What did I do those four years??

Stefan Maubach

June 2010

(slides adapted for website)

A different talk than you're used to.

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- Less explanation than I normally do (no definition of "polynomial map" etcetera!).

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- I will give you all my main results of the last four years, *no matter how much it hurts you to listen to them all !!* (So, if you stay, you must be a real friend. . .)
- Less explanation than I normally do (no definition of "polynomial map" etcetera!).
- I will talk about the things I did not achieve, and hence, I make myself a bit vulnerable. . .

History:

December 2005: You have a VENI, please come back to the Netherlands from Brownsville !

Maubach $\longleftarrow$ *very happy!*

July 2006: start of project.

Title: On the foundations of polynomial automorphisms and applications.

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1c. Summary of research proposal. Linear maps are well-known and thoroughly studied. Therefore, they are widely applied. Polynomial maps are, in some sense, the natural extension of linear maps. (...) However, contrary to linear maps, the usefulness of polynomial maps is still severely hampered by a lack of theoretical foundation, which prevents them from reaching their full potential. Some noteworthy applications may be cryptography and statistics. The main aim of this research proposal will be to address some of the important theoretical gaps: (...)

From the proposal:

Goals

(. . .) Make progress on the following:

1. (90%) Increase the theoretical foundation of polynomial maps (. . .).
2. Spot possible applications of polynomial endomorphisms in other fields of (applied) mathematics.
3. Spot the voids in theoretical knowledge, obstructing the application of polynomial endomorphisms in the above applications, and address these problems under part 1.

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Applications of polynomial maps??

Statistics: Idea: *"A stochast X is just like the variable X."*

Baby example: you receive data in $\mathbb{R}^2$:

(0,0), (1,1),(2,4),(3,9)

Statistics is often *linear*. However, transforming the data by a polynomial automorphism

$$(x, y) \longrightarrow (x, y - x^2)$$

yields much simpler data (unachievable by linear means):

(0,0),(1,0),(2,0),(3,0).

Is there a way to systematize this?

Applications: statistics

No results worth mentioning! Reason why:

- There already exists "polynomial regression", answering this partially.
- I could not interest statisticians to this problem: probably the problem is not directly applicable enough?
- Statistics is a HUGE, unoverseeable field of which I don't have enough knowledge. Easy to spend time on things that already exist !

Applications: cryptography

(Moh's idea:) public key cryptography: Make simple polynomial maps $F_1, F_2, \dots, F_n$ and compose

$$F := F_1 \circ F_2 \circ \cdots \circ F_n.$$

It is hard to find an inverse of F without finding $F_1, \dots, F_n$.
HOWEVER: (Michiel de Bondt & me:) It is easy to find preimages:

$$\text{given } \; F, q \text{ find } p \text{ such that } F(p) = q$$

(Does not kill Moh's idea completely)
So, in some sense: a **negative result !**

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I still believe in a good application of polynomial maps to *symmetric* key cryptography.

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I have another idea right now, and perhaps that works.

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Again: Cryptography is mostly an applied science. Finding applications is not always mathematics, making it somewhat hard to determine usefullness of one's results !

The other 90%; the results:

(*) (Joint with A. Crachiola), Rigid rings.

(*) (Joint with M. de Bondt), Computing preimages of polynomial maps .

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- Understanding $GA_n(\mathbb{C})$
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Locally finite polynomial maps

A linear map L satisfies a relation $L^n + a_{n-1}L^{n-1} + \ldots + a_1 L + a_0 I = 0$ ($a_i \in \mathbb{C}$); the Cayley-Hamilton theorem says so!

