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An Introduction to Partial Differential Equations

Stefanie Sonner

Preface

These lecture notes are written for the course "An Introduction to Partial Differential Equations" (NWI-WB046B) at Radboud University, Nijmegen. They provide an introduction to the vast research field of partial differential equations. Further details and many additional topics can be found in the monographs by L. Evans [4], W. Craig [1], Y. Pinchover and J. Rubinstein [9], W.A. Strauss [10] and A. Vasy [12]. To follow the course a solid understanding of analysis, calculus, linear algebra and ordinary differential equations is required.

We introduce and analyze basic types of partial differential equations. Solution methods, representation formulas for solutions and properties of solutions for classical linear equations of second order (Laplace, heat and wave equation) are discussed. Moreover, we study nonlinear partial differential equations of first order via the method of characteristics. We are mainly concerned with the existence, uniqueness and regularity of solutions. This involves the use of fundamental solutions, maximum principles and energy methods.

Except for particularly simple cases, partial differential equations cannot be solved explicitly. In the analysis of partial differential equations, we are therefore mainly concerned with proving the well-posedness and investigating the qualitative behavior of solutions. Different from ordinary differential equations, there is no general theory for partial differential equations. Typically, each particular type of partial differential equation requires an individual theory and specific methods to study the existence and properties of solutions.

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Chapter 1

Introduction

Partial differential equations (PDEs) are used to model a wide range of phenomena, in particular, in physics, engineering, chemistry, biology and finance. For instance, they are fundamental in the modern understanding of sound, fluid dynamics, elasticity, general relativity and quantum mechanics. They also play an important role in "pure mathematics", in particular, in geometry and analysis.

1.1 Basic definitions

A PDE is an equation for an unknown function u of several variables that involves partial derivatives of u. The order of the highest partial derivative is called the order of the PDE.

Definition 1.1. Let $\Omega \subset \mathbb{R}^n$ be open, $n \geq 2$ and $k \in \mathbb{N}$. An expression of the form

$$F(D^k u(x), D^{k-1}u(x), \ldots, u(x), x) = 0, \qquad x \in \Omega, \tag{1.1}$$

is called a **k-th order PDE**, where

$$F\colon \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is a given function and $u\colon \Omega \to \mathbb{R}$ is the unknown.

A **classical solution** of the PDE is a k-times continuously differentiable function $u\colon \Omega \to \mathbb{R}$ that satisfies (1.1).

Here, we use the following notation to denote the partial derivatives. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be open, $x = (x_1, \ldots, x_n) \in \Omega$ and $u : \Omega \to \mathbb{R}$ be a scalar function.

- The *partial derivatives of u at x*, are defined as

$$\frac{\partial u}{\partial x_i}(x) := \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h} \qquad \text{(if the limit exists)},$$

  for $i = 1, \ldots, n$, where e_i denotes the i-th standard basis vector of $\mathbb{R}^n$. Commonly used are also the notations $\frac{\partial u}{\partial x_i} = \partial_{x_i} u = u_{x_i}$.

- The *partial derivatives of second order* are defined as

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) := \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}\right)(x) \qquad \text{(if they exist),}$$

  for $i, j = 1, \dots, n$. Commonly used are also the notations $\frac{\partial^2 u}{\partial x_i \partial x_j} = u_{x_i x_j} = \partial^2_{x_i x_j} u$.

- *Multiindex notation*: Let $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{N}_0^n$ be a multiindex. Its *order* is defined as

$$|\alpha| := \sum_{i=1}^{n} \alpha_i,$$

  and the corresponding $|\alpha|$*-th order partial derivatives* of u are

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u(x) \qquad \text{(if they exist).}$$

  Moreover, for $k \in \mathbb{N}$ we denote by

$$D^k u(x) := \{D^\alpha u(x) \,:\, |\alpha| = k\}$$

  the collection of all k-th order partial derivatives of u in x.

