The mysteries of Affine Algebraic Geometry

Stefan Maubach

September 2010

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$$\begin{array}{rcl} k^n & \leftrightarrow & k[X_1, \dots, X_n] \\ V & \leftrightarrow & \mathcal{O}(V) := k[X_1, \dots, X_n]/I(V) \end{array}$$

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Subtopic - but of *fundamental importance* to the whole of Algebraic geometry.

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Objects, hence morphisms!

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Objects, hence morphisms!

$$F : k^n \longrightarrow k^n$$

polynomial map if $F = (F_1, \ldots, F_n)$, $F_i \in k[X_1, \ldots, X_n]$.

Example: $F = (X + Y^2, Y)$ is polynomial map $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$.

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Any linear map is a polynomial map.

Understanding polynomial automorphisms

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A map $F : k^n \longrightarrow k^n$ given by n polynomials:

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Various ways of looking at polynomial maps:

- A map $k^n \longrightarrow k^n$.
- A list of n polynomials: $F \in (k[X_1, \ldots, X_n])^n$.
- A ring automorphism of $k[X_1, \ldots, X_n]$ sending $g(X_1, \ldots, X_n)$ to $g(F_1, \ldots, F_n)$.

Understanding polynomial automorphisms

A polynomial map F is a polynomial automorphism if there is a polynomial map G such that $F(G) = (X_1, \ldots, X_n)$.

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$$\begin{aligned}(X + Y^2, Y) \circ (X - Y^2, Y) &= ([X - Y^2] + [Y]^2, [Y]) \\ &= (X - Y^2 + Y^2, Y) \\ &= (X, Y).\end{aligned}$$

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$(X^3, Y) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is not a polynomial automorphism, even though it induces a bijection of $\mathbb{R}$!

Understanding polynomial automorphisms

Remark: If k is algebraically closed, $\mathrm{char}(k) = 0$, then a polynomial endomorphism $k^n \longrightarrow k^n$ which is a bijection, is an invertible polynomial map.

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Understanding polynomial automorphisms

Group of polynomial automorphisms with coefficients in a ring R is denoted by $\mathrm{GA}_n(R)$ (similarly to $\mathrm{GL}_n(R)$).

A topic is defined by its problems.

Many problems in AAG: inspired by linear algebra!
(In some sense: AAG most "natural generalization of linear algebra"...)

Problems in AAG: Jacobian Conjecture

$\mathrm{char}(k) = 0$

L linear map;

$L \in \mathrm{GL}_n(k)$ invertible $\iff$ $\det(L) = \det(\mathrm{Jac}(L)) \in k^*$

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Jacobian Conjecture:

$$F \in \mathrm{GA}_n(k) \mathit{invertible} \Longleftarrow \det(\mathrm{Jac}(F)) \in k^*$$

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History of the Jacobian Conjecture

J.C. was advertised by Abhyankar, Bass, and others


"Visual" version of Jacobian Conjecture

Volume-preserving polynomial maps are invertible.

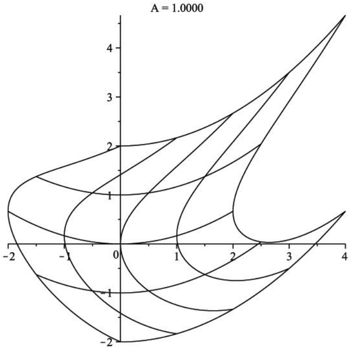

Figure: Image of raster under $(X + \frac{1}{2}Y^2, Y + \frac{1}{6}(X + \frac{1}{2}Y^2)^2)$.

Jacobian Conjecture very particular for *polynomials:*

$$F : (x, y) \longrightarrow (e^x, ye^{-x})$$

$$\mathrm{Jac}(F) = \begin{pmatrix} e^x & 0 \\ -ye^{-x} & e^{-x} \end{pmatrix}$$

$$\det(\mathrm{Jac}(F)) = 1$$

Jacobian Conjecture in $\text{char}(k) = p$:

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$$\begin{aligned} F: \quad k^1 &\longrightarrow k^1 \\ X &\longrightarrow X - X^p \end{aligned}$$

$Jac(F) = 1$ but $F(0) = F(1) = 0$.

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Jacobian Conjecture in char$(k) = p$**:** Suppose $\det(Jac(F)) = 1$ and $p \not| [k(X_1, \ldots, X_n) : k(F_1, \ldots, F_n)]$. Then F is an automorphism.

Jacobian Conjecture in $\text{char}(k) = p$:

$\text{char}(k) = 0$:

$F = (X + a_1X^2 + a_2XY + a_3Y^2, Y + b_1X^2 + b_2XY + b_3Y^2)$

$$\begin{aligned}
1 = \ & \det(Jac(F)) \\
= \ & 1+ \\
& (2a_1 + b_2)X+ \\
& (a_2 + 2b_3)Y+ \\
& (2a_1b_2 + 2a_2b_1)X^2+ \\
& (2b_2a_2 + 4a_1b_3 + 4a_3b_1)XY+ \\
& (2a_2b_3 + 2a_3b_2)Y^2
\end{aligned}$$

In char(k)=2 : (parts of) equations vanish. **Question:** What are the right equations in $\text{char}(k) = 2$? (or p?)

