The Automorphism Group of Certain Factorial Threefolds and a Cancellation Problem

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Abstract

A new class of counterexamples to a generalized cancellation problem for affine varieties is presented. Each member of the class is an affine factorial complex threefold admitting a locally trivial action of the additive group, i.e. the total space for a principal G_a bundle over a quasiaffine base. The automorphism groups for these varieties are also determined.

1 Introduction

A well known cancellation problem asks, for complex affine varieties X and Y, whether an isomorphism $X \times \mathbb{C} \cong Y \times \mathbb{C}$ implies that $X \cong Y$. For X and Y of dimension 1 a positive answer is given by [1] and for X and Y of dimension 2 counterexamples are provided by the Danielewski surfaces [2] [10] [8] [5]. On the other hand, for $X \times \mathbb{C} \cong \mathbb{C}^3$, Fujita and Miyanishi-Sugie proved that $X \cong \mathbb{C}^2$. The Danielewski surfaces can be realized as total spaces for principal bundles for G_a, the additive group of complex numbers, over the affine line with two origins. They are therefore smooth surfaces, but nonfactorial, i.e. their coordinate rings lack the unique factorization property. It is natural then to ask whether the cancellation problem has a positive solution for factorial affine varieties, or for affine total spaces of principal G_a bundles over quasiaffine varieties. We produce families of three dimensional counterexamples.

To point out the role played by principal G_a bundles, let Y be a scheme over $\mathbb{C}$, and X_1, X_2 total spaces for principal G_a bundles over Y. Then each X_i

is represented by a one cocycle in $H^1(Y, O_Y)$, and we can represent the base extension $X_1 \times_Y X_2$ by elements of $H^1(X_1, O_{X_1})$ (and $H^1(X_2, O_{X_2})$. If the X_i are affine then $H^1(X_i, O_{X_i}) = 0$ and therefore $X_1 \times \mathbb{C} \cong X_1 \times_Y X_2 \cong X_2 \times \mathbb{C}$. In particular, affine total spaces for principal G_a bundles is a natural context in which to seek potential counterexamples to the cancellation problem.

In the case of the Danielewski surfaces, not only are the bundles inequivalent, the total spaces are not homeomorphic in the natural (complex) topology on $\mathbb{C}^3$, let alone isomorphic as varieties. For a complex quasiprojective base however, a principal G_a bundle is necessarily trivial in the natural topology [20]. Thus algebraic methods are necessary to distinguish the total spaces. The Makar-Limanov invariant, which for an affine $k-$domain A is the intersection of the kernels of all locally nilpotent $k-$derivations of A, provides the necessary algebraic tool enabling the determination of the automorphism groups of certain affine threefolds, all obtained as total spaces for principal G_a bundles over the spectrum of singular but factorial complex surfaces punctured at the singular point. A class of these threefolds yield the desired counterexamples:

Example 1 *Let $X_{n,m} \subset \mathbb{C}^5$ be the affine variety defined by*

$$X^a + Y^b + Z^c, X^m U - Y^n V - 1$$

with m, n positive integers and a, b, c pairwise relatively prime positive integers satisfying $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$. Then $X_{n,m}$ is factorial,

$$X_{n,m} \times \mathbb{C} \cong X_{n',m'} \times \mathbb{C}$$

for all $(m, n), (m', n')$, but $X_{n,m} \cong X_{n',m'}$ implies that $(m, n) = (m', n')$.

We suspect that the condition $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$ can be weakened.

Principal G_a bundles with affine total space X arise from locally trivial algebraic G_a actions on X. The local triviality implies that the quotient X/G_a exists as an algebraic scheme, and gives X the structure of a principal G_a bundle over X/G_a. If X is in addition factorial, then X/G_a has the structure of a quasiaffine variety. The Makar-Limanov invariant enters the picture since every algebraic G_a action on an affine X arises as the exponential of a locally nilpotent derivation D of $\mathbb{C}[X]$. If X is factorial, then the action is locally trivial if and only if $\ker(D) \cap im(D)$ generates the unit ideal in $\mathbb{C}[X]$ [4]. An action is equivariantly trivial, i.e. X is G_a isomorphic to $Y \times \mathbb{C}$ with G_a acting by addition on the second component, if and only if $D(s) = 1$ for some $s \in \mathbb{C}[X]$. Such an s is called a slice.

