OPEN PROBLEMS AND COMMENTS

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Let X be a normal affine variety defined over the complex field $\mathbb{C}$ with an effective algebraic action of an algebraic group G. Let $\varphi : X \to X$ be an unramified endomorphism which commutes with the G-action. The following is the equivariant version of the generalized Jacobian conjecture.

Conjecture 1. *With the above settings, φ is a finite morphism.*

If X has the Euler number $\chi(X) = 1$ then the conjecture says that φ is an automorphism.

Suppose that the algebraic quotient $Y := X//G$ exists under the above setting. Then φ induces an endomorphism $\psi : Y \to Y$. We have the following result.

Theorem 2. (cf. [6]) *Let the notations and assumptions be the same as in the above conjecture. Suppose that G is a reductive algebraic group or a unipotent algebraic group and that $\Gamma(X, \mathcal{O}_X)$ is factorial if G is unipotent. Then the endomorphism $\psi : Y \to Y$ is unramified.*

The equivariant Jacobian conjecture can be decomposed into the following conjectures.

Conjecture 3. *Let X be a normal affine variety with an effective algebraic group action of G. Suppose that the algebraic quotient $Y = X//G$ exists. Let $\varphi : X \to X$ be an unramified endomorphism which commutes with the G-action on X. Let $\psi : Y \to Y$ be the induced endomorphism. Then ψ is a finite morphism.*

Conjecture 4. *Let the notations and assumptions be the same as in Conjecture 3. Suppose that the Euler number $\chi(X) = 1$ and that the induced endomorphism ψ is an automorphism. Then the endomorphism φ is an automorphism.*

Note that the equivariant generalized Jacobian conjecture is the ordinary generalized Jacobian conjecture when G is trivial. The Conjecture 4 was treated in [4] without assuming that φ is unramified (see [1] also). By Gurjar [2], the two-dimensional quotient of $\mathbb{A}^n$ under an algebraic group action of a reductive group is isomorphic to $\mathbb{A}^2/\Gamma$, where Γ is a small finite subgroup of $\mathrm{GL}(2, \mathbb{C})$. Thus, if one notes that $\chi(\mathbb{A}^2/\Gamma) = 1$, the Conjecture 3 in this case is reduced to ask whether an unramified endomorphism of $\mathbb{A}^2/\Gamma$ is an automorphism. Indeed, the generalized Jacobian conjecture for $\mathbb{A}^2/\Gamma$ is equivalent to the following (cf. [7]):

Conjecture 5. *Let φ be an unramified endomorphism of $\mathbb{A}^2$ which commutes with a linear action of a small finite subgroup Γ of $\mathrm{GL}(2,\mathbb{C})$. Then φ is an automorphism.*

There are several known results (see [6]).

Lemma 6. *The following assertions hold.*

(1) *Suppose that G is a reductive algebraic group. Suppose further that ψ is an automorphism and that a general fiber of the quotient morphism $\mu : X \to Y$ contais a dense orbit. Then φ is an automorphism.*
(2) *The generalized Jacobian conjecture holds if $\dim X = 1$.*
(3) *Let G be a reductive algebraic group. Suppose that $\dim Y = 1$ and that a general fiber of μ contains a dense orbit. Suppose further that $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$. Then φ is an automorphism.*
(4) *Let G be a unipotent algebraic group. Suppose that the induced unramified endomorphism $\psi : Y \to Y$ is an automorphism. Then φ is an automorphism.*

Lemma 7. *Suppose that the multiplicative group G_m acts linearly and effectively on the affine space $\mathbb{A}^n$. Write the G_m-action as*

$$^t(x_1, \ldots, x_n) = (t^{\alpha_1}x_1, \ldots, t^{\alpha_n}x_n)$$

with $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$. Suppose further that $\alpha_1 \ge 0$. Let m be the dimension of the G_m-fixed point locus. If the Jacobian conjecture for $\mathbb{A}^m$ holds, then $\varphi : \mathbb{A}^n \to \mathbb{A}^n$, which is a G_m-equivariant unramified endomorphism, is an automorphism.

Theorem 8. (cf. [5]) *Let an algebraic group G of positive dimension act effectively on a normal affine surface X and let $\varphi : X \to X$ be a G-equivariant unramified endomorphism. Suppose that $\Gamma(X, \mathcal{O}_X)^* = \mathbb{C}^*$ and that $\Gamma(X, \mathcal{O}_X)$ is factorial if G is unipotent. Suppose further that X is not elliptic-ruled [1]. Then φ is an automorphism.*

If one uses results obtained in [3, 9, 10], one can obtain the following rsult.

Theorem 9. *The following assertions hold.*

[1] X is said to be elliptic-ruled if X is birational to a $\mathbb{P}^1$-bundle over an elliptic curve

(1) *Let $\varphi : \mathbb{A}^3 \to \mathbb{A}^3$ be an unramified endomorphism which commutes with an effective G_m-action. If the Jacobian conjecture for $\mathbb{A}^2$ and the Conjecture 5 hold, then φ is an automorphism.*
(2) *Let $\varphi : \mathbb{A}^3 \to \mathbb{A}^3$ be an unramified endomorphism which commutes with an effective G_a-action. If the Jacobian conjecture for $\mathbb{A}^2$ holds, φ is an automorphism.*

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