As usual, we write $D^1 u(x)$ as a column vector,

$$D^1 u(x) = Du(x) = \begin{pmatrix} \partial_{x_1} u(x) \\ \vdots \\ \partial_{x_n} u(x) \end{pmatrix} = \nabla u(x) \qquad (\textit{gradient}),$$

and $D^2 u(x)$ as a matrix,

$$D^2 u(x) = \begin{pmatrix} \partial^2_{x_1 x_1} u(x) & \dots & \partial^2_{x_1 x_n} u(x) \\ \vdots & \ddots & \vdots \\ \partial^2_{x_n x_1} u(x) & \dots & \partial^2_{x_n x_n} u(x) \end{pmatrix} \qquad (\textit{Hessian matrix}).$$

Depending on the structure of the function F in (1.1) we classify PDEs as follows.

Definition 1.2.

- The PDE (1.1) is **linear** if the function F is linear in u and its derivatives, i.e. if it is of the form

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u(x) + f(x) = 0,$$

  for given functions a_α and f. Moreover, if $f \equiv 0$, the PDE is called *homogeneous* and otherwise *inhomogeneous*.

- The PDE (1.1) is **semilinear** if it is linear in the highest order derivatives, i.e. if it is of the form

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u(x) + a_0(D^{k-1} u(x), \dots, u(x), x) = 0,$$

  for given functions a_α and a_0.

- The PDE (1.1) is **quasilinear** if it is of the form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u(x),\dots,u(x),x)D^\alpha u(x) + a_0(D^{k-1}u(x),\dots,u(x),x) = 0,$$

  for given functions a_α and a_0.

- The PDE (1.1) is **fully nonlinear** if F is a nonlinear function of the highest order derivatives $D^k u$.

For linear homogeneous equations the *superposition principle* holds, i.e. if u and v are both solutions of the PDE, then the same applies to $\alpha u + \beta v$, for all $\alpha,\beta \in \mathbb{R}$. More generally, if $u_1,\dots,u_m$ are solutions, then so is any linear combination of these solutions.

Typically, the difficulty of the analysis of a PDE increases with the degree of nonlinearity.

Instead of scalar equations we can also look at *systems of PDEs* which arise in many applications. Here, several unknown functions $u_1,\dots,u_m$, $m \geq 2$, have to be determined that satisfy a system of m PDEs.

Definition 1.3. An expression of the form (1.1) is called a **k-th order system of PDEs** if $m \geq 2$ and

$$F\colon \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times \dots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m,$$

where $u = (u_1,\dots,u_m)\colon \Omega \to \mathbb{R}^m$ is the unknown. Here, $D^\alpha u = (D^\alpha u_1,\dots,D^\alpha u_m)$ and $D^k u = \{D^\alpha u \,:\, |\alpha| \leq k\}$.

A **classical solution** of the system of PDEs is a k-times continuously differentiable function $u : \Omega \to \mathbb{R}^m$ that satisfies (1.1).

1.2 Examples

We briefly discuss several examples of PDEs that illustrate the variety of applications in different fields.

Minimal surface equation

Let $\Omega \subset \mathbb{R}^2$ be open and bounded and $u : \Omega \to \mathbb{R}$. Then, the surface area of the graph of u is given by

$$J(u) = \int_\Omega \sqrt{1 + |\nabla u(x)|^2}\, dx.$$

A classical problem in the *Calculus of Variations* is the *minimal surface problem*: Minimize $J(u)$ subject to prescribed boundary conditions. That is, among all functions u that satisfy $u = g$ on the boundary $\partial\Omega$, where g is given, find the function such that the surface area of its graph is minimal.

One can show that such a minimizer u satisfies the corresponding Euler–Lagrange equation

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \qquad \text{in } \Omega,$$

where $\cdot$ denotes the inner product in $\mathbb{R}^2$. This *minimal surface equation* is a quasilinear PDE of second order.

The minimal surface problem is also known as the *Plateau problem*, named after the Belgian physicist J. A. F. Plateau (1801 - 1883). He conducted experiments with soap films by dipping wire contours in a solution of soapy water.

Reaction-diffusion equations

Reaction-diffusion equations are widely used to model phenomena in chemistry, physics and biology. They describe the changes in space and time of concentrations of chemical substances or densities of populations.

