Title: Locally finite derivations.

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Locally finite derivations.*

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Abstract: Let K be an algebraically closed field of characteristic zero. Let **A** be a finitely generated algebra over the field K, then any locally finite derivation D of **A** admits a decomposition $D_s + D_\eta$, where D_s is semisimple, D_η is locally nilpotent, and D_s commutes with D_η . The ring of generalized constants of a locally finite derivation of **A** is finitely generated. The ring of constants of a locally nilpotent homogeneous (non trivial) derivation of K[x,y,z] is a ring of polynomials in two variables. If D has at least two fixed points, then D anihilates a non-constant polynomial. In case $\mathbf{A} = K[x_1,...,x_n]$, the rank ℓ of the spectrum of a locally finite derivation is less than the height of the ideal $\boldsymbol{\nu}$ generated by $\{Dx_1,.., Dx_n\}$.

Introduction.

In 1958, M. Nagata [N] settled in the negative the problem of finite generacy of the ring of invariants of a group action on polynomial rings. Nagata exhibited an example of a subgroup G of the group of automorphisms of $k[x_1,...,x_{32}]$ with k a sufficiently big field, of any characteristic, such that ring R of invariants under the action of G is not finitely generated.

A closely related problem is the problem of deciding whether the ring of constants R(D) of a derivation D of the ring $k[x_1,...,x_n]$ is finitely generated. Recently H. Derksen showed that Nagata's example can be cast in the form R(D) for some derivation D, thus for a general derivation D the ring of constants R(D) need not be finitely generated.

There are some positive results in low dimension:
if n=1 and $D \neq 0$, then R(D)=k,

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if n=2 and $D \neq 0$, then R(D) is finitely generated [Zak], [E],
if n=3, and charact(k)$\neq$0, then R(D) is finitely generated [Zar].

We are interested in a distinguished class of derivations, namely the class of locally finite derivations. In the case of K=$\mathbb{C}$ –the field of complex numbers– locally finite derivations of $\mathbb{C}[x_1,..,x_n]$ are in one to one correspondence with smooth morphisms from $\mathbb{C}$ to Aut($\mathbb{C}[x_1,..,x_n]$) [CZ], [vdE]. It is of interest to consider the ring of constants for this type of derivations. The goal of this work is to present some result about the ring of constants of a locally nilpotent homogeneous (non trivial) derivation of $\mathbb{C}[x,y,z]$. We will show that any such ring of constants is a ring of polynomials in two variables. This is a step towards extending results of [R] to three dimensional space. We will also present other useful results about locally finite derivations of finitely generated rings.

Definitions, notation:

Let **A** be an algebra over a field K (we will consider only commutative algebras with unit). Let D:**A** $\to$**A** be a K-derivation (K-linear, satisfying Leibnitz's rule). The linearity of D allows us to endow **A** with the structure of a K[D]-module.

We say that D is *locally finite* iff **A** is a torsion K[D]-module, i.e., for any P $\in$ K[$x_1,..,x_n$], there exist a "differential operator" $0 \neq f(D) = D^n + k_1 D^{n-1} + ... + k_n \in K[D]$ such that $f(D)P=0$. We say D is *locally nilpotent* iff every element of **A** is annihilated by a power of D.

vectors

The set Ker(D) consisting of – 0 and of – eigen~~values~~ of D corresponding to the eigenvalue 0 is a subring of **A**. It is called the ring of constants of D. It will be denoted by R(D). By extension, the set $\cup_{n\geq 1}$ Ker(D^n) will be called the set of *generalized constants* of D.

The spectrum $\sigma(D)$ of D as a K-linear map is a groupoid of the additive group (K,+) [CZ]. Let ℓ be the rank of the subgroup generated by $\sigma(D)$ in (K,+). We will call ℓ the *rank* of $\sigma(D)$.

Let $\mathbf{m}$ be a maximal ideal of A. We will say that $\mathbf{m}$ is a *fixed point* of the derivation D if and only if $Df \in \mathbf{m}$ for all f $\in$ **A**. Thus, in case **A** = K[$x_1,...,x_n$], $\mathbf{m}$ = { f : $f(a_1,..,a_n) = 0$ } with $(a_1,..,a_n) \in K^n$, and $Dx_i = V_i(x_1,..,x_n)$ for i=1,..,n we have: $\mathbf{m}$ is a fixed point of D iff $V_i(a_1,..,a_n) = 0$ for i=1,..,n.

Results.

Let K be an algebraically closed field of characteristic 0. We will prove the following facts.

Theorem 1: (Jordan decomposition) Let **A** be an algebra over K, let D:**A** $\to$**A** be a locally finite K-derivation, then there exists a unique decomposition $D = D_s + D_\eta$, where D_s is semisimple, D_η is locally nilpotent, and $[D_s , D_\eta] = 0$. Furthermore, for all $\lambda \in$ K,

$$\mathrm{Ker}(D_s - \lambda) = \cup_{n\geq 1} \mathrm{Ker}((D - \lambda)^n).$$

Theorem 2: Let D:**A** $\to$**A** be a locally finite derivation, A finitely generated as K-algebra, then there exists n, and a derivation $\tilde{D}$: $K[x_1,..,x_n] \to K[x_1,..,x_n]$ such that

i) $\tilde{D}$ is locally finite,

ii) **A** is a homomorphic image of $K[x_1,..,x_n]$,

iii) the following diagram commutes:

$$\begin{tikzcd} K[x_1,..,x_n] \arrow[r, "\tilde{D}"'] \arrow[d] & K[x_1,..,x_n] \arrow[d] \\ \mathbf{A} \arrow[r, "D"'] & \mathbf{A} \end{tikzcd}$$

Theorem 3: Let D:**A** $\to$**A** be a locally finite derivation, with **A** a finitely generated K-algebra, then the ring of generalized constants of D is finitely generated.

Theorem 4: If D, D_0 : $K[x_1,..,x_n] \to K[x_1,..,x_n]$ are non-trivial derivations, D is locally finite, and $D = F(x_1,..,x_n)\, D_0$ for some non-constant polynomial F, then D_0 is locally nilpotent, and $D_0F \in$ K.

Theorem 5: If D:$K[x,y,z] \to K[x,y,z]$ is a locally nilpotent derivation, $D \neq 0$, D homogeneous with respect to a weight w (with w(x), w(y), w(z) > 0), then the ring of constants of D is of the form K[A,B] with A, B algebraically independent.

Theorem 6: Let D:$K[x_1,..,x_n] \to K[x_1,..,x_n]$ be a locally finite derivation. Then the rank ℓ of $\sigma(D)$ is not grater than the height of the ideal $\boldsymbol{\mathcal{V}}$ generated by $\{Dx_1,..., Dx_n\}$, i.e.: $\ell \leq \mathrm{ht}(\boldsymbol{\mathcal{V}})$.

greater

Theorem 7: Let D:A $\to$ A be a locally finite derivation, with A a finitely generated K-algebra. If D has at least two fixed points, then D annihilates an element of A \ K.

Proof of theorem 1. Since D is locally finite ~~then~~ A is a K[D]-torsion module, this implies the existence of a decomposition $A = \bigoplus_{\lambda \in \sigma} M_\lambda$, where $\sigma \subset K$ is the set of eigenvalues of D (as linear map), M_λ is the set of elements of A that are annihilated by a power of $(D-\lambda)$, i.e. $M_\lambda = \cup_{n\geq 1} \mathrm{Ker}((D-\lambda)^n)$. Furthermore σ, which is the spectrum of D, is closed under addition. Moreover: $M_\lambda M_\mu \subset M_{\lambda+\mu}$ for any $\lambda, \mu \in \sigma$.See [CZ].

Define D_s as $D_s(P) = \sum_\lambda \lambda P_\lambda$ if $P = \sum_\lambda P_\lambda$. Clearly D_s is a derivation of A.

Since $D(M_\lambda) \subseteq M_\lambda$ for all λ, it follows that $DD_s = D_sD = \lambda D$ on M_λ, and so D_s commutes with D. The derivation D_s is locally finite, because if an element P is writen as $P = \sum_{\lambda \in F} P_\lambda$, with F a finite set, then $\prod_{\lambda \in F} (D_s - \lambda)\, P = 0$. The derivation D_s is semisimple in the sense that the ring A admits a direct sum decomposition $A = \bigoplus_{\lambda \in \sigma} M_\lambda$, and for each λ, $M_\lambda = \mathrm{Ker}(D_s - \lambda)$.

Let $D_\eta = D - D_s$. Clearly D_η is a derivation of A, and D_η commutes with D. Moreover, if $P \in M_\lambda$ then $D_\eta P = DP - D_sP = (D-\lambda)P$, and so $D_\eta^n\, P = (D-\lambda)^n\, P = 0$ if n is sufficiently big.

Since every element of A is a finite sum of elements in some of the M_λ's, it follows that every element of A is annihilated by a power of D_η. Thus D_η is locally nilpotent.

Note that $\mathrm{Ker}(D_s - \lambda) = M_\lambda = \cup_{n\geq 1} \mathrm{Ker}((D-\lambda)^n)$.

$\tilde{D}_s$

Finally, we show uniqueness. Assume that D = \o(D,\s\up6(~))$_s$ + $\tilde{D}_\eta$, with $[\tilde{D}_s, \tilde{D}_\eta] = 0$, $\tilde{D}_s$ semisimple and $\tilde{D}_\eta$ locally nilpotent. Let $P \in M_\lambda$, and suppose $(D-\lambda)^n P = 0$, $\tilde{D}_\eta^k\, P = 0$.

