

A three dimensional UFD cancellation counterexample

Stefan Maubach

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Central question:
How to distinguish two rings?

Cancellation problems

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2006 (Finston/Maubach)

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Idea: take suitable rigid ring R , and

$A_{n,m} := R[U, V]/(x^m U - y^n V - 1)$ for some $x, y \in R$.

First something else...

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Let $f, g, h \in k[X]$ where k is an algebraically closed field.

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Then all f, g, h are constant.

Using Mason's in rings

Definition: $R := \mathbb{C}[X, Y, Z]/(X^a + Y^b + Z^c) = \mathbb{C}[x, y, z]$

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Corollary implies: $f, g, h \in \bar{K} \cap R = R^D$. So $LND(R) = \{0\}!!$

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$$0 = 1 + x^a + y^b + z^c = 1 + f^a + g^b + h^c$$

Implies (using Mason's) that f, g, h constant.

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Claim: $A_{n,m}[T] \cong A_{n',m'}[T]$ for all n, m, n', m'

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$x \in A_{m,n}$ is prime, $A_{m,n}[x^{-1}] = R[x^{-1}][V]$.

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$x \in A_{m,n}$ is prime, $A_{m,n}[x^{-1}] = R[x^{-1}][V]$. So $A_{m,n}$ UFD.

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Corollary: Let $\varphi \in \text{Aut}_{\mathbb{C}}(A_{n,m})$. Then $\varphi^{-1}E\varphi = \lambda E$ for some
 $\lambda \in \mathbb{C}[x, y, z]^* = \mathbb{C}^*$.

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Proposition: $\varphi \in \text{Aut}_{\mathbb{C}}(R)$ implies

$$\varphi(x, y, z) = (\lambda^{bc} x, \lambda^{ac} y, \lambda^{ab} z).$$

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(Corollary: $\text{Aut}_{\mathbb{C}}(R) \cong \mathbb{C}^*$ as groups.)

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Lemma: $H \in R$; so $F - \lambda u \in R$.

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 $\varphi^{-1}E\varphi = \lambda E$ for all $\varphi \in \text{Aut}_{\mathbb{C}}(A_{n,m})$, where $E := y^n \partial_U + x^m \partial_V$.
 $\text{Aut}_{\mathbb{C}}(A_{n,m}) = \text{Aut}_{\mathbb{C}}(R) \times \text{Aut}_R A_{n,m}$.

So let us take some $\varphi \in \text{Aut}_R(A_{n,m})$.

Define $\varphi(u, v) = (F, G)$.

So: $(y^n F_u + x^m F_v, y^n G_u + x^m G_v) = (\lambda y^n, \lambda x^m)$. Thus:

$$-y^n(F_u - \lambda) = x^m(F_v)$$

$$-y^n(G_u) = x^m(G_v - \lambda)$$

$$-y^n(H_u) = x^m(H_v)$$

Lemma: $H \in R$; so $F - \lambda u \in R$. Etcetera...

$$F = u + r(x, y, z)y^n,$$

$$G = v + r(x, y, z)x^m, \text{ for some } r \in R.$$

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Concluding:

$\varphi \in \text{Aut}_R(A_{n,m})$ then

$\varphi(u, v) = (u + ry^n, v + rx^m)$ for some $r \in R$.

(Incidentally, $\text{Aut}_R(A_{n,m}) \cong \langle R, + \rangle$.)

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Then: clubbing problem with algebra \implies contradiction!

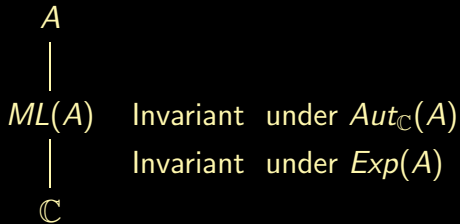
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CONCLUDING: If $\{n, m\} \neq \{n', m'\}$ then $A_{n,m} \not\cong A_{n',m'}$.

