Some open problems on k-forms of the algebraic tori and a conjecture of T. Oda

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The multiplicative group $\mathbb{C}^*$ of the complex numbers as a complex Lie group has two "real forms", namely the real Lie groups $\mathbb{R}^*_{>0}$ and SO(2). (Each of the latter groups becomes isomorphic to $\mathbb{C}^*$ through base-field extension from $\mathbb{R}$ to $\mathbb{C}$, hence the name.) On the other hand, the underlying variety of $\mathbb{C}^*$ may be identified as the hyperbola ($XY = 1$ in $\mathbb{C}^2$), and this has three real forms, i.e., the hyperbola, the real circle ($X^2 + Y^2 = 1$) and the imaginary circle ($X^2 + Y^2 = -1$), all considered as affine varieties defined over $\mathbb{R}$. All this is classical and elementary knowledge.

In a recent work [1] (a gist of which was offered in the Hanoi Conference 2006) we expanded this knowledge to higher dimensions and to general separably algebraic base-field extensions. In the present short note, we wish to present a few open questions arising from that paper and to relate these questions to a conjecture communicated to us by Tadao Oda in June, 2006. In this process, we will focus on the forms of n-dimensional *algebraic tori* $(\mathbb{G}_m)^n$ defined over a base field k of characteristic $\neq 2$ *that split under a quadratic extension* Q of k. That is, we will look at affine k-group schemes X such that, under a quadratic extension $Q = k[\sqrt{d}\,] \supset k, d \in k, d \notin k^2$, a Q-isomorphism

$$X \times_k Q \cong_Q (\mathbb{G}_m \times_k Q)^n = ((\mathbb{G}_m)_Q)^n \qquad (1)$$

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is realized. Ultimately, our aim is to find and classify all such X's.

From our work [1] let us extract and list up all known facts relevant to this aim:

(Dimension 1) *All* k-forms of $\mathbb{G}_m$ split at a quadratic extension of k, and there are exactly two 1-dimensional k-group schemes up to k-isomorphisms that split at a given $Q = k[\sqrt{d}]$: the trivial one $\mathbb{G}_m$ and the affine k-group-scheme

$$\mathbb{T}_1 := \mathrm{Spec}(k[X,Y]/\langle X^2 - d^{-1}Y^2 - 1\rangle), \tag{2}$$

whose group operation $*$ is given by

$$(x,y) * (x',y') = (xx' + d^{-1}yy', xy' + yx') \tag{3}$$

for any k-algebra R and any R-valued points $(x,y),(x',y') \in \mathbb{T}_1(R)$. The neutral point of this group-scheme is (1,0), and $(x,y)^{-1} = (x,-y)$ for any $(x,y) \in \mathbb{T}(R)$.

(Dimension 2) The k-isomorphism classes of *nontrivial* (Q/k)-forms of $(\mathbb{G}_m)^2$ correspond to the conjugacy classes of involutions $P \in \mathrm{GL}(2,\mathbb{Z})$ (i.e., integral (2×2)-matrix $P \neq I_2$ such that $P^2 = I_2$), and there are exactly 3 such conjugacy classes. So, all in all, There are 4 k-isomorphism classes of (Q/k)-forms of $(\mathbb{G}_m)^2$, each represented by: (a) the trivial form $(\mathbb{G}_m)^2$; (b) $\mathbb{T}_1 \times_k \mathbb{T}_1$; (c) $\mathbb{G}_m \times_k \mathbb{T}_1$; (d) $\mathbb{T}_2$, whose description follows just below.