Since D and $\tilde{D}_\eta$ commute, then

$$(\tilde{D}_s - \lambda)^{n+k}\, P = (D - \lambda - \tilde{D}_\eta)^{n+k}\, P = \sum_{0\leq h\leq n+k} \binom{n+k}{h} (D-\lambda)^h (-\tilde{D}_\eta)^{n+k-h}\, P$$

$= 0$

because in each term of the last sum either $h \geq n$ or $n+k-h \geq k$. But $\tilde{D}_s$ has simple eigenvalues, so $(\tilde{D}_s - \lambda)P = 0$. Thus, $\tilde{D}_s = D_s$ on M_λ for all λ, and so $\tilde{D}_s = D_s$, and $\tilde{D}_\eta =$

$\mathcal{D}_\eta$. This shows the uniqueness of the decomposition of $\mathcal{D}$.

Proof of theorem 2. Let $a_1,\ldots, a_k$ be generators of **A** as K algebra. Since $\mathcal{D}$ is locally finite, there exists m such that $\mathcal{D}^m a_i = \sum_{0\leq j<m} \alpha_{ij}\, \mathcal{D}^j a_i$ for i = 1,..., k.

In the ring $K[\{x_{ij} : 1 \leq i \leq k, 0 \leq j < m\}]$ of polynomials in kxm variables consider the derivation $\bar{\mathcal{D}}$ whose action on the generators is given by

$$\bar{\mathcal{D}} x_{iv} = \begin{cases} x_{i,v+1} & \text{if } v+1<m\ , \\ \sum_{0\leq j<m} \alpha_{ij}\, x_{ij} & \text{if } v+1=m\ . \end{cases}$$

Let π: $K[(x_{ij})] \to$ **A** be the K-algebra homomorphism such that $\pi(x_{ij}) = \mathcal{D}^j a_i$. Clearly $\pi \bar{\mathcal{D}} = \mathcal{D}\, \pi$. $\bar{\mathcal{D}}$ is locally finite since it is linear in $(x_{ij})_{ij}$.

Proof of theorem 3. If $\mathbf{A} = \bigoplus_{\lambda\in\sigma} M_\lambda$ is the decomposition induced by $\mathcal{D}$, then from theorem 1, the ring of generalized constants is $M_0 = \mathrm{Ker}(\mathcal{D}_s)$. Thus without loss of generality we may assume $\mathcal{D} = \mathcal{D}_s$. Let π_λ be the projection of **A** onto M_λ. Let $x_1,\ldots, x_n$ be generators of **A** as algebra over K, and let $x_i = \sum_\lambda x_{i\lambda}$ be the decomposition of x_i as sum of eigenvectors of $\mathcal{D}$. Note that in these sums the index λ can be restricted to be in a finite set F. Each monomial in $x_1,\ldots,x_n$ can be written as

$$x^\nu = \prod_i x_i^{\nu_i} = \sum_k c_{k\nu} \prod_{i,\lambda} x_{i\lambda}^{k_{i\lambda}}$$

where ν denotes a multi-index of non-negative integers: $\nu = (\nu_1,\ldots,\nu_n)$, k also denotes a multi-index of non-negative integers $k = (k_{i\lambda})_{1\leq i\leq n,\, \lambda\in F}$, and $c_{k\nu}$ is a natural number depending upon k and ν. But $\prod_{i,\lambda} x_{i\lambda}^{k_{i\lambda}}$ is an eigenvector of $\mathcal{D}$ corresponding to the eigenvalue $\sum_{i,\lambda} k_{i\lambda}\lambda$, therefore

$$\pi_0(x^\nu) = \sum_{k\in S} c_{k\nu} \prod_{i,\lambda} x_{i\lambda}^{k_{i\lambda}}\ ,$$

where $S = \{ (k_{i\lambda}) : \sum_{i,\lambda} k_{i\lambda}\lambda = 0\}$.

For each multi-index $k \in S$, let $P_k = \prod_{i,\lambda} x_{i\lambda}^{k_{i\lambda}}$. Since for each monomial x^ν, $\pi_0(x^\nu)$ is a linear combination of the P_k's, then the ring of generalized constants M_0 is spanned, as vector space over K, by the set $\{P_k : k \in S\}$. Theorem 3 will be proved if we can show that there exists a finite subset G of S with the property that for any $k \in S$, k is a linear combination of the elements of G with non-negative integer coefficients, i.e.: $k = \sum_{g\in G} n_g g$, with $n_g \geq 0$, integer. Indeed, if that is the case, then $P_k = \prod_{g\in G} P_g^{n_g}$, and so $M_0 = C[\{P_g : g \in G\}]$. Thus, we look at the set

$$S = \{ (k_{i\lambda})_{1\leq i\leq n,\ \lambda\in F} : \sum_{i,\lambda} k_{i\lambda} \lambda = 0\}.$$

Let $\lambda_1,..\lambda_h$ be elements of F, linearly independent over the rationals, with h maximal, so that any other element of F can be written as a linear combination of $\lambda_1,..\lambda_h$ with rational coefficients. Then the equation $\sum_{i,\lambda} k_{i\lambda} \lambda = 0$ that defines S is equivalent to h linear homogeneous equations in $(k_{i\lambda})$, with rational coefficients. Upon multiplying these equations by a common multiple of the denominators of the coefficients, we may assume the coefficients are integer. Hence we are in position to apply the following lemma, and that concludes the proof of theorem 3.

Lemma 1(existence of generators for integer linear programming homogeneous problems, or Gordan's lemma) Let $L: Z^n \rightarrow Z^m$ be a linear map. Let $S = \{ x=(x_1,...,x_n)^t \in Z^n : Lx = 0,\ x_i \geq 0 \text{ for all } i \}$. Then there exist $s_1,.., s_f \in S$ such that $S = \{ \sum_{j=1}^{f} n_j s_j : n_j \in Z, n_j \geq 0 \}$.

Proof lemma 1. Let $W = \{ x \in R^n : Lx = 0 , x_i \geq 0 \text{ for all } i, \Sigma x_i = 1\}$. W is a compact convex polyhedral set . If $W = \varnothing$, then $S = \{ 0 \}$, and there is nothing else to prove. Assume $W \neq \varnothing$. Then W is a convex polygon, and as such, W is the convex hull of a finite set : W = convex hull of $\{ \check{w}_1,\ldots, \check{w}_k \}$. We claim that the vertices of W have rational coordinates. Indeed, let $\check{v} = (v_1,..,v_n)^t \in W$ be a vertex. We proceed by induction on n. If $n = 1$, then $W = \{1\}$, and the claim follows immediately. Let then $n > 1$. Assume first that $v_i > 0$ for all i, then we claim that $\check{v}$ is the unique solution of the equations $L\check{x} = 0$, $\sum_{1\leq i\leq n} x_i = 1$, which have rational coefficients, hence $\check{v}$ have rational entries. To justify our last claim, assume $L\check{x} = 0$, $\Sigma x_i = 1$ and consider $\check{v} + \varepsilon(\check{x} - \check{v})$, and $\check{v} - \varepsilon(\check{x} - \check{v})$. If $\varepsilon > 0$ is sufficiently small, then all entries of $\check{v} \pm \varepsilon(\check{x} - \check{v})$ are positive, and so these two vectors are in W, but $\check{v}$ lies on the segment with end-points $\check{v} \pm \varepsilon(\check{x} - \check{v})$. Since $\check{v}$ is a vertex of W it follows that $\check{x} - \check{v} = 0$.

Assume next that $v_i = 0$ for some i, then applying the inductive assumption to the intersection of W with the hyperplane $x_i = 0$, we conclude that the remaining coordinates of $\check{v}$ are rational.

Let $s_i = m\, w_i$ for $i = 1,..,k$, where m is a positive integer, big enough so that $s_i \in Z^n$. Let $x \in S$, $x \neq 0$. Then $\dfrac{x}{\sum_{1\leq i\leq n} x_i} \in W$, hence it is a convex combination of $w_1,..,w_k$. Therefore, x can be written as $x = \sum_{i=1}^{k} \mu_i s_i$ with $\mu_i \geq 0$. We split μ_i as the sum of its integer part and a real number in the interval $[0,1)$: $\mu_i = [\mu_i] + \theta_i$, $0 \leq \theta_i < 1$. Then $x - \sum_{i=1}^{k} [\mu_i] s_i = \sum_{i=1}^{k} \theta_i s_i$

is in S, and has norm bounded above by $\sum_{i=1}^{k} \|s_i\|$. But the elements of S have non-negative integer coefficients, thus there is only a finite number of points in S with norm less than $\Sigma \|s_i\|$, say $s_{k+1},...s_f$. It follows that $x = \sum_{i=1}^{f} n_i s_i$ for suitable non-negative integers $n_1,..., n_f$.

Proof of theorem 4. We will show a more general version of theorem 4. We will show:

Theorem 4': Let K be an algebraically closed field of characteristic zero, **A** an algebra over K without zero divisors, D and D_0 K-derivations of **A**, $D \neq 0$, $D = F D_0$ for some $F \in$ **A**, $\bigcap_{j\geq 0} F^j$ **A** $= 0$. If D is locally finite then:

$FD_0(F)=\alpha F \Leftrightarrow D_0(F)=\alpha$ $\Leftarrow$ i) $DF = \alpha F$ for some $\alpha \in K$,

ii) the spectrum $\sigma(D)$ of D is $N_0\alpha = \{ 0, \alpha , 2\alpha , 3\alpha ,...\}$,

iii) if $\alpha = 0$, then D and D_0 are locally nilpotent,

iv) if $\alpha \neq 0$, then

a) all eigenvalues of D are simple, i.e. $\mathrm{Ker}(D - \lambda)^2 = \mathrm{Ker}(D - \lambda)$ for all $\lambda \in K$,

b) **A** $= R[F]$, where $R = \mathrm{Ker}(D)$ is the ring of constants of D,

c) D_0 is locally nilpotent.