One can see easily that no two dimensional UFD can give rise to a counterexample to generalized cancellation via non trivial G_a bundles.

Lemma 1 *Let A be a two dimensional finitely generated $\mathbb{C}$ algebra which is a UFD. If A admits a nonzero LND, then A is isomorphic to a one variable polynomial ring over a UFD subring.*

Proof. Suppose that $D \in Der(A)$ is locally nilpotent. Denote by F the set of fixed points of the G_a action on **Spec** A generated by D. By assumption, either F is empty, in which case D has a slice [7], or the dimension of F is equal to one [17]. In the latter case, F the support of a principal divisor $\mathfrak{D} = (f)$ for some $f \in A^D$, and $D(A) \subset fA$. Thus $D' := f^{-1}D$ is again locally nilpotent generating a fixed point free G_a action with a slice. ■

That the generalized cancellation problem has an affirmative solution for a polynomial ring in one variable over a one dimensional UFD follows from the results in [1] or [15, Theorem 2.9].

Since a singular point of a factorial surface is isolated, such a surface cannot be isomorphic to the product of a curve with a line. Thus

Corollary 1 *A singular factorial surface admits no nontrivial locally nilpotent dervartions.*

2 The Makar-Limanov Invariant.

The condition on the exponents a, b, c in the above example will enable us to use Mason's theorem, stated here as Theorem 1. Let k be a field of characteristic 0 and, for $f \in k[T]$, denote by $N(f)$ the number of distinct zeroes of f in an algebraic closure of k .

Theorem 1 *(e.g. [19]) Let $f, g \in k[T]$ and let $h = f + g$. Assume that f, g, h are relatively prime of positive degree. Then*

$$max\{deg(f), deg(g), deg(h)\} < N(fgh).$$

Two corollaries apply to the problem at hand.

Corollary 2 *Let $P(X, Y, Z) = X^a + Y^b + Z^c$ where $a, b, c \in \mathbb{N}$ satisfy*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1.$$

If $f, g, h \in k[T]$ satisfy

1. $P(f, g, h) = 0$ and
2. f, g, h are relatively prime.

Then at least one of f, g, h must be constant.

Proof. It is enough to consider the case that k is algebraically closed. Assume that none of f, g, h is constant. Applying Mason's theorem, the fact that $f^a + g^b + h^c = 0$ yields:

$$\begin{array}{rcl} max(a \cdot deg(f), b \cdot deg(g), c \cdot deg(h)) & < & N(f^a) + N(g^b) + N(h^c) \\ & = & N(f) + N(g) + N(h) \\ & \leq & deg(f) + deg(g) + deg(h). \end{array}$$

Suppose

$$a \cdot deg(f) \geq b \cdot deg(g),\ a \cdot deg(f) \geq c \cdot deg(h) > 0.$$

Then $\frac{\deg g}{\deg f} \leq \frac{a}{b}, \frac{\deg h}{\deg f} \leq \frac{a}{c}$ so that

$$\begin{array}{rcl} a \cdot deg(f) & = & max(a \cdot deg(f), b \cdot deg(g), c \cdot deg(h)) \\ & < & deg(f) + deg(g) + deg(h) \\ & \leq & deg(f)(1 + \frac{a}{b} + \frac{a}{c}). \end{array}$$

Thus $1 < \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, which exactly contradicts the assumption.

The cases where $b \cdot deg(g)$ or $c \cdot deg(h)$ is the largest go equivalently. ■

Corollary 3 *Let* $P(X, Y, Z) = X^a + Y^b + Z^c + \lambda$ *where* $\lambda \in k$*, and* $a, b, c \in \mathbb{N} \backslash \{0, 1, 2, 3\}$ *satisfy* $\frac{1}{a-3} + \frac{1}{b-3} + \frac{1}{c-3} \leq \frac{1}{2}$*. If* $f, g, h \in k[T]$ *satisfy*

1. $P(f, g, h) = 0$ and
2. f, g, h are relatively prime.

Then at least one of f, g, h must be constant.