Central question:
How to distinguish two rings?

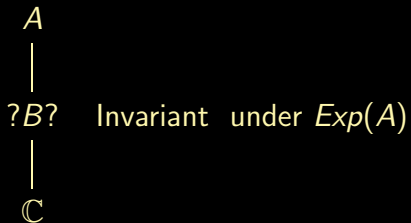
Some final considerations, and how to proceed in the future.

$$\begin{array}{c} A \\ | \\ ML(A) \\ | \\ C \end{array} \quad \text{Invariant under } Aut_C(A)$$

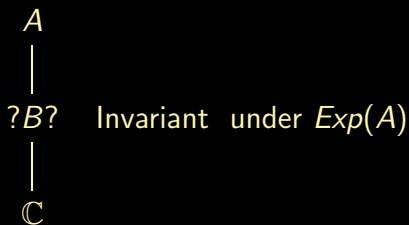


$$Exp(A) = \langle exp(D); D \in LND(A) \rangle$$

A
|
 $?B?$ Invariant under $Exp(A)$
|
 C

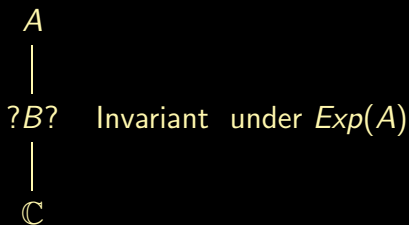


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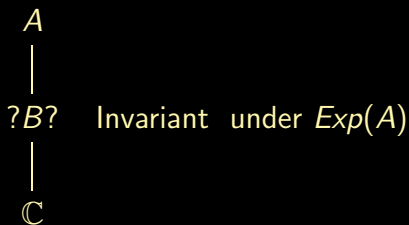
for all $D \in \text{LND}(A)$



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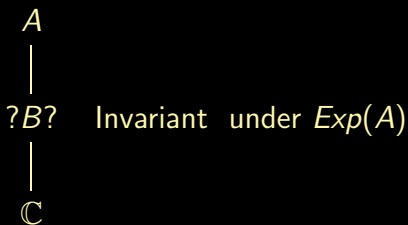
for all $D \in \text{LND}(A)$

for all $f \in B$



Lemma: B invariant under $\text{Exp}(A)$, then

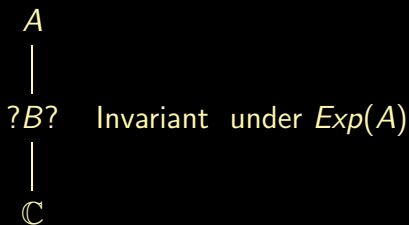
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How to distinguish such rings??

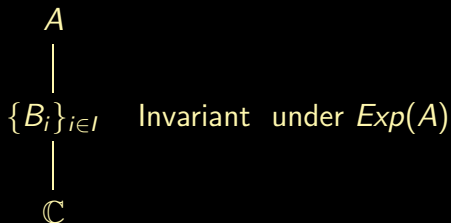
A

|

|

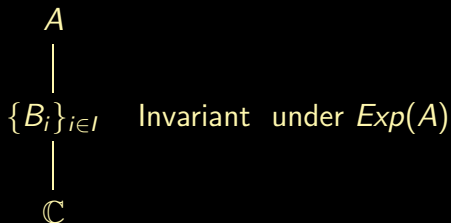
C

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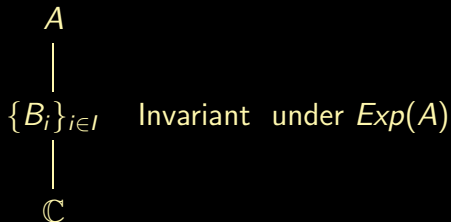
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Many interesting questions!!

Possible goal: recognize of many ideals I when $\mathbb{C}^{[n]}/I \not\cong \mathbb{C}^{[m]}$.

Just one more thing to say:

THANK YOU