Proof: Assume first that $D_0F = 0$, then $DF = 0F$, so i) holds with $\alpha = 0$. Assume next that $D_0F \neq 0$. Let **A** $= \bigoplus_{\lambda \in \sigma} M_\lambda$ be the decomposition of **A** induced by D. Let $P \in$ **A**, $P = \Sigma P_\lambda$. The support of P is the set $\mathrm{supp}(P) = \{ \lambda \in K : P_\lambda \neq 0 \}$.

We claim that $\mathrm{supp}(D_0F) = \{0\}$. This claim follows immediately from the next lemma, which is a mere reformulation of a result due to J.F. Ritt. [Zur, section 7.3].

Lemma 2: Let K be an algebraically closed field of characteristic zero, **A** an algebra over K without zero divisors, F and B in the torsion part of **A** as K[D]-module, $DF = F\,B \neq 0$. Then $\operatorname{supp}(B) = \{0\}$.

Proof of lemma 2. Suppose $\operatorname{supp}(B) \neq 0$. Since $B \neq 0$, then its support is not empty, therefore there exists a Q-linear map $f: K \to Q$, such that f assumes a positive value on supp(B). Let $\lambda \in \operatorname{supp}(B)$ and $\mu \in \operatorname{supp}(F)$ be such that $f(\lambda) = \text{maximum } f(\operatorname{supp}(B)) > 0$, and $f(\mu) = \text{maximum } f(\operatorname{supp}(F))$.

Note that $F = F_\mu$ + (terms supported where $f < \mu$), $B = B_\lambda$ + (terms supported where $f < \lambda$), hence $FB = F_\mu B_\lambda$ + (terms supported where $f < \mu+\lambda$), and since **A** has no divisors of zero then $F_\mu B_\lambda \neq 0$, so $\mu+\lambda = \max f(\operatorname{supp}(FB))$. But

$F_\mu B_\lambda$ = part of FB supported at $\mu + \lambda$ = part of DF supported at $\mu + \lambda = DF_{\mu+\lambda} = 0$, since $f(\mu+\lambda) > f(\mu)$, so $\mu+\lambda \notin \operatorname{supp}(F)$.

Proof of theorem 4'(continuation). If $F = \sum_{\lambda \in \operatorname{supp}(F)} F_\lambda$, then equating terms with like support on both members of $DF = FD_0F$, we get $DF_\lambda = F_\lambda\, D_0F$ for all $\lambda \in \operatorname{supp}(F)$, and so $(D - \lambda)F_\lambda = (D_0F - \lambda)\, F_\lambda$. A straightforward induction shows that for all $k \geq 0$

$$(*) \qquad (D - \lambda)^{k+1} F_\lambda = \sum_{0 \leq j \leq k} \binom{k}{j} D^j (D_0F - \lambda)\, (D - \lambda)^{k-j} F_\lambda .$$

Take $\alpha \in \operatorname{supp}(F)$, so that $F_\alpha \neq 0$. F_α is annihilated by a power of $(D - \alpha)$. Let a be such that $(D - \alpha)^a F_\alpha \neq 0$, but $(D - \alpha)^{a+1} F_\alpha = 0$. If $D_0F - \alpha \neq 0$, from lemma 2 we deduce that there exists an integer $b \geq 0$ such that $D^b(D_0F - \alpha) \neq 0$, but $D^{b+1}(D_0F - \alpha) = 0$. Taking $k = a+b$ and $\lambda = \alpha$ in (*), we get $0 = \binom{a+b}{b} D^b(D_0F - \alpha)\, D^a F_\alpha$ which is absurd, since neither $D^b(D_0F - \alpha)$ nor $D^a F_\alpha$ is zero, and **A** has no zero divisors. Therefore $D_0F - \alpha = 0$. This shows i).

We show next that $\sigma(D) = N_0\,\alpha$. Let $\lambda \in \sigma(D)$. Then, there exists $P \in \mathbf{A}$, $P \neq 0$, such that $DP = \lambda P$. Since $\bigcap_{j \geq 0} F^j\,\mathbf{A} = 0$, then we can find j maximal such that F^j divides P in **A**, i.e.: $P = F^j B$ for some $B \in \mathbf{A}$, j maximal. From $DP = \lambda P$, and $D_0F = \alpha$ we get $F\, D_0(F^jB) = \lambda\, F^j B$, hence $F^{j+1} D_0B = (\lambda - j\alpha)F^j B = (\lambda - j\alpha)\,P$. Thus, if $\lambda \neq j\alpha$ then we deduce that F^{j+1} divides P in **A**, contradicting the choice of j. Therefore $\lambda = j\alpha$, and so $\sigma(D) \subseteq N_0\,\alpha$. The other inclusion is immediate, since $D\,F^j = j\alpha\,F^j$, so $N_0\,\alpha \subseteq \sigma(D)$.

From $\sigma(D) = N_0\,\alpha$ it follows that if $\alpha = 0$, then the only eigenvalue of D is 0, so D is locally nilpotent. Moreover if $\alpha = 0$ then $D^k = F^k D_0^k$ for all k , thus if an element of **A** is annihilated by D^k, then it is annihilated by D_0^k as well. Since in case $\alpha = 0$ each element of **A** is annihilated by some power of D, it follows that D_0 is locally nilpotent.

Finally, we show iv). Assume $\alpha \neq 0$. We claim that $\mathrm{Ker}(D - j\alpha) = F^j R$. Clearly any element of A of the form $F^j B$, with $B \in R = \mathrm{Ker}(D)$ satisfies $(D - j\alpha)F^j B = 0$. Conversely, if $(D - j\alpha)P = 0$, write P as $P = F^k B$, with k maximal. Then $0 = (D - j\alpha)P = (k - j)\alpha F^k B + F^{k+1} D_0 B$, thus if $k \neq j$ then F^{k+1} divides $F^k B$, contradicting the maximality of k. Hence $k = j$, $DB=0$, and so $P \in F^j R$.

Now we can show that all eigenvalues of D are simple. If $(D - j\alpha)^2 P = 0$ and $(D - j\alpha)P \neq 0$, then $(D - j\alpha)P$ is in $\mathrm{Ker}(D - j\alpha)$, and so $(D - j\alpha)P = F^j B$ for some $B \neq 0$, $B \in R$. Write P as $P = F^m C$, with m maximal. Then $F^j B = (D - j\alpha)P = (D - j\alpha) F^m C$, so $F^j B = (m - j)\alpha F^m C + F^{m+1} D_0 C$. F does not divide B, indeed $B \in R$, so $\mathrm{supp}(B) = \{0\}$, whereas for any $H \in A$ the element FH is supported at $\alpha + h\alpha$, for some integer $h \geq 0$. It follows that $j \geq m$. But if $j > m$, then F^{m+1} divides $P = F^m C$, which is absurd. If $j = m$, then $B = F D_0 C$, hence F divides B, which is absurd as well. Therefore $(D - j\alpha)P = 0$.

The last two paragraphs imply that any element of **A** whose support is $\{j\alpha\}$ is in $F^j R$. But every element of **A** is a finite sum of elements with support of the form $\{j\alpha\}$ (ii)
for some j, hence $\mathbf{A} = R[F]$.
← want $D_0 F = \alpha F$

D_0 annihilates all elements in R, and $D_0^k F^j = 0$ for all $k > j$. Therefore any element of **A** is annihilated by some power of D_0, or in other words: D_0 is locally nilpotent.

interesting glitch that $K = \overline{K}$!

Proof of theorem 4 (conclusion). Let $\overline{K}$ be an algebraic closure of K. We extend D and D_0 to derivations of $\overline{K}[x_1,..,x_n]$ as $\overline{K}$-linear derivations. Since there exists $f \in K[D] \subset \overline{K}[D]$ such that $f(D)x_i = 0$ for $i=1,..,n$, and $f \neq 0$, then D is a locally finite derivation of A $= \overline{K}[x_1,..,x_n]$. We may apply theorem 4' and conclude that $DF = \alpha F$ for some $\alpha \in \overline{K}$, but F and D have coefficients in K, so $\alpha = \frac{DF}{F} \in K(x_1,..,x_n) \cap \overline{K} = K$. Furthermore, D_0 is a locally nilpotent derivation of $\overline{K}[x_1,..,x_n]$. In particular, any $P \in K[x_1,..,x_n]$ is anilated by a power of D_0. Thus, D_0 is a locally nilpotent derivation of $K[x_1,..,x_n]$.

Proof of theorem 5. First we explain the notation and statement of the theorem. By a weight in $K[x,y,z]$ we mean a function $w: K[x,y,z] \to Z \cup \{-\infty\}$ defined as follows:

$$w(c\, x^i y^j z^k) = \alpha i + \beta j + \gamma k$$

for all non-zero monomials $c\, x^i y^j z^k$ (α, β, γ are given integers, $c \in K\backslash\{0\}$),

$$w(0) = -\infty,$$

and if $P = \Sigma\, a_{ijk}\, x^i y^j z^k$, then

$$w(P) = \max_{ijk} w(a_{ijk}\, x^i y^j z^k)\ .$$

A polynomial P is said to be w-homogeneous iff there exists ω such that $P = \sum_{\alpha i+\beta j+\gamma k=\omega} a_{ijk}\, x^i y^j z^k$, and in that case we will say that P is w-homogeneous of weight ω.

A map $L: K[x,y,z] \to K[x,y,z]$ is said to be w-homogeneous of weight ω iff for every n, L maps w-homogeneous elements of weight n into w-homogeneous elements of weight $\omega + n$. We will write in this case $w(L) = \omega$.

Dan $w(D)$ goedgedefinieerd!

Occasionally we will consider more general maps: the weight w induces a grading on the ring $K[x,y,z]$, and so it makes sense to talk about w-graded $K[x,y,z]$-modules, and w-homogeneous maps between such modules. For example we can consider the exterior algebra $\Lambda^* K[x,y,z]$. In this case, since x,y,z are w-homogeneous, we may extend the weight w to $\Lambda^* K[x,y,z]$) in such a way that the exterior derivative d is w-homogeneous.