Proof. Again it is enough to consider the case that k is algebraically closed. We will arrive at a contradiction from the assumption that $f^a + g^b + h^c = \lambda$ for some nonconstant f, g, h. Taking derivatives with respect to T yields $af^{a-1}f' + bg^{b-1}g' + ch^{c-1}h' = 0$. Now we cannot apply Mason's theorem directly as there may be common factors in ff', gg', hh'. Define $w := gcd(f^{a-1}f', g^{b-1}g', h^{c-1}h')$. Using the fact that $gcd(xy, z)$ divides $gcd(x, z)gcd(y, z)$ repeatedly we see that w divides $gcd(f', g^{b-1}g', h^{c-1}h') \cdot gcd(f^{a-1}, g', h^{c-1}h') \cdot gcd(f^{a-1}, g^{b-1}, h') \cdot gcd(f^{a-1}, g^{b-1}, h^{c-1})$ and since $gcd(f, g, h) = 1$, we see that $deg(w) \leq deg(f') + deg(g') + deg(h') = deg(f) + deg(g) + deg(h) - 3$. One can apply Mason's theorem to

$$a\frac{1}{w}f^{a-1}f' + b\frac{1}{w}g^{b-1}g' + c\frac{1}{w}h^{c-1}h' = 0,$$

which, together with some calculus, yields

$$\begin{array}{rl} & 2(deg(f) + deg(g) + deg(h)) \\ \geq & N(ff'gg'hh') \\ \geq & N(ff'\frac{1}{w}gg'\frac{1}{w}hh'\frac{1}{w}) \\ \text{(Mason's)} > & max\big(deg(f^{a-1}f'\frac{1}{w}), deg(g^{b-1}g'\frac{1}{w}), deg(h^{c-1}h'\frac{1}{w})\big) \\ = & max\big(deg(f^{a-1}f'), deg(g^{b-1}g'), deg(h^{c-1}h')\big) - deg(w) \\ \geq & max\big(deg(f^{a-1}f'), deg(g^{b-1}g'), deg(h^{c-1}h')\big) \\ & \quad - deg(f) - deg(g) - deg(h) + 3 \\ = & max\big(adeg(f) - 1, bdeg(g) - 1, cdeg(h) - 1\big) \\ & \quad - deg(f) - deg(g) - deg(h) + 3 \\ \geq & max\big((a-3)deg(f), (b-3)deg(g), (c-3)deg(h)\big) + 2 \\ > & max\big((a-3)deg(f), (b-3)deg(g), (c-3)deg(h)\big) \end{array}$$

Assuming that $max((a-3)deg(f),(b-3)deg(g),(c-3)deg(h)) = (a-3)deg(f)$ (the other cases go similarly) then will yield $(a-3)deg(f) < 2(1+\frac{a-3}{b-3}+\frac{a-3}{c-3})deg(f)$ which exactly contradicts the assumption $\frac{1}{a-3}+\frac{1}{b-3}+\frac{1}{c-3} \le \frac{1}{2}$. ■

Definition 1 *1. For a k-domain B, $LND(B)$ is the set of locally nilpotent k derivations of B.*

2. Given $D \in LND(B)$, $s \in B$ is a slice for D if $D(s) = 1$.

3. Given $D \in LND(B)$, an element p of B is called a preslice if $0 = D^2(p) \neq D(p)$.

Remark 1 *A preslice always exists for a nonzero locally nilpotent derivation D. Indeed, by local nilpotency, for $b \in B - \ker(D)$, there is a positive integer n for which $0 \neq D^{n+1}(b) \in \ker(D)$. Then $p = D^n(b)$ is a preslice. If D admits a slice s, then $B = B^D[s]$, where B^D denotes $\ker(D)$, and therefore $D = \frac{\partial}{\partial s}$ [3].*

Lemma 2 *Let A be a $\mathbb{C}$-domain and $x, y, z \in A\backslash\{0\}$. Let $P = x^a + y^b + z^c + \lambda$ for some $a, b, c \in \mathbb{N}\backslash\{0,1\}$, $\lambda \in \mathbb{C}$. Let $B := A/(P)$, and assume that B is a domain (i.e. P is a prime element of A). If either*
i) $\lambda = 0$ and $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \le 1$, or
ii) $a, b, c \ge 4$ and $\frac{1}{a-3}+\frac{1}{b-3}+\frac{1}{c-3} \le \frac{1}{2}$,
then $D \in LND(B)$ implies $D(x) = D(y) = D(z) = 0$.

