_G^X$ and $\langle \cdot, \cdot \rangle_G^Y$, we have for all $x_1, x_2 \in X$ in the same π_X-fibre that:

$$\langle x_1, x_2 \rangle_G^X = \langle \varphi(x_1), \varphi(x_2) \rangle_G^Y.$$

Proof. Note that $\langle \varphi(x_1), \varphi(x_2) \rangle_G^Y$ is the unique arrow in G such that $\langle \varphi(x_1), \varphi(x_2) \rangle_G^Y \varphi(x_2) = \varphi(x_1)$. However, by G-equivariance we get $\varphi(x_1) = \varphi\left(\langle x_1, x_2 \rangle_G^X x_2\right) = \langle x_1, x_2 \rangle_G^X \varphi(x_2)$, from which the claim immediately follows. □

4.2.2 Invertibility of G-bundle morphisms

We now prove a result that generalises the fact that morphisms between principal Lie group bundles are always diffeomorphisms. In our case we shall do the proof in two separate lemmas. Note that if a G-bundle map is a diffeomorphism, then a simple calculation shows that its inverse is automatically a G-bundle morphism, too.

Lemma 4.28. *Consider a G-bundle morphism $\varphi : X \to Y$ between a pre-principal bundle $G \overset{l_X}{\curvearrowright} X \xrightarrow{\pi_X} B$ and a bundle $G \overset{l_Y}{\curvearrowright} Y \xrightarrow{\pi_Y} B$ whose underlying action $G \curvearrowright Y$ is free. Then φ is injective.*

Proof. Since $G \curvearrowright X \to B$ is pre-principal, we get a smooth division map $\langle \cdot, \cdot \rangle_G^X : X \times_B^{\pi_X, \pi_X} X \to G$. To start the proof, suppose that we have two points $x_1, x_2 \in X$ such that $\varphi(x_1) = \varphi(x_2)$. Since φ preserves the fibres, we get that

$$\pi_X(x_1) = \pi_Y \circ \varphi(x_1) = \pi_Y \circ \varphi(x_2) = \pi_X(x_2).$$

Hence the pair (x_1, x_2) defines an element in $X \times_B X$, so we get an arrow $\langle x_1, x_2 \rangle_G^X \in G$, satisfying $\langle x_1, x_2 \rangle_G^X x_2 = x_1$. If we apply φ to this equation and use its G-equivariance, we get $\varphi(x_1) = \langle x_1, x_2 \rangle_G^X \varphi(x_2)$. However, by assumption, $\varphi(x_1) = \varphi(x_2)$ and the action $G \curvearrowright Y$ is free, so we must have that $\langle x_1, x_2 \rangle_G^X$ is the identity arrow at $l_Y \circ \varphi(x_2) = l_X(x_2)$. Hence we get the desired result:

$$x_1 = \langle x_1, x_2 \rangle_G^X x_2 = \mathrm{id}_{l_X(x_2)} x_2 = x_2,$$ □

Lemma 4.29. *Let $\varphi : X \to Y$ be a G-bundle morphism from a subductive bundle $G \overset{l_X}{\curvearrowright} X \xrightarrow{\pi_X} B$ to a pre-principal bundle $G \overset{l_Y}{\curvearrowright} Y \xrightarrow{\pi_Y} B$. Then φ is a subduction.*

Proof. Denote the smooth division map of $G \curvearrowright Y \to B$ by $\langle \cdot, \cdot \rangle_G^Y$. Then φ and $\langle \cdot, \cdot \rangle_G^Y$ combine into a smooth map

$$\psi : X \times_B^{\pi_X, \pi_Y} Y \longrightarrow X; \qquad (x, y) \longmapsto \langle y, \varphi(x) \rangle_G^Y x.$$

Note that this is well-defined because if $\pi_X(x) = \pi_Y(y)$, then $\pi_Y \circ \varphi(x) = \pi_Y(y)$ as well, and moreover $l_Y \circ \varphi(x) = l_X(x)$, showing that the action on the right hand side is allowed. The G-equivariance of φ then gives

$$\varphi \circ \psi = \mathrm{pr}_2|_{X \times_B Y}.$$

Since π_X is a subduction, so is $\mathrm{pr}_2|_{X \times_B Y}$ by Lemma **2.124**, and by Lemma **2.121** it follows φ is a subduction. □

Proposition 4.30. *Any bundle morphism from a principal groupoid bundle to a pre-principal groupoid bundle is a diffeomorphism. In particular, both must then be principal.*

Proof. By Lemma **4.29** any such bundle morphism is a subduction, and since in particular the underlying action of a pre-principal bundle is free, it must also be injective by Lemma **4.28**. The result follows by Proposition **2.123**. That the second bundle is principal too follows from the fact that a bundle map preserves the fibres, so the projection of the second bundle can be written as the composition of a diffeomorphism and a subduction. □

For principal Lie group(oid) bundles, every bundle morphism is a diffeomorphism, and hence the category of principal G-bundles with bundle morphisms is a groupoid (cf. [**Hus66**, Theorem 3.2]). By Proposition **4.30** we know that the same result holds for principal diffeological groupoid bundles, where the notion of subduction replaces that of submersion. In the Lie groupoid case, submersiveness of the projection of the bundle is necessary to even define the codomain of the action map as a smooth manifold. We may ask, then, if the category of *pre*-principal groupoid bundles and bundle morphisms form a groupoid. It appears that pre-principality is not enough, and merely for set-theoretic reasons: it can be proven that a bundle morphism $\varphi : X \to Y$ is surjective if and only if $\mathrm{im}(\pi_Y) \subseteq \mathrm{im}(\pi_X)$. This suggests that the category of *pre*-principal groupoid bundles is no longer a groupoid.

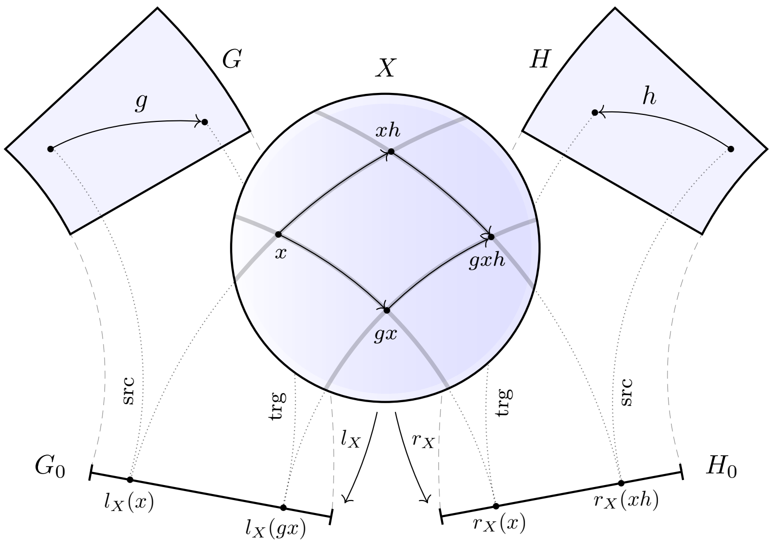
4.3 Diffeological groupoid bibundles

In this section we will introduce the notion of a type of generalised smooth morphism between diffeological groupoids. These will be called *diffeological bibundles*. These will become the morphisms in a new *bi*category **DiffeolBiBund** of diffeological groupoids. Off the bat, we point out some differences with the Lie groupoid case. There, the constructions can only be carried out for principal bundles, because it rests crucially on the existence of fibred products in the category of smooth manifolds. In the diffeological category, these assumptions become redundant, and we can obtain a bicategory of *all* bibundles.

Definition 4.31. Let $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ be two diffeological groupoids. A *(diffeological)* (G, H)-*bibundle* consists of a smooth left action $G \overset{l_X}{\curvearrowright} X$ and a smooth right action $X \overset{r_X}{\curvearrowleft} H$ such that the left moment map l_X is H-invariant and the right moment map r_X is G-invariant, and moreover that the actions commute: $(g \cdot x) \cdot h = g \cdot (x \cdot h)$, whenever defined. We draw:

$$\begin{array}{ccccc} G & \curvearrowright & X & \curvearrowleft & H \\ \downdownarrows & \swarrow_{l_X} & & {}_{r_X}\searrow & \downdownarrows \\ G_0 & & & & H_0, \end{array}$$

and we denote them by $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ in-line. Underlying each bibundle are two groupoid bundles: one left G-bundle $G \overset{l_X}{\curvearrowright} X \xrightarrow{r_X} H_0$, and one right H-bundle $G_0 \xleftarrow{l_X} X \overset{r_X}{\curvearrowleft} H$. We call these the *left*- and *right underlying bundles*, respectively. It is the properties of these underlying bundles that will determine the behaviour of the bibundle itself. We can picture this as a single space that is simultaneously a bundle over both bases of the groupoids. Since l_X is H-invariant and r_X is G-invariant, the different types of fibres intersect 'transversally':



Definition 4.32. Consider a diffeological bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$. We say this bibundle is *left pre-principal* if the left underlying bundle $G \overset{l_X}{\curvearrowright} X \xrightarrow{r_X} H_0$ is pre-principal. We say it is *right pre-principal* if the right underlying bundle $G_0 \xleftarrow{l_X} X \overset{r_X}{\curvearrowleft} H$ is pre-principal. We make similar definitions for subductiveness and principality. Notice that, in this convention, if a bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ is *left* subductive, then its *right* moment map r_X is a subduction (and vice versa)[40].

Definition 4.33. A diffeological bibundle is called:

1. *pre-biprincipal* if it is both left- and right pre-principal[41];
2. *bisubductive* if it is both left- and right subductive;
3. *biprincipal* if it is both left- and right principal.

Two diffeological groupoids G and H are called *Morita equivalent* if there exists a biprincipal bibundle between them, and in that case we write $G \simeq_{\mathrm{ME}} H$[42].

Compare this to the original definition [**MRW87**, Definition 2.1] of equivalence for locally compact Hausdorff groupoids. We will prove in Corollary **4.57** that Morita equivalence forms a genuine equivalence relation. Let us consider some elementary examples of Morita equivalences:

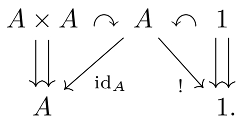
Example 4.34 (Identity bibundle)**.** Consider a diffeological groupoid $G \rightrightarrows G_0$. There exists a canonical (G, G)-bibundle structure on the space of arrows G, which is called the *identity bibundle*. This is just the left- and right multiplication of G on itself:

$$G \overset{\mathrm{trg}}{\curvearrowright} G; \qquad g_1 \cdot g_2 := g_1 \circ g_2,$$
$$G \overset{\mathrm{src}}{\curvearrowleft} G; \qquad g_2 \cdot g_1 := g_2 \circ g_1.$$

Note that the identity bibundle is always biprincipal, because the action map has a smooth inverse $(g_1, g_2) \mapsto (g_1 \circ g_2^{-1}, g_2)$. This proves that any diffeological groupoid is Morita equivalent to itself, through the identity bibundle $G \overset{\mathrm{trg}}{\curvearrowright} G \overset{\mathrm{src}}{\curvearrowleft} G$.

Example 4.35. Consider two diffeological spaces $A, B \in$ **Diffeol**, and their unit groupoids $A \rightrightarrows A$ and $B \rightrightarrows B$ (Example **3.19**). A left action $(A \rightrightarrows A) \overset{l_X}{\curvearrowright} X$ consists merely of a smooth function $l_X : X \to A$, since the groupoid contains only identity arrows, so the action has to be trivial. For a groupoid bundle $(A \rightrightarrows A_0) \overset{l_X}{\curvearrowright} X \xrightarrow{\pi} B$ to be pre-principal, its π-fibres should be the orbits of this trivial action, which are merely the points $\pi^{-1}(\{\pi(x)\}) = \{x\}$. That just means that π is injective, so if the bundle is moreover principal, then π is an injective subduction, which induces a diffeomorphism $X \cong B$. Hence, whenever $A \rightrightarrows A$ and $B \rightrightarrows B$ are Morita equivalent, it follows that there is a diffeomorphism $A \cong B$. If the spaces are diffeomorphic to begin with, it is easy to see that their unit groupoids are Morita equivalent (with overkill, this can be proved formally with a later result in Proposition **5.5**). We could write something of the form: $A \cong B$ if and only if $A \simeq_{\mathrm{ME}} B$[43].

Example 4.36. Consider again two diffeological spaces $A, B \in$ **Diffeol**, but now take their pair groupoids $A \times A \rightrightarrows A$ and $B \times B \rightrightarrows B$ (Example **3.20**). We will show that, irrespective of the underlying spaces, the pair groupoids are always Morita equivalent. We can prove this by borrowing the transitivity of Morita equivalence from Corollary **4.57** below. The third groupoid we introduce is the unit groupoid induced by the one-point space $1 := \{*\}$ with its unique diffeology (Example **2.13**). Let us denote the unique smooth map by $! : A \to 1$. We will define a biprincipal bibundle:



[40]Note: [**dHo12**, Section 4.6] defines this differently, where "[a] bundle is left (resp. right) principal if only the right (resp. left) underlying bundle is so." We suspect this may be a typo, since it apparently conflicts with their use of terminology in the proof of [**dHo12**, Theorem 4.6.3], cf. Lemma **5.11**. We stick to the terminology defined above, where *left* principality pertains to the *left* underlying bundle.

[41]The prefixes *bi-* and *pre-* commute: "bi-(pre-principal) = pre-(biprincipal)".

[42]The preferred symbol is '$\simeq$' instead of '$\sim$' to keep in line with the bicategorical terminology introduced in Appendix **A.2**.

[43]It could have been possible that somehow the flexibility of diffeology would completely trivialise the theory of Morita equivalence between Lie groupoids, by making all of them equivalent. Luckily, this example shows this not to be the case.

The left action is given by the canonical action of the pair groupoid on its base space (Example **4.4**), meaning that $(a_1, a_2) \cdot a_2 := a_1$. The unit groupoid $1 \rightrightarrows 1$, containing only identity arrows, can only act trivially on A. It is therefore clear that this does form a bibundle. Now, it is easy to see that the left underlying bundle $(A \times A) \curvearrowright^{\mathrm{id}_A} A \xrightarrow{!} 1$ is principal, since the unique !-fibre is just A. Similarly, the right underlying bundle $A \xleftarrow{\mathrm{id}_A} A \,{}^{!}\!\curvearrowleft 1$ is also trivially principal. This proves that, for all diffeological spaces $A, B \in$ **Diffeol**:

$$(A \times A \rightrightarrows A) \simeq_{\mathrm{ME}} (1 \rightrightarrows 1) \simeq_{\mathrm{ME}} (B \times B \rightrightarrows B).$$

Example 4.37. Morita equivalence of diffeological *groups* reduces to isomorphism. In other words, if G and H are diffeological groups, then

$$(G \rightrightarrows 1) \simeq_{\mathrm{ME}} (H \rightrightarrows 1) \qquad \text{if and only if} \qquad G \cong H.$$

Example 4.38. Of course, since submersions between manifolds are subductions with respect to the manifold diffeologies, we see that if two *Lie* groupoids $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are Morita equivalent in the *Lie* sense (e.g. [**CM18**, Definition 2.15]), then they are Morita equivalent in the *diffeological* sense. We remark on the converse question in Section **4.4.3**.

Example 4.39 (Gauge groupoids)**.** Let G be a diffeological group (or a Lie group), and consider a principal G-bundle $G \curvearrowright P \xrightarrow{\pi} B$ in the sense of [***Diffeology***, Article 8.11] (or in the usual sense). The action of G on P extends to the *diagonal action* $G \curvearrowright P \times P$, defined by $g(p, q) := (gp, gq)$. We denote the quotient space of this action by $P \times^G P := (P \times P)/G$, whose elements we shall denote by $[p, q] \in P \times^G P$. In this quotient space we have the identity $[gp, q] = [p, g^{-1}q]$. By transferring the groupoid structure of the pair groupoid $P \times P \rightrightarrows P$ to this quotient, we obtain the *gauge groupoid* $P \times^G P \rightrightarrows B$. The source and target maps are

$$\mathrm{src}, \mathrm{trg} : P \times^G P \longrightarrow B; \qquad \mathrm{src}([p, q]) := \pi(q), \quad \mathrm{trg}([p, q]) := \pi(p).$$

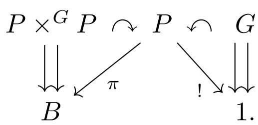
The composition of two pairs $[p, q]$ and $[r, s]$ can be defined whenever q and r are in the same fibre: $\pi(q) = \pi(r)$. In that case, since G acts principally on P, there is a unique group element $\langle q, r\rangle \in G$ such that $\langle q, r\rangle r = q$. This allows us to transfer the pair groupoid structure to this quotient:

$$[p, q] \circ [r, s] = [p, q] \circ [q, \langle q, r\rangle s] := [p, \langle q, r\rangle s].$$

It is straightforward to check that this is well-defined, and that we obtain a diffeological groupoid $P \times^G P \rightrightarrows B$. We further get a canonical action

$$P \times^G P \curvearrowright^{\pi} P; \qquad [p, q] \cdot r := \langle r, q\rangle p.$$

Similarly, seen as a groupoid $G \rightrightarrows 1$, the action $G \curvearrowright P$ can be reinterpreted as a right groupoid action $P \,{}^{!}\!\curvearrowleft G$, where $! : P \to 1$ is the unique map into the point. Since π is G-invariant, these two actions determine a diffeological bibundle:



That the right underlying bundle $B \xleftarrow{\pi} P \,{}^{!}\!\curvearrowleft G$ is principal is just to say that the original bundle $P \xrightarrow{\pi} B$ is principal. The left underlying bundle $P \times^G P \curvearrowright^{\pi} P \xrightarrow{!} 1$ is trivially subductive. Moreover, it is easy to check that the action map $([p, q], r) \mapsto (\langle r, q\rangle p, r)$ has smooth inverse $(p, q) \mapsto ([p, q], q)$, showing that it is also principal. Hence the above bibundle is biprincipal, and we get a Morita equivalence:

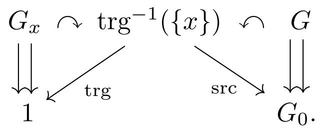
$$(P \times^G P \rightrightarrows B) \simeq_{\mathrm{ME}} (G \rightrightarrows 1).$$

The Example **4.39** can be generalised to the class of all fibration groupoids:

Example 4.40. Recall that a diffeological groupoid $G \rightrightarrows G_0$ is called *fibrating* if the characteristic map $(\mathrm{src}, \mathrm{trg}) : G \to G_0 \times G_0$ is a subduction (Definition **3.41**). In particular, this means that the groupoid is *transitive*: $G_0/G = 1$, meaning that every two points in the groupoid are connected by an arrow. We claim that, if G if fibrating, then for any point $x \in G_0$ there is a Morita equivalence to the isotropy group:

$$(G \rightrightarrows G_0) \simeq_{\mathrm{ME}} (G_x \rightrightarrows 1).$$

Proof. If we identify $1 = \{x\} \subseteq G_0$, which we may do, then the groupoid $G_x \rightrightarrows 1$ is nothing but a subgroupoid of $G \rightrightarrows G_0$. We then know that there are canonical left- and right composition actions on the space of arrows. However, the left composition of G_x is only defined for arrows $g \in G$ whose target is x. We can therefore restrict the right composition action $G\,{}^{\mathrm{src}}\!\curvearrowleft G$ to the subset $\mathrm{trg}^{-1}(\{x\}) \subseteq G$, equipped with its subset diffeology, to get a diffeological bibundle:



It is an easy exercise to show that this bibundle is always right principal, and left pre-principal. We are therefore left to show that the restricted source map $\mathrm{src} : \mathrm{trg}^{-1}(\{x\}) \to G_0$ is a subduction. For that, take a plot $\alpha : U_\alpha \to G_0$. Given the fixed point $x \in G_0$, we can also define the constant plot $\mathrm{const}_x : U_\alpha \to G_0$, which together with α gives a plot $(\alpha, \mathrm{const}_x) : U_\alpha \to G_0 \times G_0$. Since the characteristic map is a subduction, for every $t \in U_\alpha$ there exists a plot $\beta : V \to G$ defined on an open neighbourhood $t \in V \subseteq U_\alpha$ such that $\mathrm{src} \circ \beta = \alpha|_V$ and $\mathrm{trg} \circ \beta = \mathrm{const}_x|_V$. This shows that β takes values in $\mathrm{trg}^{-1}(\{x\})$, and proves that the plot α locally lifts along the restriction $\mathrm{src} : \mathrm{trg}^{-1}(\{x\}) \to G_0$. □

The above example motivates the important viewpoint in [***Diffeology***, Article 8.16], where every diffeological fibre bundle can equivalently be described by its *associated principal bundle*.

In the following we demonstrate that bibundles really do form a *generalised* type of morphism between groupoids. We do this by constructing a functorial assignment of groupoid morphisms to bibundles. That is, for every smooth functor $\varphi : (G \rightrightarrows G_0) \to (H \rightrightarrows H_0)$, we construct a diffeological bibundle $G \curvearrowright B(\varphi) \curvearrowleft H$, as follows. This construction also appears in [**Blo08**].

Construction 4.41 (Bundlisation)**.** Let $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ be two diffeological groupoids, with a smooth functor $\phi : G \to H$ between them. We define the *bundlisation of* ϕ as the following bibundle $G \overset{l_\phi}{\curvearrowright} B(\phi)\, {}^{r_\phi}\!\curvearrowleft H$. As a diffeological space, we set

$$B(\phi) := G_0 \times_{H_0}^{\phi, \mathrm{trg}} H.$$

On the right we can implement the obvious H-action by pre-composition. That means we have a right action $B(\phi) \curvearrowleft H$ along the moment map $r_\phi := \mathrm{src} \circ \mathrm{pr}_2|_{B(\phi)} : B(\phi) \to H_0$, given by

$$(x, h_1) \cdot h_2 := (x, h_1 \circ h_2),$$

whenever defined. The left G-action is defined along the first projection $l_\phi := \mathrm{pr}_1|_{B(\phi)} : B(\phi) \to G_0$, and is given by

$$g \cdot (x, h) := (\mathrm{trg}(g), \phi(g) \circ h).$$

It is easy to see this forms a diffeological bibundle.

In fact, the left moment map l_ϕ is already subduction by Lemma **2.124** and Proposition **3.17**. Moreover, given two pairs $(x_1, h_1), (x_2, h_2) \in B(\phi)$ such that $l_\phi(x_1, h_1) = l_\phi(x_2, h_2)$, i.e. $x_1 = x_2$, then we can define a division map by

$$\langle (x_1, h_1), (x_1, h_2) \rangle_H := h_1^{-1} \circ h_2.$$

It is easy to check that this makes the right underlying bundle $G_0 \overset{l_\phi}{\longleftarrow} B(\phi)\, {}^{r_\phi}\!\curvearrowleft H$ principal. Therefore: the bundlisation of *any* smooth functor is always a right principal bibundle.

Another construction that will be crucial is the following. Namely, for any bibundle we can obtain another bibundle going in the other direction. This is not to be confused with its inverse!

Construction 4.42 (Opposite bibundle)**.** Let $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ be a diffeological bibundle. The *opposite bibundle* $H \overset{l_{\overline{X}}}{\curvearrowright} \overline{X} \overset{r_{\overline{X}}}{\curvearrowleft} G$ is defined as follows. As a diffeological space, we set $\overline{X} := X$, and the moment maps are $l_{\overline{X}} := r_X$ and $r_{\overline{X}} := l_X$. The actions are defined by their opposites:

$$\begin{aligned} &H \overset{r_X}{\curvearrowright} \overline{X}; && h \cdot x := xh^{-1}, \\ &\overline{X} \overset{l_X}{\curvearrowleft} G; && x \cdot g := g^{-1}x. \end{aligned}$$

Note that the opposite of the opposite bibundle returns the original bibundle.

To end this section, we collect a lemma for further use:

Lemma 4.43. *Consider a left pre-principal bibundle* $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$*, and also the opposite G-action* $\overline{X} \overset{l_X}{\curvearrowleft} G$*. Then, whenever defined, we have:*

$$\langle x_1, x_2 g\rangle_G = \langle x_1, x_2\rangle_G \circ g.$$

Proof. This follows directly from Proposition **4.26** and the definition of the opposite action:

$$\langle x_1, x_2 g\rangle_G = \langle x_1, g^{-1}x_2\rangle_G = \left(g^{-1} \circ \langle x_2, x_1\rangle_G\right)^{-1} = \langle x_1, x_2\rangle_G \circ g. \qquad \square$$

4.3.1 Invariance of orbit spaces

As we have mentioned in Chapter **I**, it is a well known result that if two Lie groupoids $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are Morita equivalent (in the Lie groupoid sense), then there is a *homeo*morphism between their orbit spaces G_0/G and H_0/H [**CM18**, Lemma 2.19]. In general, this is the best we can get, since the orbit space may very well be singular, and may therefore have no canonical smooth structure. For diffeological groupoids this is different, since we can just endow the orbit space with the quotient diffeology. In this section we will generalise the claim for Lie groupoids to diffeology, and prove that we get a genuine *diffeo*morphism. The construction of the underlying function is inspired by the proof in [**CM18**, Lemma 2.19].

Theorem 4.44. *If $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are two Morita equivalent diffeological groupoids, then there is a diffeomorphism $G_0/G \cong H_0/H$ between their orbit spaces.*

Proof. Let $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ be the bibundle instantiating the Morita equivalence. Our first task will be to construct a function $\Phi : G_0/G \to H_0/H$ between the orbit spaces. The idea is to lift a point $a \in G_0$ of the base of the groupoid to its l_X-fibre, which by right principality is just an H-orbit in X, and then to project this orbit down to the other base H_0 along the right moment map r_X. The fact that the bundle is biprincipal ensures that this can be done in a consistent fashion.

We are dealing with *four* actions here, so we need to slightly modify our notation to avoid confusion. If $a \in G_0$ is an object in the groupoid G, we shall denote its orbit by $\mathrm{Orb}_{G_0}(a)$, which, as usual, is just the set of all points $a' \in G_0$ such that there exists an arrow $g : a \to a'$ in G. Similarly, for $b \in H_0$ we write $\mathrm{Orb}_{H_0}(b)$. On the other hand, we have two actions on X, for whose orbits we use the standard notations $\mathrm{Orb}_G(x)$ and $\mathrm{Orb}_H(x)$, where $x \in X$.

Now, start with a point $a \in G_0$, and consider its fibre $l_X^{-1}(a)$ in X. Since the bibundle is right subductive, the map l_X is surjective, so this fibre is non-empty and we can find a point $x_a \in l_X^{-1}(a)$. We claim that the expression $\mathrm{Orb}_{H_0} \circ r_X(x_a)$ is independent on the choice of the point x_a in the fibre. For that, take another point $x_a' \in l_X^{-1}(a)$. This gives the equation $l_X(x_a) = l_X(x_a')$, and since bibundle is right pre-principal, we get a unique arrow $h \in H$ such that $x_a' = x_a h$. From the definition of a right groupoid action, this in turn gives the equations $r_X(x_a') = \mathrm{src}(h)$ and $r_X(x_a) = \mathrm{trg}(h)$, which proves the claim. To summarise, whenever $x_a, x_a' \in l_X^{-1}(a)$ are two points in the same l_X-fibre, then we have:

$$\mathrm{Orb}_{H_0} \circ r_X(x_a) = \mathrm{Orb}_{H_0} \circ r_X(x_a'). \qquad (\clubsuit)$$

Next we want to show that neither is this expression dependent on the point $a \in G_0$, but rather on its orbit $\mathrm{Orb}_{G_0}(a)$. For this, take another point $b \in \mathrm{Orb}_{G_0}(a)$, so there exists some arrow $g : a \to b$ in G. Pick then $x \in l_X^{-1}(a)$ and $y \in l_X^{-1}(b)$. This means that $\mathrm{src}(g) = l_X(x)$ and $\mathrm{trg}(g) = l_X(y)$, which means that if we let g act on the point x we get a point $gx \in l_X^{-1}(b)$, in the same l_X-fibre as y. Then using equation ($\clubsuit$) applied to gx and y, and the G-invariance of the right moment map r_X, we immediately get:

$$\mathrm{Orb}_{H_0} \circ r_X(x) = \mathrm{Orb}_{H_0} \circ r_X(gx) = \mathrm{Orb}_{H_0} \circ r_X(y).$$

Using this, we can now conclude that there is a well-defined function

$$\Phi : G_0/G \longrightarrow H_0/H; \qquad \mathrm{Orb}_{G_0}(a) \longmapsto \mathrm{Orb}_{H_0} \circ r_X(x_a),$$

that is neither dependent on the point a in the orbit $\mathrm{Orb}_{G_0}(a)$, nor on the choice of the point $x_a \in l_X^{-1}(a)$ in the fibre. Note that this function exists by virtue of right subductivity (and the Axiom of Choice), which ensures that the left moment map l_X is a surjection (and for each a there exists an x_a).

Either by replacing $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ by its opposite bibundle, or by switching the words 'left' and 'right', the above argument analogously gives a function going the other way:

$$\Psi : H_0/H \longrightarrow G_0/G; \qquad \mathrm{Orb}_{H_0}(b) \longmapsto \mathrm{Orb}_{G_0} \circ l_X(y_b),$$

where now $y_b \in r_X^{-1}(b)$ is some point in the fibre of the right moment map r_X. We claim that Φ and Ψ are mutual inverses. To see this, pick a point $a \in G_0$, a point $x_a \in l_X^{-1}(a)$, a point $y_{r_X(x_a)} \in r_X^{-1}(r_X(x_a))$. Then we can write

$$\Psi \circ \Phi\left(\mathrm{Orb}_{G_0}(a)\right) = \Psi\left(\mathrm{Orb}_{H_0}(r_X(x_a))\right) = \mathrm{Orb}_{G_0}\left(l_X(y_{r_X(x_a)})\right).$$

We also have, by choice, the equation $r_X(x_a) = r_X(y_{r_X(x_a)})$, so by left pre-principality there exists an arrow $g \in G$ such that $gx_a = y_{r_X(x_a)}$. By definition of a left groupoid action, this then further gives

$$\mathrm{src}(g) = l_X(x_a) = a \qquad \text{and} \qquad \mathrm{trg}(g) = l_X(y_{r_X(x_a)}).$$

This proves that the right-hand side of the previous equation is equal to

$$\mathrm{Orb}_{G_0}\left(l_X(y_{r_X(x_a)})\right) = \mathrm{Orb}_{G_0}(a),$$

which gives $\Psi \circ \Phi = \mathrm{id}_{G_0/G}$. Through a similar argument, using right pre-principality, we obtain that $\Phi \circ \Psi = \mathrm{id}_{H_0/H}$.

