(Almost) rigid rings and infinitely generated invariants.

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Some Notations

If A is a ring, then

$\mathsf{DER}(A), \mathsf{LND}(A)$ is set of (locally nilpotent) derivations on A.

$\mathsf{LND}^*(A) := \mathsf{LND}(A) \setminus \{0\}$.

$\mathbb{C}^{[n]} := \mathbb{C}[X_1, \ldots, X_n]$

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But - it is not that easy to prove that something is rigid!

How to prove something is rigid?

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If $\widehat{A} := \mathbb{C}^{[n]}/\widehat{\mathfrak{p}}$ is rigid, then A rigid!

Example

Let

$A := \mathbb{C}[x, y, z] = \mathbb{C}[X, Y, Z]/(X^a Y^b + Z^c + XYZ + X + Y + Z)$

where $a, b, c \geq 2$. Choose a degree function on $\mathbb{C}[X, Y, Z]$ such that top degree part is $X^a Y^b + Z^c$.

Theorem: there exist no homogeneous nonzero LNDs on $\mathbb{C}[X, Y, Z]/(X^a Y^b + Z^c)$.

Corollary: there exist no nonzero LNDs on $\mathbb{C}[X, Y, Z]/(X^a Y^b + Z^c)$.

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Here K is algebraic closure of A^D. $S = p/D(p)$, where $p \in A$ is a preslice $(D(p) \neq 0, D^2(p) = 0)$.

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Let me give some examples:

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Let's do another one!

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Always Mason's theorem!

Mason's Theorem: Let $f, g, h \in K[X]$ not all constant, $\gcd(f, g, h) = 1$ and $f + g = h$. Then $deg(f) < \mathcal{N}(fgh)$ ($\mathcal{N}$ is number of zeroes).

Generalization: (de Bondt) Let $f_1, \ldots, f_n \in K[X]$ not all constant, $f_1 + \ldots + f_n = 0$, and some requirement replacing $\gcd(f, g, h) = 1$. Then $deg(f_1) < (n-2)\mathcal{N}(f_1 f_2 \cdots f_n)$.

The Typical Example: Brieskorn-Catalan-Fermat

Let $A := \mathbb{C}^{[n]}/(X_1^{d_1} + \ldots + X_n^{d_n})$, $n \geq 3$. If

$$\frac{1}{d_1} + \ldots + \frac{1}{d_n} \leq \frac{1}{n-2}$$

then A is rigid.

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IDEA: take one of those examples of LNDs on $\mathbb{C}^{[n]}$ that have infinitely generated kernel, and *force* this to be the *only* derivation that exists!

A non-finitely generated kernel

Known:

Robert's derivation:

$D_R := X^3\partial_S + Y^3\partial_T + Z^3\partial_U + X^2Y^2Z^2\partial_V$ has infinitely generated kernel.

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Let's make a ring where Robert's derivation is the only one that exists!

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Define $A := \mathbb{C}[T^2, T^3, X, Y, z] :=$
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$0 = D(z) = D(T^2(T^2X + T^3Y))$ etc... D is multiple of
$T^3\partial_X - T^2\partial_Y$.

Now easy: $A^D = \mathbb{C}[T, z, X + TY] \cap A$ not finitely generated.