Proof. Since B is a domain, and D is locally nilpotent, a preslice p exists. Set $q := D(p)$ (and thus $q \in B^D$) and observe that D extends uniquely to a locally nilpotent derivation $\tilde{D}$ of $\tilde{B} := B[q^{-1}]$. Since $\tilde{D}$ has the slice $s := p/q$ we have $\tilde{B} = \tilde{B}^D[s]$. We can identify $\tilde{D}$ with $\frac{\partial}{\partial s}$. Denote by K the quotient field of $\tilde{B}^{\frac{\partial}{\partial s}}$ (= quotient field of B^D) noting that D extends uniquely to $\frac{\partial}{\partial s}$ on $K[s]$. Write $x, y, z \in K[s]$, as $x = f(s), y = g(s), z = h(s)$ for some polynomials $f, g, h \in K[s]$. If $k = \gcd(f, g)$ then k divides h as well. Writing

$$f = k\widehat{f}, g = k\widehat{g}, h = k\widehat{h}$$

we obtain

$$(k^{bc}\widehat{f})^a + (k^{ac}\widehat{g})^b + (k^{ab}\widehat{h})^c = 0$$

and therefore

$$\widehat{f}^a + \widehat{g}^b + \widehat{h}^c = 0$$

with $\widehat{f}, \widehat{g}, \widehat{h}$ pairwise relatively prime.

In case i) we can use corollary 2, to conclude that k and at least one of $\widehat{f}, \widehat{g}, \widehat{h}$ lie in K, so that one of x, y, z lies in $\ker(D)$. But if, for instance, $D(x) = 0$, then $0 = D(y^b + z^c)$ then by the following lemma we see that $D(y) = D(z) = 0$.

Similarly in case ii) we can use corollary 3 to conclude that at least one of x, y, z must lie in $\ker(D)$. Suppose it is x. Then again $D(y^b + z^c) = 0$ where $b, c \ge 2$. ■

Lemma 3 *(Makar-Limanov [11, Lemma 2]) Let A be a domain and let $n, m \in \mathbb{N}$ satisfying $n, m \geq 2$. If $D \in LND(A)$ and $D(c_1 a^n + c_2 b^m) = 0$ where $a, b \in A$, $c_1, c_2 \in A^D$, and $c_1 a^n + c_2 b^m \neq 0$. Then $D(a) = D(b) = 0$.*

Fix $P(X, Y, Z) := X^a + Y^b + Z^c + \lambda$ in $\mathbb{C}[X, Y, Z]$ and assume that P is irreducible, i.e. that a, b, c are pairwise relatively prime.

Notation 1 *For the remainder of the paper, $R := \mathbb{C}[X, Y, Z]/(P)$, and x, y, z denote the images of X, Y, Z in R. Set $A_{n,m} := R[U, V]/(x^m U - y^n V - 1)$ where $m, n \in \mathbb{N}$, $m, n \geq 2$. The images of U, V in $A_{n,m}$ will be denoted by u, v.*

Proposition 1 *If $gcd(a, b) = gcd(a, c) = gcd(b, c) = 1$, then $A_{n,m}$ is a UFD.*

Proof. That R is a UFD in case $\lambda = 0$ is a well known result of Samuel. A slight modification of the argument in [18] yields the result for $\lambda \neq 0$. Define an R derivation D of $A_{n,m}$ by setting $D(v) = x^m$, $D(u) = y^n$). Clearly D is locally nilpotent and generates a locally trivial G_a action on the smooth variety $X_{n,m} \equiv$ **Spec** $A_{n,m}$. The quotient $X_{n,m}/G_a$ is isomorphic to the complement of a finite but nonempty subset of **Spec** R. The quotient map $X_{n,m} \to X_{n,m}/G_a$ is a Zariski fibration with both the base and fiber having trivial Picard group. By [9] we conclude that $Pic(X_{n,m})$ is also trivial and therefore $A_{n,m}$ is a UFD.

In case $\lambda = 0$ one can argue directly that $A_{n,m}$ is a UFD using Nagata's theorem [13, Theorem 20.2] . Note that x is a prime element in $A_{n,m}$:

$$\begin{aligned} A_{n,m}/(x) &\cong \mathbb{C}[Y, Z, U, V]/(Y^n V + 1) \\ &\cong \mathbb{C}[Y, Z, U][\frac{1}{Y}] \end{aligned}$$

a domain, and

$$A_{n,m}\,[x^{-1}] \cong \mathbb{C}[X, Y, Z]/(X^a + Y^b + Z^c)[x^{-1}][U]$$

is a UFD. ∎

The following is a consequence of Lemma 1. 2

Corollary 4 *If $D \in LND(A_{n,m})$ then $D(x) = D(y) = D(z) = 0$.*

Lemma 4 *Let $D \in LND(A_{n,m})$ and assume $D \neq 0$. Then $A^D_{n,m} = \mathbb{C}[x, y, z]$.*