To finish the proof, it suffices to prove that both Φ and Ψ are smooth. Again, due to the symmetry of the situation, and since the bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ is biprincipal, we shall only prove that Φ is smooth. The proof for Ψ will follow analogously. Since Orb_{G_0} is a subduction, to prove that Φ is smooth it suffices by Lemma **2.122** to prove that $\Phi \circ \mathrm{Orb}_{G_0}$ is smooth. Since the left moment map l_X is a surjection, using the Axiom of Choice we pick a section $\sigma : G_0 \to X$, which replaces our earlier notation of $\sigma(a) =: x_a$. From the way Φ is defined, we see that we get a commutative diagram:

$$\begin{array}{ccccc} G_0 & \xrightarrow{\sigma} & X & \xrightarrow{r_X} & H_0 \\ {\scriptstyle \mathrm{Orb}_{G_0}}\downarrow & & & & \downarrow{\scriptstyle \mathrm{Orb}_{H_0}} \\ G_0/G & & \xrightarrow[\Phi]{\qquad\qquad} & & H_0/H. \end{array}$$

We are therefore to show that $\mathrm{Orb}_{H_0} \circ r_X \circ \sigma$ is smooth. For this, pick a plot $\alpha : U_\alpha \to G_0$ of the base space. By right subductivity, the left moment map l_X is a subduction, so locally $\alpha|_V = l_X \circ \beta$, where β is some plot of X. Now, note that, for all $t \in V$, both the points $\beta(t)$ and $\sigma \circ l_X \circ \beta(t)$ are elements of the fibre $l_X^{-1}(l_X \circ \beta(t))$. Therefore, by equation ($\clubsuit$) we get:

$$\mathrm{Orb}_{H_0} \circ r_X \circ \sigma \circ \alpha|_V = \mathrm{Orb}_{H_0} \circ r_X \circ \sigma \circ l_X \circ \beta = \mathrm{Orb}_{H_0} \circ r_X \circ \beta.$$

The right-hand side of this equation is clearly smooth (and no longer dependent on the choice of section σ). By the Axiom of Locality for G_0, it follows that $\mathrm{Orb}_{H_0} \circ r_X \circ \sigma \circ \alpha$ is globally smooth, and since the plot α was arbitrary, this proves that $\Phi \circ \mathrm{Orb}_{G_0}$ is smooth. Hence, Φ is smooth. After an analogous argument that shows Ψ is smooth, the desired diffeomorphism between the orbit spaces follows. $\square$

We will give an alternative proof of this result (Theorem **5.18**) in Section **5.2** of the next chapter. The corresponding theorem for Lie groupoids justifies the viewpoint that Morita equivalence classes of Lie groupoids describe a smooth geometric model for possibly singular quotient spaces. For example, in [**Moe02**], this philosophy is used to study *orbifolds* as special types of Lie groupoids. Our Theorem **4.44** shows that this viewpoint extends to diffeology. More generally, a Morita equivalence class of a Lie groupoid serves as a model for a *differentiable stack* [**BX11**]. We could then define a *"diffeological stack"* to be a Morita equivalence class of diffeological groupoids, cf. [**WW19**]. The question is how much more information a diffeological stack is able to capture than a bare diffeological space. The advantage of Lie groupoids over ordinary manifolds is that they provide models for quotients that would not otherwise be manifolds. However, the quotient of a diffeological space always gets the quotient diffeology, so this advantage disappears. On the other hand, it is clear that by projecting from a diffeological groupoid to its quotient space we lose some of its structure, namely the information of the isotropy groups. In this way it is clear that a groupoid contains more information than its quotient space. The relation between diffeology, Lie groupoids and stacks has only recently begun to be explored, but (besides [**WW19**]) we are not aware of any publications on this topic.

4.3.2 Induced actions

A bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ contains enough structure to transfer a groupoid action $H \curvearrowright Y$ to a groupoid action $G \curvearrowright X \otimes_H Y$. This is called the *induced action*, and uses the balanced tensor product (Construction **4.12**):

Construction 4.45 (Induced actions)**.** Consider a diffeological bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ and a smooth action $H \overset{l_Y}{\curvearrowright} Y$. We will construct a smooth left G-action on the balanced tensor product $X \otimes_H Y$. As the left moment map, take

$$L_X : X \otimes_H Y \longrightarrow G_0; \qquad x \otimes y \longmapsto l_X(x).$$

This is well defined because l_X is H-invariant, and smooth by Lemma **2.122**. For an arrow $g \in G$ with $\mathrm{src}(g) = L_X(x \otimes y) = l_X(x)$, define the action as:

$$G \overset{L_X}{\curvearrowright} X \otimes_H Y; \qquad g \cdot (x \otimes y) := (gx) \otimes y.$$

Note that the right hand side is well defined because r_X is G-invariant, so $r_X(gx) = r_X(x) = l_Y(y)$. Since there can be no confusion, we will drop all parentheses and write $gx \otimes y$ instead. That the action is smooth follows because $(g, (x, y)) \mapsto (gx, y)$ is smooth (on the appropriate domains) and by another application of Lemma **2.122**. Hence we obtain the *induced action* $G \overset{L_X}{\curvearrowright} X \otimes_H Y$.

Now suppose that we are given a smooth H-equivariant map $\varphi : Y_1 \to Y_2$ between two smooth actions $H \overset{l_1}{\curvearrowright} Y_1$ and $H \overset{l_2}{\curvearrowright} Y_2$. We define a map

$$\mathrm{id}_X \otimes \varphi : X \otimes_H Y_1 \longrightarrow X \otimes_H Y_2; \qquad x \otimes y \longmapsto x \otimes \varphi(y).$$

The underlying map $X \times_{H_0} Y_1 \to X \times_{H_0} Y_2 : (x, y) \mapsto (x, \varphi(y))$ is clearly smooth. Then by composition of the projection onto $X \otimes_H Y_2$ and Lemma **2.122**, we find $\mathrm{id}_X \otimes \varphi$ is smooth. Moreover, it is G-equivariant:

$$\mathrm{id}_X \otimes \varphi(gx \otimes y) = gx \otimes \varphi(y) = g\left(\mathrm{id}_X \otimes \varphi(x \otimes y)\right).$$

Definition 4.46. Let $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ be a diffeological bibundle. It defines the *induced action functor*:

$$\begin{aligned} X \otimes_H - : \mathbf{Act}(H \rightrightarrows H_0) &\longrightarrow \mathbf{Act}(G \rightrightarrows G_0), \\ \left(H \overset{l_Y}{\curvearrowright} Y\right) &\longmapsto \left(G \overset{L_X}{\curvearrowright} X \otimes_H Y\right), \\ \varphi &\longmapsto \mathrm{id}_X \otimes \ \varphi. \end{aligned}$$

sending each smooth left H-action $\left(H \overset{l_Y}{\curvearrowright} Y\right) \mapsto \left(G \overset{L_X}{\curvearrowright} X \otimes_H Y\right)$ and each H-invariant map $\varphi \mapsto \mathrm{id}_X \otimes \varphi$.

4.3.3 The bicategory of diffeological groupoids and -bibundles

Definition 4.47. Let $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ and $G \overset{l_Y}{\curvearrowright} Y \overset{r_Y}{\curvearrowleft} H$ be two bibundles between the same two diffeological groupoids. A smooth map $\varphi : X \to Y$ is called a *bibundle morphism* if it is a bundle morphism between both underlying bundles. We also say that φ is *biequivariant*. Concretely, this means that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{r_X} & H_0 \\ {\scriptstyle l_X}\downarrow & \overset{\varphi}{\searrow} & \uparrow{\scriptstyle r_Y} \\ G_0 & \xleftarrow[l_Y]{} & Y, \end{array} \qquad \text{that is:} \qquad \begin{aligned} l_X &= l_Y \circ \varphi, \\ r_X &= r_Y \circ \varphi, \end{aligned}$$

and that φ is equivariant with respect to both actions. It is clear from Proposition 4.9 that the composition of biequivariant maps is again biequivariant. Since the inverse of a biequivariant map is also biequivariant, the isomorphisms of bibundles are exactly the diffeomorphic biequivariant maps.

We will now use the Construction 4.12 of the balanced tensor product to define a notion of composition for bibundles. Again, note that we perform this construction more generally for any diffeological bibundle, while in the classical case it only works for bibundles that are left- or right principal.

Construction 4.48 (Bibundle composition)**.** Consider two diffeological bibundles, $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ and $H \overset{l_Y}{\curvearrowright} Y \overset{r_Y}{\curvearrowleft} K$. We shall define on $X \otimes_H Y$ a (G, K)-bibundle structure using the induced actions from Section 4.3.2. On the left we take the induced G-action along $L_X : X \otimes_H Y \to G_0$, which we recall maps $x \otimes y \mapsto l_X(x)$, defined by

$$G \overset{L_X}{\curvearrowright} X \otimes_H Y; \qquad g(x \otimes y) := (gx) \otimes y.$$

This action is well-defined because the G- and H-actions commute. Similarly, we get an induced K-action on the right along $R_Y : X \otimes_H Y \to K_0$, which maps $x \otimes y \mapsto r_Y(y)$, given by

$$X \otimes_H Y \overset{R_Y}{\curvearrowleft} K; \qquad (x \otimes y)k := x \otimes (yk).$$

It is easy to see that these two actions form a bibundle $G \overset{L_X}{\curvearrowright} X \otimes_H Y \overset{R_Y}{\curvearrowleft} K$, which we also call the *balanced tensor product*. Note that the moment maps are smooth by Lemma 2.122.

We have seen that a diffeological groupoid $G \rightrightarrows G_0$ gives rise to a (G, G)-bibundle by acting on itself by left- and right translation (Example 4.34). These behave like identity arrows respect to the balanced tensor product:

Proposition 4.49. *Let $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ be a diffeological bibundle. Then there are biequivariant diffeomorphisms*

$$\begin{array}{ccc} G \overset{L_G}{\curvearrowright} G \otimes_G X \overset{R_X}{\curvearrowleft} H & & G \overset{L_X}{\curvearrowright} X \otimes_H H \overset{R_H}{\curvearrowleft} H \\ \Big\Downarrow {\scriptstyle \varphi} & \textit{and} & \Big\Downarrow \\ G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H. & & G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H. \end{array}$$

Proof. The idea of the proof is briefly sketched on [Blo08, Page 8]. The map $\varphi : G \otimes_G X \to X$ is defined by the action: $g \otimes x \mapsto gx$. This map is clearly well defined, and by an easy application of Lemma 2.122 also smooth. Further note that φ intertwines the left moment maps:

$$l_X \circ \varphi(g \otimes x) = l_X(gx) = \operatorname{trg}(g) = L_G(g \otimes x),$$

and similarly we find it intertwines the right moment maps. Associativity of the G-action and the fact that it commutes with the H-action directly ensure that φ is biequivariant. Moreover, we claim that the smooth map $\psi : X \to G \otimes_G X$ defined by $x \mapsto \mathrm{id}_{l_X(x)} \otimes x$ is the inverse of φ. It follows easily that $\varphi \circ \psi = \mathrm{id}_X$, and the other side follows from the defining property of the balanced tensor product:

$$\psi \circ \varphi(g \otimes x) = \psi(gx) = \mathrm{id}_{l_X(gx)} \otimes gx = (\mathrm{id}_{\operatorname{trg}(g)} \circ g) \otimes x = g \otimes x.$$

It follows from an analogous argument that the identity bibundle of H acts like a weak right inverse. □

Proposition 4.50. *Let $G\curvearrowright^{l_X} X\ ^{r_X}\!\curvearrowleft H$, $H\curvearrowright^{l_Y} Y\ ^{r_Y}\!\curvearrowleft H'$, and $H'\curvearrowright^{l_Z} Z\ ^{r_Z}\!\curvearrowleft K$ be diffeological bibundles. Then there exists a biequivariant diffeomorphism*

$$\begin{array}{c} G\curvearrowright^{L_{X\otimes_H Y}} (X\otimes_H Y)\otimes_{H'} Z\ ^{R_Z}\!\curvearrowleft K \\ \Big\Downarrow A \\ G\curvearrowright^{L_X} X\otimes_H (Y\otimes_{H'} Z)\ ^{R_{Y\otimes_{H'} Z}}\!\curvearrowleft K, \end{array} \qquad A : (x\otimes y)\otimes z \longmapsto x\otimes (y\otimes z).$$

Proof. That the map A is smooth follows by Lemma **2.122**, because the corresponding underlying map $((x,y),z)\mapsto (x,(y,z))$ is a diffeomorphism. The inverse of this diffeomorphism on the underlying fibred product induces exactly the smooth inverse of A, showing that A is a diffeomorphism. Furthermore, it is easy to check that A is biequivariant. □

Combining Propositions **4.49** and **4.50** gives that the balanced tensor product of bibundles behaves like the composition in a bicategory. We recall in Appendix **A.2** the notion of a bicategory. Given what we know about diffeological bibundles and their analogues in the Lie category, the proof of this theorem is the same as the one in [**Blo08**].

Theorem 4.51. *There is a bicategory* **DiffeolBiBund** *consisting of diffeological groupoids as objects, diffeological bibundles as morphisms with balanced tensor product as composition, and biequivariant smooth maps as 2-morphisms.*

Given that there is a bicategory of diffeological groupoids, we get an intrinsic notion of equivalence for free. This is the equivalence induced by the weak isomorphisms in **DiffeolBiBund**. We say that two diffeological groupoids are *equivalent* if and only if there exists a weakly invertible bibundle between them. Let us spell out what that means. In this bicategory, a weak inverse for a diffeological bibundle $G\curvearrowright^{l_X} X\ ^{r_X}\!\curvearrowleft H$ is yet another diffeological bibundle $H\curvearrowright^{l_Y} Y\ ^{r_Y}\!\curvearrowleft G$, such that there exists two biequivariant diffeomorphisms:

$$\begin{array}{c} G\curvearrowright^{L_X} X\otimes_H Y\ ^{R_Y}\!\curvearrowleft G \\ \Big\Downarrow \\ G\curvearrowright^{\mathrm{trg}} G\ ^{\mathrm{src}}\!\curvearrowleft G \end{array} \quad \text{and} \quad \begin{array}{c} H\curvearrowright^{L_Y} Y\otimes_G X\ ^{R_X}\!\curvearrowleft H \\ \Big\Downarrow \\ H\curvearrowright^{\mathrm{trg}} H\ ^{\mathrm{src}}\!\curvearrowleft H. \end{array}$$

In other words, the compositions $X\otimes_H Y$ and $Y\otimes_G X$ are 2-isomorphic to the respective identity bibundles: they are each other's weak inverses. However, nothing is mentioned here about principality of the underlying bundles, and it is not entirely obvious that this implies biprincipality of $G\curvearrowright^{l_X} X\ ^{r_X}\!\curvearrowleft H$. The point of the rest of this chapter is to prove that these two notions of equivalence do indeed coincide. This will show that Morita equivalence can be defined either externally as equivalence in a bicategory, or internally as through the existence of biprincipal bibundles. To prepare the proof of this theorem, in the section below we study how subductiveness and pre-principality of bibundles is preserved under balanced tensor products and biequivariant diffeomorphisms.

For later reference, we describe some algebraic properties of the division map in relation to bibundles:

Lemma 4.52. *Let $G\curvearrowright^{l_X} X\ ^{r_X}\!\curvearrowleft H$ be a left pre-principal bibundle, and denote its division map by $\langle\cdot,\cdot\rangle_G$. Then, in addition to the properties in Proposition 4.26 with respect to the G-action, we have:*

$$\langle x_1, x_2h\rangle_G = \langle x_1h^{-1}, x_2\rangle_G, \qquad \textit{or equivalently:} \qquad \langle x_1h, x_2h\rangle_G = \langle x_1, x_2\rangle_G,$$

whenever defined.

Proof. The arrow $\langle x_1h, x_2h\rangle_G \in G$ is the unique one so that $\langle x_1h, x_2h\rangle_G(x_2h) = x_1h$. Now, since the actions commute, we can multiply both sides of this equation from the right by h^{-1}, which gives $\langle x_1h, x_2h\rangle_X x_2 = x_1$, and this immediately gives our result. □

4.3.4 Properties of bibundles under composition and isomorphism

Looking at Definitions **4.17** and **4.18**, we get three different subcollections of diffeological bibundles: the (left) subductive, (left) pre-principal, and (left) principal ones. We denote these by $\mathbf{DiffeolBiBund}_{\mathrm{LS}}$, $\mathbf{DiffeolBiBund}_{\mathrm{LpP}}$, and $\mathbf{DiffeolBiBund}_{\mathrm{LP}}$, respectively. The latter one is the direct analogue of the bicategory of Lie groupoids and left principal bibundles: $\mathbf{LieGrpd}_{\mathrm{LP}}$. In all, they collect in the following hierarchy of generality:

$$\begin{array}{ccccc} \mathbf{LieGrpd}_{\mathrm{LP}} & \hookrightarrow & \mathbf{DiffeolBiBund}_{\mathrm{LP}} & \hookrightarrow & \mathbf{DiffeolBiBund}_{\mathrm{LpP}} \\ & & \big\downarrow & & \big\downarrow \\ & & \mathbf{DiffeolBiBund}_{\mathrm{LS}} & \hookrightarrow & \mathbf{DiffeolBiBund}. \end{array}$$

In this section we prove four results that will show that each of these subcollections are closed under the balanced tensor product, as well as under biequivariant diffeomorphism. This will be crucial in characterising the weakly invertible principal bibundles.

Proposition 4.53. *The balanced tensor product of two left subductive bibundles is again left subductive.*

Proof. Consider the balanced tensor product $G \curvearrowright^{L_X} X \otimes_H Y \,{}^{R_Y}\!\curvearrowleft K$ of two left subductive bibundles $G \curvearrowright^{l_X} X \,{}^{r_X}\!\curvearrowleft H$ and $H \curvearrowright^{l_Y} Y \,{}^{r_Y}\!\curvearrowleft K$. We need to show that the smooth map $R_Y : X \otimes_H Y \to K_0$ is a subduction. But, note that it fits into the following commutative diagram:

$$\begin{array}{ccc} X \times_{H_0}^{r_X, l_Y} Y & \xrightarrow{\ \pi\ } & X \otimes_H Y \\ {\scriptstyle \mathrm{pr}_2|_{X \times_{H_0} Y}} \big\downarrow & & \big\downarrow {\scriptstyle R_Y} \\ Y & \xrightarrow[\ r_Y\ ]{} & K_0. \end{array}$$

Here π is the canonical projection. The restricted projection $\mathrm{pr}_2|_{X \times_{H_0} Y}$ is a subduction by Lemma **2.124**, noting that r_X is a subduction. Moreover, r_Y is a subduction, so the bottom part of the diagram is a subduction. It follows by Lemma **2.122** that R_Y is a subduction. □

Note that, even though R_Y implicitly only depends on r_Y, as our notation would suggest, it is crucial that we also assume r_X to be a subduction. Otherwise we cannot guarantee that the restricted projection in the above diagram is a subduction, and our argument fails.

Proposition 4.54. *The balanced tensor product of two left pre-principal bibundles is left pre-principal.*

Proof. To start the proof, take two left pre-principal bibundles, with our usual notation: $G \curvearrowright^{l_X} X \,{}^{r_X}\!\curvearrowleft H$ and $H \curvearrowright^{l_Y} Y \,{}^{r_Y}\!\curvearrowleft K$. Denote their division maps by $\langle \cdot, \cdot \rangle_G^X$ and $\langle \cdot, \cdot \rangle_H^Y$, respectively. Using these, we will construct a smooth inverse of the action map of the balanced tensor product. Let us denote the action map of the balanced tensor product by

$$\Phi : G \times_{G_0}^{\mathrm{src}, L_X} (X \otimes_H Y) \longrightarrow (X \otimes_H Y) \times_{K_0}^{R_Y, R_Y} (X \otimes_H Y),$$

mapping $(g, x \otimes y) \mapsto (gx \otimes y, x \otimes y)$. After some calculations (which we describe below), we propose the following map as an inverse for Φ:

$$\begin{aligned} \Psi : (X \otimes_H Y) \times_{K_0}^{R_Y, R_Y} (X \otimes_H Y) &\longrightarrow G \times_{G_0}^{\mathrm{src}, L_X} (X \otimes_H Y); \\ (x_1 \otimes y_1, x_2 \otimes y_2) &\longmapsto \Big(\big\langle x_1 \langle y_1, y_2 \rangle_H^Y, x_2 \big\rangle_G^X, x_2 \otimes y_2 \Big). \end{aligned}$$

It is straightforward to check that every action and division occurring in this expression is well defined. We need to check that Ψ is independent on the representations of $x_1 \otimes y_1$ and $x_2 \otimes y_2$. Only the first component Ψ_1 of Ψ could be dependent on the representations, so we focus there. Suppose we have two arrows $h_1, h_2 \in H$ satisfying $\mathrm{trg}(h_i) = r_X(x_i) = l_Y(y_i)$, so that $x_i h_i \otimes h_i^{-1} y_i = x_i \otimes y_i$. For the division of y_2 and y_1 we then use Proposition **4.26** to get:

$$\langle h_1^{-1} y_1, h_2^{-1} y_2 \rangle_H^Y = h_1^{-1} \circ \langle y_1, h_2^{-1} y_2 \rangle_H^Y = h_1^{-1} \circ \big(h_2^{-1} \circ \langle y_2, y_1 \rangle_H^Y \big)^{-1} = h_1^{-1} \circ \langle y_1, y_2 \rangle_H^Y \circ h_2.$$

Then, using this and Lemma **4.52**, we get:

$$\begin{aligned}\Psi_1(x_1h_1 \otimes h_1^{-1}y_1, x_2h_2 \otimes h_2^{-1}y_2) &= \left\langle x_1h_1\langle h_1^{-1}y_1, h_2^{-1}y_2\rangle_H^Y, x_2h_2\right\rangle_G^X \\ &= \left\langle (x_1h_1)\left(h_1^{-1} \circ \langle y_1, y_2\rangle_H^Y \circ h_2\right), x_2h_2\right\rangle_G^X \\ &= \langle (x_1\langle y_1, y_2\rangle)h_2, x_2h_2\rangle_G^X \\ &= \left\langle x_1\langle y_1, y_2\rangle_H^Y, x_2\right\rangle_G^X .\end{aligned}$$

Since the second component of Ψ is by construction independent on the representation, it follows that Ψ is a well-defined function. We now need to show that Ψ is smooth. The second component is clearly smooth, because it is just the projection onto the second component of the fibred product. That the other component is smooth follows from Lemmas **2.122** and **2.125**. Writing

$$\psi : ((x_1, y_1), (x_2, y_2)) \longmapsto \langle x_1\langle y_1, y_2\rangle_H^Y, x_2\rangle_G^X$$

and $\pi : X \times_{H_0}^{r_X, l_Y} Y \to X \otimes_H Y$ for the canonical projection, we get a commutative diagram

$$\begin{array}{ccc}\left(X \times_{H_0}^{r_X,l_Y} Y\right) \times_{K_0}^{\overline{r_Y},\overline{r_Y}} \left(X \times_{H_0}^{r_X,l_Y} Y\right) & \xrightarrow{(\pi\times\pi)|_{\mathrm{dom}(\psi)}} & (X \otimes_H Y) \times_{K_0}^{R_Y,R_Y} (X \otimes_H Y) \\ & \searrow^{\psi} \quad G. \quad {}^{\Psi_1}\swarrow & \end{array}$$

Here we temporarily use the notation $\overline{r_Y} := r_Y \circ \mathrm{pr}_2|_{X\times_{H_0}Y}$, which satisfies $R_Y \circ \pi = \overline{r_Y}$. Therefore by Lemma **2.125** the top arrow in this diagram is a subduction. Since the map ψ is evidently smooth, it follows by Lemma **2.122** that the first component Ψ_1, and hence Ψ itself, must be smooth.

Thus, we are left to show that Ψ is an inverse for Φ. That Ψ is a right inverse for Φ now follows by simple calculation using Proposition **4.26** and Lemma **4.52**:

$$\Psi \circ \Phi(g, x \otimes y) = \Psi(gx \otimes y, x \otimes y) = \left(\langle gx\langle y, y\rangle_H^Y, x\rangle_G^X, x \otimes y\right) = \left(g \circ \langle x, x\rangle_G^X, x \otimes y\right) = (g, x \otimes y).$$

For the other direction, we calculate:

$$\begin{aligned}\Phi \circ \Psi(x_1 \otimes y_1, x_2 \otimes y_2) &= \Phi\left(\left\langle x_1\langle y_1, y_2\rangle_H^Y, x_2\right\rangle_G^X, x_2 \otimes y_2\right) \\ &= \left(\left\langle x_1\langle y_1, y_2\rangle_H^Y, x_2\right\rangle_G^X x_2 \otimes y_2, x_2 \otimes y_2\right) \\ &= \left(x_1\langle y_1, y_2\rangle_H^Y \otimes y_2, x_2 \otimes y_2\right) \\ &= \left(x_1 \otimes \langle y_1, y_2\rangle_H^Y y_2, x_2 \otimes y_2\right) \\ &= (x_1 \otimes y_1, x_2 \otimes y_2) .\end{aligned}$$

Here in the second to last step we use the properties of the balanced tensor product to move the arrow $\langle y_1, y_2\rangle_H^Y$ over the tensor symbol. Hence we conclude that Φ is a diffeomorphism, which proves that $G \curvearrowright^{L_X} X \otimes_H Y \,^{R_Y}\!\curvearrowleft K$ is a left pre-principal bibundle. □

We further note that both of these properties are also preserved by biequivariant diffeomorphism.

Proposition 4.55. *Left pre-principality is preserved by biequivariant diffeomorphism.*

Proof. Suppose that $\varphi : X \to Y$ is a biequivariant diffeomorphism from a left pre-principal bibundle $G \curvearrowright^{l_X} X \,^{r_X}\!\curvearrowleft H$ to another diffeological bibundle $G \curvearrowright^{l_Y} Y \,^{r_Y}\!\curvearrowleft H$. Denote their left action maps by A_X and A_Y, respectively. The following square commutes because of biequivariance:

$$\begin{array}{ccc} G \times_{G_0}^{\mathrm{src},l_X} X & \xrightarrow{A_X} & X \times_{H_0}^{r_X,r_X} X \\ {\scriptstyle (\mathrm{id}_G\times\varphi)|_{G\times_{G_0}X}}\downarrow & & \downarrow{\scriptstyle (\varphi\times\varphi)|_{X\times_{H_0}X}} \\ G \times_{G_0}^{\mathrm{src},l_Y} Y & \xrightarrow[A_Y]{} & Y \times_{H_0}^{r_Y,r_Y} Y. \end{array}$$

It is easy to see that both vertical maps are diffeomorphisms. (This also follows by Lemma **2.125** and Proposition **2.123**.) Hence it follows A_Y must be a diffeomorphism as well. □

Proposition 4.56. *Left subductiveness is preserved by biequivariant diffeomorphism.*

Proof. Suppose that $\varphi : X \to Y$ is a biequivariant diffeomorphism from a left subductive bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ to $G \overset{l_Y}{\curvearrowright} Y \overset{r_Y}{\curvearrowleft} H$. That the first bundle is left subductive means that r_X is a subduction, but since φ intertwines the moment maps, it follows immediately that $r_Y = r_X \circ \varphi^{-1}$ is a subduction as well. □

Of course, Propositions **4.53** to **4.56** all hold for their respective 'right' versions as well. This can be proved formally, without repeating the work, by using opposite bibundles.

Corollary 4.57. *Morita equivalence defines an equivalence relation between diffeological groupoids.*

Proof. Morita equivalence is reflexive by the existence of identity bibundles, which are always biprincipal (Example **4.34**). It is also easy to check that the opposite bibundle (Construction **4.42**) of a biprincipal bibundle is again biprincipal, showing that Morita equivalence is symmetric. Transitivity follows directly from Propositions **4.53** and **4.54** and their opposite versions. □

4.3.5 Weak invertibility of diffeological bibundles

We have now stated and proved all preliminary definitions and results to start our characterisation of weakly invertible bibundles. Ultimately, this is to justify that the bicategory **DiffeolBiBund** is the correct setting for Morita equivalence (as defined in Definition **4.33**). We start by giving a direct analogue of the corresponding result for Lie groupoids. The theorem that characterises Morita equivalence between Lie groupoids is the following:

Theorem 4.58. *A bibundle in* $\mathbf{LieGrpd}_{\mathrm{LP}}$ *is weakly invertible if and only if it is biprincipal.*

This can be found in the literature in multiple places, such as [**Lan01c**, Proposition 4.21] and [**Blo08**, Section 3]. In this section we will prove that the same result holds in the setting of left principal bibundles between diffeological groupoids. Most of the groundwork has been laid in the previous Section **4.3.4**. The proof consists in, first, proving that any biprincipal bibundle is weakly invertible (Proposition **4.59** directly below), and second, proving that any weakly invertible left principal bibundle is biprincipal (Proposition **4.61**).

The weak inverse of a biprincipal bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ should be a bibundle going the other way around. A natural candidate for such a bibundle is the opposite bibundle that X induces (Construction **4.42**), and the following proves that this does indeed form a weak inverse.

Proposition 4.59. *Let* $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ *be a biprincipal bibundle. Then its opposite bundle* $H \overset{r_X}{\curvearrowright} \overline{X} \overset{l_X}{\curvearrowleft} G$ *is a weak inverse.*

Proof. We construct biequivariant diffeomorphisms

$$\begin{array}{ccc} G \overset{L_X}{\curvearrowright} X \otimes_H \overline{X} \overset{R_{\overline{X}}}{\curvearrowleft} G & & H \overset{L_{\overline{X}}}{\curvearrowright} \overline{X} \otimes_G X \overset{R_X}{\curvearrowleft} H \\ {\scriptstyle \varphi_G} \Big\Downarrow & \text{and} & {\scriptstyle \varphi_H} \Big\Downarrow \\ G \overset{\mathrm{trg}}{\curvearrowright} G \overset{\mathrm{src}}{\curvearrowleft} G, & & H \overset{\mathrm{trg}}{\curvearrowright} H \overset{\mathrm{src}}{\curvearrowleft} H. \end{array}$$

Since the original bundle is pre-biprincipal, we have a smooth division map $\langle \cdot, \cdot \rangle_G : X \times_{H_0}^{r_X, r_X} \overline{X} \to G$. We define the map

$$\varphi_G : X \otimes_H \overline{X} \longrightarrow G; \qquad x_1 \otimes x_2 \longmapsto \langle x_1, x_2 \rangle_G$$

This is independent on the representation of the tensor product by Lemma **4.52**, and smooth by Lemma **2.122** since $\varphi_G \circ \pi = \langle \cdot, \cdot \rangle_G$, where π is the canonical projection onto the balanced tensor product. We check that φ_G is biequivariant. It is easy to check that φ_G intertwines the moment maps, for example:

$$\operatorname{src}\circ\varphi_G(x_1\otimes x_2) = \operatorname{src}\left(\langle x_1,x_2\rangle_G\right) = l_X(x_2) = R_{\overline{X}}(x_1\otimes x_2).$$

The left G-equivariance of φ_G follows directly out of Proposition **4.26**, and the right G-equivariance follows from Lemma **4.43**. Hence φ_G is a genuine bibundle morphism.

Since the original bundle is biprincipal, so is its opposite, and therefore by Propositions **4.53** and **4.54** it follows that both balanced tensor products are also biprincipal. Therefore φ_G is in particular a left G-equivariant bundle morphism from a principal bundle $G\curvearrowright^{L_X} X\otimes_H \overline{X} \xrightarrow{R_{\overline{X}}} G_0$ to a pre-principal bundle $G\curvearrowright^{\operatorname{trg}} G \xrightarrow{\operatorname{src}} G_0$, and hence a diffeomorphism by Proposition **4.30**. This proves that the opposite bibundle is a weak right inverse. Note that we already need full biprincipality of the original bibundle for this. To prove that it is also a weak left inverse we make an analogous construction for φ_H, which we leave to the reader. □

The converse of Proposition **4.59** is given by Proposition **4.61** below, for which we need the following lemma.