Let $I \subset \mathbb{R}$ be an open interval, $U \subset \mathbb{R}^n$ be open and $\Omega = I \times U$. Moreover, $u : \Omega \to \mathbb{R}$ is a function of time $t \in I$ and the spatial position $x \in U$. A *reaction-diffusion equation* is of the form

$$\partial_t u = d\Delta u + f(u) \qquad \text{in } I \times U,$$

where $\Delta u = \Delta_x u = \sum_{i=1}^n u_{x_i x_i}$ denotes the *Laplace operator* or *Laplacian* with respect to x and $d > 0$ is the diffusion coefficient. The first term on the right hand side of the equation models the diffusion (particles or individuals move from regions with high concentrations to regions of low concentrations) and the given function $f : \mathbb{R} \to \mathbb{R}$ describes local reactions. The reaction-diffusion equation is a semilinear PDE of second order.

More generally, we can consider *reaction diffusion systems*,

$$\partial_t u = D\Delta u + f(u) \qquad \text{in } I \times U,$$

where $u = (u_1, \ldots, u_m)$, $D \in \mathbb{R}^{m\times m}$ is a diagonal matrix with positive coefficients and $f : \mathbb{R}^m \to \mathbb{R}^m$ a given function. Reaction diffusion systems are used to model, e.g. ecological invasions, the spread of epidemics, tumor growth or reactions between several different chemical substances.

Korteweg de Vries equation

Let $I \subset \mathbb{R}$ be an open interval, $U \subset \mathbb{R}$ be open and $\Omega = I \times U$. The *Korteweg de Vries equation*

$$\partial_t u(t, x) - u(t, x)u_x(t, x) + u_{xxx}(t, x) = 0, \qquad (t, x) \in I \times U,$$

describes shallow water waves in narrow channels and can predict the formation of *solitons*, i.e. wave packets that maintain its shape and travel with a constant speed. The Korteweg de Vries equation is a semilinear PDE of third order.

The history of the Korteweg de Vries equation goes back to observations and experiments by J. S. Russell in 1834. He discovered the phenomenon of solitons when observing a boat that was first drawn along a narrow channel and then suddenly stopped. The mass of water which the boat had put in motion accumulated and rolled forward, forming a rounded, well-defined heap. Russel followed this heap on his horse for several kilometers and noticed that it seemed to travel along the channel without changing its form or speed.

Navier–Stokes equations

Let $I \subset \mathbb{R}$ be an open interval. The *Navier–Stokes equations*

$$\begin{aligned} \partial_t u + (u \cdot \nabla)u &= \nu\Delta u - \nabla p + f \qquad \text{in } I \times \mathbb{R}^n, \\ \nabla \cdot u &= 0, \end{aligned}$$

describe the motion of an incompressible fluid in $\mathbb{R}^n$, where $\nu > 0$ is the *viscosity* of the fluid and $f : \mathbb{R}^n \to \mathbb{R}^n$ the *external force*. The fluid is described by its *velocity field* $u : I \times \mathbb{R}^n \to \mathbb{R}^n$ and *pressure* $p : I \times \mathbb{R}^n \to \mathbb{R}$. The Navier–Stokes equations are a system of semilinear PDEs of second order.

They play an important role in physical and engineering applications. They are used to model, e.g. the weather, ocean currents, blood flow in arteries and air flow around a wing, and enormous computational efforts are invested to solve them numerically.

They are also of great mathematical interest and their analysis is challenging. For the system in $\mathbb{R}^3$ (and $f = 0$) the global existence of smooth solutions is still an open problem. It is one of the seven *Millennium Prize Problems* that were stated by the Clay Mathematics Institute in 2000. For a correct solution to any of the problems an award of one million US dollars is offered.

1.3 Type classification of linear second order PDEs

In this course, we mainly focus on linear, scalar PDEs of second order, i.e. equations of the form

$$\sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j}(x) + \sum_{i=1}^{n} a_i(x)u_{x_i}(x) + a_0(x)u(x) = f(x), \qquad x \in \Omega, \tag{1.2}$$

that we now further classify. By Schwarz' theorem, the Hessian matrix is symmetric if u is twice continuously differentiable and hence, we may assume that

$$a_{ij} = a_{ji} \qquad \forall i, j = 1, \ldots, n.$$

Then, the coefficients a_{ij} form a symmetric matrix

$$A(x) = \begin{pmatrix} a_{11}(x) & \ldots & a_{1n}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \ldots & a_{nn}(x) \end{pmatrix}, \qquad x \in \Omega.$$

A useful type classification of the PDE (1.2) is based on the definiteness properties of A.