Proof. $x^m D(u) - y^n D(v) = D(x^m u - y^n v) = D(1) = 0$. Since $A_{n,m}$ is a UFD we see that $D(u) = cy^n$ for some c. Thus $D(v) = x^m c$. Thus D is equivalent to the locally nilpotent derivation $D' = y^n \partial_u + x^m \partial_v$ in particular they have the same kernel. An easy application of the algorithm in [6] reveals that $\ker(D') = \mathbb{C}[x, y, z]$. ∎

Theorem 2 *$ML(A_{n,m}) = R$.*

3 The Automorphism Group

In this section we take $R := \mathbb{C}[X,Y,Z]/(X^a+Y^b+Z^c)$ with a,b,c pairwise relatively prime satisfying

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1,$$

and $A_{n,m}$ as before. The derivation

$$E := y^n\partial_u + x^m\partial_v \in Der_{\mathbb{C}}(A_{n,m}).$$

plays a special role.

Lemma 5 *Let B be a k-domain, and $\varphi \in Aut(B)$. Then $\varphi^{-1}LND(B)\varphi = LND(B)$. Also, $\varphi(ML(B)) = ML(B)$.*

Proof. If D is LND, then $\varphi^{-1}D\varphi$ is also LND. So $\varphi^{-1}LND(B)\varphi \subseteq LND(B)$ for any automorphism φ. Then

$$\varphi^{-1}(\varphi LND(B)\varphi^{-1})\varphi \subseteq \varphi^{-1}LND(B)\varphi,$$

which proves the converse inclusion.

It follows moreover that

$$\varphi(ML(B)) = \varphi\Big(\bigcap_{D\in LND(B)} \ker(D)\Big)$$

$$= \bigcap_{D\in LND(B)} \varphi(B^D)$$

$$= \bigcap_{D\in LND(B)} B^{\varphi D\varphi^{-1}}$$

which is equal to $ML(B)$ since $\varphi LND(B)\varphi^{-1} = LND(B)$. ■

Corollary 5 *Let $\varphi \in Aut_{\mathbb{C}}(A_{n,m})$. Then $\varphi^{-1}E\varphi = \lambda E$ where $\lambda \in \mathbb{C}^*$.*

Proof. $LND(A_{n,m}) = \mathbb{C}[x,y,z]E$, so by Lemma 5 $\varphi(E) = \lambda E$ for some $\lambda \in \mathbb{C}[x,y,z]^* = \mathbb{C}^*$. ■

Let $S \subset T \subset B$ be domains, T an S-algebra, and B a T-algebra. Suppose that for any $\varphi \in Aut_S B$ we have $\varphi(T) = T$. Then restriction to T defines a group homomorphism $\rho : Aut_S B \to Aut_S T$ and $Aut_S B$ is an extension of $Aut_T B$ by the image of ρ. For $S = \mathbb{C}, T = R, B = A_{n,m}$ we will show that ρ is surjective, and determine $Aut_{\mathbb{C}} R$ and $Aut_R A_{n,m}$.

The following proposition may be well known. It can be deduced from several results in [12] which are summarized in the proof.

Proposition 2 $Aut_{\mathbb{C}} R \cong \mathbb{C}^*$ *where, for* $\lambda \in \mathbb{C}^*$, $\lambda\,(x,y,z) = (\lambda^{bc}x, \lambda^{ac}y, \lambda^{ab}z)$.