Lemma 4.60. *Let $G\curvearrowright^{l_X} X\ {}^{r_X}\!\curvearrowleft H$ be a left pre-principal bibundle, and consider a right action $Y\ {}^{r_Y}\!\curvearrowleft G$. Then there is a diffeomorphism*

$$\theta : X\times_{H_0}^{r_X,R_X}(Y\otimes_G X)\longrightarrow Y\times_{G_0}^{r_Y,l_X} X; \qquad (x_1, y\otimes x_2)\longmapsto (y\langle x_2,x_1\rangle_G, x_1)\,.$$

Proof. The map θ is smooth because of a similar argument that is used in the proof of Proposition **4.54**. It is easily seen that $(y,x)\mapsto (x,y\otimes x)$ is a smooth inverse. □

Proposition 4.61. *If a left principal bibundle $G\curvearrowright^{l_X} X\ {}^{r_X}\!\curvearrowleft H$ between diffeological groupoids is weakly invertible in* **DiffeolBiBund**$_{\mathrm{LP}}$*, then it is biprincipal.*

Proof. We follow the idea of the proof of [**MM05**, Proposition 2.9]. If the bibundle is weakly invertible in **DiffeolBiBund**$_{\mathrm{LP}}$, there exists a *left principal* bibundle $H\curvearrowright^{l_Y} Y\ {}^{r_Y}\!\curvearrowleft G$ and two biequivariant diffeomorphisms $\varphi_G : X\otimes_H Y\Rightarrow G$ and $\varphi_H : Y\otimes_G X\Rightarrow H$. The idea of the proof is to show that Y is biequivariantly diffeomorphic to the opposite bibundle $\overline{X}$.

By Lemma **4.60** we get two diffeomorphisms:

$$\theta : X\times_{H_0}^{r_X,R_X}(Y\otimes_G X)\longrightarrow Y\times_{G_0}^{r_Y,l_X} X,$$
$$\widetilde{\theta} : Y\times_{G_0}^{r_Y,R_Y}(X\otimes_H Y)\longrightarrow X\times_{H_0}^{r_X,l_Y} Y.$$

We use these to construct the following smooth maps:

$$\tau : \overline{X}\longrightarrow Y; \qquad \tau(x) := \operatorname{pr}_1|_{Y\times_{G_0}X}\circ\theta\left(x,\varphi_H^{-1}(\mathrm{id}_{r_X(x)})\right),$$
$$\sigma : Y\longrightarrow \overline{X}; \qquad \sigma(y) := \operatorname{pr}_1|_{X\times_{H_0}Y}\circ\widetilde{\theta}\left(y,\varphi_G^{-1}(\mathrm{id}_{r_Y(y)})\right).$$

Using the fact that φ_G and φ_H intertwine the moment maps, it is easy to see that each term in these formulas are well defined. A simple calculation using the G-invariance of r_X and Proposition **4.26** shows that $\tau(xg)=\tau(x)g$. Similarly, we find $\tau(hx)=h\tau(x)$, $\sigma(yg)=\sigma(y)g$, and $\sigma(hy)=h\sigma(y)$. It further follows that τ and σ intertwine the moment maps, because of the codomains of θ and $\widetilde{\theta}$. Therefore τ and σ both form bibundle morphisms. Sadly, we cannot rely on Proposition **4.30**, since Y is left principal, whereas $\overline{X}$ is right principal. However, the compositions $\tau\circ\sigma : Y\to Y$ and $\sigma\circ\tau : \overline{X}\to\overline{X}$ *do* form bundle maps between principal bundles, and hence by Proposition **4.30** they are both diffeomorphisms. It has to be the case, then, that the individual maps τ and σ are diffeomorphisms as well. Therefore, the right principal bibundle $H\curvearrowright^{r_X}\overline{X}\ {}^{l_X}\!\curvearrowleft G$ is biequivariantly diffeomorphic to the left principal bibundle $H\curvearrowright^{l_Y} Y\ {}^{r_Y}\!\curvearrowleft G$, so by Propositions **4.55** and **4.56** it follows that $H\curvearrowright^{r_X}\overline{X}\curvearrowleft^{l_X} G$ is biprincipal. But that is just to say that $G\curvearrowright^{l_X} X\ {}^{r_X}\!\curvearrowleft H$ is biprincipal. □

The following is the analogue of Theorem **4.58** in the diffeological setting, and follows directly from Propositions **4.59** and **4.61**.

Theorem 4.62. *In* **DiffeolBiBund**$_{\mathrm{LP}}$*, bibundles are weakly invertible if and only if they are biprincipal. In other words, two diffeological groupoids are Morita equivalent if and only if they are bicategorically equivalent in* **DiffeolBiBund**$_{\mathrm{LP}}$*.*

Note the crucial assumption in the proof of Proposition **4.61** that the bibundle is weakly invertible *in* **DiffeolBiBund**$_{\mathrm{LP}}$, to ensure that the weak inverse is also left principal. This begs the question:

Question 4.63. *What are the necessary and sufficient conditions for an* arbitrary *bibundle to be weakly invertible in* **DiffeolBiBund***?*

Biprincipality clearly remains sufficient, as Proposition **4.59** states. But the proof of Proposition **4.61** does not answer the question, because we already assume left principality there. Question **4.63** therefore boils down to: *is this assumption essential?* Here we will prove that: *no*, it is not. We do this in steps, by showing first that weak invertibility implies bisubductiveness:

Proposition 4.64. *A weakly invertible bibundle between diffeological groupoids is bisubductive.*

Proof. Suppose we have a bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ that admits a weak inverse $H \overset{l_Y}{\curvearrowright} Y \overset{r_Y}{\curvearrowleft} G$. Let us denote the included biequivariant diffeomorphisms by $\varphi_G : X \otimes_H Y \Rightarrow G$ and $\varphi_H : Y \otimes_G X \Rightarrow H$, as usual. Since the identity bibundles of G and H are both biprincipal, it follows by Proposition **4.56** that the moment maps L_X, R_X, L_Y and R_Y are all subductions. Together with the original moment maps, we get four commutative squares, each of the form:

$$\begin{array}{ccc} X \times_{H_0}^{r_X, l_Y} Y & \xrightarrow{\ \pi\ } & X \otimes_H Y \\ {\scriptstyle \mathrm{pr}_1|_{X\times_{H_0} Y}}\downarrow & & \downarrow{\scriptstyle L_X} \\ X & \xrightarrow[\ l_X\ ]{} & G_0. \end{array}$$

Here $\pi : X \times_{H_0}^{r_X, l_Y} Y \to X \otimes_H Y$ is the quotient map of the diagonal H-action. By Lemma **2.122** it follows that, since L_X is a subduction, so is $l_X \circ \mathrm{pr}_1|_{X\times_{H_0} Y}$, and in turn by Lemma **2.121** it follows l_X is a subduction. In a similar fashion we find that r_X, l_Y and r_Y are all subductions as well. □

This proposition gets us halfway to proving that weakly invertible bibundles are biprincipal. To prove that they are pre-biprincipal, it is enough to construct smooth division maps. We will give this construction below (Construction **4.67**), which follows from a careful reverse engineering of the division map of a pre-principal bundle. Recall from Proposition **4.54** that the smooth inverse of the action map contains the information of both the G-division map and the H-division map. Specifically, the first component of the inverse is of the form $\langle x_1 \langle y_1, y_2\rangle_H^Y, x_2\rangle_G^X$, in which if we set $y_1 = y_2$, we simply reobtain the G-division map $\langle x_1, x_2\rangle_G^X$. The question is if this "reobtaining" can be done in a smooth way. This is not so obvious at first. Namely, if we vary (x_1, x_2) smoothly within $X \times_{H_0}^{r_X, r_X} X$, can we guarantee that y_1 and y_2 vary smoothly with it, while still retaining the equalities $r_X(x_i) = l_Y(y_i)$ and $y_1 = y_2$? The elaborate Construction **4.67** proves that this can indeed be done. An essential part of our argument will be supplied by the following lemma.

Lemma 4.65. *If* $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ *is a weakly invertible bibundle, with weak inverse* $H \overset{l_Y}{\curvearrowright} Y \overset{r_Y}{\curvearrowleft} G$*, then all four actions are free.*

Proof. This follows from an argument that is used in the proof of [Blo08, Proposition 3.23]. Suppose we have an arrow $h \in H$ and a point $y \in Y$ such that $hy = y$. By Proposition **4.64** it follows that in particular l_X is surjective, so we can find $x \in X$ such that $y \otimes x \in Y \otimes_G X$. Then

$$h(y \otimes x) = (hy) \otimes x = y \otimes x.$$

But by Proposition **4.55** the bundle $H \overset{L_Y}{\curvearrowright} Y \otimes_G X \xrightarrow{R_X} G_0$, which is equivariantly diffeomorphic to the identity bundle on H, is pre-principal. So, the left action $H \curvearrowright Y \otimes_G X$ is free, and hence $h = \mathrm{id}_{L_Y(y\otimes x)} = \mathrm{id}_{l_Y(y)}$, proving that $H \curvearrowright Y$ is also free. That the three other actions are free follows analogously. □

Lemma 4.66. *Let $X \overset{r_X}{\curvearrowleft} H$ and $H \overset{l_Y}{\curvearrowright} Y$ be smooth actions, so that we can form the balanced tensor product $X \otimes_H Y$. Suppose that $H \curvearrowright Y$ is free. Then $x_1 \otimes y = x_2 \otimes y$ if and only if $x_1 = x_2$. Similarly, if $X \curvearrowleft H$ is free, then $x \otimes y_1 = x \otimes y_2$ if and only if $y_1 = y_2$.*

Proof. If $x_1 = x_2$ to begin with, the implication is trivial. Suppose therefore that $x_1 \otimes y = x_2 \otimes y$, which means that there exists an arrow $h \in H$ such that $(x_1 h^{-1}, hy) = (x_2, y)$. In particular $hy = y$, which, because the action on Y is free, implies $h = \mathrm{id}_{l_Y(y)}$, and it follows that $x_1 = x_1 \mathrm{id}_{l_Y(y)}^{-1} = x_2$. □

We shall now describe how the division map arises from local data:

Construction 4.67. For this construction to work, we start with a diffeological bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$, admitting a weak inverse $H \overset{l_Y}{\curvearrowright} Y \overset{r_Y}{\curvearrowleft} G$. Consider a pointed plot $\alpha : (U_\alpha, 0) \to (X \times_{H_0}^{r_X, r_X} X, (x_1, x_2))$. We split α into the components (α_1, α_2), which in turn are pointed plots $\alpha_i : (U_\alpha, 0) \to (X, x_i)$ satisfying $r_X \circ \alpha_1 = r_X \circ \alpha_2 : U_\alpha \to H_0$. This equation gives a plot of H_0, and since by Proposition **4.64** the moment map $l_Y : Y \to H_0$ is a subduction, for every $t \in U_\alpha$ we can find a plot $\beta : V \to Y$, defined on an open neighbourhood $t \in V \subseteq U_\alpha$, such that $r_X \circ \alpha_i|_V = l_Y \circ \beta$. From this equation it follows that the smooth maps $(\alpha_i|_V, \beta) : V \to X \times_{H_0}^{r_X, l_Y} Y$ define two plots of the underlying space of the balanced tensor product. Applying the quotient map $\pi : X \times_{H_0}^{r_X, l_Y} Y \to X \otimes_H Y$, we thus get two full-fledged plots $s \mapsto \alpha_i|_V(s) \otimes \beta(s)$ of the balanced tensor product. We combine these two plots to define yet another smooth map:

$$\Omega^\alpha|_V := (\pi \circ (\alpha_1|_V, \beta), \pi \circ (\alpha_2|_V, \beta)) : V \longrightarrow (X \otimes_H Y) \times_{G_0}^{R_Y, R_Y} (X \otimes_H Y) .$$

Note that $\Omega^\alpha|_V$ lands in the right codomain because $R_Y \circ \pi \circ (\alpha_i|_V, \beta) = r_Y \circ \beta$, irrespective of $i \in \{1, 2\}$. We also note that the codomain of $\Omega^\alpha|_V$ is exactly the *do*main of the inverse $\Psi = (\Psi_1, \Psi_2)$ of the action map of the balanced tensor product $G \overset{L_X}{\curvearrowright} X \otimes_H Y \xrightarrow{R_Y} H_0$ (given explicitly in Proposition **4.54**). In particular we then get a smooth map

$$\Psi_1 \circ \Omega^\alpha|_V : V \xrightarrow{\Omega^\alpha|_V} (X \otimes_H Y) \times_{G_0}^{R_Y, R_Y} (X \otimes_H Y) \xrightarrow{\Psi_1} G.$$

We now extend this map to the entire domain U_α, and show that it is independent on the choice of plot β. For that, pick two points $t, \bar{t} \in U_\alpha$, so that by subductiveness of the left moment map l_Y we can find two plots, $\beta : V \to Y$ and $\overline{\beta} : \overline{V} \to Y$, defined on open neighbourhoods of t and $\bar{t}$, respectively, such that $r_X \circ \alpha_i|_V = l_Y \circ \beta$ and $r_X \circ \alpha_i|_{\overline{V}} = l_Y \circ \overline{\beta}$. Following the above construction, we get two smooth maps:

$$\Omega^\alpha|_V : s \longmapsto (\alpha_1|_V(s) \otimes \beta(s), \alpha_2|_V(s) \otimes \beta(s)) ,$$
$$\overline{\Omega}^\alpha|_{\overline{V}} : s \longmapsto \left(\alpha_1|_{\overline{V}}(s) \otimes \overline{\beta}(s), \alpha_2|_{\overline{V}}(s) \otimes \overline{\beta}(s)\right) .$$

We now remark an important characterisation of Ψ, as a consequence of it being a diffeomorphism and inverse to the action map. Namely, $\Psi_1(x_1 \otimes y_1, x_2 \otimes y_2) \in G$ is the *unique* arrow $g \in G$ satisfying $gx_2 \otimes y_2 = x_1 \otimes y_1$. Therefore, $\Psi_1 \circ \Omega^\alpha|_V(s) \in G$ is the unique arrow such that

$$[\Psi_1 \circ \Omega^\alpha|_V(s)] \cdot (\alpha_2|_V(s) \otimes \beta(s)) = \alpha_1|_V(s) \otimes \beta(s).$$

By Lemma **4.65** all of the four actions of the original bibundles are free. Consequently, applying Lemma **4.66**, since the second component in each term is just $\beta(s)$, this means that $\Psi_1 \circ \Omega^\alpha|_V(s)$ is the unique arrow in G such that

$$\Psi_1 \circ \Omega^\alpha|_V(s) \cdot \alpha_2|_V(s) = \alpha_1|_V(s),$$

where the tensor with $\beta(s)$ can be removed. But, for exactly the same reasons, if we take $s \in V \cap \overline{V}$, then $\Psi_1 \circ \overline{\Omega}^\alpha|_{\overline{V}}(s) \in G$ is *also* the unique arrow such that

$$\Psi_1 \circ \overline{\Omega}^\alpha|_{V \cap \overline{V}}(s) \cdot \alpha_2|_{V \cap \overline{V}}(s) = \alpha_1|_{V \cap \overline{V}}(s),$$

proving that

$$\Psi_1 \circ \Omega^\alpha|_{V \cap \overline{V}} = \Psi_1 \circ \overline{\Omega}^\alpha|_{V \cap \overline{V}}.$$

This shows that on the overlaps $V \cap \overline{V}$ the map $\Psi_1 \circ \Omega^\alpha|_{V\cap\overline{V}}$ does *not* depend on the plots β and $\overline{\beta}$. This allows us to extend $\Psi_1 \circ \Omega^\alpha|_V$, in a well-defined way, to the entire domain of U_α. We do this as follows. For every $t \in U_\alpha$ there exists a plot $\beta_t : V_t \to Y$, defined on an open neighbourhood $V_t \ni t$, such that $r_X \circ \alpha_i|_{V_t} = l_Y \circ \beta_t$. Clearly, this gives an open cover $(V_t)_{t\in U_\alpha}$ of U_α. For $t \in U_\alpha$ we then set $\Psi_1 \circ \Omega^\alpha(t) := \Psi_1 \circ \Omega^\alpha|_{V_t}(t)$. Hence we get a well-defined function $\Psi_1 \circ \Omega^\alpha : U_\alpha \to G$, which is smooth by the Axiom of Locality.

The main observation now is that, as the plot α is centred at (x_1, x_2), we get that $\Psi_1 \circ \Omega^\alpha(0)$ is the unique arrow in G such that $\Psi_1 \circ \Omega^\alpha(0) \cdot x_2 = x_1$. This is exactly the property that characterises the division $\langle x_1, x_2\rangle_G$!

Proposition 4.68. *A weakly invertible bibundle between diffeological groupoids is pre-biprincipal.*

Proof. The bulk of the work has been done in Construction **4.67**. Start with a diffeological bibundle $G\curvearrowright^{l_X} X \,{}^{r_X}\!\curvearrowleft H$ and a weak inverse $H\curvearrowright^{l_Y} Y \,{}^{r_Y}\!\curvearrowleft G$. We shall define a smooth division map $\langle\cdot,\cdot\rangle_G$ for the left G-action. For $(x_1, x_2) \in X \times^{r_X, r_X}_{H_0} X$, we know by the Axiom of Covering that the constant map $\mathrm{const}_{(x_1,x_2)} : \mathbb{R} \to X \times^{r_X,r_X}_{H_0} X$ is a plot centred at (x_1, x_2). We use the shorthand $\Psi_1 \circ \Omega^{(x_1,x_2)}$ to denote the map $\Psi_1 \circ \Omega^\alpha$ defined by the plot $\alpha = \mathrm{const}_{(x_1,x_2)}$, and then write:

$$\langle x_1, x_2\rangle_G := \Psi_1 \circ \Omega^{(x_1,x_2)}(0).$$

That just leaves us to show that this map is smooth. So take an arbitrary plot $\alpha : U_\alpha \to X \times^{r_X,r_X}_{H_0} X$ of the fibred product. We need to show that $\langle\cdot,\cdot\rangle_G \circ \alpha$ is a plot of G. For any $t \in U_\alpha$, we have that

$$\langle \alpha_1(t), \alpha_2(t)\rangle_G = \Psi_1 \circ \Omega^{\alpha(t)}(0)$$

is the unique arrow in G such that

$$\Psi_1 \circ \Omega^{\alpha(t)}(0) \cdot \mathrm{const}^2_{\alpha(t)}(0) = \mathrm{const}^1_{\alpha(t)}(0),$$

where const^i denotes the ith component of the constant plot. But then $\mathrm{const}^i_{\alpha(t)}(0) = \alpha_i(t)$, and we already know that $\Psi_1 \circ \Omega^\alpha(t) \in G$ is the unique arrow that sends $\alpha_2(t)$ to $\alpha_1(t)$, so we have:

$$\Psi_1 \circ \Omega^{\alpha(t)}(0) = \Psi_1 \circ \Omega^\alpha(t), \qquad \text{which means} \qquad \langle\cdot,\cdot\rangle_G \circ \alpha = \Psi_1 \circ \Omega^\alpha.$$

But the right hand side $\Psi_1 \circ \Omega^\alpha : U_\alpha \to G$ is a plot of G as per Construction **4.67**, proving that the map $\langle\cdot,\cdot\rangle_G$ is smooth. It is quite evident from its construction that it satisfies exactly the properties of a division map, and it is now easy to verify that

$$\big(\langle\cdot,\cdot\rangle_G, \mathrm{pr}_2|_{X\times_{H_0}X}\big) : X \times^{r_X,r_X}_{H_0} X \longrightarrow G \times^{\mathrm{src},l_X}_{G_0} X$$

is a smooth inverse of the action map (see Section **4.2.1**). The fact that it lands in the right codomain, i.e., $\mathrm{src}(\langle x_1, x_2\rangle_G) = l_X(x_2)$, follows from the properties of Ψ as the inverse of the action map of the balanced tensor product. Therefore $G\curvearrowright^{l_X} X \xrightarrow{r_X} H_0$ is a pre-principal bundle. An analogous argument will show that $G_0 \xleftarrow{l_Y} X \,{}^{r_X}\!\curvearrowleft H$ is also pre-principal, and hence we have proved the claim. ☐

The answer to Question **4.63** is then:

Theorem 4.69. *A bibundle is weakly invertible in* **DiffeolBiBund** *if and only if it is biprincipal. That means: two diffeological groupoids are Morita equivalent if and only if they are equivalent in* **DiffeolBiBund**.

Proof. One of the implications is just Proposition **4.59**. The other now follows from a combination of Propositions **4.64** and **4.68**. ☐

This significantly generalises the Lie version of this result in Theorem **4.58**, and shows that left principality of the Lie bibundles was more like a technical necessity as opposed to a meaningful assumption about the smooth structure of the bibundles.

4.4 Some applications

Now that we have constructed the bicategory **DiffeolBiBund** and have established Theorem **4.69**, we give some applications of this framework.

4.4.1 Equivalence of action categories

In the Morita theory of rings, it holds that two rings are Morita equivalent if and only if their categories of modules are equivalent. For groupoids, even discrete ones, this is no longer an "if and only if" proposition, but merely an "only if". Nevertheless, it is known that the result transfers to Lie groupoids as well [**Lan01a**, Theorem 6.6], and here we shall prove that it transfers also to diffeology.

Theorem 4.70. *Suppose $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are Morita equivalent diffeological groupoids. Then* $\mathbf{Act}(G \rightrightarrows G_0)$ *and* $\mathbf{Act}(H \rightrightarrows H_0)$ *are categorically equivalent.*

Proof. If $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are Morita equivalent, there exists a biprincipal bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$. Recall from Definition **4.10** the notion of the action categories and from Definition **4.46** that of induced action functors. We claim that

$$\begin{aligned} X \otimes_H - &: \mathbf{Act}(H \rightrightarrows H_0) \longrightarrow \mathbf{Act}(G \rightrightarrows G_0), \\ \overline{X} \otimes_G - &: \mathbf{Act}(G \rightrightarrows G_0) \longrightarrow \mathbf{Act}(H \rightrightarrows H_0) \end{aligned}$$

are mutually inverse functors up to natural isomorphism. To see this, take a left H action $H \overset{l_Y}{\curvearrowright} Y$. Then

$$\left(\overline{X} \otimes_G -\right) \circ (X \otimes_H -) \, [H \overset{l_Y}{\curvearrowright} Y] = \left(\overline{X} \otimes_G -\right) \left[G \overset{L_X}{\curvearrowright} X \otimes_H Y\right] = H \overset{L_{\overline{X}}}{\curvearrowright} \left(\overline{X} \otimes_G (X \otimes_H Y)\right).$$

Therefore, we need to construct a natural biequivariant diffeomorphism

$$\mu_Y : \overline{X} \otimes_G (X \otimes_H Y) \longrightarrow Y.$$

For this, we collect the biequivariant diffeomorphisms from Propositions **4.49**, **4.50** and **4.59**. Let us denote them by

$$\begin{aligned} &A_Y : \overline{X} \otimes_G (X \otimes_H Y) \longrightarrow \left(\overline{X} \otimes_G X\right) \otimes_H Y, \\ &\varphi_H : \overline{X} \otimes_G X \longrightarrow H, \\ &M_Y : H \otimes_H Y \longrightarrow Y, \end{aligned}$$

describing the association up to isomorphism, the division map of the bibundle, and the left action $H \curvearrowright Y$, respectively. We then define

$$\mu_Y := M_Y \circ (\varphi_H \otimes \mathrm{id}_Y) \circ A_Y.$$

Note that $(\varphi_H \otimes \mathrm{id}_Y)$ is still a biequivariant diffeomorphism. The naturality square of the natural transformation $\mu : \left(\overline{X} \otimes_G -\right) \circ (X \otimes_H -) \Rightarrow \mathrm{id}_{\mathbf{Act}(H)}$ then becomes:

$$\begin{array}{ccc} \overline{X} \otimes_G (X \otimes_H Y) & \xrightarrow{\ \mu_Y\ } & Y \\ {\scriptstyle \mathrm{id}_{\overline{X}} \otimes (\mathrm{id}_X \circ \varphi)} \big\downarrow & & \big\downarrow {\scriptstyle \varphi} \\ \overline{X} \otimes_G (X \otimes_H Z) & \xrightarrow[\ \mu_Z\ ]{} & Z, \end{array}$$

where $\varphi : Y \to Z$ is an H-equivariant smooth map. It follows from the structure of these maps that the naturality square commutes. The top right corner of the diagram becomes:

$$\begin{aligned} \varphi \circ \mu_Y \left(x_1 \otimes (x_2 \otimes y)\right) &= \varphi \circ M_Y \circ (\varphi_H \otimes \mathrm{id}_Y) \circ A_Y \left(x_1 \otimes (x_2 \otimes y)\right) \\ &= \varphi \circ M_Y \circ (\varphi_H \otimes \mathrm{id}_Y) \left((x_1 \otimes x_2) \otimes y\right) \\ &= \varphi \circ M_Y \left(\varphi_H(x_1 \otimes x_2) \otimes y\right) \\ &= \varphi \left(\varphi_H(x_1 \otimes x_2) y\right) \\ &= \varphi_H(x_1 \otimes x_2) \varphi(y), \end{aligned}$$

where the very last step follows from H-equivariance of φ. Following a similar calculation, the bottom left corner evaluates as

$$\begin{aligned}\mu_Z \circ (\mathrm{id}_{\overline{X}} \otimes (\mathrm{id}_X \otimes \varphi)) &= M_Z \circ (\varphi_H \otimes \mathrm{id}_Z) \circ A_Z \circ (\mathrm{id}_{\overline{X}} \otimes (\mathrm{id}_X \otimes \varphi)) \\ &= M_Z \circ (\varphi_H \otimes \mathrm{id}_Z) \circ ((\mathrm{id}_{\overline{X}} \otimes \mathrm{id}_X) \otimes \varphi) \\ &= M_Z \circ (\varphi_H \otimes \varphi),\end{aligned}$$

which, when evaluated, gives exactly the same as the above expression for the top right corner. This proves that μ is natural, and since every of its components is an H-equivariant diffeomorphism, it follows that μ is a natural isomorphism. The fact that $(X \otimes_H -) \circ \left(\overline{X} \otimes_G -\right)$ is naturally isomorphic to $\mathrm{id}_{\mathbf{Act}(G)}$ follows from an analogous argument. Hence the categories $\mathbf{Act}(G \rightrightarrows G_0)$ and $\mathbf{Act}(H \rightrightarrows H_0)$ are equivalent, as was to be shown. □

The category $\mathbf{Act}(G \rightrightarrows G_0)$ is itself a functional space, as we see that the space of objects can be described as a (very big) union over diffeological spaces of the form $C^\infty(G \times_{G_0} X, X)$. Each of these carries the standard functional diffeology, and using the colimit diffeology, we could try to equip the union of these spaces with a diffeology as well. We therefore pose the question:

Question 4.71. *Is there a natural diffeology on* $\mathbf{Act}(G \rightrightarrows G_0)$*, and if so, does the categorical equivalence above become smooth?*

4.4.2 Morita equivalence of fibration groupoids

In Section **3.3.2** we saw that diffeological fibre bundles can be studied through fibration groupoids. It is natural to ask whether this property is invariant under Morita equivalence:

Theorem 4.72. *Let* $G \rightrightarrows G_0$ *and* $H \rightrightarrows H_0$ *be two Morita equivalent diffeological groupoids. Then* $G \rightrightarrows G_0$ *is fibrating if and only if* $H \rightrightarrows H_0$ *is fibrating.*

Proof. Because Morita equivalence is an equivalence relation, it suffices to prove that if $G \rightrightarrows G_0$ is fibrating, then so is $H \rightrightarrows H_0$. Denoting the characteristic maps of these groupoids by $\chi_G = (\mathrm{trg}_G, \mathrm{src}_G)$ and $\chi_H = (\mathrm{trg}_H, \mathrm{src}_H)$, assume that G is fibrating, so that χ_G is a subduction. Our goal is to show χ_H is also a subduction.

To begin with, take an arbitrary plot $\alpha = (\alpha_1, \alpha_2) : U_\alpha \to H_0 \times H_0$, and fix an element $t \in U_\alpha$. We thus need to find a plot $\Phi : W \to H$, defined on an open neighbourhood $t \in W \subseteq U_\alpha$, such that $\alpha|_W = \chi_H \circ \Phi$. Morita equivalence yields a biprincipal bibundle $G \curvearrowright^{l_X} X \,^{r_X}\!\curvearrowleft H$. To construct the plot Φ, we use almost all of the structure of this bibundle.