If L is linear, and

$$L^7 + 3L^5 + L - 5I = 0.$$

Is L invertible? Yes, since $5 \neq 0$! But - this works for polynomial maps just as well !

$$F^7 + 3F^5 + F - 5I = 0 \longrightarrow F \text{ invertible}$$

Locally finite polynomial maps

A polynomial map F is called *locally finite* if there *exists* such a relation: some $p \in \mathbb{N}, a_i \in \mathbb{C}$ such that

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I knew and know that this is a very fruitful object ! You can define this class in any category C where $\mathrm{End}(C)$ is an additive group having a scalar multiplication of a field.

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I knew and know that this is a very fruitful object ! You can define this class in any category C where $\text{End}(C)$ is an additive group having a scalar multiplication of a field. I still have to think of a counterexample to this:
$Aut(C)$ is generated by the locally finite *automorphisms*.

Locally finite polynomial maps: results

Theorem: If F is l.f. then $F = F_s \circ F_u = F_u \circ F_s$ where F_u is unipotent, and F_s is semisimple.

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Furthermore, $F_u = \exp(D_n)$ where D_n is a locally nilpotent derivation.

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Theorem: In dim. 2: F l.f. define

$$cc(F) := \{G^{-1}FG \mid G \text{ automorphism}\}.$$

Then $cc(F)$ is closed (in $End(\mathbb{C}^2)$) if and only if F is semisimple. (Compare with linear maps: $cc(L)$ closed if and only if L is semisimple.)

Locally finite polynomial maps: unwritten stuff

Question: if $F : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and l.f., what can F be?

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Theorem: $F : \mathbb{R} \longrightarrow \mathbb{R}$ differentiable, invertible. If $F' > 0$ then $F(x) = ax + b$ for some $a, b \in \mathbb{R}$. If $F' < 0$ then there are many different possibilities (like $f := g^{-1} \circ (-x) \circ g$ satisfies $f^2 - I = 0$).

Locally finite polynomial maps: weird stuff

Question (Jelonek): What if not

$$\sum_{i=0}^{d} a_i F^i = 0$$

for some $a_i \in \mathbb{C}$, but

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Answer: If $F(0) = 0$ and eigenvalues λ of linear part $|\lambda| < 1$ then: yeah, sure!

Corollary: Automorphism group generated by "power series zeros" and affine maps.

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Understanding $\mathrm{GA}_n(\mathbb{C})$

Famous automorphism: Nagata:

$$N := (X - 2Y(XZ + Y^2) - Z(XZ + Y^2)^2, Y + Z(XZ + Y^2), Z)$$

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But $\sqrt{-1}N$ *is* linearizable.

What magic is happening here????

generalizing, generalizing, and then. . .

Theorem: Let $D, E \in LFD(\mathbb{C})$ and suppose $[D, E] = \lambda D$, $\lambda \in \mathbb{C}^*$. If $\exp(E)$ is linear, then $\exp(E)\exp(bD)$ is linearizable as long as $\lambda \notin 2\pi i\mathbb{Z}$.

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Helps in finding explicit formula that work over any field:
Corollary: If $k \neq \mathbb{F}_2$ then there exists $L \in \mathrm{GL}_3(k)$ such that LN is linearizable.

Retrospective links:

Poincaré-Siegel: $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ holomorphic, $F(0) = 0$, then for almost all $\lambda \in \mathbb{C}^*$, λF is locally linearizable.

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Meister's conjecture (leadiing up to the solution to the Markus-Yamabe conjecture:) For which polynomial automorphisms F does there exist $\lambda \in C^*$ such that λF is linearizable?

Coordinates over $\mathbb{C}[z]$:

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Abhyankar-Sataye conjecture: If $f \in k[x_1, \ldots, x_n]$ such that $k[x_1, \ldots, x_n]/(f) \cong k[y_1, \ldots, y_{n-1}]$, then f is a coordinate.

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(2) $k[x, y, z]/(f) \cong k[x_1, x_2]$ and $f(x, y, a)$ is a coordinate for ONE $a \in \mathbb{C}$.

Corollary: If $f(x, y, z)$ is a coordinate which is also a $k(z)$-coordinate, then f is a $k[z]$-coordinate.

