The Nagata Automorphism is Shifted Linearizable

Stefan Maubach, Pierre-Marie Poloni

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Still open conjectures:

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Nagata's automorphism:

$N := (X - 2Y\Delta - Z\Delta^2, Y + Z\Delta, Z)$ where $\Delta = XZ + Y^2$.

In fact:

$N = \exp(\Delta\partial)$ where $\partial = -2Y\frac{\partial}{\partial X} + Z\frac{\partial}{\partial Y}$.

Let's define:

$N^\lambda = \exp(\lambda\Delta\partial)$ where $\lambda \in \mathbb{C}$.

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What is going on?

After generalizing, generalizing and generalizing it all came down to the following: two noncommuting locally finite derivations D, E forming a Lie algebra.

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$$\exp(\beta E)D = e^{\beta} D \exp(\beta E)$$

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Proof.

Use the well-known formulae

$$\exp(A)B\exp(-A) = \exp([A, -]) \circ B$$

where A, B are elements of a Lie algebra. Put in $A = \beta E, B = D$ you get

$$(\exp[\beta E, -]) \circ D = e^{\beta} D.$$

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Use lemma 1 to show that

$$\exp(\beta E) D^i = (e^{\beta})^i D^i \exp(\beta E).$$

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We are going to apply this to the situation that D is a homogeneous locally finite(nilpotent) derivation (like Nagata's derivation). We will make a semisimple derivation E using the fact that D is homogeneous.

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D homogeneous means: D(homogeneous) =homogeneous. D homogeneous then exists $k \in \mathbb{Z}$: $D(A_d) \subseteq A_{d+k}$. We say that D is *homogeneous of degree* k.

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Theorem: If $D \in \mathrm{LFD}_n(\mathbb{C})$ is homogeneous of degree $k \neq 0$ w.r.t. a monomial grading, then $\exp(D)$ is shifted linearizable.

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Proof.

Follows from Lemma and Corollary 2: $\exp(E)$ is a linear map: the diagonal map $(e^{w_1}X_1, \ldots, e^{w_n}X_n)$. □

Applying this to Nagata

Goal: find linear maps L for which LN is linearizable, and determine for which L LN is not linearizable. We will do this for some particular linear maps, that behave nice w.r.t. Nagata.

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For deg: $E := sX\partial_X + tY\partial_Y + (-s + 2t)Z\partial_Z$, D of degree $s + 3t$.

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$$\exp(E)\exp(D) = (aX, bY, cZ) \circ N$$

where $ac = b^2, abc \neq 0$.

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$$\exp(E)\exp(D) = (aX, bY, cZ) \circ N$$

where $ac = b^2, abc \neq 0$. Requirement $s = -3t$ translates to $bc = 1$.

Applying this to Nagata

$\partial = -2Y\partial_X + Z\partial_Y$, $\Delta = XZ + Y^2$, $D := \Delta\delta$.

D homogeneous w.r.t *deg*, then

$deg(X, Y, Z) = s(1, 0, -1) + t(0, 1, 2)$, $s, t \in \mathbb{C}$.

$E := sX\partial_X + tY\partial_Y + (-s + 2t)Z\partial_Z$, D of degree $s + 3t$.

If $s = -3t$, then D of degree 0.

We thus are considering

$$\exp(E)\exp(D) = (aX, bY, cZ) \circ N$$

Define $L_{(a,b,c)} := (aX, bY, cZ)$. where $ac = b^2, abc \neq 0$. As long as $bc \neq 1$ then we can linearize!

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$$L_{(a,b,c)} N$$

is linearizable to $L_{(a,b,c)}$, for applying the formula we get

$$N^{-\frac{bc}{1-bc}} (L_{(a,b,c)} N) N^{\frac{bc}{1-bc}}.$$

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What is the case that $L_{(a,b,c)} = sI = (sX, sY, sZ)$?

$ss = s^2, sss \neq 0, ss \neq 1$. Then

$$L_{(s,s,s)}N$$

is linearizable to $L_{(s,s,s)}$. Applying the formula we get

$$N^{-\frac{ss}{1-ss}}(L_{(s,s,s)}N)N^{\frac{ss}{1-ss}}.$$

$s \neq 0, s \neq 1, -1$. Then

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is linearizable to sI. Applying the formula we get

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We also give a (new) proof that if $bc = 1$ then one cannot linearize $L_{(b^3,b,b^{-1})}N$. So indeed. $N, -N$ not linearizable. And $2N, iN$ are linearizable.

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Well, is $\mathrm{TA}_n(k) \subset \mathrm{GLIN}_n(k)$? YES if $k \neq \mathbb{F}_2$. NO if $k = \mathbb{F}_2$.

$N \in \mathrm{GLIN}_n(k)$ except if $k = \mathbb{F}_2, \mathbb{F}_3$.

In case $k = \mathbb{F}_2, \mathbb{F}_3$ we don't know...

Meister's Linearization Problem:

For which $F \in \mathrm{GA}_n(\mathbb{C})$ does there exist some $s \in \mathbb{C}^*$ such that sF is linearizable?

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******* THANK YOU *******