The right moment map $r_X : X \to H_0$ is a subduction, so for each of the components α_i of α we get a plot $\beta_i : U_i \to X$, defined on an open neighbourhood $t \in U_i \subseteq U_\alpha$, such that $\alpha_i|_{U_i} = r_X \circ \beta_i$. Define $U := U_1 \cap U_2$, which is another open neighbourhood of $t \in U_\alpha$, and introduce the notation

$$\beta := (\beta_1|_U, \beta_2|_U) : U \longrightarrow X \times X.$$

Composing with the left moment map $l_X : X \to G_0$, we get another plot $(l_X \times l_X) \circ \beta : U \to G_0 \times G_0$. It is here that we use that $G \rightrightarrows G_0$ is fibrating. Because of that, we can find an open neighbourhood $t \in V \subseteq U \subseteq U_\alpha$ and a plot $\Omega : V \to G$ such that

$$\chi_G \circ \Omega = (l_X \times l_X) \circ \beta|_V . \qquad (♥)$$

This means that $\mathrm{trg}_G \circ \Omega = l_X \circ \beta_1|_V$ and $\mathrm{src}_G \circ \Omega = l_X \circ \beta_2|_V$. Let $\varphi_G : X \otimes_H \overline{X} \Rightarrow G$ be the biequivariant diffeomorphism from Proposition **4.59**. Thus we get a plot $\varphi_G^{-1} \circ \Omega : V \to X \otimes_H \overline{X}$. The canonical projection $\pi_H : X \times_{H_0}^{r_X, r_X} \overline{X} \to X \otimes_H \overline{X}$ of the diagonal H-action is a subduction, so we can find an open neighbourhood $t \in W \subseteq V$ and a plot $\omega : W \to X \times_{H_0}^{r_X, r_X} \overline{X}$ such that

$$\pi_H \circ \omega = \varphi_G^{-1} \circ \Omega|_W . \qquad (♣)$$

Note that the plot ω decomposes into its components $\omega_1, \omega_2 : W \to X$, which satisfy $r_X \circ \omega_1 = r_X \circ \omega_2$. Using the biequivariance of φ_G and the defining relation $L_X \circ \pi_H = l_X \circ \mathrm{pr}_1|_{X \times_{H_0} \overline{X}}$ we find:

$$l_X \circ \beta_1|_W = \mathrm{trg}_G \circ \Omega|_W = L_X \circ \varphi_G^{-1} \circ \Omega|_W = L_X \circ \pi_H \circ \omega = l_X \circ \mathrm{pr}_1|_{X \times_{H_0} \overline{X}} \circ \omega = l_X \circ \omega_1,$$

where the first equality follows from the equation (♥), and the third one from (♣). Similarly, we find $l_X \circ \beta_2|_W = l_X \circ \omega_2$. These two equalities give two well-defined plots, one for each $i \in \{1, 2\}$, given by

$$\beta_i|_W \otimes \omega_i := \pi_G \circ (\beta_i|_W, \omega_i) : W \xrightarrow{(\beta_i|_W, \omega_i)} \overline{X} \times_{G_0}^{l_X, l_X} X \xrightarrow{\pi_G} \overline{X} \otimes_G X,$$

where $\pi_G : \overline{X} \times_{G_0}^{l_X, l_X} X \to \overline{X} \otimes_G X$ is the canonical projection of the diagonal G-action. We can now apply the biequivariant diffeomorphism $\varphi_H : \overline{X} \otimes_G X \Rightarrow H$ from Proposition **4.59** to get two plots in H. It is from these two plots that we will create Φ. Here it is absolutely essential that we have constructed the plot ω such that $r_X \circ \omega_1 = r_X \circ \omega_2$, because that means that the sources of these two plots in H will be equal, and hence they can be composed if we first invert one of them component-wise. To see this, use the biequivariance of φ_H to calculate

$$\mathrm{src}_H \circ \varphi_H \circ (\beta_i|_W \otimes \omega_i) = R_X \circ (\beta_i|_W \otimes \omega_i) = r_X \circ \mathrm{pr}_2|_{\overline{X} \times_{G_0} X} \circ (\beta_i|_W, \omega_i) = r_X \circ \omega_i,$$

and similarly:

$$\mathrm{trg}_H \circ \varphi_H \circ (\beta_i|_W \otimes \omega_i) = L_{\overline{X}} \circ (\beta_i|_W \otimes \omega_i) = r_X \circ \mathrm{pr}_1|_{\overline{X} \times_{G_0} X} \circ (\beta_i|_W, \omega_i) = r_X \circ \beta_i|_W = \alpha_i|_W.$$

Of course, if we switch $\beta_i|_W \otimes \omega_i$ to $\omega_i \otimes \beta_i|_W$, which is defined in the obvious way, then the right-hand sides of the above two equations will switch. So, for every $s \in W$, the expression $\varphi_H\,(\omega_2(s) \otimes \beta_2(s))$ is an arrow in H from $r_X \circ \beta_2(s) = \alpha_2(s)$ to $r_X \circ \omega_2(s)$, and $\varphi_H\,(\beta_1(s) \otimes \omega_1(s))$ is an arrow from $r_X \circ \omega_1(s) = r_X \omega_2(s)$ to $r_X \circ \beta_1(s) = \alpha_1(s)$, which can hence be composed to give an arrow from $\alpha_2(s)$ to $\alpha_1(s)$. This is exactly the kind of arrow we want. Therefore, for every $s \in W$, we get a commutative triangle in the groupoid H, which defines for us the plot $\Phi : W \to H$:

$$\begin{array}{ccc} \alpha_2(s) & \overset{\Phi(s)}{\dashrightarrow} & \alpha_1(s) \\ & {\scriptstyle \varphi_H(\omega_2(s) \otimes \beta_2(s))} \searrow \quad \nearrow {\scriptstyle \varphi_H(\beta_1(s) \otimes \omega_1(s))} & \\ & r_X \circ \omega_1(s). & \end{array}$$

The map Φ is clearly smooth, because inversion and multiplication in H are smooth. Hence we have defined the plot Φ, and by the above diagram it is clear that it satisfies

$$\chi_H \circ \Phi = (\mathrm{trg}_H \circ \Phi, \mathrm{src}_H \circ \Phi) = \alpha|_W.$$

Thus we may at last conclude that χ_H is a subduction, and hence that $H \rightrightarrows H_0$ is also fibrating. □

4.4.3 Diffeological bibundles between Lie groupoids

If $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are two *Lie* groupoids, such that there exists a *diffeological* biprincipal bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ between them, what does that say about Morita equivalence of G and H in the Lie category? Does the inclusion pseudofunctor $\mathbf{LieGrpd}_{\mathrm{LP}} \hookrightarrow \mathbf{DiffeolBiBund}$ reflect equivalences[44]? Does X have to be a manifold? In attempting to answer these questions we will study a special class of diffeological groupoids:

Definition 4.73. We say a diffeological groupoid $G \rightrightarrows G_0$ is *locally subductive* if its source and target maps are local subductions[45]. Clearly, every Lie groupoid is a locally subductive diffeological groupoid.

[44]Recall that a functor $F : \mathbf{C} \to \mathbf{D}$ is said to *reflect* some property of an object (or arrow), if whenever FC has that property in $\mathbf{D}$, then C has that same property in $\mathbf{C}$. The functor F is said to *reflect isomorphism* if whenever $FC \cong FD$ are isomorphic in $\mathbf{D}$, then $C \cong D$ in $\mathbf{C}$. The definition extends to the reflection fo weak isomorphisms by pseudofunctors.

[45]Given the fact that local subductions on smooth manifolds are the same as submersions (Proposition **2.128**), it would be tempting to call such groupoids *"diffeological Lie groupoids"*. Sadly, this would conflict with earlier established terminology of so-called *diffeological Lie groups* in [***Diffeology***, Article 7.1] and [**Les03**; **Mag18**].

Looking at the structure of the proofs in the preceding parts of this Chapter **IV**, it appears that everything seems to work just as well if we strengthen our fundamental definitions of groupoids and bundles from being subductive to being *locally* subductive. In doing so, we would get a theory of locally subductive groupoids, locally subductive groupoid bundles, and the corresponding notions for bibundles and Morita equivalence, which, as it appears, would follow the same story as we have so far presented. In this subtheory, since local subductions are the submersions on smooth manifolds (Proposition **2.128**), the restriction from diffeological spaces to smooth manifolds would then precisely return the Lie groupoid theory as it is known in the literature.

But it is not evident that reducing to Lie groupoids within the general diffeological framework (where the bibundles are not necessarily locally subductive) also returns the Lie groupoid theory. This is mainly because it is not clear that the condition of subductiveness of bundles should reduce to submersiveness automatically. The results below prove that this *does* have to be the case. In this way we will prove that for locally subductive groupoids, our choice of starting with *subductive* bundles faithfully returns what we would have found if we decided to use the notion of *local subductivness* (Proposition **4.79**). In hindsight, this provides more justification for our choice of starting with subductions instead of local subductions. One consequence of this choice is that it allows for groupoid bundles that are truly *pseudo*-bundles, in the sense of [**Per16**]. The notion of pseudo-bundles seems to be the correct notion in the setting of diffeology to truly generalise all bundle constructions on manifolds, at least if we want to treat (internal) tangent bundles as such. For instance, the canonical projection of the internal tangent bundle on the cross (Example **2.18**) is not a *local* subduction, and can therefore not be a diffeological fibration. If we had defined principality of a groupoid bundle to include *local* subductiveness, these examples would not be treatable by our theory of Morita equivalence.

Lemma 4.74. *Let $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ be a diffeological bibundle, where $H \rightrightarrows H_0$ is a locally subductive groupoid. Then the canonical projection map $\pi_H : X \times_{H_0}^{r_X, r_X} \overline{X} \to X \otimes_H \overline{X}$ is a local subduction.*

Proof. Let $\alpha : (U_\alpha, 0) \to (X \otimes_H \overline{X}, x_1 \otimes x_2)$ be a pointed plot of the balanced tensor product. Since π_H is already a subduction, we can find a plot $\beta : V \to X \times_{H_0} \overline{X}$, defined on an open neighbourhood $0 \in V \subseteq U_\alpha$ of the origin, such that $\alpha|_V = \pi_H \circ \beta$. This plot decomposes into two plots $\beta_1, \beta_2 \in \mathcal{D}_X$ on X, satisfying $r_X \circ \beta_1 = r_X \circ \beta_2$. We use the notation $\alpha|_V = \beta_1 \otimes \beta_2$. In particular, we get an equality $x_1 \otimes x_2 = \beta_1(0) \otimes \beta_2(0)$ inside the balanced tensor product, which means that we can find an arrow $h \in H$ such that $\beta_i(0) = x_i h$. The target must be $\mathrm{trg}(h) = r_X(x_1) = r_X(x_2)$. This arrow allows us to write a pointed plot $r_X \circ \beta_i : (V, 0) \to (H_0, \mathrm{trg}(h^{-1}))$, so that now we can use that $H \rightrightarrows H_0$ is locally subductive. Since the target map of H is a local subduction, we can find a pointed plot $\Omega : (W, 0) \to (H, h^{-1})$ such that $r_X \circ \beta_i|_W = \mathrm{trg}_H \circ \Omega$. This relation means that, for every $t \in W$, we have a well-defined action $\beta_i(t) \cdot \Omega(t) \in X$. Hence we get a pointed plot

$$\Psi : (W, 0) \longrightarrow (X \times_{H_0}^{r_X, r_X} \overline{X}, (x_1, x_2)); \qquad t \longmapsto (\beta_1(t)\Omega(t), \beta_2(t)\Omega(t)) .$$

It then follows by the definition of the balanced tensor product that

$$\pi_H \circ \Psi(t) = \beta_1|_W(t)\Omega(t) \otimes \beta_2|_W(t)\Omega(t) = \beta_1|_W(t) \otimes \beta_2|_W(t) = \alpha|_W(t),$$

proving that π_H is a local subduction. □

Lemma 4.75. *If $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ is a biprincipal bibundle between locally subductive groupoids, then the moment maps l_X and r_X are local subductions as well.*

Proof. If $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ is biprincipal, we get two biequivariant diffeomorphisms $\varphi_G : X \otimes_H \overline{X} \to G$ and $\varphi_H : \overline{X} \otimes_G X \to H$ (Proposition **4.59**). It follows that the local subductivity of the source and target maps of G and H transfer to the four moment maps of the balanced tensor products. For example, the left moment map $L_X : X \otimes_H \overline{X} \to G_0$ can be written as $L_X = \mathrm{trg}_G \circ \varphi_G$, where the right hand side is clearly a local subduction. We know as well that L_X fits into a commutative square with the original moment map l_X:

$$\begin{array}{ccc} X \times_{H_0}^{r_X, r_X} \overline{X} & \xrightarrow{\pi_H} & X \otimes_H \overline{X} \\ {\scriptstyle \mathrm{pr}_1|_{X \times_{H_0} \overline{X}}} \downarrow & & \downarrow {\scriptstyle L_X} \\ X & \xrightarrow[l_X]{} & G_0. \end{array}$$

Since local subductions compose, and since by Lemma **4.74** the projection π_H is a local subduction, we find that the upper right corner $L_X \circ \pi_H$ must be a local subduction. Hence the composition $l_X \circ \mathrm{pr}_1|_{X \times_{H_0} \overline{X}}$ is a local subduction, which by Lemma **2.131** gives the local subductiveness of l_X. That r_X is a local subduction follows from a similar argument. □

This lemma provides evidence that the diffeological bibundle theory reduces to the Lie groupoid theory in the correct way. We see that the underlying groupoid bundles of any diffeological biprincipal bibundle between Lie groupoids have to be locally subductive. Therefore locally subductive Lie groupoid principal bundles defined on smooth manifolds return the traditional definition of principal Lie groupoid bundles (see e.g. [**dHo12**, Section 3.6]).

The lemma also suggests that, if we refine our notion of principality something we might call *pure-principality*, by passing from subductions to local subductions, then biprincipality between locally subductive groupoids means the same thing as this new notion of pure-principality. Let us make this precise.

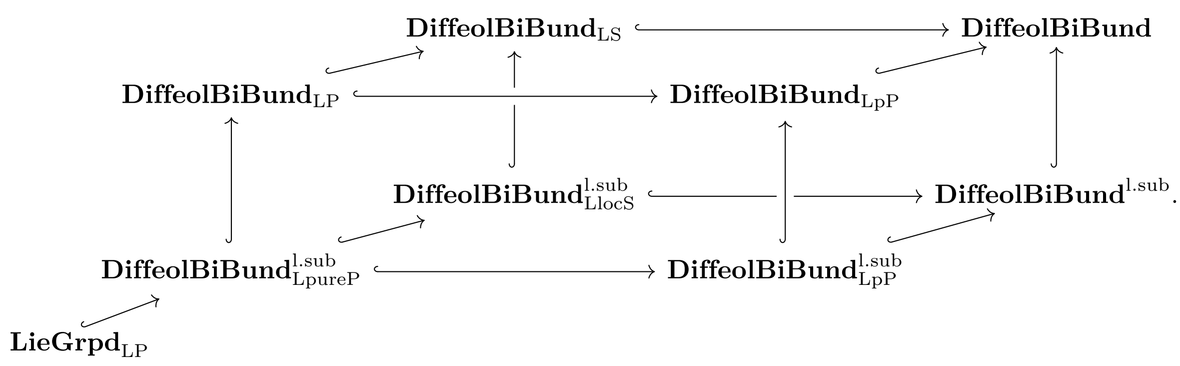
Definition 4.76. A diffeological groupoid bundle $(G \rightrightarrows G_0) \curvearrowright^{l_X} X \xrightarrow{\pi} B$ is called *locally subductive* if π is a local subduction. We say the bundle is *purely-principal* if it is both locally subductive and pre-principal.

Pure-principality can then be defined for bibundles, naturally refining Definitions **4.32** and **4.33**.

Definition 4.77. A diffeological bibundle $G \curvearrowright^{l_X} X \,{}^{r_X}\!\curvearrowleft H$ is called *left locally subductive* if the left underlying bundle $G \curvearrowright^{l_X} X \xrightarrow{r_X} H_0$ is locally subductive. This just means that r_X is a local subduction. Let the definition extend naturally to its opposite version as well, which we call *right local subductiveness.*

The bibundle is called *left purely-principal* if the left underlying bundle is purely-principal. Again, a similar definition holds for *right pure-principality.*

If we denote the bicategory of locally subductive groupoids and bibundles by $\mathbf{DiffeolBiBund}^{\mathrm{l.sub}}$, we get the following hierarchy of types of bibundles:



Definition 4.78. A diffeological bibundle $G \curvearrowright^{l_X} X \,{}^{r_X}\!\curvearrowleft H$ is called *locally bisubductive* if it is both left- and right locally subductive[46]. We call the bibundle *purely-biprincipal* if it is both left- and right purely-principal[47]. Two diffeological groupoids are called *purely-Morita equivalent* if there exists a purely-biprincipal bibundle between them.

[46] As with pre-biprincipality, we have a commutativity relation: "bi-(locally subductive) = locally (bisubductive)".
[47] Also here we have commutativity: "bi-(purely-principal) = purely-(biprincipal)".

Clearly, pure-Morita equivalence implies ordinary Morita equivalence, since local subductions are, in particular, subductions. The question is if the other implication holds as well. We suspect that it does not. For locally subductive groupoids they are the same, however, because with this new terminology, Lemma **4.75** can be restated as follows:

Proposition 4.79. *Two locally subductive groupoids are Morita equivalent if and only if they are purely-Morita equivalent.*

Especially in light of the existence of subductions that are not local subductions (Example **2.129**), and the fact that the proof of Lemma **4.75** relies so heavily on the assumption that the groupoids are locally subductive, it seems that the ordinary diffeological Morita equivalence of Definition **4.33** is not equivalent to pure-Morita equivalence in general. We do not know to what extent the theory of pure-groupoid bundles between arbitrary diffeological groupoids differs from the one we have presented here.

Given this discussion, we leave this section with an open question:

Question 4.80. *Does the inclusion pseudofunctor* $\mathbf{LieGrpd}_{\mathrm{LP}} \hookrightarrow \mathbf{DiffeolBiBund}$ *reflect weak isomorphism? That is: does diffeological Morita equivalence reduce to Lie Morita equivalence on Lie groupoids?*

Since G and H are both manifolds, it follows that $X \otimes_H \overline{X}$ and $\overline{X} \otimes_G X$ are also manifolds. We suspect that this may somehow imply that X itself must be a smooth manifold. As mentioned before, [***Diffeology***, Article 4.6] gives a characterisation for when a quotient of a diffeological space by an equivalence relation is a smooth manifold. Since the balanced tensor products are quotients of diffeological spaces, one may try to use this result to obtain a special family of plots for their underlying fibred products. This could potentially be used to define an atlas on X. Intuitively, X should look locally like a product of the orbits of G and H. That is, if $x \in X$ then there should be a D-open neighbourhood that looks like $\mathrm{Orb}_G(x) \times \mathrm{Orb}_H(x)$. Since the orbits of a Lie groupoid are immersed submanifolds [**CM18**, Proposition 2.4], can we prove a similar result for X?

Chapter V
The calculus of fractions approach to Morita equivalence

Chapter **IV** has so far been devoted to constructing and studying the bicategory **DiffeolBiBund** of diffeological groupoids and bibundles. There is another, equivalent way of constructing this bicategory (for Lie groupoids) that occurs in the literature, and gives the same notion of Morita equivalence. This alternative approach uses the idea of a *localised category* (see e.g. [**Rob12**] for the general theory). Intuitively, a localisation supplements a category with additional arrows that ensure the existence of additional inverses. The motivation for this is similar to that for the introduction of *C^*-correspondences*, in order to generalise the ordinary $*$-homomorphisms between C*-algebras. This motivation originates from a noncommutative geometric viewpoint: there are non-isomorphic C*-algebras (in the sense of $*$-isomorphisms) that describe an 'equivalent' noncommutative space (see e.g. [**vSu15**, Chapter 2] for an explanation of this). This means that the category of C*-algebras and $*$-homomorphism does not have enough structure to *reflect* certain desired isomorphisms. By generalising the notion of a $*$-homomorphism to a C*-correspondence, this glitch is remedied to some extent.

The situation is similar for groupoids. There are many non-isomorphic diffeological groupoids (or Lie groupoids for that matter) whose orbits paces are isomorphic. We have seen, for example, that the orbit space of a pair groupoid $A \times A \rightrightarrows A$ (Example **3.20**) of an arbitrary diffeological space $A \in$ **Diffeol** is always just a single-point space. However, the pair groupoids $A \times A \rightrightarrows A$ and $B \times B \rightrightarrows B$ are isomorphic in the category **DiffeolGrpd** (i.e., in the sense that there exists an invertible smooth functor between them) if and only if A and B are diffeomorphic. By the existence of non-diffeomorphic diffeological spaces, this shows there are non-isomorphic diffeological groupoids that nevertheless describe the same underlying orbit space. On the other hand, we have seen in Example **4.36** that any two pair groupoids, while not isomorphic, are Morita equivalent. This shows that the notion of a biprincipal bibundle is a more appropriate notion than that of an invertible smooth functor to describe the geometry of the orbit spaces.

The localisation of categories is an abstract way to obtain this result. The general idea of a category of fractions is the following. Instead of arrows $C \to D$ in a category $\mathbf{C}$, the morphisms from C to D are described by so-called *spans* $C \leftarrow E \to D$, where E is a third object in $\mathbf{C}$ (this is also related to the notion of an *anafunctor*). We call these *generalised morphisms*. Generally, one of the arrows in a such a span is often taken to be part of a special class of arrows W, called the *weak equivalences*. The *calculus of fractions* $C[W^{-1}]$ is then the category whose objects are just those of $\mathbf{C}$, but whose arrows are the generalised morphisms $C \leftarrow E \to D$ (and whose 2-morphisms fit into an obvious commutative diagram). The motivating example is when $\mathbf{C} = \mathbf{Cat}$, the category of all (small) categories, and W is the class of all categorical equivalences. The notion of a calculus of fractions is also important in homotopy theory [**GZ67**].

The main goal of this section is to provide the technical details of an appropriate notion of *weak equivalence* for diffeological groupoids, but will not focus on the categorical aspects of the calculus of fractions specifically. Our discussion therefore merely aims to provide a foundation for further work of a rigorous construction and study of the calculus of fractions for diffeological groupoids. We suspect that, just as the biprincipality of bibundles between diffeological groupoids behaves differently than in the case of Lie groupoids, here the notion of *weak equivalence* has a slightly different rôle than for Lie groupoids. We show that the diffeological bibundles are just geometric models for the resulting generalised arrows (Section **5.1.3**). Here we follow mostly [**dHo12**; **Li15**; **MM03**; **MM05**], where the construction of this bicategory of generalised morphisms is constructed for Lie groupoids. This also appears in e.g. [**ALR07**]. See also [**Pro96**] for a rigorous treatment and comparison of the calculus of fractions for topological groupoids. This relation appears already early on in [**Pra89**].

5.1 Weak equivalences and generalised morphisms

Just as in the generalisation of the bibundle theory from Lie groupoids to diffeological groupoids, we generalise here by replacing submersions by subductions. The first crucial definition then becomes:

Definition 5.1. A smooth functor $\phi : G \to H$ between diffeological groupoids is called a *weak equivalence* (also called *Morita maps* in [dHo12], or a *essential equivalence*) if the following two conditions are satisfied:

1. *(Essential surjectivity)* The following map is a subduction[48]:

$$\mathrm{trg} \circ \mathrm{pr}_1 : H \times_{H_0}^{\mathrm{src},\phi_0} G_0 \longrightarrow H_0; \qquad (h, x) \longmapsto \mathrm{trg}(h).$$

2. *(Full faithfulness)* The following square is a pullback diagram in **Diffeol**:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle (\mathrm{src}_G, \mathrm{trg}_G)}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{src}_H, \mathrm{trg}_H)} \\ G_0 \times G_0 & \xrightarrow[\phi_0 \times \phi_0]{} & H_0 \times H_0. \end{array}$$

Unpacking these definitions, we see immediately that these two definitions are indeed adaptations of the classical notions of essential surjectivity and full faithfulness from ordinary category to the smooth setting. First of all, let us deal with the apparent ambiguity in the essential surjectivity condition, and show that it does not matter if we swap the source and target maps.

Proposition 5.2. *Let $\phi : G \to H$ be a smooth functor. Then*

$$\mathrm{trg} \circ \mathrm{pr}_1 : H \times_{H_0}^{\mathrm{src},\phi_0} G_0 \longrightarrow H_0$$

is a subduction if and only if

$$\mathrm{src} \circ \mathrm{pr}_1 : H \times_{H_0}^{\mathrm{trg},\phi_0} G_0 \longrightarrow H_0$$

is a subduction.

Proof. It suffices to prove only one of the implications, since the other one will be completely analogous due to the symmetry between the source and target maps. Suppose first that $\mathrm{src} \circ \mathrm{pr}_1|_{H \times_{H_0} G_0}$ is a subduction, and consider a plot $\alpha : U_\alpha \to H_0$, and fix a point $t \in U_\alpha$ in the domain. Using subductiveness, we can find two plots $\beta_1 : V \to H$ and $\beta_2 : V \to G_0$, related by $\mathrm{trg} \circ \beta_1 = \phi_0 \circ \beta_2$, and satisfying $\alpha|_V = \mathrm{src} \circ \beta_1$. The inversion map $\mathrm{inv} : H \to H$ is smooth, and it helps us give a plot $(\mathrm{inv} \circ \beta_1, \beta_2) : V \to H \times_{H_0}^{\mathrm{src},\phi_0} G_0$, which now satisfies

$$\mathrm{trg} \circ \mathrm{pr}_1 \circ (\mathrm{inv} \circ \beta_1, \beta_2) = \mathrm{trg} \circ \mathrm{inv} \circ \beta_1 = \mathrm{src} \circ \beta_1 = \alpha|_V.$$

This shows that $(\mathrm{inv} \circ \beta_1, \beta_2)$ is the desired lift, and proves that $\mathrm{trg} \circ \mathrm{pr}_1|_{H \times_{H_0} G_0}$ is also a subduction. ☐

With that out of the way, we can now define the arrows in a localised category of diffeological groupoids.

Definition 5.3. Consider two diffeological groupoids $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$. A *generalised smooth morphism* from G to H is a span of smooth functors $G \xleftarrow{\phi} K \xrightarrow{\psi} H$, where K is a third diffeological groupoid. (In category theory, such spans are also sometimes called *anafunctors*.) Note that the groupoid K is part of the data. In the case of Lie groupoids it is typical to assume that ϕ (or ψ) is a weak equivalence. This is to ensure that the composition of these general morphisms can be defined (cf. Construction 5.8). For diffeological groupoids the existence of this construction is ensured without any assumptions on ϕ and ψ. The analogy is to the principality of our bibundles, which in the Lie

[48]Note that, even though $\mathrm{pr}_1 : H \times G_0 \to H$ and trg are both subductions, we must heed our earlier precautions, and remark that we are dealing here with the restriction $\mathrm{trg} \circ \mathrm{pr}_1|_{H \times_{H_0}^{\mathrm{src},\phi_0} G_0}$, which is not automatically subductive.

case is necessary, but in the diffeological case is not. This analogy will be spelled out more explicitly in Section **5.1.3**.

Two diffeological groupoids $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ are called *Morita equivalent* if there exists a span $G \xleftarrow{\phi} K \xrightarrow{\psi} H$ where both ϕ and ψ are weak equivalences.

In Theorem **5.14** we will prove that this notion of Morita equivalence is the same as the definition of Morita equivalence in terms of biprincipal bibundles. Note in particular that weak equivalences themselves induce Morita equivalences:

Proposition 5.4. *If there exists a weak equivalence $\phi : G \to H$ between diffeological groupoids, then G and H are Morita equivalent.*

Proof. Since the identity functor $\mathrm{id}_H : H \to H$ is a weak equivalence, if there exists a weak equivalence $\phi : G \to H$, then G and H are already Morita equivalent through the span $G \xleftarrow{\phi} H \xrightarrow{\mathrm{id}_H} H$. □

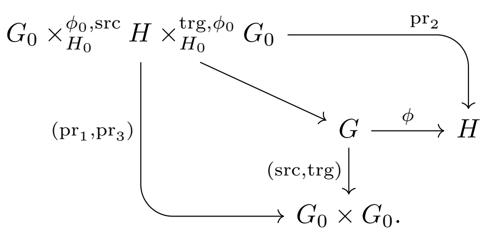
5.1.1 Technical properties of weak equivalences

Recall from Definition **3.18** that if ϕ and ψ are two smooth functors $G \to H$ between diffeological groupoids, then a natural transformation $T : \phi \to \psi$ is smooth if the underlying map $G_0 \to H$ is smooth. In particular, every natural transformation between functors whose codomain is a groupoid is automatically a natural isomorphism. In the 2-category **DiffeolGrpd** of diffeological groupoids and smooth functors, all smooth natural transformations are hence natural isomorphisms. This 2-category then gives us a notion of equivalence as follows: a smooth functor $\phi : G \to H$ is called a *smooth categorical equivalence* (*strong equivalence* in [**MM03**]) if there exists another smooth functor $\psi : H \to G$ and two smooth natural isomorphisms $T : \phi \circ \psi \to \mathrm{id}_H$ and $S : \psi \circ \phi \to \mathrm{id}_G$. In the smooth context, this notion of equivalence is no longer the same as a functor being essentially surjective and fully faithful, yet we still have:

Proposition 5.5. *Let $\phi : G \to H$ be a smooth categorical equivalence. Then ϕ is a weak equivalence.*

Proof. In that case, we know there exists a smooth functor $\psi : H \to G$ and a smooth natural isomorphism $T : \phi \circ \psi \to \mathrm{id}_H$. We prove first that ϕ is essentially surjective, so pick a plot $\alpha : U_\alpha \to H_0$. Each $t \in U_\alpha$ thus gives an arrow $T_{\alpha(t)} : \phi_0\psi_0\alpha(t) \to \alpha(t)$ in H, and it is easy to see that we get a smooth plot $t \mapsto (T_\alpha(t), \psi_0\alpha(t))$ of $H \times^{\mathrm{src},\phi_0}_{H_0} G_0$ that lifts α (globally) along $\mathrm{trg} \circ \mathrm{pr}_1$.

For full faithfulness, it suffices to construct a diffeomorphism $G \to G_0 \times^{\phi_0,\mathrm{src}}_{H_0} H \times^{\mathrm{trg},\phi_0}_{H_0} G_0$ that fits into a commutative diagram:



This is because $G_0 \times^{\phi_0,\mathrm{src}}_{H_0} H \times^{\mathrm{trg},\phi_0}_{H_0} G_0$ is exactly the pullback of the square in Definition **5.1**. This diffeomorphism is given by $g \mapsto (\mathrm{src}(g), \phi g, \mathrm{trg} g)$, and has smooth inverse

$$(x, h, y) \longmapsto T_y \circ \psi h \circ T_x^{-1},$$

which clearly makes the diagram commute. □

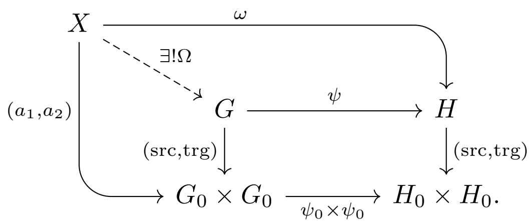
Proposition 5.6. *Let $\phi, \psi : G \to H$ be two smooth functors between diffeological groupoids, admitting a smooth natural transformation $T : \phi \to \psi$. Then ϕ is a weak equivalence if and only if ψ is a weak equivalence.*

Proof. Since T is a natural isomorphism, it suffices to prove one of the implications. Let us therefore start with a weak equivalence ϕ. We begin by proving that ψ is essentially surjective. For that, take a plot $\alpha : U_\alpha \to H_0$ of the object space. Since ϕ is essentially surjective, we can find a plot $(\beta_1, \beta_2) : V \to H \times_{H_0}^{\mathrm{src},\phi_0} G_0$, defined on some open neighbourhood $t \in V \subseteq U_\alpha$ of an arbitrary point in the domain of α, and satisfying $\mathrm{trg} \circ \beta_1 = \alpha|_V$. From this data, using the smooth natural map T, we can construct a lift of $\mathrm{trg} \circ \mathrm{pr}_1 : H \times_{H_0}^{\mathrm{src},\psi_0} G_0 \to H_0$. Abusing notation, let us denote the underlying smooth function of the natural transformation just by $T : G_0 \to H$. The plot β_2 then gives a plot $T \circ \beta_2 : V \to H$. We want a smooth family of arrows in H whose targets are given by $\alpha|_V(t)$, and whose source is controlled by β_2. For this, we define:

$$\omega_1 : V \longrightarrow H; \qquad t \mapsto \beta_1(t) \circ T_{\beta_2(t)}^{-1}.$$

This is clearly a smooth map, since T and the composition and inverse in H are smooth. An easy calculation then shows that $\mathrm{src} \circ \omega_1 = \psi_0 \circ \beta_2(t)$, since $T_{\beta_2(t)} : \phi_0\beta_2(t) \to \psi_0\beta_2(t)$, and similarly we get $\mathrm{trg} \circ \omega_1 = \mathrm{trg} \circ \beta_1 = \alpha|_V$. This shows that we get a plot $(\omega_1, \beta_2) : V \to H \times_{H_0}^{\mathrm{src},\psi_0} G_0$ that defines a local lift of α along $\mathrm{trg} \circ \mathrm{pr}_1 : H \times_{H_0}^{\mathrm{src},\psi_0} G_0 \to H_0$. Therefore ψ is essentially surjective.