Now we present the proof of theorem 5. Without loss of generality we may assume that $w(x)$, $w(y)$, and $w(z)$ have no common factors. Note that D and w extend to the field $K(x,y,z)$ (w is to be extended according to $w(N/G) = w(N) - w(G)$). We will say that a fraction $r \in K(x,y,z)$ is w-homogeneous iff r can be written as $r = N/G$, with N and G w-homogeneous.

We consider separately the cases $w(D) < 0$, and $w(D) \geq 0$.

da $w(x^i y^j z^k) > w(x)$ als $i+j+k > 0$

Case $w(D) < 0$. Without loss of generality, we may assume that $0 < w(x) \leq w(y) \leq w(z)$. If we let $V_1 = Dx$, $V_2 = Dy$, and $V_3 = Dz$, then V_1 is homogeneous of weight $w(D) + w(x) < w(x)$, hence V_1 must be a constant, i.e., V_1 must be of the form $V_1 = c$. V_2 is homogeneous of weight $w(D) + w(y) < w(y)$, hence V_2 must be of the form $V_2 = f(x)$. V_3 is homogeneous of weight $w(D) + w(z) < w(z)$, hence V_3 must be of the form $V_3 = g(x,y)$.

Sub-case $V_1 \neq 0$. If $V_1 \neq 0$ we may rescale x and assume $V_1 = 1$, then $0 = w(Dx) = w(D) + w(x)$, so $w(x) = -w(D) > 0$. $f(x)$ is either 0 or a w-homogeneous polynomial of weight $w(D) + w(y) = w(y) - w(x)$, hence f is of the form $f(x) = c_2\, x^n$, with

$D = \partial_x + c_2 x^n \partial_y + g(x,y)\partial_z$

$n = \dfrac{w(y) - w(x)}{w(x)}$ and c_2 a constant (possibly equal to 0).

We look at the flow corresponding to D:

$$x' = 1, \qquad y' = c_2\, x^n, \qquad z' = g(x,y).$$

This system can be integrated. If $\Phi(x,y,z,t)$ denotes the solution at time t that is at (x,y,z) when t = 0, then

$$\Phi(x,y,z,t) = \Big(x + t,\ y + c_2\frac{(x+t)^{n+1} - x^{n+1}}{n+1},\ z + \int_0^t g\Big(x+s,\ y + c_2\frac{(x+s)^{n+1} - x^{n+1}}{n+1}\Big)\, ds$$

$$\Big).$$

A polynomial P is constant with respect to D iff for all t $P(x,y,z) = P(\Phi(x,y,z,t))$. In particular, taking $t = -x$ we get that for any P constant with respect to D,

$$P(x,y,z) = P\Big(0,\ y - c_2\frac{x^{n+1}}{n+1},\ z + \int_0^{-x} g\Big(x+s,\ y + c_2\frac{(x+s)^{n+1} - x^{n+1}}{n+1}\Big)\, ds\ \Big) \in K[A,B],$$

with $A = y - c_2\dfrac{x^{n+1}}{n+1}$ and $B = z + \displaystyle\int_0^{-x} g\Big(x+s,\ y + c_2\frac{(x+s)^{n+1} - x^{n+1}}{n+1}\Big)\, ds$.

$-D = g\partial_z$, Du $\ker D = \ker \partial_z = k(x,y)$

Sub-case $V_1 = 0$, $Dy = 0$. If $V_1 = 0$, and $f(x) = 0$, from $D \neq 0$ we deduce $g(x,y) \neq 0$. We find that the flow corresponding to D is $\Phi(x,y,z,t) = (x, y, z + g(x,y)t)$. Therefore, if P is constant with respect to D and we set $t = -\dfrac{z}{g(x,y)}$ in $P(x,y,z) = P(\Phi(x,y,z,t))$ we deduce that $P(x,y,z) = P(x,y,0)$. Hence the ring of constants of D is $K[x,y]$.

$D = cx^n\partial_y + g(x,y)\partial_z$

Sub-case $V_1 = 0$, $Dy \neq 0$. If $V_1 = 0$, and $f(x) \neq 0$, we can also find the flow corresponding to D. If $G(x,y)$ is defined by the conditions $G(x,0) = 0$, $\dfrac{\partial G}{\partial y}(x,y) = g(x,y)$, then the flow is

$$\Phi(x,y,z,t) = \Big(x,\ y + t\, f(x),\ z + \frac{G(x,y+tf(x)) - G(x,y)}{f(x)}\Big).$$

Therefore, if P is constant with respect to D and we set $t = -\dfrac{y}{f(x)}$ in $P(x,y,z) = P(\Phi(x,y,z,t))$ we deduce that $P(x,y,z) = P\Big(x, 0, z - \dfrac{G(x,y)}{f(x)}\Big)$.

Let $G(x,y) = \displaystyle\sum_{s=1}^{N} g_s(x)\, y^s$. Cancelling common factors in x, we write $\dfrac{G(x,y)}{f(x)}$

$$= \frac{\sum_{s=1}^{N} \tilde{g}_s(x)\, y^s}{\tilde{f}(x)}$$. Then, the polynomial $B = z\tilde{f}(x) + \sum_{s=1}^{N} \tilde{g}_s(x)\, y^s$ is irreducible in $K[x,y,z]$. Note that $\mathcal{D}B = 0$. We claim that in this sub-case, the ring of constants of $\mathcal{D}$ is $K[x,B]$. We justify our claim by induction on degree of P with respect to z:

Let $P \in \operatorname{Ker}(\mathcal{D})$, $P \notin K$. If $\deg_z P = 0$, then $P(x,y,z) = P(x, 0, z - \frac{G(x,y)}{f(x)}) = P(x,0,0)$ since P does not depend on z, thus $P \in C[x,B]$.

If $\deg_z P > 0$, then we write $P(x,y,z)$ as $P = P(x,0,0) + U(x,y,z)$, with $U(x,0,0) = 0$. Since $\mathcal{D}x = 0$, then $P(x,0,0)$, and consequently U, is in $\operatorname{Ker}(\mathcal{D})$. Moreover $\deg_z U = \deg_z P$. Thus, to show that P is in $K[x,B]$ it is enough to consider the case $P(x,0,0) = 0$. Assume then that $P(x,0,z) = z\,H(x, y, z)$ for some polynomial H. It follows that

$P(x,y,z) = P(x,0, z - \frac{G(x,y)}{f(x)}) = P(x,0, \frac{B}{\tilde{f}(x)}) = \frac{B}{\tilde{f}(x)} H(x, 0, \frac{B}{\tilde{f}(x)})$, hence B divides P in $K(x)[y,z]$, but B is irreducible in $K[x,y,z]$, so B divides P in $K[x,y,z]$, i.e: $P(x,y,z) = B(x,y,z)\,L(x,y,z)$. Since P and B are in $\operatorname{Ker}(\mathcal{D})$, then so is L. Furthermore, $\deg_z L = \deg_z P - 1$, hence by induction we conclude that $L \in K[x,B]$, and therefore $P \in K[x,B]$.

This takes care of theorem 5 in case $w(\mathcal{D}) < 0$.

gewicht nul?

Case $w(\mathcal{D}) \geq 0$. Let $\mathbb{V} = \{f \in K(x,y,z) : \mathcal{D}f = 0\}$, $\mathbb{H} = \{ f \in K(x,y,z) : \mathcal{D}f = 0$ and f is w-homogeneous$\}$.
$= \{f \in \mathbb{V} \mid f$ w-homogeneous$\}$

We can apply the "t+ε+ℓ" theorem [Zur, section 8]. Since $\mathcal{D}$ is locally nilpotent, then $\ell = 0$. Since $\mathcal{D} \neq 0$, then there exists $g \in K(x,y,z)$ such that $\mathcal{D}g \neq 0$. If $k \geq 1$ is such that $\mathcal{D}^{k+1}g = 0 \neq \mathcal{D}^k g$, then $\mathcal{D}(\frac{\mathcal{D}^{k-1}g}{\mathcal{D}^k g}) = 1$, so $\varepsilon = 1$. Therefore for t = (transcendence degree of $\mathbb{V}$ over K) we have $t = n - (\varepsilon+\ell) = 3 - (1) = 2$.

$\mathbb{V}$ is the quotient field of the ring of constants of $\mathcal{D}$ in $K[x,y,z]$, because if $\xi \in \mathbb{V}$, $\xi = A/B$, then $A = \xi B$, and since ξ is constant with respect to $\mathcal{D}$, then $\mathcal{D}^k A = \xi\, \mathcal{D}^k B$ for all k. Take $k \geq 0$, maximal such that $\mathcal{D}^k B \neq 0$, then $\mathcal{D}^{k+1}B = 0$, and so $\mathcal{D}^{k+1}A = 0$. Therefore $\mathcal{D}^k B$, and $\mathcal{D}^k A$ are in $R = \mathbb{V} \cap K[x,y,z]$ = ring of constants of $\mathcal{D}$ in $K[x,y,z]$. But $\xi = \frac{\mathcal{D}^k A}{\mathcal{D}^k B}$, so $\mathbb{V}$ = quotient field of R .

maar $\xi \in \mathbb{H}$

For future reference, note that if ξ is w-homogeneous, and $\xi \in \mathbb{V}$, then ξ can be written as quotient of two w-homogeneous elements in R. Indeed, if ξ is w-homogeneous, then in the previous argument we can take A and B w-homogeneous. Since $\mathcal{D}$ is itself homogeneous, then $\mathcal{D}^k A$, $\mathcal{D}^k B$ are w-homogeneous for all k. Thus $\xi = \frac{\mathcal{D}^k A}{\mathcal{D}^k B}$ is quotient of w-homogeneous elements of R.