Proof. Let $\widetilde{X}$ be the quasihomogeneous factorial affine surface with coordinate ring R (whose unique singular point is the origin 0) and $X \equiv \widetilde{X} - \{0\}$. Note that $Aut(X) \cong Aut(\widetilde{X}\ )$. That the mapping

$$\begin{aligned} G_m \times X &\to X \\ (\lambda\ , (x, y, z)) &\mapsto (\lambda^{bc}x, \lambda^{ac}y, \lambda^{ab}z) \end{aligned}$$

gives an action is clear. The quotient mapping $\pi : X \to B$, $(B \equiv X/G)$ is an A^1_* fibration, i.e. all π fibers are geometrically $\mathbb{C}^*$, and there are precisely three singular fibers F_a, F_b, F_c,of multiplicity a, b, c respectively. In fact $B \cong \mathbb{P}^1$, and any automorphism $\varphi : X \to X$ preserves the fibration, i.e. yields a group homomorphism

$$f : Aut(X) \to Aut(\mathbb{P}^1).$$

However, relative primeness of a, b, c forces φ to stabilize the singular fibers and moreover $F_a = \pi^{-1}(\pi(F_a)), F_b = \pi^{-1}(\pi(F_b)), F_c = \pi^{-1}(\pi(F_c))$. Thus $\pi(F_a), \pi(F_b), \pi(F_c)$ are fixed by $f(\varphi)$, and we see that f is the trivial homomorphism [12, Cor. 4.6]. Theorem 6.2 of [12] gives the exact sequence

$$0 \to G_m \to Aut(X) \to im(f)$$

as asserted. ■

Lemma 6 *The restriction homomorphism* $Aut_{\mathbb{C}}A_{n,m} \to Aut_{\mathbb{C}}R$ *is surjective.*

Proof. Let $X_{n,m}$ be the affine variety with coordinate ring $A_{n,m}$. Observe that the mapping

$$\begin{aligned} G_m \times X_{n,m} &\to X_{n,m} \\ (\mu, (x, y, z, u, v)) &\mapsto (\mu^{bc}x, \mu^{ac}y, \mu^{ab}z, \mu^{-mbc}u, \mu^{-nac}v) \end{aligned}$$

is an action inducing the G_m action on X given above. ■

Lemma 7 $\varphi \in Aut_R A_{n,m}$ *if and only if* φ *is an* R*-homomorphism satisfying* $\varphi(u, v) = (f(x, y, z)y^n + u, f(x, y, z)x^m + v)$ *for some* $f \in \mathbb{C}[x, y, z]$*. Consequently,* $Aut_R A_{n,m} \cong < R, + >$ *as groups.*

Proof. We know by corollary 5 that $\varphi^{-1}(E)\varphi = \lambda E$ for some $\lambda \in \mathbb{C}^*$. Define $(F, G) := \varphi(u, v)$. Also, $\varphi(x, y, z) = (x, y, z)$. So now

$$\begin{aligned} (\lambda y^n, \lambda x^m) &= \varphi(\lambda y^n, \lambda x^m) \\ &= \varphi\lambda E(u, v) \\ &= \varphi(\varphi^{-1}E\varphi)(u, v) \\ &= E(F, G) \\ &= (y^n F_u + x^m F_v, y^n G_u + x^m G_v) \end{aligned}$$

where the subscript denotes partial derivative.

Let us consider the first equation,

$$\lambda y^n = y^n F_u + x^m F_v.$$

Defining $H := F - \lambda u$, we see that $-y^n H_u = x^m H_v$. By the following lemma 8 we see that $H \in R$, so

$$F = p(x, y, z) + \lambda u.$$

The second equation yields $\lambda x^m = y^n G_u + x^m G_v$. Defining $H := G - \lambda v$, yields $-x^m H_v = y^n H_u$, which by the following lemma 8 yields $H = q(x, y, z)$ and thus $G = q(x, y, z) + \lambda v$. Now

$$\begin{array}{rcl} 0 & = & \varphi(x^m u - y^n v - 1) \\ & = & x^m \varphi(u) - y^n \varphi(v) - 1 \\ & = & x^m F - y^n G - 1 \\ & = & x^m (p + \lambda u) - y^n (q + \lambda v) - 1 \\ & = & x^m p - y^n q + \lambda(x^m u - y^n v) - 1 \\ & = & x^m p - y^n q + \lambda - 1. \end{array}$$

Thus $\lambda = 1$ and $p = y^n f(x, y, z)$ and $q = x^m f(x, y, z)$ for some f. It is not difficult to check that the constructed objects are well-defined homomorphisms which are isomorphisms. ■