Enough about the Jacobian Problem! Another problem:

Cancellation problem

Cancellation problem: introduction

V, W vector spaces, if $V \times k \cong W \times k$ then $V \cong W$.

V vector space, then $V \times k \cong k^{n+1}$ implies $V \cong k^n$.

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V, W varieties, if $V \times k \cong W \times k$ then $V \cong W$?
Cancellation problem: V variety. $V \times k \cong k^{n+1}$, is $V \cong k^n$?

Cancellation $V \times k \cong W \times k$

counterexamples

1972(?): Hoechster: over $\mathbb{R}$

Cancellation $V \times k \cong W \times k$ counterexamples

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1986(?): Danielewski: $V : xz + y^2 + 1 = 0$, $W : x^2z + y^2 + 1$

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(Not a UFD)

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2008: Finston & M. : "Best" counterexamples so far (UFD, over $\mathbb{C}$, lowest possible dimension):

$$V_{n,m} := \{(x, y, z, u, v) \mid x^2 + y^3 + z^7 = 0, x^m u - y^n v - 1 = 0\}$$

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2010: better examples by Dubouloz/Moser/Poloni...

Cancellation $V \times k \cong W \times k$ counterexamples

Still looking for an example where $V = k^n$!

Abhyankar-Sataye conjecture/ coordinates

Denote $\mathbb{C}[X_1, \ldots, X_n]$ as $\mathbb{C}^{[n]}$.

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Denote $\mathbb{C}[X_1, \ldots, X_n]$ as $\mathbb{C}^{[n]}$. $f \in \mathbb{C}^{[n]}$ is called a *coordinate* if there exist $f_2, \ldots, f_n$ such that

$$\mathbb{C}[f, f_2, f_3, \ldots, f_n] = \mathbb{C}[X_1, \ldots, X_n]$$

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Or equivalently: $(f, f_2, \ldots, f_n)$ is a polynomial automorphism.

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Unnamed problem: How to recognise if $f \in \mathbb{C}^{[n]}$ is a coordinate? Is $x + xz^2 + zy^2$ a coordinate?

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AS(2) is true.

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Linearization Problem: Let $F^s = I$ some s. Is F linearizable?

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Proven for $n = 2$.

The Automorphism Group

(This whole talk: $n \geq 2$)

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- Map $(aX_1 + bX_j, X_2, \ldots, X_n)$ $(a \in k^*, b \in k)$

The Automorphism Group

(This whole talk: $n \geq 2$)

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$GA_n(k)$ is generated by ???

Elementary map: $(X_1 + f(X_2, \ldots, X_n), X_2, \ldots, X_n)$,
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$TA_n(k) := < J_n(k), Aff_n(k) >$

In dimension 1: we understand the automorphism group.

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In dimension 2: famous Jung-van der Kulk-theorem:

$$\mathrm{GA}_2(\mathbb{K}) = \mathrm{TA}_2(\mathbb{K}) = \mathit{Aff}_2(\mathbb{K}) \rtimes \mathrm{J}_2(\mathbb{K})$$

Jung-van der Kulk is the reason that we can do a lot in dimension 2 !

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AMS E.H. Moore Research Article Prize


Ivan Shestakov (center) and Ualbai Umirbaev (right) with Jim Arthur.

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Thus: N is tame over $k[z, z^{-1}]$, i.e. N in $\mathrm{TA}_2(k[z, z^{-1}])$.

Nagata proved: N is NOT tame over $k[z]$, i.e. N not in $\mathrm{TA}_2(k[z])$.

Stably tameness

N tame in one dimension higher:

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&(X, Y, Z, W - \tfrac{1}{2}\Delta)\circ \\
&(X - 2YW - ZW^2, Y + ZW, Z, W)\circ \\
&(X, Y, Z, W + \tfrac{1}{2}\Delta) \\
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\end{aligned}
$$

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iN is linearizable.

Theorem: (Maubach, Poloni, '09) sN is linearizable unless $s = 1, -1$.

(Part of a deeper theorem - on a Lie algebra...)

Over finite fields

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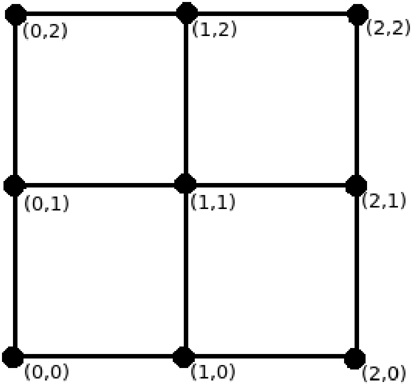

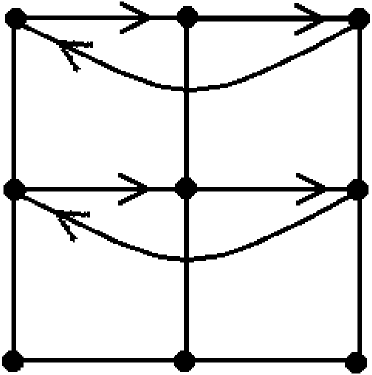

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Simpler question: what is $\pi_q(\mathsf{TA}_n(\mathbb{F}_q))$?