Also for future reference, note that R is a homogeneous subring of K[x,y,z], in the sense that if $A \in R$ decomposes in K[x,y,z] as $A = \Sigma A_i$ with A_i w-homogeneous, then $A_i \in R$ for all i, because $\mathcal{D}$ is w-homogeneous, so $\mathcal{D}A = 0 = \Sigma \mathcal{D}A_i$, hence each homogeneous component must vanish: $\mathcal{D}A_i = 0$ for all i.

Lemma 3: $\mathbb{H}$ is a subfield of $\mathbb{V}$ of transcendence degree 1 over K.

Proof: Since $\mathbb{V}$ has transcendence degree 2 over K, there exist $\xi, \eta \in \mathbb{V}$, and $N \in K(x,y,z)$ such that ξ, η and N are algebraically independent. Consider the derivation of K(x,y,z) given by

$$\mathcal{D}N \frac{J(\xi,\eta,\,.\,)}{J(\xi,\eta,N)}$$

(here and elsewhere, J(a,b,c) = jacobian of a,b,c with respect to x, y, z $= \frac{\partial(a,b,c)}{\partial(x,y,z)}$). This derivation coincides with $\mathcal{D}$ on ξ,η, and N. Therefore

$$\mathcal{D} = \mathcal{D}N \frac{J(\xi,\eta,\,.\,)}{J(\xi,\eta,N)}.$$

In particular, $J(\xi,\eta,\,.) \neq 0$, or alternatively, $d\xi \wedge d\eta \neq 0$.

Since $\mathbb{V}$ is the quotient field of R, we can write $\xi = \frac{A}{G}$, $\eta = \frac{B}{G}$ for some A,B,G in R. But $d\xi \wedge d\eta = \frac{dA \wedge dB}{G^2} - \frac{B}{G^3} dA \wedge dG + \frac{A}{G^3} dB \wedge dG \neq 0$, hence either $dA \wedge dB$, $dA \wedge dG$, or $dB \wedge dG$ is not zero. Without loss of generality we may assume that $dA \wedge dB \neq 0$. But d is a w-homogeneous operation, so by decomposing A and B as sum of w-homogeneous polynomials and using the linearity of d and $\wedge$ we find that there exist $U,V \in R$, w-homogeneous and such that $dU \wedge dV \neq 0$, or equivalently, U,V algebraically independent. If $w(U) = 0$, then $U \in \mathbb{H}$. If $w(V) = 0$, then $V \in \mathbb{H}$. If $w(U) \neq 0$ or $w(V) \neq 0$, then $\frac{U^{w(V)}}{V^{w(U)}} \in \mathbb{H}\backslash K$. In either case, there is $f \in \mathbb{H}\backslash K$, so transcendence degree $\mathbb{H}/K \geq 1$.

If $\xi \in \mathbb{V} \backslash \mathbb{H}$ is algebraic over $\mathbb{H}$ then $w(\xi) = 0$. Indeed, otherwise we may assume without loss of generality that $w(\xi) > 0$ (replace ξ by ξ^{-1} if necessary). If ξ

satisfies $\xi^n = \sum_{j=0}^{n-1} c_j \xi^j$, with $c_j \in \mathbb{H}$. Then $nw(\xi) = w(\xi^n) = w(\Sigma c_j \xi^j) \leq \max_j w(c_j \xi^j) \leq (n-1)w(\xi)$, which is absurd. We claim that transcendence degree $\mathbb{V}/\mathbb{H} \geq 1$. From the previous observation it is enough to find an element of $\mathbb{V}$ of non-zero weight. But this is easy: because $w \geq 0$, $w \neq 0$, so there is some homogeneous polynomial A with $w(A) > 0$. Let $k \geq 0$ be such that $D^k A \neq 0 = D^{k+1} A$, then $D^k A \in \mathbb{V}$, and $w(D^k A) = kw(D) + w(A) \geq w(A) > 0$.

Finally $2 = \text{trans.deg } \mathbb{V}/K = \text{trans.deg } \mathbb{V}/\mathbb{H} + \text{trans.deg } \mathbb{H}/K \geq 1 + 1$, so $\text{trans.deg } \mathbb{H}/K = 1$.

Corollary: $\mathbb{H} = K(\xi)$ for some $\xi \in \mathbb{H} \backslash K$.

Proof of corollary: $K \subset \mathbb{H} \subset K(x,y,z)$, and $\text{trans.deg } \mathbb{H}/K = 1$. The conclusion follows from Lüroth theorem.

def in Cor

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Proof of theorem 5 (continuation)**:** Let $\xi = N/G$ with $N, G \in R = \text{Ker}(D)$, w-homogeneous. Any common factor of N and G in K[x,y,z] must be w-homogeneous, and must be in R, hence we may assume that N and G have no common factors in K[x,y,z].

Rosticklen vorrizen

R is a subring of K[x,y,z], hence a domain. Moreover, since an element is irreducible in R iff it is irreducible in K[x,y,z], then R is a unique factorization domain. We look next at the irreducible elements of R, since the generators of R must lie among them.

Lemma 4: with the notation just introduced, setting $\omega = w(N)$:

i) for every irreducible w-homogeneous $P \in R$, there exists a natural number λ, and $\alpha, \beta \in K$ such that $P^\lambda = \alpha N + \beta G$,

ii) each of N and G has one and only one irreducible factor, i.e: there are A, B irreducible w-homogeneous elements of R, such that $N = A^p$, and $G = B^q$ for some $p, q \geq 1$, integers,

iii) p and q in ii) are relatively prime,

iv) let $\Delta = \text{g.c.d}(w(A), w(B))$, then for every irreducible w-homogeneous $P \in R \setminus K[A,B]$, the exponent λ such that $P^\lambda = \alpha N + \beta G$ is a non-trivial factor of Δ. Different irreducibles have relatively prime corresponding exponents.

v) if η is any generator of $\mathbb{H}$, then the weight of the numerator of η (when η is written as quotient of two polynomials without common factors) is $\omega = w(N)$.

vi) the exponent λ corresponding to a w-homogeneous irreducible element $P \in R$ according to i) is given by $\lambda = \dfrac{\omega}{w(P)}$, and so it is independent of the choice of generators of $\mathbb{H}$.

Proof of lemma 4.

i) If $P \in R \setminus K$ is an irreducible polynomial, then $\dfrac{P^{w(G)}}{G^{w(P)}} \in \mathbb{H}$, hence it can be written as $\dfrac{P^{w(G)}}{G^{w(P)}} = c \ \Pi\ (\xi - \beta_i)^{n_i} = c\, G^n\, \Pi\ (N - \beta_i G)^{n_i}$ for some n, $n_i \in \mathbb{Z}$, and some c, $\beta_i \in K$. Therefore, $P^{w(G)} = c\, G^m\, \Pi\ (N - \beta_i G)^{n_i}$ where $m = n + w(P)$. Since N and G have no common factors, then G, $N - \beta_1 G$, $N - \beta_2 G$,... have no common factors if $\beta_1 \neq \beta_2$, etc. Therefore at most one of m, n_1,n_2, etc. is different from zero. (gebraucht P irreducibel!)

Note $w(G) > 0$. Indeed: G is a polynomial, and $w(x)$,$w(y)$,$w(z) > 0$, so $w(G) \geq 0$. If $w(G) = 0$, then $w(N) = 0$, because $\xi \in \mathbb{H}$, so ξ is w-homogeneous of weight 0. Thus $N, G \in K$, contradicting the fact that $\mathbb{H} = K(\xi)$ has transcendence degree 1 over K.

From the last remark, $P^{w(G)} \notin K$, and so either m or some n_i is different from zero. Thus, $P^{w(G)} = (\alpha N + \beta G)^k$ for some α, β, k. But P is irreducible, so the only possible irreducible factor of $\alpha N + \beta G$ is P, i.e., $P^\lambda = \alpha N + \beta G$ for some $\lambda \geq 1$. $\in \mathbb{N}$

ii) As we mentioned in i) $w(G) > 0$. (deg G not constant) It follows that G has at least one irreducible factor B. (B divides G) From i) $B^q = \alpha N + \beta G$ for some constants α, β, and $q \geq 1$. If $\alpha \neq 0$, then B divides N as well as G, which is absurd, since N, G have no common factors. Thus $\alpha = 0$, and after rescaling B we get $G = B^q$.

A similar argument shows that $N = A^p$ for some irreducible polynomial A.

iii) If $p = \tilde{p}\, d$ and $q = \tilde{q}\, d$, then $\xi = \dfrac{N}{G} = (\dfrac{A^{\tilde{p}}}{B^{\tilde{q}}})^d$. Let $\eta = \dfrac{A^{\tilde{p}}}{B^{\tilde{q}}}$. Clearly η is w-homogeneous of weight 0, constant with respect to D, thus $\eta \in \mathbb{H}$. Also $K(\eta) \subseteq \mathbb{H} = K(\xi) \subseteq K(\eta)$, so $\mathbb{H} = K(\eta)$. Therefore $d = |K(\eta):K(\xi)| = |\mathbb{H}:\mathbb{H}| = 1$, i.e. p and q are relatively prime.

iv) Let $P \in R \setminus K[A,B]$ be an irreducible, w-homogeneous polynomial. From i) we know that $P^\lambda = \alpha N + \beta G = \alpha A^p + \beta B^q$. Since $P \notin K[A,B]$, then $\lambda > 1$, and $\alpha\beta \neq 0$. Note that $\dfrac{P^\lambda}{B^q} = \alpha\, \xi + \beta$

$\dfrac{P^\lambda}{\xi} = \alpha \dfrac{N}{\xi} + \beta$

$= \alpha \zeta + \beta$

$\dfrac{P^\lambda}{B^q}$ $\quad g = B^q$

u and $d \neq 0$ a$\mathbb{H} = k(B)$)

iii) generates $\mathbb{H}$. Then the argument in ii) applied to P^λ / B^q implies that λ and q are relatively prime. Similarly, λ and p are relatively prime. But $\lambda\, w(P) = w(\alpha N + \beta G) = q w(B) = p w(A)$, hence λ divides $w(A)$ and $w(B)$, i.e., λ divides Δ.