Definition 1.4. We call the linear second order PDE (1.2) **elliptic** if $A(x)$ is positive or negative definite, **parabolic** if $A(x)$ is singular ($\det A(x) = 0$) and **hyperbolic** if one eigenvalue of $A(x)$ has a different sign than all the others (where eigenvalues are counted according to their multiplicity).

The following three examples are the archetypes of linear second order PDEs. We will study them in detail in the following chapters. Each equation requires a different approach and has essentially different properties.

Example 1.5.

- **Laplace equation**

$$\Delta u = u_{x_1 x_1} + \ldots + u_{x_n x_n} = 0 \qquad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^n$ is open, $u : \Omega \to \mathbb{R}$ and Δ is the *Laplace operator* or *Laplacian*.

We have $A(x) = \mathrm{Id} \in \mathbb{R}^{n \times n}$ and thus, the PDE is elliptic.

- **Heat equation**

$$u_t - \Delta u = 0 \qquad \text{in } \Omega = I \times U,$$

where $t \in I$ denotes time, $x \in U$ space, $I \subset \mathbb{R}$ is an open interval and $U \subset \mathbb{R}^n$ is open. Moreover, $u : I \times U \to \mathbb{R}$ and $\Delta u = \Delta_x u$ is the Laplace operator with respect to x.

We obtain a singular matrix $A(t, x) = \begin{pmatrix} 0 & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}$, $\mathrm{Id} \in \mathbb{R}^{n \times n}$, and thus, the PDE is parabolic.

- **Wave equation**

$$u_{tt} - \Delta u = 0 \qquad \text{in } \Omega = I \times U,$$

where we use the same notation as for the heat equation.

In this case, we have $A(t, x) = \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{Id} \end{pmatrix}$, and thus, the PDE hyperbolic.

1.4 Strategies for studying PDEs

A *classical solution* of a k-th order PDE is a k-times continuously differentiable function that satisfies the PDE pointwise in $\Omega \subset \mathbb{R}^n$. Often, a PDE possesses families of solutions, but the solution u is uniquely determined if values of u and/or its derivatives are specified on the boundary $\partial\Omega$ of Ω. A PDE together with these *boundary conditions* is called a *boundary-value problem*. In applications that involve time we typically consider sets if the form $\Omega = I \times U$, $I = (t_0, t_1) \subset \mathbb{R}$, $U \subset \mathbb{R}^n$ open. In this special case, the values of u and/or its derivatives specified at the initial time t_0 are called *initial conditions* and the values specified on ∂U *boundary conditions*.

In the ideal case, we find explicit solutions for a given PDE, but this is only possible in few particularly simple cases. This classical approach to PDEs that dominated the 19th century was to develop methods for deriving explicit representation formulas for solutions. If such formulas cannot be found, we aim at proving the existence and studying qualitative properties of solutions. In particular, we say that a problem is **well-posed** if the following properties hold:

- There exists a solution.
- The solution is unique.
- The solution depends continuously on the given data (e.g. parameters, boundary or initial values).

The continuous dependence on data is particularly important in applications, since the solution should change only slightly if we vary the data specifying the problem only slightly.

For many PDEs the notion of classical solutions is too restrictive and such solutions do not exist. However, one can weaken the concept of solutions and consider so-called *weak solutions* or *distributional solutions* which are less regular and satisfy the PDE in a generalized sense. For instance, PDEs describing the occurrence of shocks (essentially, the appearance of discontinuities in the derivatives), require this notion. Moreover, even if classical solutions exist, it is often easier to prove the existence of weak solutions first and then to show that the solutions have a higher regularity and are, in fact, classical solutions of the problem.

Different from ordinary differential equations there is no general theory or approach for the solvability of PDEs, except for very few specific cases. Typically, research in PDEs focuses on various, particular PDEs that are relevant in applications and on the development of specific methods for the problem at hand.

In general, the difficulty of the analysis of a PDE increases with the degree of nonlinearity, with the order k of the PDE, with the number of variables n and with the number of equations m (i.e. systems of PDEs are typically more difficult to analyze than scalar equations).