As an affine k-scheme, $\mathbb{T}_2 := \mathrm{Spec}(B_2)$, where

$$B_2 := k[X,Y,Z,Z^{-1}]/\langle X^2 - dY^2 - Z\rangle = k[x,y,z,z^{-1}]. \tag{4}$$

One can easily check that $Q \otimes_k B_2 \cong k[T_1, T_1^{-1}, T_2, T_2^{-1}]$, so that the underlying scheme of $\mathbb{T}_2$ is a (Q/k)-form of that of $(\mathbb{G}_m)^2$. As for the group structure, for any k-algebra R and R-valued points $(x,y,z,z^{-1}),(x',y',z',z'^{-1}) \in \mathbb{T}_2(R)$, the group multiplication is to be given by

$$(x, y, z, z^{-1}) \cdot (x', y', z', z'^{-1}) = (xx' + dyy', xy' + x'y, zz', (zz')^{-1}). \tag{5}$$

The neutral point is $(1, 0, 1, 1)$ and $(x, y, z, z^{-1})^{-1} = (z^{-1}x, -z^{-1}y, z, z^{-1})$.

The results as outlined above are obtained by the classical method of Galois cohomology as detailed in Serre's book [3]. According to this theory, the (K/k)-forms of $(\mathbb{G}_m)^n$ for any finite Galois extension with $G = \mathrm{Gal}(K/k)$ is parametrized by $\mathrm{H}^1(G, \mathrm{Aut}_K((\mathbb{G}_m)^n))$. Since $\mathrm{Aut}_K((\mathbb{G}_m)^n) = \mathrm{GL}(n, \mathbb{Z})$ regardless of K and G operates trivially on $\mathrm{GL}(n,\mathbb{Z})$, we have

$$\mathrm{H}^1(G, \mathrm{Aut}_K((\mathbb{G}_m)^n)) = \mathrm{Hom}(G, \mathrm{GL}(n, \mathbb{Z}))/\!\approx,$$

where $\approx$ denotes the conjugacy relation. This led us to study the finite subgroups of $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{GL}(2,\mathbb{Z})$, and aided by the elementary part of modular group theory we were able to reach results as above.

Going up to the next stage at dimension 3 level, let us present our main question:

Problem 1 *Let $Q = k[\theta]$ be a quadratic extension of k. Find, up to k-isomorphisms, all (Q/k)-forms of $(\mathbb{G}_m)^3$.*

If one is to employ the orthodox technique of Galois cohomology as done in [1], then it is natural to solve the next problem first before embarking on Problem 1, namely

Problem 2 *Determine all involutions in $\mathrm{GL}(3,\mathbb{Z})$ up to conjugacy.*

Once Problem 2 is settled, one can then proceed to find the invariant k-subalgebra of the action of Galois group $\mathbb{Z}/2\mathbb{Z}$ on $Q[T_1, T_1^{-1}, T_2, T_2^{-1}, T_3, T_3^{-1}]$ *twisted by each involution.* We would then get closer to the solution of our main Problem 1. This 'classical' approach, however, is easier said than done, since Problem 2 appears to be rather difficult.

It may well be that an alternative approach suggested by Tadao Oda could be more effective. This goes as follows: Let $\mathbb{Z}[\epsilon] := \mathbb{Z} \oplus \mathbb{Z}\epsilon, \epsilon^2 = 1$. By studying finitely-generated $\mathbb{Z}[\epsilon]$-modules free over $\mathbb{Z}$, Oda formulated a conjecture to the effect that such a module would be a direct sum of $\mathbb{Z}[\epsilon], \mathbb{Z}[\epsilon]/\mathbb{Z}(1-\epsilon)$ and $\mathbb{Z}[\epsilon]/(1+\epsilon)$.

Problem 3 *Settle Oda's Conjecture as above stated.*

One can show that, if the conjecture is established as true, each (Q/k)-form of $(\mathbb{G}_m)^3$ is a direct product of $\mathbb{G}_m$, $\mathbb{T}_1$, and $\mathbb{T}_2$. This would mean that, at dimension 3, no essentially new (Q/k)-forms of $(\mathbb{G}_m)^3$ should occur, which may be somewhat too optimistic.

References

[1] T. Kambayashi, On forms of the Laurent polynomial rings and algebraic tori in dimensions 1 and 2, PREPRINT 2007

[2] J.-P. Serre, Local Fields, Grad. Texts in Math. No.67, Springer-Verlag, New York 1975

[3] J.-P. Serre, Cohomologie Galoisienne, Lect. Notes in Math. No. 5, 5^e éd, Springer-Verlag 1994