If P and T are distinct irreducible w-homogeneous elements of R, P not a constant multiple of T, then we can write $P^\lambda = \alpha N + \beta G$, and $T^\mu = \gamma N + \delta G$, thus $\frac{P^\lambda}{T^\mu} = \frac{\alpha\xi + \beta}{\gamma\xi + \delta}$

iv) generates $\mathbb{H}$. The argument of part ii) applied to $\frac{P^\lambda}{T^\mu}$ implies that λ, μ have no common factors.

v) If η is any generator of $\mathbb{H}$, then from ii) $\eta = \frac{P^\lambda}{T^\mu}$ for some w-homogeneous irreducibles P, T. From i), $P^\lambda = \alpha N + \beta G$, and $T^\mu = \gamma N + \delta G$, for suitable constants α,etc. But N and G are w-homogeneous of the same weight, so $w(P^\lambda) = w(\alpha N + \beta G) = w(N)$.

vi) This is obvious.

Proof of theorem 5 (continuation)**:** From lemma 4 it follows that most irreducible w-homogeneous elements of R are of the form $\alpha N + \beta G$, and so they have weight $\omega = w(N)$, but there may be some "exceptional" w-homogeneous irreducibles whose weights are proper factors of ω. The last part of lemma 4 implies that the number of distinct (mod the relation $P \approx \lambda P$ for $\lambda \in K^x$) such "exceptional" irreducibles is finite, actually it is less than the number of distinct prime factors of the integer ω.

Lemma 5: The number of distinct "exceptional" w-homogeneous irreducibles is at most 3.

Proof of lemma 5: suppose there at least three exceptional w-homogeneous irreducibles T,U,V. We will show that their corresponding exponents $\frac{\omega}{w(T)}$, $\frac{\omega}{w(U)}$, $\frac{\omega}{w(V)}$ must be 2,3, and 5. Hence no other w-homogeneous irreducible could exist because if it did then first: by part iv) of lemma 4, its corresponding exponent would have to be relatively prime to 2, 3, and 5, and second: applying the first part of this lemma to T,U, and the fourth irreducible, the corresponding exponent of the later would have to be one of 2, 3, or 5.

Let t, u, v be the corresponding exponents of T,U,V.Then T^t / U^u is a generator of $\mathbb{H}$, and so $V^v = \alpha T^t + \beta U^u$. Thus, after rescaling T and U, we may assume that

$$T^t + U^u + V^v = 0.$$

We take exterior derivative of this identity, and so we obtain the system:

$$\begin{bmatrix} T & U & V \\ tdT & udU & vdV \end{bmatrix} \begin{bmatrix} T^{t-1} \\ U^{u-1} \\ V^{v-1} \end{bmatrix} = 0 .$$

If $t\dfrac{dT}{T} = u\dfrac{dU}{U} = v\dfrac{dV}{V}$ then $\dfrac{T^t}{U^u}, \dfrac{T^t}{V^v}, \dfrac{V^v}{U^u}$ are constants, which is absurd since either of them generates $\mathbb{H}$. Thus we may assume for example that

$$\begin{bmatrix} T & U & V \\ tT_x & uU_x & vV_x \end{bmatrix}$$

has rank 2 (working over the field K(x,y,z)). We solve the system

$$\begin{bmatrix} T & U & V \\ tT_x & uU_x & vV_x \end{bmatrix} \begin{bmatrix} T^{t-1} \\ U^{u-1} \\ V^{v-1} \end{bmatrix} = 0$$

by the usual method and find that $T^{t-1}, U^{u-1}, V^{v-1}$ are proportional to $vUC_x - uVU_x$, $tVT_x - vTV_x$, and $uTU_x - tUT_x$ respectively, i.e.:

$$\begin{bmatrix} T^{t-1} \\ U^{u-1} \\ V^{v-1} \end{bmatrix} = \frac{L}{M} \begin{bmatrix} vUV_x - uVU_x \\ tVT_x - vTV_x \\ uTU_x - tUT_x \end{bmatrix}$$

need

with $L,M \in K[x,y,z]$ relatively prime. It follows that L divides $T^{t-1}, U^{u-1}, V^{v-1}$, which have no common factor, hence we may take $L = 1$. Therefore $M\,T^{t-1} = vUV_x - uVU_x$, and so

(and $t,u,v \geq 1$ as all T,U,V [illegible]

[illegible]

$= w(T^{t-1}) \leq$

$$(t-1)\frac{\omega}{t} = (t-1)\,w(T) \leq w(MT^{t-1}) = w(vUV_x - uVU_x) = w(U) + w(V) - w(x) \leq$$

$\leq w(U) + w(V) - 1 = \dfrac{\omega}{u} + \dfrac{\omega}{v} - 1$, from where we deduce

$$(*) \qquad 1 + \frac{1}{\omega} \leq \frac{1}{t} + \frac{1}{u} + \frac{1}{v} .$$

At this point recall that $t,u,v \geq 2$, since T, U, V are exceptional irreducibles. Also t, u, v are pairwise relatively prime (lemma 4, iv)). Assume for the sake of the argument that $2 \leq t < u < v$. If $t \geq 3$, then $u \geq 4$, $v \geq 5$, and so $\dfrac{1}{t} + \dfrac{1}{u} + \dfrac{1}{v} < 1$, making (*) impossible. Therefore $t = 2$.

If $u > 3$, then $u \geq 5$ (u, and $t = 2$ have no common factors!), and $v \geq 7$, so again

$\dfrac{1}{t} + \dfrac{1}{u} + \dfrac{1}{v} < 1$, which is absurd. Therefore $u = 3$.

If v > 5 then v ≥ 7, and once again $\frac{1}{t} + \frac{1}{u} + \frac{1}{v} < 1$. Therefore v = 5.

Proof of theorem 5 (continuation): Assume first that there is only one exceptional w-homogeneous irreducible element of R , say T. Then any other w-homogeneous irreducible has weight ω. In particular either A or B has weight ω. Therefore we can find U w-homogeneous irreducible of weight ω. Then T^λ / U is a generator of $\mathbb{H}$, where λ = ω / w(T). If P is any w-homogeneous irreducible, then P is a multiple of T (the only exceptional irreducible) or $P = \alpha T^\lambda + \beta U$, since P not a multiple of T implies w(P) = ω. Any element of R is a finite sum of w-homogeneous elements, which in turn are products of w-homogeneous irreducible elements. But the later are in K[T,U]. Therefore R = K[T,U].

Assume next that there are exactly two exceptional w-homogeneous irreducible elements of R , say T and V, then T^λ / V^μ is a generator of $\mathbb{H}$, where λ = ω / w(T) and μ = ω / w(V) . Moreover, any other w-homogeneous irreducible element of R is either a multiple of T, of V, or a linear combination of T^λ, and V^μ. In either case, any w-homogeneous irreducible element of R is in K[T,V]. Thus R = K[T,V].

The proof will be finished once we show that there cannot be three exceptional irreducibles in R.

Lemma 6: There cannot be three exceptional w-homogeneous irreducible elements in R.

Proof of lemma 6: If there are three exceptional irreducibles, then as shown in the proof of lemma 5, we may assume that they satisfy $T^2 + U^3 + V^5 = 0$. In this case $T^2 U^{-3}$ generates $\mathbb{H}$, and R = K[T,U,V].

We go back to the idea in the proof of lemma 5, and consider the system

$$\begin{bmatrix} T & U & V \\ 2dT & 3dU & 5dV \end{bmatrix} \begin{bmatrix} T \\ U^2 \\ V^4 \end{bmatrix} = 0 .$$

We claim that rank\b\bc\|(\a\al\co3\hs9(T ,U ,V ,$2T_x$,$3U_x$,$5V_x$)) = 2 . Indeed, if $2\frac{T_x}{T} = 3\frac{U_x}{U} = 5\frac{V_x}{V}$ then $\frac{T^2}{U^3}$, $\frac{T^2}{V^5}$, and $\frac{V^5}{U^3}$ are independent of x. Since T,U are irreducibles in K[x,y,z], it follows that T,U are themselves independent of x. Moreover T,U are algebraically independent. Let N ∈ K[x,y,z] be such that J(T,U,N) ≠ 0. Then as in the proof of lemma 3, we deduce

$$D = DN \frac{J(T,U,\,.\,)}{J(T,U,N)} = DN \frac{\partial(T,U)}{\partial(y,z)} \frac{1}{J(T,U,N)} \frac{\partial}{\partial x} .$$

waarom?

Hence $R = \mathrm{Ker}(D) = \mathrm{Ker}(\frac{\partial}{\partial x}) = K[y,z]$. But $R \approx K[x,y,z]/\langle x^2+y^3+z^5\rangle$, which is not isomorphic to the ring of polynomials in two variables. Thus, $\mathrm{rank}\begin{bmatrix} T & U & V \\ 2T_x & 3U_x & 5V_x \end{bmatrix} = 2$, as we had claimed.

In a similar way we deduce $\mathrm{rank}\begin{bmatrix} T & U & V \\ 2T_y & 3U_y & 5V_y \end{bmatrix} = \mathrm{rank}\begin{bmatrix} T & U & V \\ 2T_z & 3U_z & 5V_z \end{bmatrix} = 2$. As in lemma 5, this implies that

$$\begin{bmatrix} T \\ U^2 \\ V^4 \end{bmatrix} = \frac{1}{M_1}\begin{bmatrix} 5UV_x - 3VU_x \\ 2VT_x - 5TV_x \\ 3TU_x - 2UT_x \end{bmatrix}$$

for some polynomial M_1, and similar equations with derivatives with respect to y, and z. Also as in lemma 5 , considering the weight w , from $M_1 T = 5UV_x - 3VU_x$ we obtain

(*) $w(M_1) + w(T) \leq w(U) + w(V) - w(x)$ and similar equations for y, and z.