We now prove that ψ is fully faithful. The naturality of T means that for every arrow $g \in G$ we can decompose $\phi(g) = T_{\mathrm{trg}(g)}^{-1} \circ \psi(g) \circ T_{\mathrm{src}(g)}$. We will use this trick to prove that the following inner square is a pullback:



We start here with two smooth maps $\omega : X \to H$ and $(a_1, a_2) : X \to G_0 \times G_0$ making the outer square commute. This means that $\omega(x)$ is an arrow in H with source $\psi_0 a_1(x)$ and target $\psi_0 a_2(x)$. Therefore, using the above decomposition, we define

$$\overline{\omega} : X \longrightarrow H; \qquad x \longmapsto T_{a_2(x)}^{-1} \circ \omega(x) \circ T_{a_1(x)},$$

which satisfies $\mathrm{src} \circ \overline{\omega} = \phi_0 a_1(x)$ and $\mathrm{trg} \circ \overline{\omega} = \phi_0 a_2(x)$, or in other words:

$$(\mathrm{src}, \mathrm{trg}) \circ \overline{\omega} = (\phi_0 \times \phi_0) \circ (a_1, a_2),$$

and since ϕ is fully faithful, there exists a unique smooth map $\overline{\Omega} : X \to G$ such that $(\mathrm{src}, \mathrm{trg}) \circ \Omega = (a_1, a_2)$ and $\phi \circ \Omega = \overline{\omega}$. Transforming back using the natural map T, using $\psi(g) = T_{\mathrm{trg}(g)} \circ \phi(g) \circ T_{\mathrm{src}(g)}^{-1}$, we find:

$$\psi\Omega(x) = T_{a_2(x)} \circ \phi\Omega(x) \circ T_{a_1(x)}^{-1} = T_{a_2(x)} \circ \overline{\omega}(x) \circ T_{a_1(x)}^{-1} = \omega(x),$$

which proves that Ω is also the unique map completing the pullback square of ψ. Hence we conclude that ψ is also a weak equivalence. □

Proposition 5.7. *The composition of weak equivalences is again a weak equivalence.*

Proof. Let $\phi : G \to H$ and $\psi : H \to K$ be two weak equivalences between diffeological groupoids. We prove that $\psi \circ \phi$ is also a weak equivalence. We start with essential surjectivity, for which we need to show that

$$\mathrm{trg} \circ \mathrm{pr}_1 : K \times_{K_0}^{\mathrm{src},\psi_0\phi_0} G_0 \longrightarrow K_0$$

is a subduction. Let $\alpha : U_\alpha \to K_0$ be a plot of the object space of the third groupoid, and fix some point $t \in U_\alpha$. Since ψ is essentially surjective, there exists a plot $(\beta_1, \beta_2) : V \to K \times_{K_0}^{\mathrm{src},\psi_0} H_0$ defined on some open neighbourhood $t \in V \subseteq U_\alpha$, which lifts $\alpha|_V = \mathrm{trg} \circ \beta_1$. In particular we have a plot $\beta_2 : V \to H_0$, so by the essential surjectivity of the first functor ϕ we get another plot $\omega : W \to H \times_{H_0}^{\mathrm{src},\phi_0} G_0$ defined on an open neighbourhood $t \in W \subseteq V$, and satisfying $\mathrm{trg} \circ \omega_1 = \beta_2|_W$. We now have all the data we need to define a lift for the essential surjectivity of $\psi \circ \phi$. Namely, we define a plot

$$\Omega : W \longrightarrow K; \qquad s \longmapsto \beta_1(t) \circ \psi\omega_1(t).$$

Note that this composition is well-defined in K, because

$$\mathrm{src} \circ \beta_1|_W = \psi_0 \circ \beta_2|_W = \psi_0 \circ \mathrm{trg} \circ \omega_1.$$

It is now easy to verify that $(\Omega, \omega_2) : W \to K \times_{K_0}^{\mathrm{src},\psi_0\phi_0} G_0$ defines the appropriate lift, since $\mathrm{trg} \circ \Omega = \alpha|_W$.

For the proof of full faithfulness, we note that we get a concatenation of two pullback squares:

$$\begin{array}{ccccc} G & \xrightarrow{\phi} & H & \xrightarrow{\psi} & K \\ {\scriptstyle (\mathrm{src},\mathrm{trg})}\downarrow & \lrcorner & {\scriptstyle (\mathrm{src},\mathrm{trg})}\downarrow & \lrcorner & \downarrow{\scriptstyle (\mathrm{src},\mathrm{trg})} \\ G_0 \times G_0 & \xrightarrow[\phi_0 \times \phi_0]{} & H_0 \times H_0 & \xrightarrow[\psi_0 \times \psi_0]{} & K_0 \times K_0, \end{array}$$

which proves through elementary category theory that the outer square, exactly the one we need, is also a pullback. This shows that $\psi \circ \phi$ is fully faithful, and hence we are done. □

5.1.2 Weak pullbacks

One of the crucial ingredients in the calculus of fractions approach is that of the *weak pullback*. This is, for every pair of smooth functors $\phi : G \to K$ and $\psi : H \to K$, another diffeological groupoid $G \times_K^{\phi,\psi} H$, that satisfies a certain universal pullback condition up to natural isomorphism. We adapt the construction and notation as in [**MM03**, Section 5.3], and generalise to the diffeological setting.

Construction 5.8. Consider two smooth functors $\phi : G \to K$ and $\psi : H \to K$ between diffeological groupoids. We construct the *weak pullback* (also called *weak fibred product* in [**MM03**], or *homotopy pullback* in [**dHo12**]). The space of objects of this diffeological groupoid is

$$\left(G \times_K^{\phi,\psi} H\right)_0 := G_0 \times_{K_0}^{\phi_0,\mathrm{src}} K \times_{K_0}^{\mathrm{trg},\psi_0} H.$$

This is the space of triples (x, k, y), which we think of as arrows in K of the form $k : \phi_0 x \to \psi_0 y$, where $x \in G_0$ and $y \in H_0$. Given two such arrows, $k_1 : \phi_0 x_1 \to \psi_0 y_1$ and $k_2 : \phi_0 x_2 \to \psi_0 y_2$, an arrow in $G \times_K^{\phi;\psi} H$ is a pair $(g, h) \in G \times H$ fitting into a commutative square:

$$\begin{array}{ccc} \phi_0 x_1 & \xrightarrow{k_1} & \psi_0 y_1 \\ {\scriptstyle \phi g}\downarrow & & \downarrow{\scriptstyle \psi h} \\ \phi_0 x_2 & \xrightarrow[k_2]{} & \psi_1 y_2. \end{array}$$

Note that the arrow k_2 is completely determined by this commuting square, as it is equal to the composition $k_2 = \psi h \circ k_1 \circ \phi g^{-1}$. Explicitly, the space of arrows can then be written as triples (g, k, h) in the following fibred product:

$$G \times_K^{\phi,\psi} H := G \times_{K_0}^{\mathrm{src}\phi,\mathrm{src}} K \times_{K_0}^{\mathrm{trg},\mathrm{src}\psi} H.$$

The structure maps are then defined in the following way:

$$\mathrm{src}(g, k, h) := (\mathrm{src}(g), k, \mathrm{src}(h)) \qquad \text{and} \qquad \mathrm{trg}(g, k, h) := (\mathrm{trg}(g), \psi h \circ k \circ \phi g^{-1}, \mathrm{trg}(h)).$$

Note the way the arrow in K changes according to the decomposition of k_2 by the commutative square above. The composition in $G \times_K^{\phi,\psi} H$ is then defined as the component-wise composition in G and H:

$$(g_2, k_2, h_1) \circ (g_1, k_1, h_1) := (g_2 \circ g_1, k_1, h_2 \circ h_1).$$

It is clear that this defines a diffeological groupoid $G \times_K^{\phi,\psi} H$.

Associated to the weak pullback $G \times_K^{\phi,\psi} H$ we get two *projection functors*, defined as follows. The projection onto G is a smooth functor $\mathrm{pr}_1 : G \times_K^{\phi,\psi} H \to G$, whose underlying maps are both just the projection onto the first component of the fibred products. Similarly, we get a smooth projection functor $\mathrm{pr}_3 : G \times_K^{\phi,\psi} H \to H$. We then get a 2-commutative square in the 2-category **DiffeolGrpd**:

$$\begin{array}{ccc} G \times_K^{\phi,\psi} H & \xrightarrow{\mathrm{pr}_3} & H \\ {\scriptstyle \mathrm{pr}_1}\downarrow & \overset{T}{\nearrow} & \downarrow{\scriptstyle \psi} \\ G & \xrightarrow[\phi]{} & K, \end{array}$$

where the natural transformation $T : \phi \circ \mathrm{pr}_1 \to \psi \circ \mathrm{pr}_3$ is given by $T_{(x,k,y)} := k$.

Proposition 5.9. *Weak equivalences are preserved under weak pullbacks.*

Proof. Let $\phi : G \to K$ and $\psi : H \to K$ be two smooth functors between diffeological groupoids such that ψ is a weak equivalence. The claim is that the projection functor $\mathrm{pr}_1 : G \times_K^{\phi,\psi} H \to G$ is also a weak equivalence. First, let us show that it is essentially surjective, for which we need to prove that the map

$$\mathrm{trg} \circ \mathrm{pr}_1 : G \times_{G_0}^{\mathrm{src},\mathrm{pr}_1} \left(G \times_K^{\phi,\psi} H\right)_0 \longrightarrow G_0; \qquad (g,(x,k,y)) \longmapsto \mathrm{trg}(g)$$

is a subduction. As always, let us start with a plot $\alpha : U_\alpha \to G_0$, which induces a plot $\phi_0 \circ \alpha : U_\alpha \to K_0$. Since ψ is a weak equivalence (and using Proposition **5.2**), for any point $t \in U_\alpha$ in the domain of this plot we can find a plot $\beta : V \to K \times_{K_0}^{\mathrm{trg},\psi_0} H_0$ such that $\mathrm{src} \circ \beta_1 = \phi_0 \circ \alpha|_V$, where $t \in V \subseteq U_\alpha$ is some open neighbourhood. The codomain of β ensures moreover that $\mathrm{trg} \circ \beta_1 = \psi_0 \circ \beta_2$, so that for every $t \in V$ we have an arrow $\beta_1(t) : \phi_0\alpha(t) \to \psi_0\beta_2(t)$. These are exactly the types of arrows that form the object space of $G \times_K^{\phi,\psi} H$, and so we can define a plot

$$\Omega : V \longrightarrow G \times_{G_0}^{\mathrm{src},\mathrm{pr}_1} \left(G \times_K^{\phi,\psi} H\right)_0; \qquad t \longmapsto \left(\mathrm{id}_{\alpha|_V(t)}, (\alpha|_V(t), \beta_1(t), \beta_2(t))\right).$$

It is easy to check that $\mathrm{trg} \circ \mathrm{pr}_1 \circ \Omega = \alpha|_V$, as desired. This proves that $\mathrm{pr}_1 : G \times_K^{\phi,\psi} H \to G$ is essentially surjective. We are left to show that the commuting diagram

$$\begin{array}{ccc} G \times_K^{\phi,\psi} H & \xrightarrow{\mathrm{pr}_1} & G \\ {\scriptstyle (\mathrm{src},\mathrm{trg})}\downarrow & & \downarrow{\scriptstyle (\mathrm{src},\mathrm{trg})} \\ (G \times_K^{\phi,\psi} H)_0 \times (G \times_K^{\phi,\psi} H)_0 & \xrightarrow[\mathrm{pr}_1 \times \mathrm{pr}_1]{} & G_0 \times G_0 \end{array}$$

is a pullback in the category of diffeological spaces. For this, consider the following commutative diagram:

$$\begin{array}{cccc} M & & \xrightarrow{\quad\alpha\quad} & \\ & G \times_K^{\phi,\psi} H & \xrightarrow{\mathrm{pr}_1} & G \\ {\scriptstyle (\beta,\gamma)}\downarrow & {\scriptstyle (\mathrm{src},\mathrm{trg})}\downarrow & & \downarrow{\scriptstyle (\mathrm{src},\mathrm{trg})} \\ \hookrightarrow & (G \times_K^{\phi,\psi} H)_0 \times (G \times_K^{\phi,\psi} H)_0 & \xrightarrow[\mathrm{pr}_1 \times \mathrm{pr}_1]{} & G_0 \times G_0. \end{array}$$

In other words, we have smooth functions

$$\alpha : M \longrightarrow G \qquad \text{and} \qquad \beta, \gamma : M \longrightarrow (G \times_K^{\phi,\psi} H)_0.$$

The latter two are of the form $\beta = (\beta^G, \beta^K, \beta^H)$ and $\gamma = (\gamma^G, \gamma^K, \gamma^H)$. Commutativity of the outside square then gives $\mathrm{src} \circ \alpha(m) = \beta^G(m)$ and $\mathrm{trg} \circ \alpha(m) = \gamma^G(m)$ for all $m \in M$.

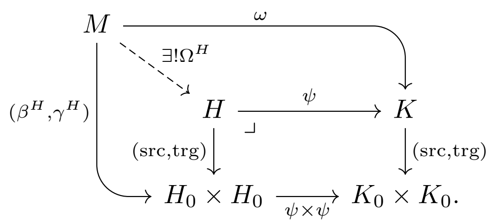
We construct a unique smooth map $\Omega : M \to G \times_K^{\phi,\psi} H$ completing the diagram. This map is necessarily of the form $\Omega = (\Omega^G, \Omega^K, \Omega^H)$, for some smooth functions $\Omega^G : M \to G$, $\Omega^K : M \to K$ and $\Omega^H : M \to H$. In order for the diagram to commute, we need to have $\mathrm{src} \circ \Omega(m) = \beta(m)$ and $\mathrm{trg} \circ \Omega(m) = \gamma(m)$, and $\mathrm{pr}_1 \circ \Omega(m) = \Omega^G(m) = \alpha(m)$. From the latter it is clear we should set $\Omega^G = \alpha$. Expanding on the first equation, we can also see that we need to set $\Omega^K = \beta^K$. We are thus left to construct the smooth function Ω^H in such a way that

$$\begin{array}{ccc} \phi\beta^G(m) & \xrightarrow{\phi\alpha(m)} & \phi\gamma^G(m) \\ {\scriptstyle \beta^K(m)}\big\downarrow & & \big\downarrow{\scriptstyle \gamma^H(m)} \\ \psi\beta^H(m) & \xrightarrow[\psi\Omega^H(m)]{} & \psi\gamma^H(m) \end{array}$$

commutes for all $m \in M$. Since every arrow is invertible in a groupoid, the bottom arrow in this square (were it to commute) could be expressed in terms of the other three. With this observation in mind, define the function

$$\omega : M \longrightarrow K; \qquad m \longmapsto \gamma^K(m) \circ \phi\alpha(m) \circ \beta^K(m)^{-1},$$

which gives to each $m \in M$ an arrow $\psi\beta^H(m) \to \psi\gamma^H(m)$ in K. It is evident that ω is smooth, as it is a composition of the structure maps of K and the smooth maps ϕ, α, β^K and γ^K. At this point we use the fact that ψ is a weak equivalence. Namely, we have a commutative diagram:



The resulting smooth map $\Omega = (\Omega^G, \Omega^K, \Omega^H) = (\alpha, \beta^K, \Omega^H)$ therefore provides the unique smooth map $M \to G \times_K^{\phi,\psi} H$ completing the diagram, and proving that the original square is a pullback in the category of diffeological spaces. We conclude that the smooth projection functor $\mathrm{pr}_1 : G \times_K^{\phi,\psi} H \to G$ is a weak equivalence, as was to be proved. □

5.1.3 A dictionary between bibundles and generalised morphisms

We now give two constructions that allow us to translate between bibundles and generalised morphisms. These constructions can be used to prove that there is a bicategorical equivalence between **DiffeolBiBund** and the bicategory of generalised morphisms between diffeological groupoids (with an appropriately defined notion of 2-morphism). We do not prove this here, but we suspect that their proofs are similar to the corresponding claim for Lie groupoids (cf. [**Li15**, Section 1.5] and references therein).

The idea is that a generalised morphism, through *bundlisation* (Construction **4.41**), will give rise to a bibundle, and that a bibundle, through a *simultaneous action groupoid*, will give rise to a generalised morphism. Before we describe the general case, let us give a demonstration with a simple example. For every diffeological groupoid $G \rightrightarrows G_0$ we have the distinguished smooth identity functor $\mathrm{id}_G : G \to G$, which gives the generalised identity morphism $G \xleftarrow{\mathrm{id}_G} G \xrightarrow{\mathrm{id}_G} G$, and from Example **4.34** we know there is also an identity bibundle $G \curvearrowright^{\mathrm{trg}} G \,^{\mathrm{src}}\!\curvearrowleft G$. Now, note that the bundlisation $B(\mathrm{id}_G)$ of the identity functor is just the space of pairs $(\mathrm{trg}(g), g)$, where $g \in G$. The corresponding left G-action is given by:

$$G \curvearrowright^{\mathrm{pr}_1} B(\mathrm{id}_G); \qquad g \cdot (\mathrm{trg}(h), h) := (\mathrm{trg}(g), g \circ h),$$

which is just a left multiplication that keeps track of the target of the arrow. Similarly, we have

$$B(\mathrm{id}_G) \,^{\mathrm{src} \circ \mathrm{pr}_2}\!\curvearrowleft G; \qquad (\mathrm{trg}(g), g) \cdot h := (\mathrm{trg}(g), g \circ h),$$

which is just right multiplication. It is easy to see that the second projection $\mathrm{pr}_2 : B(\mathrm{id}_G) \to G$ defines a biequivariant diffeomorphism:

$$\begin{array}{c} G \overset{\mathrm{pr}_1}{\curvearrowright} B(\mathrm{id}_G) \overset{\mathrm{src}}{\curvearrowleft} G \\ \mathrm{pr}_2 \Big\Downarrow \\ G \overset{\mathrm{trg}}{\curvearrowright} G \overset{\mathrm{src}}{\curvearrowleft} G. \end{array}$$

This shows that the generalised morphism $G \xleftarrow{\mathrm{id}_G} G \xrightarrow{\mathrm{id}_G} G$ is 'equivalent' to the identity bibundle $G \overset{\mathrm{trg}}{\curvearrowright} G \overset{\mathrm{src}}{\curvearrowleft} G$. On the other hand, if we start out with the identity bibundle $G \overset{\mathrm{trg}}{\curvearrowright} G \overset{\mathrm{src}}{\curvearrowleft} G$, we get two groupoid actions. Both define an action groupoid: the left multiplication gives $G \ltimes G \rightrightarrows G$, and the right multiplication gives $G \rtimes G \rightrightarrows G$. But since the left- and right multiplication commute, we in fact get a *simultaneous action groupoid* $G \ltimes G \rtimes G \rightrightarrows G$, whose morphisms are triples (g, k, h) such that the composition $g \circ k \circ h$ is well-defined. We then define

$$\mathrm{src}(g, k, h) := k \qquad \text{and} \qquad \mathrm{trg}(g, k, h) := g \circ k \circ h,$$

generalising the definition of an action groupoid. With the obvious composition

$$(g', g \circ k \circ h, h') \circ (g, k, h) := (g' \circ g, k, h \circ h'),$$

this again becomes a diffeological groupoid. By projecting to the components of this simultaneous action groupoid through the functors

$$\mathrm{pr}_1 : G \ltimes G \rtimes G \longrightarrow G; \qquad \mathrm{pr}_1(g, k, h) := g, \quad (\mathrm{pr}_1)_0(k) := \mathrm{trg}(g),$$

and

$$\mathrm{pr}_3 : G \ltimes G \rtimes G \longrightarrow G; \qquad \mathrm{pr}_3(g, k, h) := h^{-1}, \quad (\mathrm{pr}_3)_0(k) := \mathrm{src}(k),$$

we get a generalised morphism:

$$G \xleftarrow{\mathrm{pr}_1} G \ltimes G \rtimes G \xrightarrow{\mathrm{pr}_3} G.$$

we can then define a functor

$$I : G \longrightarrow G \ltimes G \rtimes G; \qquad I(g) := (g, \mathrm{id}_{\mathrm{src}(g)}, g^{-1}), \quad I_0(x) := \mathrm{id}_x,$$

that fits into a commutative diagram of the following sort:

$$\begin{array}{ccccc} & & G \ltimes G \rtimes G & & \\ & \overset{\mathrm{pr}_1}{\swarrow} & \uparrow I & \overset{\mathrm{pr}_3}{\searrow} & \\ G & & & & G. \\ & \underset{\mathrm{id}_G}{\nwarrow} & & \underset{\mathrm{id}_G}{\nearrow} & \\ & & G & & \end{array}$$

This establishes the 'equivalence' between the identity bibundle $G \overset{\mathrm{trg}}{\curvearrowright} G \overset{\mathrm{src}}{\curvearrowleft} G$ and the generalised morphism $G \xleftarrow{\mathrm{id}_G} G \xrightarrow{\mathrm{id}_G} G$ in the other direction. We will not define in general what a morphism of generalised morphisms is, but it should look something like the commutative diagram above. Let us now treat the general constructions.

We start by constructing a bibundle from a generalised morphism. Recall the notion of *bundlisation* from Construction **4.41**, where for every smooth functor $\phi : G \to H$ we get a right principal bibundle $G \overset{l_\phi}{\curvearrowright} B(\phi) \overset{r_\phi}{\curvearrowleft} H$, where $B(\phi) := G_0 \times^{\phi_0, \mathrm{trg}}_{H_0} H$, the moment maps are defined as

$$l_\phi := \mathrm{pr}_1|_{B(\phi)} \qquad \text{and} \qquad r_\phi := \mathrm{src} \circ \mathrm{pr}_2|_{B(\phi)},$$

and the actions are given by

$$\begin{aligned} G \overset{l_\phi}{\curvearrowright} B(\phi); \qquad & g \cdot (x, h) := (\mathrm{trg}(g), \phi g \circ h), \\ B(\phi) \overset{r_\phi}{\curvearrowleft} H; \qquad & (x, h_1) \cdot h_2 := (x, h_1 \circ h_2). \end{aligned}$$

The following lemma tells us exactly when this bibundle is left-, and hence *bi*principal. For Lie groupoids, this is proven in e.g. [**Mrc96**, Proposition II.1.6].

Lemma 5.10. *Consider a smooth functor $\phi : G \to H$ and its bundlisation $G \curvearrowleft^{l_\phi} B(\phi)\, {}^{r_\phi}\!\curvearrowright H$. Then:*

1. *The bundlisation $G \curvearrowleft^{l_\phi} B(\phi)\, {}^{r_\phi}\!\curvearrowright H$ is left subductive if and only if ϕ is essentially surjective.*
2. *The bundlisation $G \curvearrowleft^{l_\phi} B(\phi)\, {}^{r_\phi}\!\curvearrowright H$ is left pre-principal if and only if ϕ is fully faithful.*
3. *Together: $G \curvearrowleft^{l_\phi} B(\phi)\, {}^{r_\phi}\!\curvearrowright H$ is biprincipal if and only if ϕ is a weak equivalence.*

Proof. (*1*). To prove that the bundlisation is left subductive, we need to show that the right moment map $r_\phi : B(\phi) \to H_0$ is a subduction. However, we note that r_ϕ is, up to a canonical diffeomorphism, exactly the function $\mathrm{src} \circ \mathrm{pr}_1 : H \times_{H_0}^{\mathrm{trg},\phi_0} G_0 \to H_0$ appearing in the characterisation Proposition **5.2** of essentially surjective functors. From that it is easy to see that r_ϕ is subductive if and only if ϕ is essentially surjective.

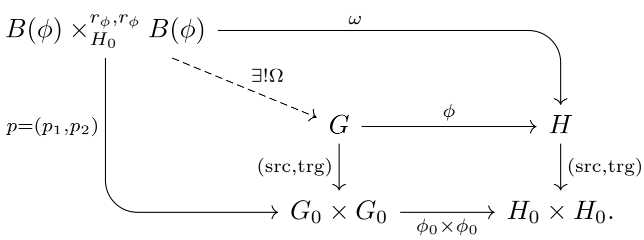
(*2*). To begin, let us start with the assumption that ϕ is fully faithful. We need to construct a smooth inverse for the action map

$$A : G \times_{G_0}^{\mathrm{src},l_\phi} B(\phi) \longrightarrow B(\phi) \times_{H_0}^{r_\phi,r_\phi} B(\phi); \qquad (g,(x,h)) \longmapsto ((\mathrm{trg}(g), \phi g \circ h),(x,h)).$$

To do that, note that the space $B(\phi) \times_{H_0}^{r_\phi,r_\phi} B(\phi)$ contains elements of the form $((x_1,h_1),(x_2,h_2))$, satisfying $\mathrm{src}(h_1) = \mathrm{src}(h_2)$ and $\mathrm{trg}(h_i) = \phi_0 x_i$. Such a quadruple therefore defines $h_2 \circ h_1^{-1} : \phi_0 x_1 \to \phi_0 x_2$ in H. Constructing an inverse for the action map A then amounts to finding (in a smooth way) a unique arrow $g \in G$ such that $\phi g = h_2 \circ h_1^{-1}$. We therefore define the following two smooth maps:

$$\omega : B(\phi) \times_{H_0}^{r_\phi,r_\phi} B(\phi) \longrightarrow H; \qquad ((x_1,h_1),(x_2,h_2)) \longmapsto h_2 \circ h_1^{-1},$$
$$p : B(\phi) \times_{H_0}^{r_\phi,r_\phi} B(\phi) \longrightarrow G_0 \times G_0; \qquad ((x_1,h_1),(x_2,h_2)) \longmapsto (x_1,x_2).$$

It is easy to see that $(\mathrm{src},\mathrm{trg}) \circ \omega = (\phi_0 \times \phi_0) \circ p$, so since ϕ is fully faithful we get the following completion of a pullback diagram:



The smooth map Ω resembles an division map, and we can therefore use it construct an inverse for the action map. Before we do that, we must describe some of its properties. For that, we define another smooth map

$$\overline{\Omega} : G \times_{G_0}^{\mathrm{src},l_\phi} B(\phi) \longrightarrow G; \qquad (g,(x,h)) \longmapsto g^{-1}.$$

It is easy to check that $\overline{\Omega}$ is a smooth map satisfying $\phi \circ \overline{\Omega} = \omega \circ A$ and $(\mathrm{src},\mathrm{trg}) \circ \overline{\Omega} = p \circ A$. However, again using that ϕ is fully faithful, we can see that $\Omega \circ A$ is supposed to be that unique such map. We therefore get $\Omega \circ A = \overline{\Omega}$, which gives the following important equation:

$$\Omega((\mathrm{trg}(g), \phi g \circ h),(x,h)) = \Omega \circ A(g,(x,h)) = \overline{\Omega}(g,(x,h)) = g^{-1}. \qquad (\clubsuit)$$

We are now ready to construct a smooth inverse for the action map A. We define:

$$\Psi := \left(\Omega^{-1}, \mathrm{pr}_1|_{B(\phi)\times_{H_0}B(\phi)}\right) : B(\phi) \times_{H_0}^{r_\phi,r_\phi} B(\phi) \longrightarrow G \times_{G_0}^{\mathrm{src},l_\phi} B(\phi),$$

where we denote by $\Omega^{-1} := \mathrm{inv} \circ \Omega$ the point-wise inverse of Ω. All that is left to prove is that Ψ forms an inverse for A. On the one hand, for $(g,(x,h)) \in G \times_{G_0}^{\mathrm{src},l_\phi} B(\phi)$ we get:

$$\Psi \circ A(g,(x,h)) = \left(\Omega^{-1} \circ A(g,(x,h)),(x,h)\right) \overset{(\clubsuit)}{=} (g,(x,h)),$$

as desired. And on the other hand, for $((x_1,h_1),(x_2,h_2)) \in B(\phi) \times_{H_0}^{r_\phi,r_\phi} B(\phi)$ we get:

$$\begin{aligned} A \circ \Psi\left((x_1,h_1),(x_2,h_2)\right) &= A\left(\Omega^{-1}((x_1,h_1),(x_2,h_2)),(x_2,h_2)\right) \\ &= \left(\left(\mathrm{src}\circ\Omega((x_1,h_1),(x_2,h_2)), \phi\Omega^{-1}((x_1,h_1),(x_2,h_2))\circ h_2\right),(x_2,h_2)\right) \\ &= \left(\left(p_1((x_1,h_1),(x_2,h_2)), f((x_1,h_1),(x_2,h_2))^{-1}\circ h_2\right),(x_2,h_2)\right) \\ &= \left((x_1,(h_2\circ h_1^{-1})^{-1}\circ h_2),(x_2,h_2)\right) \\ &= ((x_1,h_1),(x_2,h_2)). \end{aligned}$$

We may conclude that if ϕ is fully faithful, then the bundlisation is left pre-principal.

To finish the proof of claim (*2*), we need to prove that if $G \curvearrowright^{l_\phi} B(\phi)\, {}^{r_\phi}\!\curvearrowleft H$ is left pre-principal, then ϕ is fully faithful. To show this, consider a commuting diagram of diffeological spaces:

$$\begin{array}{ccccc} X & & \xrightarrow{\quad\omega\quad} & & \\ & \searrow^{p=(p_1,p_2)} & & & \downarrow \\ & & G & \xrightarrow{\phi} & H \\ & & \downarrow{\scriptstyle(\mathrm{src},\mathrm{trg})} & & \downarrow{\scriptstyle(\mathrm{src},\mathrm{trg})} \\ & & G_0\times G_0 & \xrightarrow[\phi_0\times\phi_0]{} & H_0\times H_0. \end{array}$$

Given the data of the two smooth maps p and ω, for each $x \in X$ we get an arrow $\omega(x) : \phi_0 p_1(x) \to \phi_0 p_2(x)$ in H. We use this data to define the following smooth map:

$$\Gamma : X \longrightarrow B(\phi) \times_{H_0}^{r_\phi,r_\phi} B(\phi); \qquad x \longmapsto \left((p_2(x),\omega(x)),(p_1(x),\mathrm{id}_{\phi_0 p_1(x)})\right).$$

It is easy to verify that Γ lands in the right space, since $\mathrm{src}\circ\omega = \phi_0\circ p_1$ and $\mathrm{trg}\circ\omega = \phi_0 \circ p_2$. Since the bundlisation is left pre-principal, the action map has a smooth inverse, and we get a division map $\langle\cdot,\cdot\rangle_G$, which we use to construct a smooth map

$$\Omega : X \xrightarrow{\ \Gamma\ } B(\phi)\times_{H_0}^{r_\phi,r_\phi} B(\phi) \xrightarrow{\ \langle\cdot,\cdot\rangle_G\ } G.$$

We claim that Ω completes the above commutative diagram. For that, we may first observe that from the definition of Γ, we get the following equation:

$$\mathrm{src}\circ\Omega = \mathrm{src}\circ\langle\cdot,\cdot\rangle_G\circ\Gamma = l_\phi\circ \mathrm{pr}_2|_{B(\phi)\times_{H_0}B(\phi)}\circ\Gamma = p_1.$$

A similar reasoning will show that $\mathrm{trg}\circ\Omega = p_2$. Moreover, for any $x\in X$, the division map sends $\Gamma(x)$ to the unique arrow $g(x) \in G$ such that

$$g(x)\cdot(p_1(x),\mathrm{id}_{\phi_0 p_1(x)}) = (\mathrm{trg}(g(x)),\phi g(x)) = (p_2(x),\omega(x)).$$

Looking at the second component of this equation, we see that we must have $\phi\circ\Omega = \omega$, as required. This proves that Ω completes the diagram, and from its construction it is evident that it is the unique smooth map doing so. This proves that ϕ is fully faithful, and together with the previous paragraph, this completes the proof of claim (*2*).

