Polynomial automorphisms over finite fields and Locally Finite Polynomial Maps

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Notations:

	Linear	Polynomial
All	$ML_n(k)$	$MA_n(k)$
Invertible	$GL_n(k)$	$GA_n(k)$

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Why this bold claim? Polynomial maps seem to have similar properties as linear maps (much more so than holomorphic maps for example). Well. . . to be honest, most are conjectures. . . Let's look at a few of these conjectures!

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Jacobian Conjecture in dimension n (JC(n)): (char$(k) = 0$)
Let $F \in MA_n(k)$. Then

$$\det(Jac(F)) \in k^* \Rightarrow F \text{ is invertible.}$$

Let V be a vector space. Then

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$GA_n(k)$ is generated by ???

Elementary map: $(X_1 + f(X_2, \ldots, X_n), X_2, \ldots, X_n)$,
invertible with inverse

$$(X_1 - f(X_2, \ldots, X_n), X_2, \ldots, X_n).$$

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Triangular map: $(X + f(Y, Z), Y + g(Z), Z + c)$

$$= (X, Y, Z + c)(X, Y + g(Z), Z)(X + f(X, Y), Y, Z)$$

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$n = 3$:(Shestakov-Umirbaev, 2004)

If $\mathrm{char}(k) = 0$, then Nagata's map not tame, i.e.

$GA_3(k) \neq TA_3(k)$

There are many conjectures about other possible generating sets:

$\mathsf{GA}_n(k)$

$\mathsf{TA}_n(k)$

$\mathrm{GA}_n(k)$

$\cup|$

$\mathrm{ELND}_n(k) \quad := < \mathit{Aff}_n(k), \exp(D) \mid D$ locally nilpotent derivation

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$\mathrm{GLIN}_n(k)$ $:=$ normalizer of $\mathrm{GL_n(k)}$

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$\mathrm{TA}_n(k)$

$$
\begin{array}{ll}
\mathsf{GA}_n(k) & \\
\cup| & \\
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Simpler question: what is $\mathcal{E}(\mathrm{TA}_n(\mathbb{F}_q))$?

Why simpler? Because we have a set of generators!

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$T_n(\mathbb{F}_q)$ is generated by $\mathrm{GL_n}(\mathrm{\mathbb{F}_q})$ (for which we have a finite set of generators) and maps of the form

$$\sigma_f := (X_1 + f, X_2, \ldots, X_n)$$

where $f \in \mathbb{F}_q[X_2, \ldots, X_n]$.

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$$\sigma_\alpha := \sigma_{f_\alpha}.$$

which is a finite set.

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Hence, take the following set:

$$\sigma_\alpha := (X_1 + f_\alpha, X_2, \ldots, X_n)$$
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$$\tau := (aX_1, X_2, \ldots, X_n)$$

where $< a >= \mathbb{F}_q^*$.

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So: Start looking for an odd automorphism!!! (Or prove they don't exist)

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So far: we did not find an odd automorphism. Perhaps we didn't look hard enough! Perhaps all polynomial automorphisms are even - but why?

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Then study the bijection of $\mathbb{F}_9^3$ given by Nagata - is this bijection in the group $\mathcal{E}_9(\mathrm{TA}_3(\mathbb{F}_3))$? We put it all in the computer (joint work with R. Willems):... (drums)... unfortunately, yes $\mathcal{E}_9(N)$ is in $\mathcal{E}_9(\mathrm{TA}_3(\mathbb{F}_3))$. Also, $\mathcal{E}_{p^m}(N)$ is in $\mathcal{E}_{p^m}(\mathrm{TA}_e(\mathbb{F}_p)$ if $p = 2, m \leq 3$ or $p = 3, m \leq 2$. About as much as the computer can handle - we are doing computations in the symmetric group with 512! or 729! elements! (Next options would be 4096!, 19683! or 15625!...) (Also studied Anick's example for $p = m = 2, n = 4$.)

Another "characteristic 2" anomaly: compare

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$\mathrm{GLIN}_n(k) = \mathrm{GTAM}_n(k)$.

$\mathrm{char}(k) = 2$: is $\mathrm{GLIN}_2(k) \subsetneqq \mathrm{GTAM}_2(k)$?

Which maps of the form $(X + f(Y), Y)$ can we find in $\mathrm{GLIN}_2(\mathbb{F}_2)$?

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After some trial-and-error: $f(Y) \in \mathbb{F}_2[Y^2 + Y] + \mathbb{F}_2 Y + \mathbb{F}_2$.

Note, equivalent are:

- $f \in \mathbb{F}_2[Y^2 + Y]$,
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In particular - we couldn't make $(X + Y^3, Y)$. And indeed, using Jung-v/d Kulk: these are all maps of the form $(X + f(Y), Y)$ that we can make.

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YES! We can make all affine ones (not that hard).

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Now $(X + Y^i Z, Y, Z)(X, Y, Z + 1)(X + Y^i Z, Y, Z) =$
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So: $\mathrm{GTAM}_n(k) \subset \mathrm{GLIN}_{n+1}(k)$.

But - we run into other monomials that we cannot make:

$(X + YZ, Y, Z)$

We are looking for a useful *invariant* of $\mathrm{GLIN}_n(\mathbb{F}_2)$ which distinguishes it from $\mathrm{GTAM}_n(\mathbb{F}_2)$.

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Very good: the Cayley-Hamilton theorem (characteristic polynomials of linear maps etc.).

Now, let's try to make a Cayley-Hamilton theorem for polynomial maps! (Perhaps the constant term can replace that stupid $\det(Jac(F)) = 1$ requirement!)

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Let's be a little less ambitious and study this set. LFPE's should resemble linear maps more than general polynomial maps!

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Corollary: if R is a field, there is a unique minimum polynomial

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But: the minimum polynomial may change if G is not linear!

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Conjecture: in dimension n,
F is LFPE $\iff deg(F^m) \leq deg(F)^{n-1}$ for all $m \in \mathbb{N}$.

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then $\sum a_i F^i = 0 \iff \sum a_i F_{j,\alpha}^{(i)} = 0 \forall j, \alpha$.

How did we prove that?

If $F^i = (F_1^{(i)}, \ldots, F_n^{(i)})$ and $F_j^{(i)} = \sum F_{j,\alpha}^{(i)} X^\alpha$,
then $\sum a_i F^i = 0 \iff \sum a_i F_{j,\alpha}^{(i)} = 0 \forall j, \alpha$.
If $\{F_{j,\alpha}^{(i)}\}_{i \in \mathbb{N}}$ is such a sequence, then it is a linear recurrent sequence belonging to $\sum a_i T^i$, etc. . .

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Locally nilpotent $\Rightarrow$ Locally finite

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So: we can make many examples of LFPEs!

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?$\Leftarrow$?

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if $c \in \mathbb{C}$, then no natural choice $\log(c)$.

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So there's some funny stuff you might be able to read off $\mathfrak{m}_F$!

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***** THANK YOU *****

(for watching 263 slides. . .)