Commuting Derivations

Stefan Maubach

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Notations:

	Linear	Polynomial
All	$ML_n(\mathbb{C})$	$MA_n(\mathbb{C})$
Invertible	$GL_n(\mathbb{C})$	$GA_n(\mathbb{C})$

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Why this bold claim?Polynomial maps seem to have similar properties as linear maps (much more so than holomorphic maps for example). Well. . . to be honest, most are conjectures. . . Let's look at a few of these conjectures!

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Jacobian Conjecture in dimension n (JC(n)):

Let $F \in MA_n(\mathbb{C})$. Then

$$\det(Jac(F)) \in \mathbb{C}^* \Rightarrow F \text{ is invertible.}$$

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Cancelation Problem:

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Elementary map: $(X_1 + f(X_2, \ldots, X_n), X_2, \ldots, X_n)$, invertible with inverse

$$(X_1 - f(X_2, \ldots, X_n), X_2, \ldots, X_n).$$

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Triangular map: $(X + f(Y, Z), Y + g(Z), Z + c)$

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$n = 3$:(Shestakov-Umirbaev, 2004)

Nagata's map not tame, i.e. $GA_3(\mathbb{C}) \neq TA_3(\mathbb{C})$

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$$\begin{aligned} exp(D) = & \ (exp(D)(X), exp(D)(Y), exp(D)(Z)) \\ = & \ (X + Y^2 + YZ + \tfrac{1}{6}Z^2, Y + Z, Z) \end{aligned}$$

Kernels of derivations

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If D has a slice, an element s such that $D(s) = 1$, (think of ∂_X) then $A = A^D[s]$. ($\mathbb{C}[X, Y, Z]^{\partial_X}[X] = \mathbb{C}[X, Y, Z]$.

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$$ML(A) := \bigcap_{D \in \mathrm{LND}(A)} A^D.$$

Notice:

$$\begin{aligned} ML(\mathbb{C}[X,Y,Z]) \subseteq\ & \mathbb{C}[X,Y,Z]^{\partial_X} \cap \mathbb{C}[X,Y,Z]^{\partial_Y} \cap \mathbb{C}[X,Y,Z]^{\partial_Z} \\ & \mathbb{C}[Y,Z] \cap \mathbb{C}[X,Z] \cap \mathbb{C}[X,Y] = \mathbb{C}. \end{aligned}$$

The Makar-Limanov invariant

Example: $A := \mathbb{C}[X, Y, Z]/(X^2Y - P(Z))$. $ML(A) = \mathbb{C}[X]$, hence A is not a polynomial ring.
Hence $X^2Y - P(Z) = 0$ is not isomorphic to $\mathbb{C}^3$.

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Strength of ML invariant comes through these "ML-techniques". Sometimes they work, sometimes they don't. But - ML technique is not omnipotent!

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Most rings have no locally nilpotent derivations.

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Weak Abhyankar-Sataye Conjecture: Let $A := k[X_1, \ldots, X_{n+1}]$, and let $f \in A$ be such that $k(f)[X_1, \ldots, X_n] \cong_{k(f)} k(f)[Y_1, \ldots, Y_{n-1}]$. Then f is a coordinate in A.

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$$\mathcal{A}_i := \cap_{j \neq i} A^{D_j}, \mathcal{P}_i = \{p_i \in A_i \mid D_i(p_i) \in k[f]\}.$$

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Proof.

(3) Elegant, but too long for a presentation. □

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Can we construct such E_i, given D_i, which are optimal in some way?

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$D_1, \ldots, D_n \in LND(A)$, commuting, linearly independent over A.

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Or, if this equality does not hold always, what type of rings A do have equality?

Final remark:

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Commuting derivations may be the key to distinguish polynomial rings from UFDs.

and of course. . .

THANK YOU

(for watching at 174 slides!)