(*3*). The third claim is a direct corollary of the first two. □

Starting with a generalised smooth morphism $G \xleftarrow{\phi} K \xrightarrow{\psi} H$, we get two right principal bibundles through bundlisation. If we take the opposite bundle of the bundlisation $B(\phi)$ of ϕ, we get a left principal bibundle $G\curvearrowright^{r_\phi} \overline{B(\phi)}\, {}^{l_\phi}\!\curvearrowleft K$, which we can compose through the balanced tensor product with the bundlisation of ψ:

$$G\curvearrowright^{L} \overline{B(\phi)}\otimes_K B(\psi)\, {}^{R}\!\curvearrowleft H.$$

The following lemma then tells us something about the principality of this bibundle:

Lemma 5.11. *Let $G \xleftarrow{\phi} K \xrightarrow{\psi} H$ be a smooth generalised morphism between diffeological groupoids. If ϕ is a weak equivalence, then $G\curvearrowright^{L} \overline{B(\phi)}\otimes_K B(\psi)\, {}^{R}\!\curvearrowleft H$ is right principal.*

Proof. We know that bundlisations are always right principal. If ϕ is a weak equivalence, Lemma **5.10** shows that the bundlisation $K \overset{l_\phi}{\curvearrowright} B(\phi) \overset{r_\phi}{\curvearrowleft} G$ is moreover biprincipal. It follows by Proposition **4.54** that the balanced tensor product $G \overset{L}{\curvearrowright} \overline{B(\phi)} \otimes_K B(\psi) \overset{R}{\curvearrowleft} H$ is right principal. □

In the other direction, we want to associate a generalised smooth morphism $G \overset{\phi}{\leftarrow} K \overset{\psi}{\rightarrow} H$ to any diffeological bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$. We can do this with the following construction:

Construction 5.12. Consider a diffeological bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$. We construct the *simultaneous action groupoid* (or *biaction groupoid* in [**Li15**]) $G \ltimes X \rtimes H \rightrightarrows X$ as follows. The space of morphisms contains exactly those triples (g, x, h) such that both the actions gx and xh are simultaneously defined:

$$G \ltimes X \rtimes H := G \times_{G_0}^{\mathrm{src}, l_X} X \times_{H_0}^{r_X, \mathrm{trg}} H.$$

We then define the source and target maps in an obvious way:

$$\mathrm{src}, \mathrm{trg} : G \ltimes X \rtimes H \longrightarrow X; \qquad \mathrm{src}(g, x, h) := x, \qquad \mathrm{trg}(g, x, h) := gxh.$$

With component-wise composition, which it gets from G and H, this clearly defines a diffeological groupoid over X.

The simultaneous action groupoid comes with two canonical projection functors. The first is

$$\pi_G : G \ltimes X \rtimes H \longrightarrow X; \qquad \pi_G(g, x, h) := g, \quad (\pi_G)_0(x) := l_X(x).$$

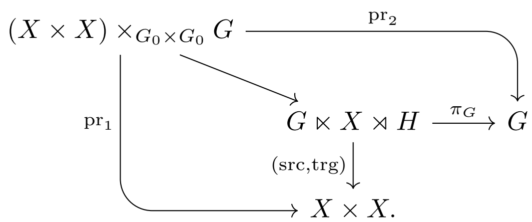
Similarly, we get a projection functor $\pi_H : G \ltimes X \rtimes H \to H$, which projects the third component to the inverse in H, and whose underlying map on objects is the right moment map. Therefore, to any diffeological bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ we can associate a generalised smooth morphism

$$G \xleftarrow{\pi_G} G \ltimes X \rtimes H \xrightarrow{\pi_H} H.$$

Lemma 5.13. *Let $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ be a diffeological bibundle. Then the canonical projection functor $\pi_G : G \ltimes X \rtimes H \to G$ is a weak equivalence if and only if the bibundle is right principal.*

Proof. We start the proof by assuming that the bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ is right principal, and prove that the projection functor $\pi_G : G \ltimes X \rtimes H \to G$ is a weak equivalence. The essential surjectivity of π_G actually follows quite immediately, because $(\pi_G)_0 = l_X$ is then a subduction, so the restricted projection $\mathrm{pr}_1|_{G \times_{G_0}^{\mathrm{src}, l_X} X}$ is still a subduction by Lemma **2.124**. Since the target of a diffeological groupoid is always a subduction, it follows that $\mathrm{trg} \circ \mathrm{pr}_1 : G \times_{G_0}^{\mathrm{src}, l_X} X \to G_0$ is a subduction as well.

That π_G is fully faithful is also straightforward. For this is suffices to construct a diffeomorphism $\Phi : (X \times X) \times_{G_0 \times G_0}^{l_X \times l_X, (\mathrm{src}, \mathrm{trg})} G \to G \ltimes X \rtimes H$ fitting into the following diagram:



We claim that $\Phi : ((x, y), g) \mapsto (g, x, \langle gx, y \rangle_H)$ defines this diffeomorphism. This map, first of all *exists* because the bibundle is right principal, is obviously smooth, clearly makes the above diagram commute, and we claim that $\Psi : (g, x, h) \mapsto ((x, gxh), g)$ defines its inverse. Ψ is also smooth, and it lands in the right codomain because $_X(gxh) = l_X(gx) = \mathrm{trg}(g)$. It is easy to check from the properties of the division map $\langle \cdot, \cdot \rangle_H$ that Φ and Ψ are mutual inverses. This shows that π_G defines a weak equivalence between $G \ltimes X \rtimes H$ and G.

For the other implication, we start with the assumption that $\pi_G : G \ltimes X \rtimes H \to G$ is a weak equivalence, and show that the bibundle $G \overset{l_X}{\curvearrowright} X \overset{r_X}{\curvearrowleft} H$ is right principal. For that we show that the left moment map $l_X : X \to G_0$ is a subduction. Take a plot $\alpha : U_\alpha \to G_0$ and $t \in U_\alpha$. Since π_G is essentially surjective, we can find a plot $\beta : V \to G \times_{G_0}^{\mathrm{src}, l_X} X$, defined on an open neighbourhood $t \in V \subseteq U_\alpha$, such that $\mathrm{trg} \circ \beta_1 = \alpha|_V$ and $\mathrm{src} \circ \beta_1 = {}_X \circ \beta_2$. The latter relation implies that each arrow $\beta_1(t) \in G$ is allowed to act on $\beta_2(t) \in X$, and hence we get a plot $\Omega : V \to X$ defined by $\Omega(t) := \beta_1(t) \cdot \beta_2(t)$. This defines exactly the lift we are looking for:

$$l_X \circ \Omega(t) = l_x(\beta_1(t)\beta_2(t)) = \mathrm{trg}(\beta_1(t)) = \alpha|_V(t),$$

showing that l_X is a subduction.

To show that the bibundle is right pre-principal, and hence to finish the proof, we use the full faithfulness of π_G to construct a division map for the right underlying bundle $G_0 \xleftarrow{l_X} X\, {}^{r_X}\!\curvearrowleft H$. For $G \ltimes X \rtimes H$ to be a pullback as in the diagram of Definition **5.1**, there is a diffeomorphism

$$\Phi : (X \times X) \times_{G_0 \times G_0} G \longrightarrow G \ltimes X \rtimes H,$$

like the one we constructed in the previous part of the proof. The division map should be of the form $\langle \cdot, \cdot \rangle_H : X \times_{G_0}^{l_X, l_X} X \to H$, so we start with a pair $(x, y) \in X \times_{G_0}^{l_X, l_X} X$. Then the triple $((x, y), \mathrm{id}_{l_X(x)})$ defines an element of the fibred product $(X \times X)_{G_0 \times G_0} G$, so we get an arrow $\Phi((x, y), \mathrm{id}_{l_X(x)}) \in G \ltimes X \rtimes H$ in the simultaneous action groupoid. In general, this arrow will be of the form (g, z, h), for $z \in X$. But since $(\mathrm{src}, \mathrm{trg}) \circ \Phi = \mathrm{pr}_1$ we find that $z = x$ and $gxh = y$. Moreover, we have that $\pi_G \circ \Phi = \mathrm{pr}_2$, so that $g = \mathrm{id}_{l_X(x)}$, which in turn implies $xh = y$. It is this unique arrow $h \in H$ that we define to be $\langle x, y \rangle_H$. In general, this procedure gives a smooth map

$$X \times_{G_0}^{l_X, l_X} X \longrightarrow X \times_{H_0}^{r_X, \mathrm{trg}} H; \qquad (x, y) \longmapsto \left(x, \mathrm{pr}_3 \circ \Phi((x, y), \mathrm{id}_{l_X(x)})\right),$$

where the second component is just $\mathrm{pr}_3 \circ \Phi((x, y), \mathrm{id}_{l_X(x)}) = \langle x, y \rangle_H$. Given the above discussion, it is straightforward to verify that this defines an inverse of the action map of the underlying bundle $G_0 \xleftarrow{l_X} X\, {}^{r_X}\!\curvearrowleft H$, which proves that it must the pre-principal. We may therefore conclude that the bibundle $G \curvearrowright^{l_X} X\, {}^{r_X}\!\curvearrowleft H$ is right principal. □

A direct consequence of Lemmas **5.11** and **5.13** is that the notion of Morita equivalence in terms of generalised morphisms is the same as that in terms of bibundles:

Theorem 5.14. *Two diffeological groupoids are Morita equivalent in the sense of Definition 5.3 if and only if they are Morita equivalent in the sense of Definition 4.33.*

5.2 Invariance of isotropy groups and orbit spaces

One advantage of the calculus of fractions is that the following results are more natural to prove. This is because some properties of groupoids are already invariant under weak equivalences, and it follows that they then also have to be invariant under Morita equivalence. In this way we discuss the invariance of isotropy groups and orbit spaces.

Proposition 5.15. *Let $\phi : G \to H$ be a fully faithful smooth functor (in the sense of Definition 5.1). Then for every two objects $x, y \in G_0$ there is a diffeomorphism*

$$\mathrm{Hom}_G(x, y) \cong \mathrm{Hom}_H(\phi_0 x, \phi_0 y).$$

In particular, there are diffeomorphisms $G_x \cong H_{\phi_0 x}$ between the isotropy groups of G and H.

Proof. If $\phi : G \to H$ is fully faithful, then there is a pullback of diffeological spaces:

$$\begin{array}{ccc} G & \xrightarrow{\quad\phi\quad} & H \\ {\scriptstyle(\mathrm{src}_G, \mathrm{trg}_G)}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle(\mathrm{src}_H, \mathrm{trg}_H)} \\ G_0 \times G_0 & \xrightarrow[\phi_0 \times \phi_0]{} & H_0 \times H_0. \end{array}$$

This manifests itself in a diffeomorphism

$$\Phi : G \longrightarrow (G_0 \times G_0) \times_{H_0 \times H_0} H; \qquad g \longmapsto ((\mathrm{src}_G(g), \mathrm{trg}_G(g)), \phi(g)),$$

which in turn induces a diffeomorphism between the subset $\mathrm{Hom}_G(x,y) \subseteq G$ and the image $\Phi(\mathrm{Hom}_G(x,y))$. But the latter contains exactly the triples of the form $((x,y),\phi(g))$, where $g \in \mathrm{Hom}_G(x,y)$. It follows easily that the projection $\mathrm{pr}_2 : (G_0 \times G_0) \times_{H_0 \times H_0} H \to H$ of the fibred product then maps this image further into $\mathrm{pr}_2 \circ \Phi(\mathrm{Hom}_G(x,y)) \subseteq \mathrm{Hom}_H(\phi_0 x, \phi_0 y)$. The desired diffeomorphism is then given by

$$\mathrm{pr}_2 \circ \Phi|_{\mathrm{Hom}_G(x,y)} : \mathrm{Hom}_G(x,y) \longrightarrow \mathrm{Hom}_H(\phi_0 x, \phi_0 y).$$ □

A corollary of this result is that the orbit spaces of Morita equivalent groupoids are diffeomorphic, providing an alternative proof for Theorem 4.44. To see this, we first demonstrate how a smooth functor $\phi : G \to H$ gives rise to an underlying map of the orbit spaces.

Construction 5.16. Fix a smooth functor $\phi : G \to H$ between diffeological groupoids. We will define a smooth map $\phi_* : G_0/G \to H_0/H$ of orbit spaces. Denote the orbit projection maps by $\mathrm{Orb}_G : G_0 \to G_0/G$ and $\mathrm{Orb}_H : H_0 \to H_0/H$. We then define $\phi_*(\mathrm{Orb}_G(x)) := \mathrm{Orb}_H(\phi_0 x)$. This is well defined by functoriality, since if there exists an arrow $g : x \to y$ in G, then ϕg gives the corresponding arrow $\phi_0 x \to \phi_0 y$ in H. Moreover, ϕ_* fits into the following commutative diagram:

$$\begin{array}{ccc} G_0 & \xrightarrow{\mathrm{Orb}_G} & G_0/G \\ {\scriptstyle \phi_0}\downarrow & & \downarrow{\scriptstyle \phi_*} \\ H_0 & \xrightarrow[\mathrm{Orb}_H]{} & H_0/H. \end{array}$$

Since both of the orbit projection maps are (by definition) subductions, it follows by Lemma 2.122 that ϕ_* is smooth if and only if $\phi_* \circ \mathrm{Orb}_G = \mathrm{Orb}_H \circ \phi_0$ is smooth, which is clearly the case. Let us call $\phi_* : G_0/G \to H_0/H$ the *orbit map* induced by ϕ.

The following proposition generalises a result about the orbit maps between Lie groupoids. Note first that, for a general Lie groupoid $G \rightrightarrows G_0$, the orbit space G_0/G has no canonical smooth structure. The projection down to the orbits still exists, but it is merely a continuous function. The following then generalises [**CM18**, Lemma 2.19] to the diffeological setting.

Proposition 5.17. *The orbit map $\phi_* : G_0/G \to H_0/H$ of a weak equivalence ϕ is a diffeomorphism.*

Proof. We know that any smooth functor gives rise to a smooth orbit map, so all we are left to show is that ϕ_* has a smooth inverse. For that, we will first show that it is bijective.

Suppose that $x, y \in G_0$ are two objects such that $\phi_*(\mathrm{Orb}_G(x)) = \phi_*(\mathrm{Orb}_G(y))$. That means there exists an arrow $h : \phi_0 x \to \phi_0 y$ in H. But, since Proposition 5.15 gives a diffeomorphism between $\mathrm{Hom}_X(x,y)$ and $\mathrm{Hom}_H(\phi_0 x, \phi_0 y)$, we get also get an arrow $g : x \to y$ in G. Hence $\mathrm{Orb}_G(x) = \mathrm{Orb}_G(y)$, which shows that ϕ_* is injective.

The surjectivity of ϕ_* will follow from the essential surjectivity of ϕ. Namely, take some orbit $\mathrm{Orb}_H(z) \in H_0/H$, represented by an object $z \in H_0$. Then essential surjectivity tells us we can find a pair $(h, x) \in H \times_{H_0}^{\mathrm{src},\phi_0} G_0$ such that $h : \phi_0 x \to z$. In particular we then find

$$\mathrm{Orb}_H(z) = \mathrm{Orb}_H(\phi_0 x) = \phi_*(\mathrm{Orb}_G(x)),$$

showing that the orbit map is surjective.

Together, this means that there is a set-theoretic inverse function $f : H_0/H \to G_0/G$. We need to show that this is smooth. For that, note that by Lemma 2.122 it suffices to prove that $f \circ \mathrm{Orb}_H$ is smooth, so we pick a plot $\alpha : U_\alpha \to H_0$. Then since ϕ is essentially surjective, we can find a plot $\beta : V \to H \times_{H_0}^{\mathrm{src},\phi_0} G_0$, defined on some open neighbourhood $t \in V \subseteq U_\alpha$, such that $\mathrm{src} \circ \beta_1 = \phi_0 \circ \beta_2$ and $\mathrm{trg} \circ \beta_1 = \alpha|_V$. Since $\mathrm{Orb}_H \circ \mathrm{trg} = \mathrm{Orb}_H \circ \mathrm{src}$, we then find:

$$f \circ \mathrm{Orb}_H \circ \alpha|_V = f \circ \mathrm{Orb}_H \circ \mathrm{src} \circ \beta_1 = f \circ \mathrm{Orb}_H \circ \phi_0 \circ \beta_2 = f \circ \phi_* \circ \mathrm{Orb}_G \circ \beta_2 = \mathrm{Orb}_G \circ \beta_2,$$

which is clearly smooth. It follows by the Axiom of Locality for the diffeology on G_0/G that this is smooth on U_α. This proves that the inverse f is smooth, and hence that ϕ_* is a diffeomorphism. □

Theorem 5.18. *Let $G \rightrightarrows G_0$ and $H \rightrightarrows H_0$ be two Morita equivalent diffeological groupoids. Then there is a diffeomorphism $G_0/G \cong H_0/H$ between orbit spaces.*

Proof. If G and H are Morita equivalent, there exists two weak equivalences $G \xleftarrow{\phi} K \xrightarrow{\psi} H$, whose orbit maps by Proposition **5.17** induce diffeomorphisms $G_0/G \cong K_0/K \cong H_0/H$. □

As we discussed in Section **4.3.1**, this theorem motivates the point of view that a Morita equivalence class of groupoids $G \rightrightarrows G_0$ forms a geometric model for the quotient G_0/G. In diffeology this viewpoint loses some power, because, unlike in the Lie case, the quotient is already a diffeological space. We will use this theorem in our study of the local structure of diffeological spaces in Section **6.1**.

For Lie groupoids there is a characterisation of weak equivalences in terms of the orbit spaces and so-called *normal representations*. Each orbit in a Lie groupoid has a well-defined linear representation over the normal bundle of the orbit. If $\phi : G \to H$ is a smooth functor between Lie groupoids whose orbit map induces a homeomorphism between orbit spaces, and isomorphisms (of representations) between the normal representations, then ϕ is a weak equivalence (see for example [**dHo12**, Theorem 4.3.1]). This proof cannot be simply generalised to the diffeological setting, because it relies explicitly on the theory of tangent bundles on smooth manifolds. Sadly, we do not as yet have a mature theory of tangency on diffeological spaces, so we have no direct analogue of the normal representation of a diffeological groupoid. We therefore have an open question:

Question 5.19. *Is there a natural additional condition that together with a diffeomorphism $G_0/G \cong H_0/H$ implies $G \simeq_{\mathrm{ME}} H$?*

5.2.1 A remark on the elementary structure of groupoids and Morita equivalence

Using the Axiom of Choice, a set-theoretic groupoid can be decomposed into a disjoint union of isotropy groups. The idea is to use the Axiom of Choice to pick a point $x_i \in G_0$ for each orbit $i \in G_0/G$, and to consider the disjoint union of the isotropy groups of the family $(x_i)_{i \in G_0/G}$. If we denote the resulting *reduced isotropy groupoid* by $\check{I}_G \rightrightarrows G_0/G$, then there is a genuine categorical equivalence

$$(\check{I}_G \rightrightarrows G_0/G) \simeq (G \rightrightarrows G_0)$$

between set-theoretic groupoids. This result does not extend to the smooth setting, since, for one, the Axiom of Choice does not guarantee that the section $G_0/G \to G_0 : i \mapsto x_i$ has to be smooth. It is unknown if this result generalises to diffeological groupoids with respect to Morita equivalence. It is still possible to define a notion of reduced isotropy groupoid. Using the construction from Example **3.23**, where each subset $A \subseteq G_0$ of a diffeological groupoid $G \rightrightarrows G_0$ defines the *restricted groupoid* $G|_A \rightrightarrows A$, the reduced isotropy groupoid could be defined as the restriction

$$(\check{I}_G \rightrightarrows O_G) := (G|_{O_G} \rightrightarrows O_G)$$

with respect to the set $O_G := (x_i)_{i \in G_0/G}$. Note that this is in general not even possible in the context of Lie groupoids, since the subset diffeology on O_G, even if G_0 is a manifold, may be highly singular. We do not know what the necessary and sufficient conditions on O_G are to realise a Morita equivalence $(\check{I}_G \rightrightarrows O_G) \simeq_{\mathrm{ME}} (G \rightrightarrows G_0)$. Note that such a Morita equivalence will also depend on the *choice* O_G, and we expect that different choices may lead to non-equivalent reduced isotropy groupoid.

In the following, we discuss some weaker results related to Morita equivalence of subgroupoids. We observe first that if the inclusion functor of a groupoid induces a Morita equivalence, then the subgroupoid has to be full:

Proposition 5.20. *If the inclusion $H \hookrightarrow G$ of a subgroupoid is a weak equivalence, then $H = G|_{H_0}$.*

This shows that there is an interesting class of cases of Morita equivalences to full subgroupoids. In particular, we may ask when the inclusion functor $G|_A \hookrightarrow G$ induces a Morita equivalence. We were unable to find general necessary and sufficient conditions, but we mention the following two results:

Proposition 5.21. *If the inclusion functor $G|_A \hookrightarrow G$ is a weak equivalence, then the set A intersects every orbit of G.*

Proof. If ϕ is a weak equivalence, its essential surjectivity gives a surjection $\mathrm{trg} \circ \mathrm{pr}_1 : G \times_{G_0}^{\mathrm{src}, i_A} A \to G_0$. Hence, for every $x \in G_0$ we can find an arrow $g : a \to x$, for some $a \in A$. This shows that $a \in \mathrm{Orb}_G(x)$, proving that A intersects every orbit. □

The following proposition proves and generalises a claim made (without proof) in [**Wan18**, Example 1.1.9] for Lie groupoids:

Proposition 5.22. *Let $G \rightrightarrows G_0$ be a locally subductive groupoid (Definition 4.73), and consider a D-open subset $A \subseteq G$. If A intersects every orbit of G, then there is a Morita equivalence to the restriction groupoid:*

$$(G|_A \rightrightarrows A) \simeq_{\mathrm{ME}} (G \rightrightarrows G_0).$$

Proof. In the language of bibundles, the subgroupoid $G|_A \rightrightarrows A$ induces a bibundle

$$G|_A \curvearrowright^{\mathrm{trg}} \mathrm{trg}^{-1}(A) {}^{\mathrm{src}}\curvearrowleft G$$

as a restriction of the usual left- and right multiplication actions (cf. Example **4.40**). It is easy to check that this bibundle is always left principal, as well as right pre-principal (even when $G \rightrightarrows G_0$ is not locally subductive, and when A is not D-open). To show that this bibundle is biprincipal, it therefore suffices to show that $\mathrm{src} : \mathrm{trg}^{-1}(A) \to G_0$ is a subduction. For that, take an arbitrary plot $\alpha : U_\alpha \to G_0$ mapping the origin $0 \in U_\alpha$ to, say, $\alpha(0) = x$. Since the source map is surjective, we can find $g \in G$ such that $x = \mathrm{src}(g)$. Moreover, since A intersects every orbit in G, there exists an arrow $h : \mathrm{trg}(g) \to a$ in G, for some $a \in A$. We can now view α as a pointed pot $(U_\alpha, 0) \to (G_0, \mathrm{src}(h \circ g))$. Since we assume the source map is a local subduction, this allows us to find another pointed plot $\beta : (V, 0) \to (G, h \circ g)$ such that $\alpha|_V = \mathrm{src} \circ \beta$. But now, we note that by construction $h \circ g \in \mathrm{trg}^{-1}(A)$, so that the image of β intersects $\mathrm{trg}^{-1}(A)$. But we assumed that A was D-open, so $\mathrm{trg}^{-1}(A)$ is D-open in G, and subsequently $W := \beta^{-1}(\mathrm{trg}^{-1}(A))$ is a non-empty open subset of V. Hence we have found a plot $\beta|_W : W \to \mathrm{trg}^{-1}(A)$ such that $\mathrm{src} \circ \beta|_W = \alpha|_W$, showing the desired result. □

The problem for arbitrary diffeological groupoids $G \rightrightarrows G_0$ and arbitrary subsets A lies in the fact that there seems to be no canonical smooth way to lift plots along the map $\mathrm{src} : \mathrm{trg}^{-1}(A) \to G_0$. If A intersects every orbit in G, it is certainly possible to do this set-theoretically. We do not know if there are more general conditions on A that induce a Morita equivalence in the above sense. It does not seem true in general that the reduced isotropy groupoid $\breve{I}_G \rightrightarrows O_G$ should enjoy such a Morita equivalence.

Chapter VI

Germ groupoids

6.1 Germ groupoids and atlases

In this chapter we will study an interesting class of groupoids that lie beyond the realm of Lie groupoids: the *groupoid of germs* of a diffeological space X. This groupoid is constructed from the space $\mathrm{Diff}_{\mathrm{loc}}(X)$ of *local diffeomorphisms* on X, defined below. This is no longer a diffeological *group*, because not every pair of local diffeomorphisms can be composed, since their domains and images might not intersect. But we will prove that it defines a diffeological *groupoid*. This material is based on [**IZL18**, Section 2], which we already partly discussed in Section **2.4.1**, where the groundwork of this section was laid.

Definition 6.1. A *local diffeomorphism* is an injective locally smooth function $f \in C^\infty_{\mathrm{loc}}(X, Y)$ (Definition **2.98**), such that the inverse $f^{-1} : \mathrm{im}(f) \to X$ is also locally smooth. In particular this means that $\mathrm{im}(f)$ has to be D-open in Y, and we get a genuine diffeomorphism $\mathrm{dom}(f) \to \mathrm{im}(f)$. The space of local diffeomorphisms from X to Y is denoted $\mathrm{Diff}_{\mathrm{loc}}(X, Y)$, and the space of local diffeomorphisms on X is denoted $\mathrm{Diff}_{\mathrm{loc}}(X)$.

So the local diffeomorphisms are just the functions defined on D-open subsets that form diffeomorphisms onto their D-open images. Our terminology is slightly non-standard, because a *local diffeomorphism* usually means a function $f : M \to N$ such that every point $x \in M$ has an open neighbourhood $x \in U \subseteq M$ such that $f|_U$ is a diffeomorphism onto $f(U)$. What we call local diffeomorphisms, on the other hand, are the maps that define diffeomorphisms on D-open subsets of a space onto their image, and are not defined on the entire space. We call functions of the former sort *étale maps*. That is, we call a function $f : X \to Y$ between diffeological spaces an *étale map* if for every $x \in X$ there exists a D-open neighbourhood $x \in A \subseteq X$ such that $f(A)$ is D-open and $f|_A : A \to f(A)$ is a diffeomorphism. Recall as well from Section **2.4.1** that the same remarks apply to our definition of *locally smooth* functions, which are only defined partially on D-open subsets. A function $f : X \to Y$ that allows smooth restrictions $f|_U$ on a D-open cover might be called *locally smooth everywhere*.

Naturally, $\mathrm{Diff}_{\mathrm{loc}}(X)$ lies in $C^\infty_{\mathrm{loc}}(X, X)$, from which it inherits some structure, but there are several things to be checked. First, we note that the local composition map is well-defined on $\mathrm{Diff}_{\mathrm{loc}}(X)$:

Proposition 6.2. *If* $f, g \in \mathrm{Diff}_{\mathrm{loc}}(X)$ *are two local diffeomorphisms such that* $f^{-1}(\mathrm{dom}(g))$ *is non-empty, then* $\mathrm{comp}_{\mathrm{loc}}(g, f) \in \mathrm{Diff}_{\mathrm{loc}}(X)$.

Subsequently, we need to put a good diffeology on $\mathrm{Diff}_{\mathrm{loc}}(X)$. Recall Definition **3.7** and the preceding discussion. There we saw that the subset diffeology on $\mathrm{Diff}(X)$, which it inherits from the standard functional diffeology on $C^\infty(X, X)$, does not make the inversion map smooth. We had to alter the diffeology on $\mathrm{Diff}(X)$ to accomplish this. The same problems arise here: the subset diffeology that $\mathrm{Diff}_{\mathrm{loc}}(X)$ inherits from the standard local functional diffeology on $C^\infty_{\mathrm{loc}}(X, X)$ is unsatisfactory. We therefore need to define a refinement so that the local inversion map

$$\mathrm{inv}_{\mathrm{loc}} : \mathrm{Diff}_{\mathrm{loc}}(X) \longrightarrow \mathrm{Diff}_{\mathrm{loc}}(X); \qquad f \longmapsto f^{-1}|_{\mathrm{im}(f)}$$

becomes smooth. The following is then a generalisation of the standard diffeomorphism diffeology (Definition **3.7**):

Definition 6.3. Let X be a diffeological space, and denote by $\mathcal{D}$ the subset diffeology on $\mathrm{Diff}_{\mathrm{loc}}(X)$ inherited by the standard local functional diffeology on $C^\infty_{\mathrm{loc}}(X, X)$. The *standard local diffeomorphism diffeology* on $\mathrm{Diff}_{\mathrm{loc}}(X)$ is the coarsest diffeology such that the evaluation map is locally smooth on $\mathcal{E}_{X,X} \cap (\mathrm{Diff}_{\mathrm{loc}}(X) \times X)$, and such that the local inversion map $\mathrm{inv}_{\mathrm{loc}}$ is smooth. This diffeology can be expressed as the intersection $\mathrm{inv}^*_{\mathrm{loc}}(\mathcal{D}) \cap \mathcal{D}$. Therefore, a parametrisation $\Omega : U_\Omega \to \mathrm{Diff}_{\mathrm{loc}}(X)$ is a plot in the standard local diffeomorphism diffeology if and only if Ω and $\Omega^{-1} := \mathrm{inv}_{\mathrm{loc}} \circ \Omega$ are both plots in the standard local functional diffeology. Note that the local composition map

$$\mathrm{comp}_{\mathrm{loc}} : \mathrm{Diff}_{\mathrm{loc}}(X) \longrightarrow \mathrm{Diff}_{\mathrm{loc}}(X)$$

is also smooth with respect to this diffeology.