Why simpler? Because we have a set of generators!

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$\mathsf{TA}_n(\mathbb{F}_q) =< \mathsf{GL}_n(\mathbb{F}_q), \sigma_f >$ where f runs over $\mathbb{F}_q[X_2, \ldots, X_n]$ and $\sigma_f := (X_1 + f, X_2, \ldots, X_n)$.

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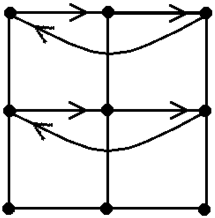
We make finite subset $\mathcal{S} \subset \mathbb{F}_q[X_2, \ldots, X_n]$ and define

$$\mathcal{G} :=< \mathsf{GL}_n(\mathbb{F}_q), \sigma_f \ ; \ f \in \mathcal{S} >$$

such that

$$\pi_q(\mathsf{TA}_n(\mathbb{F}_q)) = \pi_q(\mathcal{G}).$$



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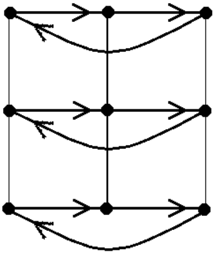
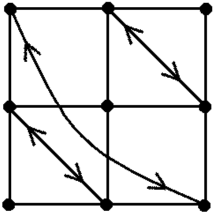
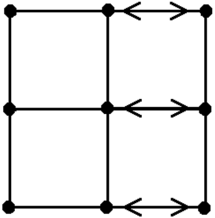


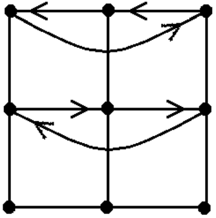
$F_3 = (x+1, y)$



$F_4 = (y, x)$



$F_5 = (2x, y)$



$F_2 = (x+y, y)$

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If $q = 4, 8, 16, \ldots$ we don't succeed to find a 2-cycle. In fact- all generators of $\mathrm{TA}_n(\mathbb{F}_q)$ turn out to be even, i.e. $\pi_q(\mathrm{TA}_n(\mathbb{F}_q)) \subseteq \mathrm{Alt}(q^n)$!

But: there's another theorem:

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Suppose $F \in \mathrm{GA}_n(\mathbb{F}_4)$ such that $\pi(F)$ odd permutation, then $\pi(F) \notin \pi(\mathrm{TA}_n(\mathbb{F}_4))$, so $\mathrm{GA}_n(\mathbb{F}_4) \neq \mathrm{TA}_n(\mathbb{F}_4)$!

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So: Start looking for an odd automorphism!!! (Or prove they don't exist)

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...drumroll... Nagata is EVEN if and only if $q = 4, 8, 16, \ldots$ and ODD otherwise... so far: no odd example found!

Different approach?

Is there perhaps a combinatorial reason why $\pi(\mathsf{GA}_n(\mathbb{F}_4)$ has only even permutations??

Losing less information: embedding $\mathbb{F}_q$ into $\mathbb{F}_{q^m}$.

$$\mathsf{GA}_n(\mathbb{F}_q) \subset \mathsf{GA}_n(\mathbb{F}_{q^m}) \xrightarrow{\pi_{q^m}} \mathsf{sym}(q^{mn}).$$

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However:

Mimicking Nagata's map:

Theorem: (M) [- general stuff -]

Corollary: For every extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$, there exists $T_m \in \mathrm{TA}_3(\mathbb{F}_{q^m})$ such that T_m "mimicks" N, i.e.

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Theorem states: for *practical* purposes, tame is almost always enough!

Nagata can be mimicked by a tame map for every $q = p^m$ - i.e. exists $F \in TA_3(\mathbb{F}_p)$ such that $\pi_q N = \pi_q F$.

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This almost works - a bit more wiggling necessary (And for the general case, even more work.)

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and indeed:

$$\exp(tD)(P) = P(X_1 + t, X_2, \ldots, X_n)$$

Additive group actions

D is a locally nilpotent derivation:

$D(fg) = fD(g) + D(f)g$, $D(f+g) = D(f) + D(g)$ (derivation)

For all f, there exists an m_f such that $D^{m_f}(f) = 0$. (locally nilpotent)

Example:

$$\begin{aligned} & \tfrac{\partial}{\partial t} P(X_1 + t, X_2, \ldots, X_n)|_{t=0} \\ = \; & \tfrac{\partial P}{\partial X_1}(X_1, X_2, \ldots, X_n) \end{aligned}$$

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$\delta(\Delta) = 0$ where $\Delta = XZ + Y^2$.

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Hence: $D := \Delta\delta$ is also an LND:

$D^3(X) = D^2(\Delta \cdot -2Y) = \Delta \cdot -2 \cdot D^2(Y) = \Delta \cdot -2 \cdot D(Z) = 0$

etc.

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Examine $t = 1$:

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Examine $t = 1$: Nagata's automorphism!

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THANK YOU

(for enduring 177 slides. . .)