We claim that $\omega = 30$ (See lemma 4)

Justification of the claim $\omega = 30$. Since 2, 3, and 5 are factors of ω, then 30 divides ω. Let $\omega = 30\,\Delta$. Then from lemma 4,vi) we deduce that Δ divides the weight of T,U,V, and of any other w-homogeneous irreducible element of R. Therefore Δ divides the weight of any w-homogeneous element of $\mathbb{V}$ = quotient field of R. The claim will be justified if we show that there exists a w-homogeneous element of $\mathbb{V}$ of weight 1. We post-pone this step for later. We state the required result for future reference.

Lemma 7: Under the assumptions of theorem 5, and the assumption $R = K[T,U,V] \approx K[x,y,z]/\langle x^2+y^3+z^5\rangle$, there exists a rational w-homogeneous function, in the kernel D, of weight 1.

$w(U) = \frac{\omega}{3} = 10$, $w(V) = \frac{\omega}{5} = 6$

(m.b.v. (*))

From $\omega = 30$, and $w(T) = \omega/2$, etc., we get $w(M_1) + 15 \leq 10 + 6 - w(x)$, or $w(M_1) \leq 1 - w(x)$. But $w(x) \geq 1$, $w(M_1) \geq 0$, thus $w(x) = 1$, and $w(M_1) = 0$, and similar equation for y, and z, so $w(y) = w(z) = 1$, and $w(M_2) = w(M_3) = 0$, i.e.: w = degree, and M_1, M_2, M_3 are scalars.

We can rewrite the system giving T,U,V in terms of M_i, etc. as

$$(M_1dx + M_2dy + M_3dz)\begin{bmatrix} T \\ U^2 \\ V^4 \end{bmatrix} = \begin{bmatrix} 5UdV - 3VdU \\ 2VdT - 5TdV \\ 3TdU - 2UdT \end{bmatrix}.$$

We make now a linear change of coordinates, taking $\tilde{x} = M_1x + M_2y + M_3z$ as one of the coordinates. If $\tilde{z}$ is a coordinate different from $\tilde{x}$, then the differential system above gives:

$$0 = \frac{\partial \tilde{x}}{\partial \tilde{z}} \begin{bmatrix} T \\ U^2 \\ V^4 \end{bmatrix} = \begin{bmatrix} 5U \frac{\partial V}{\partial \tilde{z}} - 3V \frac{\partial U}{\partial \tilde{z}} \\ 2V \frac{\partial T}{\partial \tilde{z}} - 5T \frac{\partial V}{\partial \tilde{z}} \\ 3T \frac{\partial U}{\partial \tilde{z}} - 2U \frac{\partial T}{\partial \tilde{z}} \end{bmatrix},$$

hence

$$\frac{T^2}{U^3}, \frac{T^2}{V^5}, \text{ and } \frac{V^5}{U^3}$$

are independent of $\tilde{z}$. But we have showed in the first part of lemma 6 that indepence of these quotients with respect to $\tilde{z}$ would imply R is the algebra of polynomials in $\tilde{x}$ and $\tilde{y}$, contradicting the assumption that $R \approx K[x,y,z]/\langle x^2+y^3+z^5 \rangle$. Thus there cannot be three exceptional w-homogeneous irreducible polynomials in R. This ends the proof of lemma 6.

Proof of theorem 5 (conclusion). There remains only one step to finish the proof of the theorem.

Proof of lemma 7: We will use the flow defined by D. We will write x_1, x_2, x_3 instead of x, y, z. Let $V_i = Dx_i$. Let $\Phi(x_1,x_2,x_3,t) = (\Phi_1,\Phi_2,\Phi_3)$ be the solution of the system of ordinary differential equations:

$$\frac{d\Phi_i}{dt} = V_i(\Phi_1,\Phi_2,\Phi_3) \qquad i = 1,2,3,$$

with initial condition $\quad \Phi(x_1,x_2,x_3,0) = (x_1,x_2,x_3)$.

Since D is locally nilpotent, then $\Phi_i(a,b,c,t) = \sum_{j\geq 0} \frac{D^j x_i}{j!} t^j$, where the polynomials $D^j x_i$ are evaluated at (a,b,c). The fact that x_i and D are w-homogeneous implies that $D^j x_i$ is w-homogeneous of weight $jw(D) + w(x_i)$.

Let $N \in K[x_1,x_2,x_3]$ be w-homogenous, irreducible, such that $G = DN \neq 0$, but $D^2N = 0$. Let $r_1, r_2, r_3 \in K(x,y,z)$ be defined as the coordinates of Φ_0 where $\Phi_0(x_1,x_2,x_3) = (r_1, r_2, r_3) = \Phi(x_1,x_2,x_3, -\frac{N}{G})$.

Since N is w-homogeneous, then $\frac{N}{G}$ is w-homogeneous, and $w(\frac{N}{G}) = -w(D)$. Thus, each r_i is a finite sum of w-homogeneous rational functions, all of the same weight equal to $w(x_i)$. Therefore r_i is w-homogeneous of weight $w(x_i)$.

imme $\frac{d}{dt} N(\Phi(x,t)) = DN(\Phi(x,t)) = G(\Phi(x,t))$

$N(\Phi(x,0)) = N(x)$

$N(\Phi(x,t)) =$

Claim 1: r_1, r_2, r_3 are constants with respect to D.

Proof: By definition, for any polynomial $P \in K[x,y,z]$, $\frac{d}{dt} P(\Phi(x_1,x_2,x_3,t)) =$ $N+G\cdot\frac{-N}{G}=0$

$DP(\Phi(x_1,x_2,x_3,t))$. Therefore, $N(\Phi(x_1,x_2,x_3,t)) = N(x_1,x_2,x_3) + G(x_1,x_2,x_3)\, t$, and $G(\Phi(x_1,x_2,x_3,t)) = G(x_1,x_2,x_3)$. In particular $N(r_1, r_2, r_3) = 0$. $= N(\Phi(x_1,x_2,x_3,-\frac{N}{G}))$ But then $\Phi_0(\Phi(x_1,x_2,x_3,t)) = \Phi(\Phi(x_1,x_2,x_3,t), -\frac{N}{G}(\Phi(x_1,x_2,x_3,t))) = \Phi(x_1,x_2,x_3,t - \frac{N}{G}(\Phi(x_1,x_2,x_3,t))) = \Phi(x_1,x_2,x_3, -\frac{N}{G}(x_1,x_2,x_3)) = \Phi_0(x_1,x_2,x_3)$, hence Φ_0 is constant with respect to D.

Claim 2: $R = K[x_1,x_2,x_3] \cap K[r_1, r_2, r_3]$.

Proof: If $P \in K[x_1,x_2,x_3] \cap K[r_1, r_2, r_3]$, then P is a polynomial, and P is constant with respect to D, hence $P \in R$. Reciprocally, if $P \in R$, then certainly $P \in K[x,y,z]$, also $P(x_1,x_2,x_3) = P(\Phi(x_1,x_2,x_3,t))$ for all t, in particular for $t = -\frac{N}{G}$, thus $P = P(r_1, r_2, r_3) \in K[r_1, r_2, r_3]$.

$\exp(tD)(P(x_1,x_2,x_3))$

$(I+tD+\cdots)P = P$

Claim 3: Set $\Delta = \omega/30$, then $\Delta = \text{g.c.d}\{\, w(h) : h \in \mathbb{V},\ h \neq 0,\ h \text{ is w-homogeneous}\}$.

Proof: R is generated by T, U, V, and $w(T) = \omega/2 = 15\,\Delta$, $w(U) = \omega/3 = 10\,\Delta$, $w(V) = \omega/5 = 6\,\Delta$. Any other w-homogeneous irreducible polynomial in R has weight $\omega = 30\,\Delta$. It follows that the weight of any w-homogeneous element of $\mathbb{V}$ is a multiple of Δ. Moreover, $w(\frac{UV}{T}) = (10 + 6 - 15)\,\Delta$, so $\Delta = \text{g.c.d}\{\, w(h) : h \in \mathbb{V},\ h \neq 0,\ h \text{ w-homogeneous}\}$.

Claim 4: $r_i \neq 0$ for $i = 1, 2, 3$.

Proof: Assume $r_1 = 0$. Then $r_2 \neq 0$, otherwise $R \subseteq K(r_3)$, but then $\mathbb{V}$, which equals the quotient field of R, would be contained in $K(r_3)$, contradicting the fact that transcendence degree $\mathbb{V}/K = 2$. Likewise, if $r_1 = 0$ then $r_3 \neq 0$. Let $\Delta_1 = \text{g.c.d.}\{w(x_2), w(x_3)\}$. If $\Delta_1 = n_2 w(x_2) + n_3 w(x_3)$, then $r_2^{n_2} r_3^{n_3}$ is w-homogeneous of weight Δ_1. But $\Delta = \text{g.c.d}\{\, w(h) : h \in \mathbb{V},\ h \neq 0,\ h \text{ w-homogeneous}\}$. Therefore Δ divides $\Delta_1 = \omega(r_2^{n_2} r_3^{n_3})$. Conversely, since we are assuming $r_1 = 0$, then for any $P \in R$, $P(x,y,z) = P(0, r_2, r_3)$. If in addition P is w-homogeneous, then P is a w-homogeneous polynomial in r_2, r_3. Therefore $w(P)$ is a linear combination with integer coefficients of $w(r_i) = w(x_i)$ ($i = 2, 3$). Hence $w(P)$ is a multiple of Δ_1. This implies that Δ_1 divides Δ, so $\Delta = \Delta_1$. ?