Now we will set out to construct a diffeological groupoid over X, whose arrows are the *germs* of local diffeomorphisms on X. A germ captures the local behaviour of a locally smooth function around a point, and they are defined as follows:

Definition 6.4. Given two locally smooth functions $f : A \to Y$ and $g : B \to Y$ in $C^\infty_{\mathrm{loc}}(X, Y)$ and a point $x \in A \cap B$, we say that f and g *have the same germ at* x if there exists a D-open neighbourhood $x \in V \subseteq A \cap B$ such that $f|_V = g|_V$. In that case we write $f \sim_x g$. This clearly defines an equivalence relation on all locally smooth functions whose domain contains x, and we define the *germ of* f *at* x as the equivalence class

$$[f]_x := \{g \in C^\infty_{\mathrm{loc}}(X, Y) : f \sim_x g\}.$$

Note that the value $f(x)$ is already determined by the germ $[f]_x$. Namely, if $g \in C^\infty_{\mathrm{loc}}(X, Y)$ is another representation for this germ, i.e., $[g]_x = [f]_x$, then there exists a D-open neighbourhood $x \in V$ in the domains of f and g, so that $f(x) = f|_V(x) = g|_V(x) = g(x)$. But the germ $[f]_x$ contains *more* information than just the value $f(x)$, since it encodes the smooth behaviour of f around x. Using this fact, we can define a notion of composition for germs:

Lemma 6.5. *The* composition of germs $[g]_{f(x)} \circ [f]_x := [\mathrm{comp}_{\mathrm{loc}}(g, f)]$ *is well-defined.*

Proof. We just need to show that the right-hand side does not depend on the representatives of $[f]_x$ and $[g]_{f(x)}$. So, take two other representatives: $[f]_x = [f']_x$ and $[g]_{f(x)} = [g']_{f(x)}$. Note that by the above discussion, the point $f(x) = f'(x)$ is already well defined. We can then find a D-open neighbourhood $x \in U \subseteq \mathrm{dom}(f) \cap \mathrm{dom}(f')$ such that $f|_U = f'|_U$, and another D-open neighbourhood $f(x) \in V \subseteq \mathrm{dom}(g) \cap \mathrm{dom}(g')$ such that $g|_V = g'|_V$. Then define a third D-open set, $W := f^{-1}(V) \cap U$. This contains x, because $x \in U$ and $f(x) \in V$. Further, it is contained in the intersection of $f^{-1}(\mathrm{dom}(g))$ and $f'^{-1}(\mathrm{dom}(g'))$, and we easily verify that $\mathrm{comp}_{\mathrm{loc}}(g, f)|_W = \mathrm{comp}_{\mathrm{loc}}(g', f')|_W$. Hence the germ on the right hand side in the lemma is completely determined by the germs of f and g, and the composition is thus well-defined. □

Construction 6.6 (Germ groupoid)**.** We will now describe in detail the diffeological groupoid structure of the germs of local diffeomorphisms on a diffeological space [**IZL18**]. Note that, in general, we can take the germ of any element in $\mathcal{E}_{X,Y}$, which contains pairs $(f, x) \in C^\infty_{\mathrm{loc}}(X, Y) \times X$ such that $x \in \mathrm{dom}(f)$. But, since we just want the local diffeomorphisms on X, we only want germs of pairs in the following domain:

$$\mathcal{E}_X^{\mathrm{Diff}} := \mathcal{E}_{X,X} \cap (\mathrm{Diff}_{\mathrm{loc}}(X) \times X) = \coprod_{f \in \mathrm{Diff}_{\mathrm{loc}}(X)} \mathrm{dom}(f) = \{(f, x) \in \mathrm{Diff}_{\mathrm{loc}}(X) \times X : x \in \mathrm{dom}(f)\}.$$

This is the domain on which the germs of local diffeomorphisms exist, and we define the *germ map* as:

$$\mathrm{germ} : \mathcal{E}_X^{\mathrm{Diff}} \longrightarrow \mathbf{Germ}(X); \qquad (f, x) \longmapsto [f]_x,$$

where $\mathbf{Germ}(X)$ is defined as the space of all germs of local diffeomorphisms on X, which is precisely the image of this map. Note that $\mathcal{E}_X^{\mathrm{Diff}}$ inherits a diffeology $\mathcal{D}_{\mathcal{E}_X^{\mathrm{Diff}}}$ as a subset of $\mathrm{Diff}_{\mathrm{loc}}(X) \times X$, where $\mathrm{Diff}_{\mathrm{loc}}(X)$ is endowed with the standard local diffeomorphism diffeology (Definition **6.3**). Then we define a diffeology of $\mathbf{Germ}(X)$ as the pushforward along the germ map:

$$\mathcal{D}_{\mathbf{Germ}(X)} := \mathrm{germ}_*(\mathcal{D}_{\mathcal{E}_X^{\mathrm{Diff}}}).$$

This turns the germ map into a subduction. We shall now define the structure maps of a diffeological groupoid $\mathbf{Germ}(X) \rightrightarrows X$, called the *groupoid of germs* (or *germ groupoid*). The idea is that they are all, in some way or another, projections along the germ map of the operations on $\mathrm{Diff}_{\mathrm{loc}}(X)$ and $\mathcal{E}_X^{\mathrm{Diff}}$.

- The source and target maps are defined as

$$\mathrm{src}, \mathrm{trg} : \mathbf{Germ}(X) \longrightarrow X; \qquad \mathrm{src}([f]_x) := x, \quad \mathrm{trg}([f]_x) := f(x).$$

These are well-defined, because the germ $[f]_x$ defines uniquely the points x and $f(x)$, independently of the representative. Note that these two maps fit into commutative diagrams:

$$\begin{array}{ccc} \mathcal{E}_X^{\mathsf{Diff}} & \xrightarrow{\text{germ}} & \mathbf{Germ}(X) \\ & {\scriptstyle \mathrm{pr}_2|_{\mathcal{E}_X^{\mathsf{Diff}}}} \searrow \quad \swarrow {\scriptstyle \mathrm{src}} & \\ & X & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{E}_X^{\mathsf{Diff}} & \xrightarrow{\text{germ}} & \mathbf{Germ}(X) \\ & {\scriptstyle \mathrm{ev}|_{\mathcal{E}_X^{\mathsf{Diff}}}} \searrow \quad \swarrow {\scriptstyle \mathrm{trg}} & \\ & X. & \end{array}$$

Since the germ map is a subduction, it follows by Lemma **2.122** that the source and target maps are smooth.

- The unit map $\mathrm{u} : X \to \mathbf{Germ}(X)$ just sends a point $x \in X$ to the germ $[\mathrm{id}_X]_x$ of the identity map on X. It can be realised as the composition of the inclusion function $\iota : X \to \mathcal{E}_X^{\mathsf{Diff}}; x \mapsto (\mathrm{id}_X, x)$ with the germ function, and is hence smooth.
- The inversion map $\mathrm{inv} : \mathbf{Germ}(X) \to \mathbf{Germ}(X)$ sends a germ $[f]_x$ to $[\mathrm{inv}_{\mathrm{loc}}(f)]_{f(x)} = [f^{-1}|_{\mathrm{im}(f)}]_{f(x)}$. At the level of local diffeomorphisms we can realise this function as

$$(\mathrm{inv}_{\mathrm{loc}} \circ \mathrm{pr}_1, \mathrm{ev})|_{\mathcal{E}_X^{\mathsf{Diff}}} : \mathcal{E}_X^{\mathsf{Diff}} \longrightarrow \mathcal{E}_X^{\mathsf{Diff}}; \qquad (f, x) \longmapsto \left(f^{-1}|_{\mathrm{im}(f)}, f(x)\right).$$

This map is smooth since the evaluation map, projection, and local inversion maps are smooth on $\mathcal{E}_X^{\mathsf{Diff}}$. It fits into the following commutative diagram:

$$\begin{array}{ccc} \mathcal{E}_X^{\mathsf{Diff}} & \xrightarrow{(\mathrm{inv}_{\mathrm{loc}} \circ \mathrm{pr}_1, \mathrm{ev})} & \mathcal{E}_X^{\mathsf{Diff}} \\ {\scriptstyle \text{germ}} \downarrow & & \downarrow {\scriptstyle \text{germ}} \\ \mathbf{Germ}(X) & \xrightarrow[\mathrm{inv}]{} & \mathbf{Germ}(X). \end{array}$$

Again using Lemma **2.122**, the inversion map on $\mathbf{Germ}(X)$ is smooth if and only if

$$\mathrm{inv} \circ \text{germ} : \mathcal{E}_X^{\mathsf{Diff}} \longrightarrow \mathbf{Germ}(X)$$

is smooth, but the latter is the composition in the top right corner of the diagram, and hence is smooth.

- Lastly, we need to define the composition in $\mathbf{Germ}(X)$. This is just the composition of germs:

$$\mathrm{m} : \mathbf{Germ}(X) \times_X^{\mathrm{src},\mathrm{trg}} \mathbf{Germ}(X) \longrightarrow \mathbf{Germ}(X); \qquad \left([g]_{f(x)}, [f]_x\right) \longmapsto [\mathrm{comp}_{\mathrm{loc}}(g, f)]_x.$$

In terms of local diffeomorphisms, this can be written as:

$$C : \mathcal{E}_X^{\mathsf{Diff}} \times_X^{\mathrm{pr}_2,\mathrm{ev}} \mathcal{E}_X^{\mathsf{Diff}} \longrightarrow \mathcal{E}_X^{\mathsf{Diff}}; \qquad ((g, f(x)), (f, x)) \longmapsto (\mathrm{comp}_{\mathrm{loc}}(g, f), x).$$

It is clear that C is just the composition of some projection maps and the local composition map. Since the germ map satisfies $\mathrm{src} \circ \text{germ} = \mathrm{pr}_2|_{\mathcal{E}_X^{\mathsf{Diff}}}$ and $\mathrm{trg} \circ \text{germ} = \mathrm{ev}|_{\mathcal{E}_X^{\mathsf{Diff}}}$, we get a commutative diagram:

$$\begin{array}{ccc} \mathcal{E}_X^{\mathsf{Diff}} \times_X^{\mathrm{pr}_2,\mathrm{ev}} \mathcal{E}_X^{\mathsf{Diff}} & \xrightarrow{\text{germ} \times \text{germ}} & \mathbf{Germ}(X) \times_X^{\mathrm{src},\mathrm{trg}} \mathbf{Germ}(X) \\ {\scriptstyle C} \downarrow & & \downarrow {\scriptstyle \mathrm{m}} \\ \mathcal{E}_X^{\mathsf{Diff}} & \xrightarrow[\text{germ}]{} & \mathbf{Germ}(X). \end{array}$$

Moreover, the product $\text{germ} \times \text{germ}$, when restricted to the appropriate domain, is guaranteed to still be a subduction by Lemma **2.125**. Using Lemma **2.122** for a third time, it follows that the multiplication map is smooth if and only if $\text{germ} \circ C$ is smooth, which is plainly the case.

From this discussion we finally conclude:

Proposition 6.7. *The germ groupoid* $\mathbf{Germ}(X) \rightrightarrows X$ *is a diffeological groupoid.*

In [**MM03**, Exercise 5.18] we find the following example:

Example 6.8. Let M be a smooth m-dimensional manifold. Then $\mathbf{Germ}(M) \simeq_{\mathrm{ME}} \mathbf{Germ}(\mathbb{R}^m)$.

6.1.1 Atlases on diffeological spaces

Consider a smooth m-dimensional manifold M. An *atlas* $\mathscr{A}$ on M consists of *charts*, which are local homeomorphisms $\varphi : U \to \mathbb{R}^m$, defined on open subsets $U \subseteq M$, and subject to the condition that the *transition functions* between charts have to be smooth. Now, when M is endowed with its differentiable structure, the charts become local diffeomorphisms. Therefore a smooth m-dimensional manifold is a type of space that is locally diffeomorphic (*"locally looks like"*) $\mathbb{R}^m$. Here $\mathbb{R}^m$ could be called the *model space* of the manifold. But what happens if we change the model space, or even allow for a wide range of model spaces? In this section we sketch a simple theory of atlases on diffeological spaces, and use the technology of germ groupoids to study their equivalence. These ideas are based in part on the discussion of *diffeological manifolds* in [***Diffeology***, Chapter 4], the description of orbifolds and their corresponding groupoids in [**MM03**], and the more general framework of atlases and their groupoids in [**Los94**; **Los15**][49]. The idea is that if we have a family of charts (i.e. local diffeomorphisms) that covers the space X, then X is completely determined by the groupoid of germs of transition functions between the charts. That is, the space is globally determined by its local behaviour. We develop these ideas here for diffeological spaces, extending on [***Diffeology***, Article 4.19].

Definition 6.9. For this section, we fix a diffeological space X, whose local structure we want to study. A family $\mathcal{M} \subseteq$ **Diffeol** of diffeological spaces is called a *modelling family*. Elements $A \in \mathcal{M}$ are called *models*, which '$\mathcal{M}$' is meant to stand for. An $\mathcal{M}$-*chart* on X is a local diffeomorphism $\varphi \in \mathrm{Diff}_{\mathrm{loc}}(A, X)$, where $A \in \mathcal{M}$ is a modelling space. An $\mathcal{M}$-*atlas* on X is a family $\mathscr{A}$ of $\mathcal{M}$-charts on X whose images cover X. If there exists an $\mathcal{M}$-atlas on X, this means that every point $x \in X$ has a D-open neighbourhood that is locally diffeomorphic to some model space $A \in \mathcal{M}$. In some cases we may just refer to $\mathscr{A}$ as an *atlas*, leaving the model family implicit. If the modelling family $\mathcal{M}$ contains just one element A, we may denote atlases of that type also by A-*atlases*. Every diffeological space X therefore has a canonical X-atlas, containing just the identity map id_X.

Our philosophy is different from that of [**Los15**] and the traditional purpose of atlases of smooth manifolds. There, the point of an atlas is usually to *define* the smooth structure of a space. Here we consider a *given* diffeological space X, and study atlases on it. The question of what types of atlases are allowed for any given diffeological space of course still tells us something about its local behaviour. But if we want to study diffeological spaces from the ground up using atlases, we would suggest taking a more general definition, as in [**Los15**, Definition 2.2]. In particular, we assume that the elements of an atlas are already local diffeomorphisms, while we would rather want it to *define* a structure on a space that makes the charts into local diffeomorphisms. The fact that all of our charts are local diffeomorphisms also ensures that transition maps exist between each chart and are smooth. An approach that is closer to [**Los15**, Definition 2.2] would be to have an atlas that contains merely locally *smooth* charts, together having some universal property. In any case, we continue here with the definition of an atlas as given above.

Given an $\mathcal{M}$-atlas $\mathscr{A}$ on X, we denote the disjoint union of charts by

$$\mathcal{E}_{\mathscr{A}} := \coprod_{\varphi \in \mathscr{A}} \mathrm{dom}(\varphi),$$

endowed with the coproduct diffeology. An element in $\mathcal{E}_{\mathscr{A}}$ is a pair (φ, t), where $\varphi \in \mathscr{A}$ is a chart and $t \in \mathrm{dom}(\varphi)$ is a point in the domain. We call such pairs *pointed charts*. The diffeological space X is recovered completely from the pointed charts by the following proposition:

Proposition 6.10. *Let $\mathscr{A}$ be an $\mathcal{M}$-atlas on X. Then the evaluation map* $\mathrm{ev} : \mathcal{E}_{\mathscr{A}} \to X$ *is a subduction, and we get a diffeomorphism:*

$$\mathcal{E}_{\mathscr{A}}/\mathrm{ev} = \coprod_{\varphi \in \mathscr{A}} \mathrm{dom}(\varphi) \Big/ \mathrm{ev} \cong X.$$

[49] In [**Los94**] Losik introduces a general notion of **C**-*atlas*, where **C** is a category endowed with a particular type of functor $\mathbf{C} \to \mathbf{Set}$. This framework actually subsumes diffeology when **C** is chosen to be **Eucl** or **Mnfd**.

Proof. This argument is similar to the proof in Proposition **2.58**. That the evaluation map is smooth follows immediately from the definition of the coproduct diffeology on $\mathcal{E}_{\mathscr{A}}$ and the fact that charts are smooth. For subductiveness, take a plot $\alpha : U_\alpha \to X$, together with a point $t \in U_\alpha$. Since the atlas $\mathscr{A}$ covers X, there exists a local diffeomorphism $\varphi : \mathrm{dom}(\varphi) \to X$ such that $\alpha(t) \in \mathrm{im}(\varphi)$. Now the image $\mathrm{im}(\varphi)$ is D-open in X, so we get an open subset $V := \alpha^{-1}(\mathrm{im}(\varphi)) \subseteq U_\alpha$, on which α takes values in $\mathrm{im}(\varphi)$. Since φ is a local diffeomorphism, we get a smooth map $\varphi^{-1}|_{\mathrm{im}(\varphi)} : \mathrm{im}(\varphi) \to \mathrm{dom}(\varphi)$, and if we denote the canonical inclusion as $\iota_\varphi : \mathrm{dom}(\varphi) \to \mathcal{E}_{\mathscr{A}}$, then we get a plot

$$\iota_\varphi \circ \varphi^{-1}|_{\mathrm{im}(\varphi)} \circ \alpha|_V : V \longrightarrow \mathcal{E}_{\mathscr{A}}; \qquad s \longmapsto (\varphi, \varphi^{-1}(\alpha(s))).$$

It is easy to see that this plot then defines a local lift along the evaluation map:

$$\mathrm{ev} \circ \iota_\varphi \circ \varphi^{-1}|_{\mathrm{im}(\varphi)} \circ \alpha|_V = \alpha|_V,$$

which proves the claim. □

The disjoint union of the domains of charts $\mathcal{E}_{\mathscr{A}}$ is, in particular, itself a diffeological space. Hence we can consider local diffeomorphisms $\Phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathcal{E}_{\mathscr{A}})$, and even the germ groupoid $\mathbf{Germ}(\mathcal{E}_{\mathscr{A}}) \rightrightarrows \mathcal{E}_{\mathscr{A}}$. A local diffeomorphism such as Φ does two things: its first component $\Phi_1 : \mathcal{E}_{\mathscr{A}} \to \mathscr{A}$ maps charts $\varphi \in \mathscr{A}$ to other charts $\psi \in \mathscr{A}$, and its second component maps points in the domain $\mathrm{dom}(\varphi)$ to points in $\mathrm{dom}(\psi)$. For the purposes of illustration, take a local diffeomorphism of the form

$$\Phi : \{\varphi\} \times \mathrm{dom}(\varphi) \longrightarrow \{\psi\} \times \mathrm{dom}(\psi),$$

for two charts $\varphi, \psi \in \mathscr{A}$. Then the first component is just the constant function $\Phi_1(\varphi, t) = \psi$, but the second component can be interpreted as a local diffeomorphism between the domains of the charts: $f : \mathrm{dom}(\varphi) \to \mathrm{dom}(\psi); t \mapsto \Phi_2(\varphi, t)$. This is something like a transition function, which we depict as:

$$\begin{array}{ccc} \mathrm{dom}(\varphi) & \xrightarrow{f := \Phi_2(\varphi, -)} & \mathrm{dom}(\psi) \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & X. & \end{array}$$

If we write out $\psi = \Phi_1(\varphi, t)$, the commutativity of this diagram, which states that f is *actually* a transition function between the charts, is then equivalent to the equation

$$\Phi_1(\varphi, t)(\Phi_2(\varphi, t)) = \varphi(t).$$

But this in turn is equivalent to the simple equation $\mathrm{ev} \circ \Phi = \mathrm{ev}|_{\mathrm{dom}(\Phi)}$. All transition functions between the charts of an atlas $\mathscr{A}$ are therefore captured by exactly those local diffeomorphisms $\Phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathcal{E}_{\mathscr{A}})$ that satisfy $\mathrm{ev} \circ \Phi = \mathrm{ev}|_{\mathrm{dom}(\Phi)}$. It is easy to see that local diffeomorphisms of $\mathcal{E}_{\mathscr{A}}$ satisfying that property are closed under local inversion and local composition. We can therefore define:

Definition 6.11. Let $\mathscr{A}$ be an $\mathcal{M}$-atlas on X. The *groupoid of transition functions* $\mathbf{Trans}(\mathscr{A}) \rightrightarrows \mathcal{E}_{\mathscr{A}}$ is the subgroupoid of the germ groupoid $\mathbf{Germ}(\mathcal{E}_{\mathscr{A}}) \rightrightarrows \mathcal{E}_{\mathscr{A}}$ whose germs are represented by the local diffeomorphisms $\Phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathcal{E}_{\mathscr{A}})$ satisfying $\mathrm{ev} \circ \Phi = \mathrm{ev}|_{\mathrm{dom}(\Phi)}$. As such, $\mathbf{Trans}(\mathscr{A}) \rightrightarrows \mathcal{E}_{\mathscr{A}}$ becomes a diffeological groupoid with the subset diffeology on $\mathbf{Trans}(\mathscr{A}) \subseteq \mathbf{Germ}(\mathcal{E}_{\mathscr{A}})$.

The arrows in the groupoid of transition functions are germs $[\Phi]_{(\varphi,t)}$, with $\mathrm{src}([\Phi]_{(\varphi,t)}) = (\varphi, t)$ and $\mathrm{trg}([\Phi]_{(\varphi,t)}) = (\psi, s)$. Since $\mathrm{ev} \circ \Phi = \mathrm{ev}|_{\mathrm{dom}(\Phi)}$ we then find that

$$\psi(s) = \mathrm{ev} \circ \mathrm{trg}([\Phi]_{(\varphi,t)}) = \mathrm{ev} \circ \Phi(\varphi, t) = \varphi(t),$$

so that the value of $s \in \mathrm{dom}(\psi)$ is already completely determined as $s = \psi^{-1} \circ \varphi(t)$.

Lemma 6.12. *For any atlas $\mathscr{A}$ of X, we have a diffeomorphism $\mathcal{E}_{\mathscr{A}}/\mathbf{Trans}(\mathscr{A}) \cong \mathcal{E}_{\mathscr{A}}/\mathrm{ev}$.*

Proof. The orbits in $\mathbf{Trans}(\mathscr{A})$ are the families of pointed charts that are connected by transition functions. Let us denote the orbit of a pointed chart $(\varphi, t) \in \mathcal{E}_{\mathscr{A}}$ in $\mathbf{Trans}(\mathscr{A})$ by $\mathrm{Orb}(\varphi, t)$, and let us denote its equivalence class in $\mathcal{E}_{\mathscr{A}}/\mathrm{ev}$ by $[\varphi, t]_{\mathrm{ev}}$. We shall describe why the orbits in $\mathbf{Trans}(\mathscr{A})$ and the ev-fibres are the same.

First, suppose that there is a germ $[\Phi]_{(\varphi,t)} \in \mathbf{Trans}(\mathscr{A})$, meaning that the pointed charts (φ, t) and $(\psi, s) := \Phi(\varphi, t)$ are in the same orbit. By the remarks preceding this lemma, it follows that $\varphi(t) = \psi(s)$, which shows that $[\varphi, t]_{\mathrm{ev}} = [\psi, s]_{\mathrm{ev}}$.

On the other hand, if we have two pointed charts (φ, t) and (ψ, s) in the same ev-fibre, we need to construct a germ $[\Phi]_{(\varphi,t)}$ in $\mathbf{Trans}(\mathscr{A})$ such that $\mathrm{trg}([\Phi]_{(\varphi,t)}) = (\psi, s)$. Since $\varphi(t) = \psi(s)$, the intersection of the images $\mathrm{im}(\varphi) \cap \mathrm{im}(\psi)$ is a non-empty D-open subset of X. We therefore get a commutative triangle:

$$\begin{array}{ccc} \mathrm{dom}(\varphi) \supseteq \varphi^{-1}(\mathrm{im}(\psi)) & \xrightarrow{\psi^{-1}\circ\varphi|_{\varphi^{-1}(\mathrm{im}(\psi))}} & \psi^{-1}(\mathrm{im}(\varphi)) \subseteq \mathrm{dom}(\psi) \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \mathrm{im}(\varphi) \cap \mathrm{im}(\psi), & \end{array}$$

where $\psi^{-1}\circ\varphi|_{\varphi^{-1}(\mathrm{im}(\psi))} \in \mathrm{Diff}_{\mathrm{loc}}(\mathrm{dom}(\varphi), \mathrm{dom}(\psi))$. We then get a local diffeomorphism $\Phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathcal{E}_{\mathscr{A}})$ defined by

$$\Phi : \{\varphi\} \times \mathrm{dom}(\varphi) \longrightarrow \{\psi\} \times \mathrm{dom}(\psi); \qquad (\varphi, r) \longmapsto (\psi, \psi^{-1} \circ \varphi|_{\varphi^{-1}(\mathrm{im}(\psi))}(r)).$$

Note that Φ is indeed a local diffeomorphism: first, the subset $\{\varphi\} \times \mathrm{dom}(\varphi)$ is D-open in $\mathcal{E}_{\mathscr{A}}$ because the canonical inclusion $\iota_\varphi : \mathrm{dom}(\varphi) \hookrightarrow \mathcal{E}_{\mathscr{A}}$ is an induction and $\mathrm{dom}(\varphi)$ itself is D-open in its ambient space. Furthermore, we have $\mathrm{dom}(\Phi) = \{\varphi\} \times \varphi^{-1}(\mathrm{im}(\psi))$, which itself is open in $\{\varphi\} \times \mathrm{dom}(\varphi)$ because $\varphi^{-1}(\mathrm{im}(\psi))$ is open in $\mathrm{dom}(\varphi)$. That Φ is a diffeomorphism on this domain is clear, since the charts φ and ψ themselves are local diffeomorphisms. We also clearly have $\mathrm{ev} \circ \Phi = \mathrm{ev}|_{\mathrm{dom}(\Phi)}$, and so we get an arrow $[\Phi]_{(\varphi,t)}$ in $\mathbf{Trans}(\mathscr{A})$ whose source is (φ, t), and whose target is $\Phi(\varphi, t) = (\psi, \psi^{-1} \circ \varphi(t)) = (\psi, s)$.

This proves that the orbits in $\mathbf{Trans}(\mathscr{A})$ are the same as the ev-fibres in $\mathcal{E}_{\mathscr{A}}$, and hence we have a well-defined function $f : \mathcal{E}_{\mathscr{A}}/\mathbf{Trans}(\mathscr{A}) \to \mathcal{E}_{\mathscr{A}}/\mathrm{ev}$, mapping $\mathrm{Orb}(\varphi, t) \mapsto [\varphi, t]_{\mathrm{ev}}$. This function fits into a commutative square

$$\begin{array}{ccc} \mathcal{E}_{\mathscr{A}} & \xrightarrow{\mathrm{id}_{\mathcal{E}_{\mathscr{A}}}} & \mathcal{E}_{\mathscr{A}} \\ {\scriptstyle\mathrm{Orb}}\downarrow & & \downarrow{\scriptstyle[\cdot,\cdot]_{\mathrm{ev}}} \\ \mathcal{E}_{\mathscr{A}}/\mathbf{Trans}(\mathscr{A}) & \xrightarrow[f]{} & \mathcal{E}_{\mathscr{A}}/\mathrm{ev}. \end{array}$$

In this diagram, both vertical maps are subductions, so by Lemma **2.122** it follows that both f and its inverse are smooth. This provides the diffeomorphism we were looking for. □

In particular, together with Proposition **6.10**, if $\mathscr{A}$ is an atlas for X, then the orbit space of the groupoid of transition functions $\mathbf{Trans}(\mathscr{A}) \rightrightarrows \mathcal{E}_{\mathscr{A}}$ just exactly gives X back: $\mathcal{E}_{\mathscr{A}}/\mathbf{Trans}(\mathscr{A}) \cong X$. It also implies the following:

Corollary 6.13. *Consider $X, Y \in$ **Diffeol**, and let $\mathscr{A}_X$ be an atlas on X, and $\mathscr{A}_Y$ an atlas on Y. If there is a Morita equivalence $\mathbf{Trans}(\mathscr{A}_X) \simeq_{\mathrm{ME}} \mathbf{Trans}(\mathscr{A}_Y)$, then $X \cong Y$.*

Proof. If there is a Morita equivalence $\mathbf{Trans}(\mathscr{A}_X) \simeq_{\mathrm{ME}} \mathbf{Trans}(\mathscr{A}_Y)$, it follows by Theorem **5.18** that there is a diffeomorphism $\mathcal{E}_{\mathscr{A}_X}/\mathbf{Trans}(\mathscr{A}_X) \cong \mathcal{E}_{\mathscr{A}_Y}/\mathbf{Trans}(\mathscr{A}_Y)$ between the orbit spaces, which together with Proposition **6.10** and Lemma **6.12** gives the desired diffeomorphism between X and Y. □

We will now work towards the claim that the converse holds as well. For that it becomes important what type of modelling family we allow. First we shall prove a converse where we have one fixed modelling family $\mathcal{M}$.

Definition 6.14. Each modelling family $\mathcal{M} \subseteq$ **Diffeol** defines a *maximal* $\mathcal{M}$*-atlas* on X, containing any other $\mathcal{M}$-atlas:

$$\mathscr{A}^X_{\mathcal{M}} := \bigcup_{A \in \mathcal{M}} \mathrm{Diff}_{\mathrm{loc}}(A, X).$$

We denote the corresponding space of domains by $\mathcal{E}^X_{\mathcal{M}} := \coprod_{\varphi \in \mathscr{A}^X_{\mathcal{M}}} \mathrm{dom}(\varphi)$, which is the disjoint union of all domains of local diffeomorphisms defined on model spaces in $\mathcal{M}$ into X.

Lemma 6.15. *If $\mathscr{A}$ in an $\mathcal{M}$-atlas on X, and $\mathscr{A}^X_{\mathcal{M}}$ is the maximal $\mathcal{M}$-atlas on X, then there is a Morita equivalence* **Trans**$(\mathscr{A}) \simeq_{\mathrm{ME}}$ **Trans**$(\mathscr{A}^X_{\mathcal{M}})$.

Proof. We claim that the canonical induction $\iota : \mathcal{E}_{\mathscr{A}} \hookrightarrow \mathcal{E}^X_{\mathcal{M}}$ induces a weak equivalence functor $I :$ **Trans**$(\mathscr{A}) \to$ **Trans**$(\mathscr{A}^X_{\mathcal{M}})$. If successful, this is sufficient by Proposition **5.4**. On objects, we let I act as the inclusion $\iota : \mathcal{E}_{\mathscr{A}} \to \mathcal{E}^X_{\mathcal{M}}$. Because it is an induction, the inclusion map defines a local diffeomorphism $\iota \in \mathrm{Diff}_{\mathrm{loc}}(\mathcal{E}_{\mathscr{A}}, \mathcal{E}^X_{\mathcal{M}})$ with domain $\mathrm{dom}(\iota) = \mathcal{E}_{\mathscr{A}}$. We therefore get a germ $[\iota]_{(\varphi,t)}$ for any pointed chart $(\varphi, t) \in \mathcal{E}_{\mathscr{A}}$, which has an inverse germ $[\iota^{-1}|_{\mathrm{im}(\iota)}]_{\iota(\varphi,t)}$. Using the composition of germs, we can thus define

$$I : \textbf{Trans}(\mathscr{A}) \longrightarrow \textbf{Trans}(\mathscr{A}^X_{\mathcal{M}}); \qquad [\Phi]_{(\varphi,t)} \longmapsto [\iota]_{\Phi(\varphi,t)} \circ [\Phi]_{(\varphi,t)} \circ [\iota^{-1}]_{\iota(\varphi,t)} = [\iota \circ \Phi \circ \iota^{-1}]_{\iota(\varphi,t)}.$$

(We are being sloppy with our notation here, but since we are dealing with germs it does not matter to which domain we restrict the maps.) This map is smooth because it is just a combination of the structure maps of the germ groupoid, which are all smooth (Construction **6.6**). Functoriality is also clear, since I is just a conjugation.