Lemma 8 *If $H \in A_{n,m}$ such that $-y^n H_u = x^m H_v$, then $H \in R$.*

Proof. We can find polynomials $p_i(v) \in R[v] = \mathbb{C}[x, y, z][v]$ such that $H = \sum_{i=0}^{d} p_i u^i$ for some $d \in \mathbb{N}$. Requiring $deg_z(p_i) < c$ for each $i \in \mathbb{N}^*$, and $deg_x(p_i) < m$ for each $i \in \mathbb{N}^*, i \neq 1$, then the p_i are unique (because of the equality $x^m u = y^n v + 1$ and $z^c = -x^a - y^b$). The equation $-y^n H_u = x^m H_v$ yields

$$\sum_{i=0}^{d-1} -(i+1) y^n p_{i+1} u^i = \sum_{i=0}^{d} x^m p_{i,v} u^i$$

where $p_{i,v} \equiv \frac{\partial p_i}{\partial v}$. Substitute $y^n v + 1$ for $x^m u$ to obtain a unique representation:

$$\sum_{i=0}^{d-1} -(i+1) y^n p_{i+1} u^i = \quad x^m p_{0,v} + \sum_{i=0}^{d-1} (y^n v + 1) p_{i+1,v} u^i,$$

so

$$-y^n p_1 = \quad x^m p_{0,v} + (y^n v + 1) p_{1,v}$$

and

$$-(i+1) y^n p_{i+1} = \quad (y^n v + 1) p_{i+1,v}$$

for each $i \geq 1$.

Let $i \geq 1$ and assume that p_{i+1} has degree k with respect to v. Let $\alpha(x, y, z)$ be the top coefficient of p_{i+1}, seen as a polynomial in v. Then $-(i+1) y^n \alpha = y^n k \alpha$, but that gives a contradiction. So for each $i \geq 1 : p_{i+1} = 0$. This leaves the equation $0 = x^m p_{0,v}$ which means that $p_0 \in \mathbb{C}[x, y, z]$. Thus $H = p_0 u^0 \in \mathbb{C}[x, y, z]$. ■

We conclude this section with a statement of the theorem just proved:

Theorem 3 *$Aut_{\mathbb{C}} A_{n,m}$ is generated by the maps*

1. $(x, y, z, u, v) \mapsto (x, y, z, f(x, y, z)y^n + u, f(x, y, z)x^m + v)$ for $f \in R$,
2. $(x, y, z, u, v) \mapsto (\mu^{bc}x, \mu^{ac}y, \mu^{ab}z, \mu^{-mbc}u, \mu^{-nac}v)$ for $\lambda \in \mathbb{C}^*$.

Thus $Aut_{\mathbb{C}} A_{n,m} \cong \mathbb{C}^* \ltimes < R, + >$.
Note that $Aut_{\mathbb{C}} A_{n,m}$ is nonabelian.

4 Examples

Example 2 *Let $R = \mathbb{C}[X, Y, Z]/(X^a + Y^b + Z^c)$ where a, b, c are pairwise relatively prime positive integers satisfying $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$. Then $A_{n,m} \times \mathbb{C} \cong A_{n',m'} \times \mathbb{C}$ for all $(n, m), (n', m')$ but $A_{n,m} \cong A_{n',m'}$ if and only if $(n, m) = (n', m')$. Hence the $X_{n,m} \equiv \mathbf{Spec} A_{n,m}$ are the desired counterexamples to the generalized affine cancellation problem.*

Proof. Since the $\mathbf{Spec} A_{n,m}$ are all total spaces for principal G_a bundles over $\mathbf{Spec} R - \{(0, 0)\}$, the first assertion is clear. Write $A_{n,m} = R[u, v]$ where $x^m u - y^n v = 1$, and $A_{n',m'} = R[u', v']$ where $x^{m'} u' - y^{n'} v' = 1$.Since $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$, $ML(A_{n,m}) = R$ and an isomorphism $\Phi : A_{n,m} \cong A_{n',m'}$ will restrict to an automorphism of R. Thus, possibly after a composition with an automorphism of R,

$$\Phi(x) = x, \ \Phi(y) = y, \ \Phi(z) = z.$$

Let $D \in LND(A_{n,m})$ (resp. $D' \in LND(A_{n',m'})$) satisfy

$$D : v \mapsto x^m \mapsto 0, u \mapsto y^n \mapsto 0$$
$$D' : v' \mapsto x^{m'} \mapsto 0, u' \mapsto y^{n'} \mapsto 0.$$

Since $LND(A_{n,m}) = RD$ and D, D' are irreducible derivations, the locally nilpotent derivation $\Phi^{-1} D' \Phi = rD$ for some $r \in R^* = \mathbb{C}^*$.