The special automorphism group of $R[t]/(t^m)[x_1, \ldots, x_n]$

Theorem: $\mathrm{SAut}_{R[t]} R[t][x_1, \ldots, x_n] \longrightarrow \mathrm{SAut}_{R_m} R_m[x_1, \ldots, x_n]$ is surjective.

Surprisingly: used by people in good papers! (Berson-Wright, Dubouloz-Moser)

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The topics of the VENI-proposal:

- Locally finite polynomial maps
- Understanding $\mathrm{GA}_n(\mathbb{C})$
- Recognising $\mathbb{C}^n$ through commuting derivations
- Polynomial maps over finite fields

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- Locally finite polynomial maps
- Understanding $\mathrm{GA}_n(\mathbb{C})$
- Distinguishing varieties from each other (and $\mathbb{C}^n$)
- Polynomial maps over finite fields

Cancellation counterexamples

"Best" generlized cancellation counterexamples (untill May 2010):

$$V_{n,m} := \{(x, y, z, u, v) \mid x^2 + y^3 + z^7 = 0, x^m u - y^n v - 1 = 0\}$$

$$V_{n,m} \times \mathbb{C} \cong V_{p,q} \times \mathbb{C}$$

but

$$V_{n,m} \ncong V_{p,q}$$

unless $(n, m) = (p, q)$.

Unipotent group actions

Unipotent group action of dim. n acting on X of dimension $n+1$:
Then $\mathcal{O}(X)^U = k[f]$ for some f, and if

$$\mathcal{O}(X)/(f-c) \cong \mathbb{C}[X_1, \ldots, X_n]$$

for all $c \in \mathbb{C}$, then $\mathcal{O}(X) \cong \mathbb{C}[X_1, \ldots, X_n, X_{n+1}]$.

Rigid rings

Rigid means: no additive group actions.

When is rigid:

$X^aY^b - Z^c$

$X^aY^b + Z^c + T^d$

$X^a + Y^b + Z^c + T^d$

etc.

Classifying maximal subrings - succeeding and still failing

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On-and-off worked on it for over a year.

Finally, paper done, with nice, elegant results! On a tuesday.

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Bummer... $\longrightarrow$ trashcan...

Infinitely generated invariants

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Question of Russell-Gurjar-Masuda-Miyanishi: can Derksen (and Makar-Limanov) invariant be infinitely generated?

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Question of Russell-Gurjar-Masuda-Miyanishi: can Derksen (and Makar-Limanov) invariant be infinitely generated?

Non-UFD example:

$$\mathbb{C}[a,b][x,y,z]/(a^3-b^2, z^2-(ax+by)^2-1)$$

UFD-example:

$$\mathbb{C}[X_1,\ldots,X_7]/(X_1^{d_1}+X_2^{d_2}+X_3^{d_3}+L_1^{d_4}+L_2^{d_5}+L_3^{d_6})$$

where the d_i are large (to use Mason's theorem!), and L_i chosen smartly (coming from a counterexample to Hilbert's 14th problem).

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Polynomial automorphisms over finite fields

Together with Roel Willems - I will not give spoilers for his final PhD presentation ! But: interesting results (yielded 1 paper, 1 more will follow).

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Conclusion

For 90%, I did exactly what I said I'd do!

Conclusion

For 90%, I did exactly what I said I'd do!

(And for 100%, I at least tried!)

Kudos to you for sitting through this, for now. . .

. . . come the pictures !

Working here for four years led me to meet nice, and sometimes important, mathematicians:


MARCEL
MARCEAU
QUINA
FRANCE
Wijnkeldertje
ondergronds gezellig

But, the most important mathematician I met

But, the most important mathematician I met is YOU !

and everyone else in the audience (and whom I couldn't find pictures of)!

THANK YOU

for being my colleague/student/friend these four years!

(and for watching 93 slides)