As in the proof of lemma 6, before the claim $\omega = 30$, we have

$$\begin{bmatrix} T \\ U^2 \\ V^4 \end{bmatrix} = \frac{1}{M_2}\begin{bmatrix} 5UV_y - 3VU_y \\ 2VT_y - 5TV_y \\ 3TU_y - 2UT_y \end{bmatrix}$$

for some polynomial M_2, and similar equations for the derivative with respect to z. Also as in lemma 5 , considering the weight w , from $M_2T = 5UV_y - 3VU_y$ we obtain $w(M_2) + w(T) \le w(U) + w(V) - w(y)$ and a similar inequality for z. In particular : $w(M_2) + 15\,\Delta \le$

y $10\,\Delta + 6\,\Delta - w(\cancel{x})$, so $w(M_2) \le \Delta - w(y)$, but $w(y)$ $\neq 0$ is a multiple of Δ, thus $w(M_2) = 0$, and $w(y) = \Delta$. Likewise $w(M_3) = 0$, and $w(z) = \Delta$. But if M_2, M_3 have weight 0, then they

* are in K. We consider the following w-homogeneous change of coordinates $\tilde{x} = x$, $\tilde{y} = M_2\, y + M_3\, z$, and $\tilde{z} = M_3\, z$. Then , $\dfrac{\partial}{\partial \tilde{z}} = \dfrac{1}{M_3}\dfrac{\partial}{\partial z} - \dfrac{1}{M_2}\dfrac{\partial}{\partial y}$ so

$$\begin{bmatrix} 5U\dfrac{\partial V}{\partial \tilde{z}} - 3V\dfrac{\partial U}{\partial \tilde{z}} \\ 2V\dfrac{\partial T}{\partial \tilde{z}} - 5T\dfrac{\partial V}{\partial \tilde{z}} \\ 3T\dfrac{\partial U}{\partial \tilde{z}} - 2U\dfrac{\partial T}{\partial \tilde{z}} \end{bmatrix} = \frac{1}{M_3}\begin{bmatrix} 5UV_z - 3VU_z \\ 2VT_z - 5TV_z \\ 3TU_z - 2UT_z \end{bmatrix} - \frac{1}{M_2}\begin{bmatrix} 5UV_y - 3VU_y \\ 2VT_y - 5TV_y \\ 3TU_y - 2UT_y \end{bmatrix} =$$

$$= \begin{bmatrix} T \\ U^2 \\ V^4 \end{bmatrix} - \begin{bmatrix} T \\ U^2 \\ V^4 \end{bmatrix} = 0 .$$

This implies that $\dfrac{V^5}{U^3}$, and $\dfrac{V^5}{T^2}$ are independent of $\tilde{z}$. But T, U, and V are irreducible, so they must be independent of $\tilde{z}$ as well. In the first part of lemma 6 we showed that this would imply $R = K[\tilde{x},\tilde{y}]$ contradicting the fact that $R \approx K[x,y,z]/\langle x^2+y^3+z^5\rangle$. Thus $r_1 \neq 0$. In a similar way, $r_2 \neq 0 \neq r_3$.

dd [illegible]

Now we can finish the proof of lemma 7. $w(r_i) = w(x_i)$ for i=1,2,3. Moreover, $\{w(x_i)\}_{i=1,2,3}$ have no common factors, hence $1 = n_1\, w(x_1) + n_2\, w(x_2) + n_3\, w(x_3)$ for some integers n_i . Then $r_1^{n_1}\, r_2^{n_2}\, r_3^{n_3}$ is a w-homogeneous element of $\mathbb{V}$, of weight 1. Therefore $\omega/30 = \Delta = \text{g.c.d}\{\ w(h) : h \in \mathbb{V},\ h \neq 0,\ h \text{ w-homogeneous}\}$ divides 1, so $\omega = 30$.

Proof of theorem 6. We will show a more general version of theorem 6. In order to state the stronger version we introduce some notation. Let $K[x_1,..,x_n] = \bigoplus_{\lambda \in \sigma} M_\lambda$ be the decomposition induced by the locally finite derivation $D = \Sigma_i V_i \partial_i$. Let $\mathcal{V}$ be the ideal generated by $V_1,.., V_n$. Consider the following tower of fields:

$$K \underset{t_1}{\rightarrow} q(R) \underset{t_2}{\rightarrow} q(M_0) \rightarrow K(x_1,..,x_n),$$

where $R = \mathrm{Ker}(D)$, $M_0 = \cup_{n\geq 0} \mathrm{Ker}(D^n)$, q(.) = quotient field of (.).

Let t_1 = transcendence degree $q(R) / K$,

t_2 = transcendence degree $q(M_0) / q(R)$,

ℓ = rank of the spectrum of D,

$$\varepsilon = \begin{cases} 1 \text{ if there exists } T \in K(x_1,..,x_n) \text{ such that } DT = 1; \\ 0 \text{ otherwise.} \end{cases}$$

It is shown in [Zur] that $t_1 + t_2 + \varepsilon + \ell = n$.

Theorem 6': With the notation introduced above, $t_2 + \ell \leq \mathrm{ht}(\mathcal{V})$.

Lemma 8. With the notation as above, $\mathrm{tr.deg.}q(M_0) / q(R) \leq \varepsilon$.

Proof. If $\varepsilon = 0$, then $R = M_0$. Indeed, if $D^{j+1}f = 0 \neq D^jf$ with $j \geq 1$, then $g = D^jf \in R$, and so $T = \frac{D^{j-1}f}{D^jf}$ is such that $DT = 1$, contradicting $\varepsilon = 0$. This shows $\mathrm{Ker}(D^j) = \mathrm{Ker}(D)$ for $j \geq 1$. Hence $M_0 = R$, $\mathrm{tr.deg.}q(M_0)/q(R) = 0$.

If $\varepsilon = 1$, and $\mathrm{tr.deg.}q(M_0)/q(R) = 0$, then there is nothing to prove.

If $\varepsilon = 1$, and $\mathrm{tr.deg.}\ q(M_0)/q(R) \geq 1$, then let $T \in K(x_1,..,x_n)$ be such that $DT = 1$. A straightforward argument shows that $M_0 \subset q(R)(T)$, hence $\mathrm{tr.deg.}\ q(M_0)/q(R) = 1$.

Corollary. Krull dimension $M_0 \leq \varepsilon + t_1$.

Proof. From Theorem 3 we know that M_0 is a noetherian ring. Moreover, M_0 contains the field K, and it is of finite transcendence dimension over K. Therefore, Krull dimension $M_0 = \mathrm{tr.deg.}\ q(M_0)/K = \underbrace{\mathrm{tr.deg.}\ q(M_0)/q(R)}_{\leq \varepsilon} + \underbrace{\mathrm{tr.deg.}\ q(R)/K}_{t_1} \leq \varepsilon + t_1$.

Proof of theorem 6'. Let $\mathcal{P}$ be a prime ideal of $K[x_1,..,x_n]$, $\mathcal{P} \supset \mathcal{V}$, with $\mathrm{ht}(\mathcal{P}) = \mathrm{ht}(\mathcal{V})$. Then, $n = \text{Krull dim } K[x_1,..,x_n] = \text{Krull dim}(K[x_1,..,x_n]/\mathcal{P}) + \mathrm{ht}(\mathcal{P})$. But if $\lambda \neq 0$

$D(P_\lambda) = \lambda P_\lambda$ [illegible] $\lambda \neq 0$, $P_\lambda = \frac{1}{\lambda} D(P_\lambda) = D(\frac{1}{\lambda} P_\lambda) \subset D(M_\lambda)$ $= \oplus M_\lambda / \mathcal{P} \cong M_0 / M_0 \cap \mathcal{P}$ [illegible]

then $M_\lambda \subset \mathcal{P}$, because $M_\lambda \subset D(M_\lambda) \subset \mathcal{V} \subset \mathcal{P}$. Therefore, $K[x_1,..,x_n]/\mathcal{P} \approx M_0/(M_0 \cap \mathcal{P})$,and so Krull $\dim(K[x_1,..,x_n]/\mathcal{P}) \le$ Krull dim $M_0 \le \varepsilon + t_1$. Thus, $t_1 + t_2 + \varepsilon + \ell = n \le \varepsilon + t_1 + ht(\mathcal{P})$, from where we get $t_2 + \ell \le ht(\mathcal{V})$.

Proof of theorem 7. Suppose $\mathcal{A}$ and $\mathbf{B}$ are fixed points of D. Let $\mathcal{J} = \mathcal{A} \cap \mathbf{B}$. Let $\mathbf{A} = \bigoplus_{\lambda \in \sigma} M_\lambda$ be the decomposition of $\mathbf{A}$ induced by D. Since whenever $\lambda \neq 0$ we have $M_\lambda \subset D(M_\lambda) \subset \mathcal{J}$, it follows that $\mathbf{A}/\mathcal{J} \approx M_0/(M_0 \cap \mathcal{J})$. But $\mathbf{A}/\mathcal{J} \approx KxK$.

We claim that if $R = K$, then $M_0 = K$ as well. Indeed, if $R = K$, and $M_0 \neq K$, then there exists some element $f \in \mathbf{A}$ such that $D^2 f = 0 \neq Df$, hence $Df \in R = K$. Then $T = (Df)^{-1} f \in \mathbf{A}$, and $DT = 1$. But D admits fixed points $\mathcal{A}$ and $\mathbf{B}$, so $1 = DT \in D(\mathbf{A}) \subset \mathcal{A} \cap \mathbf{B}$, contradicting maximality of $\mathcal{A}$ and $\mathbf{B}$. Therefore if $R = K$, it follows that $M_0 = K$. In particular, if $R = K$, then $M_0/(M_0 \cap \mathcal{J})$ is either 0 or K, neither of which is isomorphic to KxK as ring. Hence, $KxK \approx \mathbf{A}/\mathcal{J} \approx M_0/(M_0 \cap \mathcal{J})$ implies $R \neq K$. In other words, if D admits at least two fixed points then D annihilates some element of $\mathbf{A}$ other than those elements in K.

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