Set $K = qf(R)$, identify $K \otimes_R A_{n,m}$ with $K[v]$, $K \otimes_R A_{n',m'} = K[v']$, and note that $K[\Phi(v)] = K[v']$. Thus

$$\Phi(v) = \alpha v' + \beta \text{ for some } \alpha, \beta \in K.$$

A calculation reveals that

$$\Phi^{-1} D' \Phi(v) = \Phi^{-1}(\alpha) x^{m'} = r x^m.$$

so that $\alpha x^{m'} = r x^m$.

We obtain

$$\Phi(v) = x^{m-m'} v' + \beta$$

from which we conclude that $D'^2(\Phi(v)) = 0$. A symmetric argument yields that $D'^2(\Phi(u)) = 0$ as well. Thus

$$\Phi(u) = r_1u' + r_2v' + r_3$$
$$\Phi(v) = s_1u' + s_2v' + s_3$$

with $r_i, s_j \in R$, and $r_1s_2 - r_2s_1 \in R^*$.

If $m > m'$, then $\beta \in K \cap A_{n',m'} = R$, so that $s_1 \in xR$, $s_2 = \mu' x^{m-m'}$, and $s_3 = \beta$. But in this case

$$r_1s_2 - r_2s_1 \in xR \subsetneq R^*.$$

Thus $m \leq m'$, but the identical argument with the roles of Φ and Φ^{-1}reversed will show $m = m'$, and the symmetric argument with the roles of u and v reversed will show $n = n'$. ■

References

[1] S. Abhyankar, P. Eakin, W. Heinzer: *On the uniqueness of the coefficient ring in a polynomial ring.* J. Algebra 23 (1972) 310-342.

[2] W. Danielewski: *On the cancellation problem and automorphism group of affine varieties.* Unpublished (1989). .

[3] J.K. Deveney, D.R. Finston: *Fields of G_a invariants are ruled.* Canad. Math. Bull. 37 (1994) 37-41.

[4] J.K. Deveney, D.R. Finston: *Local triviality of proper G_a actions.* J. Algebra 221 (1999) 692-704.

[5] T. tom Dieck: *On the topology of complex surfaces with an action of the additive group of complex numbers.* Mathematica Göttingensis 28 (1992)

[6] A. van den Essen: *An algorithm to compute the invariant ring of a G_a action on an affine variety.* J. Symbolic Computation 16 (1993) 531-555.

[7] A. Fauntleroy, A. Magid: *Quasi-affine surfaces with G_a actions.* Proc. Amer. Math. Soc. 68 (1978) 265–270.

[8] K.H. Fieseler: *On complex affine surfaces with $\mathbb{C}^+$ action.* Comment. Math. Helv. 69 (1994) 5-27.

[9] A. Magid: *The Picard sequences of a fibration.* Proc. AMS 53 (1975), no. 1, 37–40.

[10] L. Makar-Limanov: On the group of automorphisms of a surface $x^ny = p(z)$. Israel J. Math. 121 (2001) 113-123.

[11] L. Makar-Limanov: *Again $x + x^2y + z^2 + t^3$.*Contemporary Mathematics 369,177-182. American Mathematical Society, Providence (2005)

[12] K. Masuda, M. Miyanishi: *Etale endomorphisms of algebraic surfaces.* Math. Ann. 319 (2001) 493-516.

[13] H. Matsumura: Commutative Ring Theory. Cambridge Studies in Advanced Mathematics 8. Cambridge 1989.

[14] M. Miyanishi: *Algebraic characterizations of the affine 3-space.* Proceedings of the Algebraic Geometry Seminar, (Singapore) (1988) 53-67.

[15] M. Miyanishi, Y. Nakai: *Some remarks on strongly invariant rings.* Osaka J. Math 12 (1975) 1-17.

[16] G. Müller: *Symmetries of surface singularities.* J. London Math. Soc. 59 (1999) 491-506.

[17] R. Rentschler: *Opérations du groupe additif sur le plane affine.*C.R. Acad. Sci 267 (1968) 384-387.

[18] P. Samuel: *Unique factorization.* Amer. Math. Monthly 75 (1968) 945-952.

[19] N. Snyder: *An Alternate Proof of Mason's Theorem.* Elemente der Mathematik 55 (2000) 93-94.

[20] N. Steenrod: Fibre Bundles. Princeton University Press, Princeton, 1951.