To show that I is a weak equivalence, we start with a demonstration of essential surjectivity, for which we take a plot $\Omega : U_\Omega \to \mathcal{E}^X_{\mathcal{M}}$. For the rest of this argument, fix a point $t \in U_\Omega$. By definition of the coproduct diffeology on $\mathcal{E}^X_{\mathcal{M}}$, we can find an open neighbourhood $t \in U \subseteq U_\Omega$ such that $\Omega|_U = \iota_\psi \circ \alpha$, where $\psi \in \mathscr{A}^X_{\mathcal{M}}$ is a chart and $\alpha : U \to \mathrm{dom}(\psi)$ a plot of the domain. We also get a plot $\mathrm{ev} \circ \Omega : U_\Omega \to X$, and since the map $\mathrm{ev} : \mathcal{E}_{\mathscr{A}} \to X$ is a subduction (Proposition **6.10**), this means we can find another open neighbourhood $t \in V \subseteq U_\Omega$ and a plot $\omega : V \to \mathcal{E}_{\mathscr{A}}$ such that $\mathrm{ev} \circ \Omega|_V = \mathrm{ev} \circ \omega$. The plot ω now takes values in the pointed charts of $\mathscr{A}$, whereas Ω takes values in the pointed charts of the maximal $\mathcal{M}$-atlas on X. Again using the defining property of the coproduct diffeology, this time of $\mathcal{E}_{\mathscr{A}}$, we can find an open neighbourhood $t \in W \subseteq V$ such that $\omega|_W = \iota_\varphi \circ \beta$, where $\varphi \in \mathscr{A}$ is a chart and $\beta : W \to \mathrm{dom}(\varphi)$ a plot of the domain. Combining this with the equation $\mathrm{ev} \circ \Omega|_V = \mathrm{ev} \circ \omega$, we get:

$$\varphi \circ \beta|_{U \cap W} = \mathrm{ev} \circ \omega|_{U \cap W} = \mathrm{ev} \circ \Omega|_{U \cap W} = \psi \circ \alpha|_{U \cap W}.$$

We shall use this equation to define a local diffeomorphism $\Psi(t) \in \mathrm{Diff}_{\mathrm{loc}}(\mathcal{E}^X_{\mathcal{M}})$, for any $t \in U \cap W =: U_\Psi$, that defines an arrow in **Trans**$(\mathscr{A}^X_{\mathcal{M}})$ from $\omega(t) = (\varphi, \beta(t))$ to $\Omega(t) = (\psi, \alpha(t))$. The local diffeomorphism $\Psi(t)$ is defined by the transition function of the charts φ and ψ:

$$\Psi(t) : \{\varphi\} \times \varphi^{-1}(\mathrm{im}(\psi)) \longrightarrow \{\psi\} \times \psi^{-1}(\mathrm{im}(\varphi)); \qquad (\varphi, s) \longmapsto (\psi, \psi^{-1} \circ \varphi(s)).$$

It is clear this is a local diffeomorphism, as we have already discussed previously. In this way, we get a (constant) parametrisation $\Psi : U_\Psi \to \mathrm{Diff}_{\mathrm{loc}}(\mathcal{E}^X_{\mathcal{M}})$. This is a plot because $\mathcal{U}_\Psi = U_\Psi \times (\{\varphi\} \times \varphi^{-1}(\mathrm{im}(\psi)))$, which is clearly D-open in $U_\Psi \times \mathcal{E}^X_{\mathcal{M}}$, and then $\mathrm{ev} \circ (\Psi \times \mathrm{id}_{\mathcal{E}^X_{\mathcal{M}}})|_{\mathcal{U}_\Psi}$ is smooth because it is just the map

$$\mathrm{ev} \circ (\Psi \times \mathrm{id}_{\mathcal{E}^X_{\mathcal{M}}})|_{\mathcal{U}_\Psi} : (t, (\varphi, s)) \longmapsto (\psi, \psi^{-1} \circ \varphi(s)).$$

For the same reasons, we can see that the pointwise inverse parametrisation Ψ^{-1} is also smooth. It is now clear that, since for every $t \in U_\Psi$ we have $\varphi \circ \beta(t) = \psi \circ \alpha(t)$, we have an inclusion $\mathrm{im}(\beta) \subseteq \varphi^{-1}(\mathrm{im}(\psi))$, and hence $\omega(t) = (\varphi, \beta(t)) \in \mathrm{dom}(\Psi(t))$. This means that for every $t \in U_\Psi$ the pair $(\Psi(t), \iota \circ \omega(t))$ has a germ, and we get a well-defined plot:

$$\Gamma := \mathrm{germ} \circ (\Psi, \iota \circ \omega) : U_\Psi \longrightarrow \textbf{Trans}(\mathscr{A}^X_{\mathcal{M}}); \qquad \Gamma(t) = [\Psi(t)]_{\iota\omega(t)}.$$

To conclude this step, we now have a plot $(\Gamma, \omega|_{U_\Psi}) : U_\Psi \to$ **Trans**$(\mathscr{A}^X_{\mathcal{M}}) \times^{\mathrm{src}, \iota}_{\mathcal{E}^X_{\mathcal{M}}} \mathcal{E}_{\mathscr{A}}$ such that

$$\mathrm{trg} \circ \Gamma(t) = \Phi(t)(\iota \circ \omega(t)) = \Phi(t)(\varphi, \beta(t)) = (\psi, \psi^{-1} \circ \varphi \circ \beta(t)) = (\psi, \psi^{-1} \circ \psi \circ \alpha(t)) = \Omega|_{U_\Psi}(t).$$

This proves that the functor I is essentially surjective. What this means, essentially, is that every (smoothly parametrised) chart in the maximal $\mathcal{M}$-atlas on X can be connected by a transition function to a chart in $\mathscr{A}$.

Next, and lastly, we need to prove that I is fully faithful. For that, we construct a unique smooth map $\Omega : P \to \mathbf{Trans}(\mathscr{A})$ completing the following diagram:

$$\begin{array}{ccccc} P & & \xrightarrow{\quad\quad\Gamma\quad\quad} & & \downarrow \\ & \overset{\exists!\Omega}{\searrow} & & & \\ {\scriptstyle (c_1,c_2)}\downarrow & & \mathbf{Trans}(\mathscr{A}) & \xrightarrow{I} & \mathbf{Trans}(\mathscr{A}^X_{\mathcal{M}}) \\ & & {\scriptstyle (\mathrm{src},\mathrm{trg})}\downarrow & & \downarrow{\scriptstyle (\mathrm{src},\mathrm{trg})} \\ \longrightarrow & & \mathcal{E}_{\mathscr{A}} \times \mathcal{E}_{\mathscr{A}} & \xrightarrow[\iota\times\iota]{} & \mathcal{E}^X_{\mathcal{M}} \times \mathcal{E}^X_{\mathcal{M}}. \end{array}$$

For each point $p \in P$ we get a germ $\Gamma(p)$ on $\mathcal{E}^X_{\mathcal{M}}$, with source $\iota \circ c_1(p)$ and target $\iota \circ c_2(p)$. Recalling the remarks around the construction of I, we can similarly define

$$\Omega : P \longrightarrow \mathbf{Trans}(\mathscr{A}); \qquad \Omega(p) := [\iota^{-1}]_{\iota c_2(p)} \circ \Gamma(p) \circ [\iota]_{c_1(p)}.$$

From its definition it easy to see that $I \circ \Omega = \Gamma$, and in fact this equation determines Ω uniquely. We clearly also have $\mathrm{src} \circ \Omega = c_1$ and $\mathrm{trg} \circ \Omega = c_2$, so that it completes the entire diagram in a unique way. The inner square is therefore a pullback in the category of diffeological spaces, and this is what we needed to show that I is fully faithful. Concluding: the functor I is a weak equivalence, and the Morita equivalence $\mathbf{Trans}(\mathscr{A}) \simeq_{\mathrm{ME}} \mathbf{Trans}(\mathscr{A}^X_{\mathcal{M}})$ follows by Proposition **5.4**. □

Corollary 6.16. *For any two $\mathcal{M}$-atlases $\mathscr{A}_1$ and $\mathscr{A}_2$ on X we have* $\mathbf{Trans}(\mathscr{A}_1) \simeq_{\mathrm{ME}} \mathbf{Trans}(\mathscr{A}_2)$.

Proof. By Lemma **6.15** it follows that the transition groupoid of any $\mathcal{M}$-atlas on X is Morita equivalent to the transition groupoid of the maximal $\mathcal{M}$-atlas, so the corollary follows by transitivity. □

We now know that every $\mathcal{M}$-atlas on a space X defines a unique Morita equivalence class of transition groupoids, namely just the one defined by the maximal $\mathcal{M}$-atlas $\mathbf{Trans}(\mathscr{A}^X_{\mathcal{M}})$. But what can we say if we have two $\mathcal{M}$-atlases on two diffeological spaces X and Y? The following lemma proves that if X and Y are diffeomorphic, then the atlases still induce Morita equivalent transition groupoids[50].

Lemma 6.17. *Let $X, Y \in$* **Diffeol** *be two diffeological spaces, both admitting maximal $\mathcal{M}$-atlases $\mathscr{A}^X_{\mathcal{M}}$ and $\mathscr{A}^Y_{\mathcal{M}}$. If $X \cong Y$, then there is a Morita equivalence* $\mathbf{Trans}(\mathscr{A}^X_{\mathcal{M}}) \simeq_{\mathrm{ME}} \mathbf{Trans}(\mathscr{A}^Y_{\mathcal{M}})$.

Proof. If X and Y are diffeomorphic, we can find a diffeomorphism $f : X \to Y$. This diffeomorphism allows us to translate between charts on X and charts on Y. Specifically, given a pointed chart $(\varphi, t) \in \mathcal{E}^X_{\mathcal{M}}$ on X, we get a pointed chart $(f \circ \varphi, t) \in \mathcal{E}^Y_{\mathcal{M}}$ on Y. Since the composition of local diffeomorphisms is smooth, we get a smooth map $F_0 : \mathcal{E}^X_{\mathcal{M}} \to \mathcal{E}^Y_{\mathcal{M}}$, given by $(\varphi, t) \mapsto (f \circ \varphi, t)$. In fact, this map is a diffeomorphism, whose inverse is just $F_0^{-1} : (\psi, s) \mapsto (f^{-1} \circ \psi, s)$. This diffeomorphism further induces

$$F : \mathbf{Trans}(\mathscr{A}^X_{\mathcal{M}}) \longrightarrow \mathbf{Trans}(\mathscr{A}^Y_{\mathcal{M}}); \qquad [\Phi]_{(\varphi,t)} \longmapsto [F_0 \circ \Phi \circ F_0^{-1}]_{F_0(\varphi,t)}.$$

To see that F is well defined, note first that this expression is clearly independent on the germ of Φ, since we can rewrite the expression as a composition of germs (which is well-defined by Lemma **6.5**).

[50] Here our framework is less rich than that in [**Los15**]. In Theorem 4.2 Losik defines for each smooth $f : X \to Y$ a right principal bibundle from $\mathbf{Trans}(\mathscr{A}_X)$ to $\mathbf{Trans}(\mathscr{A}_Y)$. This is allowed, because Losik's atlases do not merely contain local diffeomorphisms, so the function f can be used to push charts $\varphi \in \mathscr{A}_X$ forward to $f \circ \varphi \in \mathscr{A}_Y$. We have tried to replicate this proof in the diffeological setting, but found no solid argument why this bibundle should be right principal. Here we therefore present only the simpler result in which case f is a diffeomorphism.

We further need to check that $F_0 \circ \Phi \circ F_0^{-1}$ defines a transition function. For that, let $(\psi, s) \in \mathcal{E}^Y_{\mathcal{M}}$ be a pointed chart on Y, and calculate:

$$\begin{aligned}
\mathrm{ev} \circ F_0 \circ \Phi \circ F_0^{-1}(\psi, s) &= \mathrm{ev} \circ F_0 \circ \Phi(f^{-1} \circ \psi, s) \\
&= \mathrm{ev} \circ \big(f \circ \Phi_1(f^{-1} \circ \psi, s), \Phi_2(f^{-1} \circ \psi, s)\big) \\
&= f \circ \Phi_1(f^{-1} \circ \psi, s) \big(\Phi_2(f^{-1} \circ \psi, s)\big) \\
&= f \circ \mathrm{ev} \circ \Phi(f^{-1} \circ \psi, s) \\
&= f \circ \mathrm{ev}(f^{-1} \circ \psi, s) = \psi(s).
\end{aligned}$$

The map F is clearly smooth on the level of local diffeomorphisms, since it is just a composition, and hence it must also be smooth on the level of germs. Its functoriality is also easy to check. We therefore have a smooth functor from $\mathbf{Trans}(\mathscr{A}^X_{\mathcal{M}})$ to $\mathbf{Trans}(\mathscr{A}^Y_{\mathcal{M}})$. But, if we repeat this construction for the function f^{-1}, we just get a smooth functor F^{-1} which is the inverse of F. Therefore F is a smooth categorical isomorphism, and by Proposition **5.5** it follows that F induces a weak-, and hence Morita-equivalence between $\mathbf{Trans}(\mathscr{A}^X_{\mathcal{M}})$ and $\mathbf{Trans}(\mathscr{A}^Y_{\mathcal{M}})$. ☐

Combining Corollary **6.16** and Lemma **6.17** we find that the Morita equivalence classes of $\mathcal{M}$-atlases on diffeological spaces can distinguish between diffeological spaces:

Proposition 6.18. *Let $X, Y \in$ **Diffeol** be two diffeological spaces with $\mathcal{M}$-atlases $\mathscr{A}_X$ and $\mathscr{A}_Y$, respectively. Then $\mathbf{Trans}(\mathscr{A}_X) \simeq_{\mathrm{ME}} \mathbf{Trans}(\mathscr{A}_Y)$ if and only if $X \cong Y$.*

See also [**MM03**, Proposition 5.29], which is an analogous claim for orbifold atlases. Proposition **6.18** shows that Morita equivalence classes of transition groupoids of $\mathcal{M}$-atlases correspond exactly to the diffeological spaces that are modelled on $\mathcal{M}$. The next simple trick shows that we also have Morita equivalences between atlases based on different model families:

Lemma 6.19. *Suppose that X is a diffeological space that simultaneously admits a maximal $\mathcal{M}$-atlas $\mathscr{A}^X_{\mathcal{M}}$, and a maximal $\mathcal{N}$-atlas $\mathscr{A}^X_{\mathcal{N}}$. Then there is a Morita equivalence $\mathbf{Trans}(\mathscr{A}^X_{\mathcal{M}}) \simeq_{\mathrm{ME}} \mathbf{Trans}(\mathscr{A}^X_{\mathcal{N}})$.*

Proof. Both atlases $\mathscr{A}^X_{\mathcal{M}}$ and $\mathscr{A}^X_{\mathcal{N}}$ are $\mathcal{M} \cup \mathcal{N}$-atlases on X, so their transition groupoids are Morita equivalent to $\mathbf{Trans}(\mathscr{A}^X_{\mathcal{M}\cup\mathcal{N}})$ by Lemma **6.15**. The results then follows by transitivity. ☐

This result can also be obtained from the viewpoint of bibundles by looking at the proof of the theorem in [**IZL18**, Section 8]. Indeed, since X admits an $\mathcal{M}$-atlas, it also admits a maximal $\mathcal{M} \cup \mathcal{N}$-atlas, denoted $\mathscr{A}^X_{\mathcal{M}\cup\mathcal{N}}$. We then have two canonical inclusions $\iota_{\mathcal{M}} : \mathcal{E}^X_{\mathcal{M}} \hookrightarrow \mathcal{E}^X_{\mathcal{M}\cup\mathcal{N}}$ and $\iota_{\mathcal{N}} : \mathcal{E}_{\mathcal{N}} \hookrightarrow \mathcal{E}^X_{\mathcal{M}\cup\mathcal{N}}$. The claim is that there exists a biprincipal bibundle through a diffeological space $\Gamma \subseteq \mathbf{Trans}(\mathscr{A}^X_{\mathcal{M}\cup\mathcal{N}})$. This space Γ is exactly the set of germs represented by transition functions $\Phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathcal{E}^X_{\mathcal{M}\cup\mathcal{N}})$ such that $\mathrm{dom}(\Phi) \subseteq \iota_{\mathcal{M}}(\mathcal{E}^X_{\mathcal{M}})$ and $\mathrm{im}(\Phi) \subseteq \iota_{\mathcal{N}}(\mathcal{E}^X_{\mathcal{N}})$. That is, they are the germs of transition functions from the $\mathcal{M}$-charts to the $\mathcal{N}$-charts. The left- and right actions are then just the composition of germs of transition functions between the right types of charts. Hence Γ translates between the $\mathcal{M}$- and $\mathcal{N}$-charts.

The fact that the type of the atlas does not matter (Lemma **6.19**), we can slightly generalise Proposition **6.18** to the following theorem:

Theorem 6.20. *Let $X, Y \in$ **Diffeol** be two diffeological spaces with atlases $\mathscr{A}_X$ and $\mathscr{A}_Y$, respectively. Then $\mathbf{Trans}(\mathscr{A}_X) \simeq_{\mathrm{ME}} \mathbf{Trans}(\mathscr{A}_Y)$ if and only if $X \cong Y$.*

Proof. If the transition groupoids are Morita equivalent, the claim follows by Corollary **6.13**. For the converse, suppose that $\mathscr{A}_X$ is an $\mathcal{M}_X$-atlas and $\mathscr{A}_Y$ is an $\mathcal{M}_Y$-atlas. Both atlases can then be seen as $\mathcal{M}_X \cup \mathcal{M}_Y$-atlases, so the result follows by Lemma **6.17**. ☐

This result may become particularly interesting if we can extend the (pseudo)functor that assigns to each Lie groupoid $G \rightrightarrows G_0$ its groupoid C*-algebra $C^*(G)$ to the (bi)category of diffeological groupoids (and bibundles). This functor preserves Morita equivalence [**Lan01b**, Theorem 2], i.e., the groupoid C*-algebras of two Morita equivalent Lie groupoids are themselves Morita equivalent (in the sense of Rieffel [**Rie74**]). Theorem **6.20** could then become a tool to study the relation between diffeology and noncommutative geometry, extending on the relation between the two theories as discussed in [**IZL18**; **IZP20**]. There is work in progress on defining the groupoid C^*-algebra of a diffeological groupoid [**ASZ19**]. It might be interesting to also compare the diffeological groupoid appraoch to orbifolds in [**IZL18**] to the theory of spectral triples of orbifolds in [**Har14**].

Another application of this framework of atlases could be to transfer the structure of model spaces to a global diffeological space. The main example of this is the notion of a *diffeological manifold*, as we have already discussed in Example **2.21**. The model spaces for a diffeological manifold all have a (diffeological) vector space structure. Another example could be to consider a model family of measure spaces, and to somehow transfer these measures to the global space. In order to develop these ideas, we should also study what a Morita equivalence $\mathbf{Trans}(\mathscr{A}_{\mathcal{M}}) \simeq_{\mathrm{ME}} \mathbf{Trans}(\mathscr{A}_{\mathcal{N}})$ implies in terms of a relation between the $\mathcal{M}$-model spaces and $\mathcal{N}$-model spaces.

A Categories and groupoids

The author learned category theory from Jaap van Oosten, whose lecture notes [vOo16] we recommend. The golden standard is [Mac71] for pure category theory, and [MM94] for a focus on sheaves. We also recommend [Lei14] for a more modern and introductory text. We establish our definitions and notation in this appendix. We will not concern ourselves with set-theoretic foundational issues.

Definition A.1. A *category* $\mathbf{C}$ consists of: a collection $\mathrm{ob}(\mathbf{C})$ of *objects*, and for each pair $C, D \in \mathrm{ob}(\mathbf{C})$ of such objects, a collection $\mathrm{Hom}_{\mathbf{C}}(C, D)$ of *arrows* (or *morphisms*) *from* C *to* D. Together with these arrows and objects, for every three objects $B, C, D \in \mathrm{ob}(\mathbf{C})$ there is a function

$$\mathrm{Hom}_{\mathbf{C}}(C, D) \times \mathrm{Hom}_{\mathbf{C}}(B, C) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(B, D); \qquad (f, g) \longmapsto f \circ g,$$

called the *composition in* $\mathbf{C}$, and for every object $C \in \mathrm{ob}(\mathbf{C})$ there is a distinguished *identity arrow* $\mathrm{id}_C \in \mathrm{Hom}_{\mathbf{C}}(C, C)$, satisfying the following two axioms:

1. *(Associativity)* For every $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathbf{C}}(B, C)$, and $h \in \mathrm{Hom}_{\mathbf{C}}(C, D)$, we have that $h \circ (g \circ f) = (h \circ g) \circ f$.
2. *(Identity Law)* For every $f \in \mathrm{Hom}_{\mathbf{C}}(C, D)$ we have $\mathrm{id}_D \circ f = f$, and $f \circ \mathrm{id}_C = f$.

We shall usually think of a category $\mathbf{C}$ as the union of all arrows between its objects. But beware: it is common that the notation $C \in \mathbf{C}$ is taken to mean $C \in \mathrm{ob}(\mathbf{C})$. Furthermore, we adopt the standard functional notation by writing $f : C \to D$ or $C \xrightarrow{f} D$ for an arrow $f \in \mathrm{Hom}_{\mathbf{C}}(C, D)$. Given $f : C \to D$, the object C will be called the *source* or *domain* of f, and D will be called the *target* or *codomain*.

Definition A.2. Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. A *functor* $F : \mathbf{C} \to \mathbf{D}$ comprises a function $F_0 : \mathrm{ob}(\mathbf{C}) \to \mathrm{ob}(\mathbf{D})$, sometimes written as $C \mapsto F_0C$ without the parentheses, and for each pair of objects $C, D \in \mathrm{ob}(\mathbf{C})$ a function $F : \mathrm{Hom}_{\mathbf{C}}(C, D) \to \mathrm{Hom}_{\mathbf{D}}(F_0C, F_0D)$ such that:

1. *(Compositionality)* For every pair of composable arrows $f : C \to D$ and $g : B \to C$ in $\mathbf{C}$, we have $F(f \circ g) = F(f) \circ F(g)$.
2. *(Unitality)* For every object $C \in \mathrm{ob}(\mathbf{C})$ we have $F(\mathrm{id}_C) = \mathrm{id}_{F_0X}$.

If we denote the collection of all arrows in a category $\mathbf{C}$ by $\mathrm{ar}(\mathbf{C})$, then we get two functions $\mathrm{src}, \mathrm{trg} : \mathrm{ar}(\mathbf{C}) \to \mathrm{ob}(\mathbf{C})$, sending each arrow in $\mathbf{C}$ to its source and target, respectively. Then, a functor $F : \mathbf{C} \to \mathbf{D}$ consists of a function $F_0 : \mathrm{ob}(\mathbf{C}) \to \mathrm{ob}(\mathbf{D})$ and another function $F : \mathrm{ar}(\mathbf{C}) \to \mathrm{ar}(\mathbf{D})$, satisfying $\mathrm{src} \circ F = F_0 \circ \mathrm{src}$ and $\mathrm{trg} \circ F = F_0 \circ \mathrm{trg}$, in addition to the compositionality and unitality axioms in Definition A.2.

Definition A.3. Consider two functors $F, G : \mathbf{C} \to \mathbf{D}$. A *natural transformation* $\mu : F \to G$ is a family of arrows $(\mu_C : F_0C \to G_0C)_{C \in \mathrm{ob}(\mathbf{C})}$, such that for every arrow $f : C \to D$ in $\mathbf{C}$ we have a commuting *naturality square*:

$$\begin{array}{ccc} F_0C & \xrightarrow{\mu_C} & G_0C \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F_0D & \xrightarrow[\mu_D]{} & G_0D. \end{array}$$

Definition A.4. A *groupoid* is a category in which every arrow has an inverse.

The working definition of a groupoid in terms of structure maps is explained in Section 3.2.

A.1 Cartesian closedness

Definition A.5. A category $\mathbf{C}$ is called *Cartesian closed* if it has finite products, and for every object $D \in \mathbf{C}$ the product functor $- \times D : \mathbf{C} \to \mathbf{C}$ has a right adjoint. This right adjoint is denoted $\mathbf{C}(D, -) : \mathbf{C} \to \mathbf{C}$, and is called the *internal hom-functor*. The adjunction manifests itself as a natural family of bijections

$$\mathrm{Hom}_{\mathbf{C}}(C \times D, E) \cong \mathrm{Hom}_{\mathbf{C}}(C, \mathbf{C}(D, E)).$$

Internal hom-funtors are also sometimes denoted $(-)^D : \mathbf{C} \to \mathbf{C}$, in which the above natural bijections become $\mathrm{Hom}_{\mathbf{C}}(C \times D, E) \cong \mathrm{Hom}_{\mathbf{C}}(C, E^D)$. This originates from set theory, where the notation X^Y means the set of all functions $Y \to X$. The adjunction $- \times Y \dashv (-)^Y$ then means that there is a natural bijection between functions $X \times Y \to Z$ and functions $X \to Z^Y$.

Definition A.6. Consider an object B in some category $\mathbf{C}$. The *slice category* $\mathbf{C}/B$ is a category which is constituted as follows. Its objects are the arrows in $\mathbf{C}$ with codomain B. An arrow in $\mathbf{C}/B$, between two such objects $p : E \to B$ and $q : F \to B$, is a third arrow $f : E \to F$ in $\mathbf{C}$ making a commutative triangle: $q \circ f = p$.

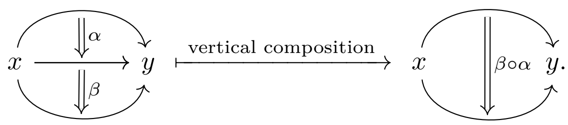
Definition A.7. A category $\mathbf{C}$ is called *locally Cartesian closed* if for every object $B \in \mathbf{C}$ the slice category $\mathbf{C}/B$ is Cartesian closed.

Proposition A.8. *If* $\mathbf{C}$ *has all pullbacks, then each slice category of* $\mathbf{C}$ *has all finite products.*

A.2 The idea of a bicategory

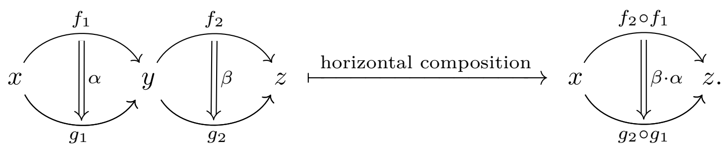
For the precise definition of a bicategory we refer to [**Mac71**] or the recent book [**JY20**], and for an informal introduction we refer to [**Lac07**]. The notion of a *2-category* arises in this thesis simply because diffeological groupoids are, in particular, categories. It is an elementary fact that the category **Cat** of all (small) categories with functors and natural transformations is a 2-category[51]. This structure carries over to diffeological groupoids: it is clear that the identity functor is smooth, and that composition of smooth functors is smooth. Smoothness of the structure maps moreover ensures that composition and inversion of smooth natural transformations are smooth.

A 2-category contains, besides objects and morphisms as in an ordinary category, an additional level of structure: morphisms between morphisms. These are called *2-morphisms*, or *2-arrows*. They behave nicely with respect to their composition in the sense that for any two objects x and y in a 2-category $\mathbf{C}$, their class of 1-morphisms $\mathrm{Hom}_{\mathbf{C}}(x, y)$ form a genuine category when taking the 2-morphisms of $\mathbf{C}$ as their arrows. In this way a 2-category can also be seen as a family of categories. Composition in $\mathrm{Hom}_{\mathbf{C}}(x, y)$ of 2-morphisms is called *vertical composition*. This may be depicted diagrammatically as



There is also a *horizontal composition*, commuting with vertical composition, in the guise of a functor

$$\mathrm{Hom}_{\mathbf{C}}(x, y) \times \mathrm{Hom}_{\mathbf{C}}(y, z) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(x, z) :$$



The additional structure allows us to think of isomorphism between arrows, called 2-isomorphism, instead of strict equality. This leads to, among other things, the notion of a *2-commutative* (or *weakly commutative*) diagram, which is a diagram that commutes only up to 2-isomorphism. For example, we would say the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[k]{} & D \end{array}$$

[51] *"***Cat** *is the mother of all 2-categories, just as* **Set** *is the mother of all categories,"* [**Lac07**].

2-commutes if there exists a 2-isomorphism $\alpha : g \circ f \Rightarrow k \circ h$, in which case we also write $g \circ f \cong k \circ h$. The notion of commuting only up to 2-isomorphism occurs already when studying the equivalence of categories. Recall that a functor $F : \mathbf{C} \to \mathbf{D}$ is called an *equivalence of categories* if there exists another functor $G : \mathbf{D} \to \mathbf{C}$ such that there are natural isomorphisms $F \circ G \cong \mathrm{id}_{\mathbf{D}}$ and $G \circ F \cong \mathrm{id}_{\mathbf{C}}$. In other words, F is invertible *up to 2-isomorphism*.

For our (and many other) purposes, the notion of 2-category is too restrictive. The 2-categories we describe are often called *strict 2-categories*. What we need instead are *weak 2-categories*, also known as *bicategories*. Intuitively we may think of a bicategory as a category where every axiom holds merely up to 2-isomorphism. In particular, a bicategory is a 2-category where composition is neither strictly associative, nor unital. Therefore, in a bicategory, whenever we have three composable arrows $f : x \to y$, $g : y \to z$ and $h : z \to w$, say, we can only hope to have 2-isomorphisms

$$(f \circ g) \circ h \Longrightarrow f \circ (g \circ h), \qquad f \circ \mathrm{id}_x \Longrightarrow f, \qquad \text{and} \qquad \mathrm{id}_y \circ f \Longrightarrow f,$$

instead of full-fledged equalities. These canonical 2-isomorphisms are subject to various *coherence axioms*. We omit them here. For the precise definition of a bicategory and more details, we refer to [**Mac71**]. Also see [**Lac07**]. It should be noted that a strict 2-category is a special case of a bicategory, where the three 2-isomorphisms in the previous equation are always just the identity maps.

In a bicategory there are three degrees of sameness for objects. The strictest form is simply equality: $x = y$. Then there is the familiar notion of *isomorphism*: $x \cong y$, which means there are two arrows $f : x \to y$ and $f^{-1} : y \to x$ satisfying $f \circ f^{-1} = \mathrm{id}_y$ and $f^{-1} \circ f = \mathrm{id}_x$. The map f is then known as a *strict 1-isomorphism*, or just as an *isomorphism*. These two concepts make sense in any category, but in a bicategory we have an additional notion: the objects x and y are called *weakly isomorphic*, or *equivalent*, sometimes denoted $x \simeq y$, if there are arrows $f : x \to y$ and $g : y \to x$ satisfying $f \circ g \cong \mathrm{id}_y$ and $g \circ f \cong \mathrm{id}_x$. That is to say, when f is invertible up to 2-isomorphism. In that case f is called a *weak 1-isomorphism*, or just an *equivalence*, and g is called its *weak inverse*. Note that this generalises the notion of equivalence between categories to the objects of an arbitrary bicategory. It is the notions of equivalence and weak inverse that we use in our study of diffeological groupoids and bibundles.

There are several notions of strictness one may employ to define morphisms between bicategories (cf. [**Lac07**, Section 3]). The weakest one is a *lax functor* F, which is not even functorial or unital up to 2-isomorphism, but only has canonical 2-morphisms $F(g) \circ F(f) \Rightarrow F(g \circ f)$ and $\mathrm{id}_{F(x)} \Rightarrow F(\mathrm{id}_x)$, called *comparison maps*. Whenever these comparison maps are invertible, we speak of a *pseudofunctor*. Intuitively we think of pseudofunctors as functors that preserve composition and units up to 2-isomorphism. (Stricter still, but not important for us, a pseudofunctor whose comparison maps are identities is called a *strict 2-functor*.) Both lax- and pseudofunctors do act strictly functorially with respect to the compositions of 2-arrows.

Proposition A.9. *Pseudofunctors preserve equivalences.*

Proof. Let $F : \mathbf{C} \to \mathbf{D}$ be a pseudofunctor between bicategories. Suppose that x and y are equivalent objects in $\mathbf{C}$, so that there exist arrows $f : x \to y$ and $g : y \to x$ with 2-isomorphisms $u_y : f \circ g \Rightarrow \mathrm{id}_y$ and $u_x : g \circ f \Rightarrow \mathrm{id}_x$. Suppose we denote the canonical 2-isomorphisms $F(g) \circ F(f) \Rightarrow F(g \circ f)$ and $\mathrm{id}_{F(x)} \Rightarrow F(\mathrm{id}_x)$ by $F_{g,f}$ and F_x, respectively. Then the horizontal composition

$$F_x^{-1} \circ F(u_x) \circ F_{g,f} : F(g) \circ F(f) \Longrightarrow \mathrm{id}_{F(x)}$$

is a 2-isomorphism; similarly we have a 2-isomorphism $F(f) \circ F(g) \Rightarrow \mathrm{id}_{F(y)}$. Hence $F(f) : F(x) \to F(y)$ is an equivalence in $\mathbf{D}$. □

This proposition confirms (or ensures, depending on the method of proof) that Morita equivalence in Lie groupoids is preserved for their C*-algebras, because $C^* : \mathbf{LieGrpd}_{\mathrm{LP}} \to \mathbf{C^*Corr}$, the assignment that sends each Lie groupoid to its groupoid C*-algebra, and each Lie groupoid left principal bibundle to its C*-correspondence, is a pseudofunctor. See also the remarks made at the very end of Chapter **VI**, or [**Lan01b**, Theorem 2].

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