A three dimensional UFD cancellation counterexample

Stefan Maubach

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Central question:

How to distinguish two rings?

Cancelation problems

Cancelation problem: A $\mathbb{C}$-algebra.

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2006 (Finston/Maubach)	$dim(A) = 3, A$ $\mathbb{C}$-algebra UFD

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Idea: take suitable rigid ring R, and

$A_{n,m} := R[U, V]/(x^m U - y^n V - 1)$ for some $x, y \in R$.

First something else. . .

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f, g, h relatively prime.

Then $max\{deg(f), deg(g), deg(h)\} < \#Z(fgh)$.

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Then all f, g, h are constant.

Using Mason's in rings

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Corollary implies: $f, g, h \in \overline{K} \cap R = R^D$. So $LND(R) = \{0\}$!!

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$0 = 1 + x^a + y^b + z^c = 1 + f^a + g^b + h^c$

Implies (using Mason's) that f, g, h constant.

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Claim: $A_{n,m}[T] \cong A_{n',m'}[T]$ for all n, m, n', m'

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Question: if $\{n, m\} \neq \{n', m'\}$, is

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Oops, have to choose $gcd(a, b) = gcd(a, c) = gcd(b, c) = 1$

for otherwise: (case $gcd(a, b) = 2$):

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Corollary: Let $\varphi \in Aut_{\mathbb{C}}(A_{n,m})$. Then $\varphi^{-1} E \varphi = \lambda E$ for some $\lambda \in \mathbb{C}[x, y, z]^* = \mathbb{C}^*$.

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Proposition: $\varphi \in Aut_{\mathbb{C}}(R)$ implies $\varphi(x,y,z) = (\lambda^{bc}x, \lambda^{ac}y, \lambda^{ab}z)$.

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(Corollary: $Aut_{\mathbb{C}}(R) \cong \mathbb{C}^*$ as groups.)

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Lemma: $H \in R$; so $F - \lambda u \in R$. Etcetera...

$F = u + r(x,y,z)y^n$,

$G = v + r(x,y,z)x^m$, for some $r \in R$.

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Concluding:

$\varphi \in Aut_R(A_{n,m})$ then
$\varphi(u, v) = (u + ry^n, v + rx^m)$ for some $r \in R$.
(Incidentally, $Aut_R(A_{n,m}) \cong < R, + >$.)

$R := \mathbb{C}[X, Y, Z]/(X^a + Y^b + Z^c)$, $A_{m,n} := R[U, V](x^m U - y^n V - 1)$, $A_{m,n}$ is UFD

$\varphi^{-1} E \varphi = \lambda E$ for all $\varphi \in Aut_{\mathbb{C}}(A_{n,m})$, where $E := y^n \partial_U + x^m \partial_V$.

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Now suppose $A_{m,n} \overset{\varphi}{\cong} A_{m',n'}$ where $\{m, n\} \neq \{m', n'\}$.

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$$
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| & & | \\
R & \xrightarrow{\varphi|_R} & R
\end{array}
$$

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May assume $\varphi|_R = Id_R$!

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Then: clubbing problem with algebra $\Longrightarrow$ contradiction!

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CONCLUDING: If $\{n, m\} \neq \{n', m'\}$ then $A_{n,m} \ncong A_{n',m'}$.

Central question:

How to distinguish two rings?

Some final considerations, and how to proceed in the future.

$$\begin{array}{cl} A & \\ | & \\ ML(A) & \quad \text{Invariant under } Aut_{\mathbb{C}}(A) \\ | & \\ \mathbb{C} & \end{array}$$

$$
\begin{array}{ll}
A & \\
| & \\
ML(A) & \text{Invariant under } Aut_{\mathbb{C}}(A) \\
| & \text{Invariant under } Exp(A) \\
\mathbb{C} &
\end{array}
$$

$$Exp(A) =< exp(D); D \in LND(A) >$$

$$\begin{array}{c} A \\ | \\ ?B? \quad \text{Invariant under } Exp(A) \\ | \\ \mathbb{C} \end{array}$$

$$A$$
$$|$$
$$?B? \quad \text{Invariant under } Exp(A)$$
$$|$$
$$\mathbb{C}$$

Lemma: B invariant under $Exp(A)$, then

A

|

$?B?$ Invariant under $Exp(A)$

|

$\mathbb{C}$

Lemma: B invariant under $Exp(A)$, then

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$$|$$
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Lemma: B invariant under $Exp(A)$, then

for all $D \in LND(A)$

for all $f \in B$

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Lemma: B invariant under $Exp(A)$, then

$$\left.\begin{array}{l} \text{for all } D \in LND(A) \\ \text{for all } f \in B \end{array}\right\} \Rightarrow D(f) \in B$$

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Many rings A do not have such an invariant subring (not $\mathbb{C}$ or A)! (Example: $\mathbb{C}[X, Y, Z, T]/(XY - ZT - 1)$.)

$$A$$

$$|$$

$?B?$ Invariant under $Exp(A)$

$$|$$

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How to distinguish such rings??

$$A$$

$$|$$

$$|$$

$$\mathbb{C}$$

(Very rough outline of) Program:

$$\begin{array}{c} A \\ | \\ \{B_i\}_{i \in I} \\ | \\ \mathbb{C} \end{array} \quad \text{Invariant under } Exp(A)$$

(Very rough outline of) Program:

Study sets of subrings invariant under $Aut_{\mathbb{C}}(A)$ or a suitable subgroup.(Compare Galois theory.)

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Many interesting questions!!

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(Very rough outline of) Program:

Study sets of subrings invariant under $Aut_{\mathbb{C}}(A)$ or a suitable subgroup.(Compare Galois theory.)

Many interesting questions!!

Possible goal: recognize of many ideals I when $\mathbb{C}^{[n]}/I \not\cong \mathbb{C}^{[m]}$.

Just one more thing to say:

THANK YOU