In this course we mainly focus on simple prototypes for linear second order PDEs (Laplace, Poisson, heat and wave equation) and on nonlinear PDEs of first order. Typical questions we address are the following:

- existence and uniqueness of solutions
- qualitative properties of solutions (e.g. regularity, dependence on data)
- explicit representation formulas for solutions
- limitations of classical solutions

1.5 Further notation

We denote the inner product in $\mathbb{R}^n$ by $\cdot$, the norm by $|\cdot|$ and b^T and A^T denote the transpose of a vector $b \in \mathbb{R}^n$ or a matrix $A \in \mathbb{R}^{n\times m}$. Moreover, we denote the open ball with center $x \in \mathbb{R}^n$ and radius $r > 0$ by $B_r(x) = \{y \in \mathbb{R}^n \,:\, |x - y| < r\}$.

When we write $\Omega \subset \mathbb{R}^n$, then $\Omega = \mathbb{R}^n$ or $\Omega \subsetneq \mathbb{R}^n$. For $\Omega \subset \mathbb{R}^n$, we denote by $\overline{\Omega}$ its closure and by $\partial\Omega$ its boundary. We introduce the following spaces of continuous functions on Ω

$$\begin{aligned} C(\Omega) &= \{u\colon \Omega \to \mathbb{R} \,:\, u \text{ continuous}\}, \\ C(\overline{\Omega}) &= \{u \in C(\Omega) \,:\, u \text{ can be continuously extended to } \partial\Omega\}. \end{aligned}$$

Analogously, the spaces $C(\Omega; \mathbb{R}^m)$ and $C(\overline{\Omega}; \mathbb{R}^m)$, $m \geq 2$, are defined for vector-valued functions $u : \Omega \to \mathbb{R}^m$.

Let now $\Omega \subset \mathbb{R}^n$ be open. For $k \in \mathbb{N}$ we denote the space of k-times continuously differentiable functions by

$$\begin{aligned} C^k(\Omega) &= \{u\colon \Omega \to \mathbb{R} \,:\, u \text{ is } k\text{-times continuously differentiable}\}, \\ C^k(\overline{\Omega}) &= \{u \in C^k(\Omega) \,:\, D^\alpha u \text{ can be continuously extended to } \partial\Omega \text{ for } |\alpha| \leq k\}. \end{aligned}$$

Analogously, we define the spaces $C^k(\Omega; \mathbb{R}^m)$ and $C^k(\overline{\Omega}; \mathbb{R}^m)$, $m \geq 2$, for vector-valued functions $u : \Omega \to \mathbb{R}^m$.

1.6 Exercises

E1.1 **Classification of PDEs**

Determine the order and type (linear, semilinear, quasilinear, fully nonlinear) of each of the following PDEs:

- Klein–Gordon equation

$$-u_{tt} + \Delta u = m^2 u \qquad \text{in } (0,\infty) \times \mathbb{R}^n,\ m > 0$$

- Burger's equation

$$u_t + u u_x = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}$$

- Monge–Ampére equation

$$\det(D^2 u) = 0 \qquad \text{in } \mathbb{R}^n$$

- Airy's equation

$$u_t + u_{xxx} = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}$$

- Eikonal equation

$$|Du| = 1 \qquad \text{in } \mathbb{R}^n$$

- Porous medium equation

$$u_t - \Delta(u^m) = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}^n,\ m > 1$$

Here, $t > 0$ denotes time, $x \in \mathbb{R}^n$ space, Δ is the Laplacian w.r.t. x and ∇ the gradient w.r.t x.

E1.2 **Minimal Surface Equation**

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary $\partial\Omega$. For $v \in C^1(\overline{\Omega})$ the n-dimensional surface area of its graph $\{(x, v(x)) : x \in \overline{\Omega}\} \subset \mathbb{R}^{n+1}$ is given by

$$J(v) = \int_\Omega \sqrt{1 + |\nabla v(x)|^2}\, dx.$$

Moreover, let $g : \partial\Omega \to \mathbb{R}$ be a given continuous function and suppose that a minimizer u of the functional J exists within the set

$$\{v \ :\ v \in C^1(\overline{\Omega}),\ v = g \text{ on } \partial\Omega\}$$

and it satisfies $u \in C^2(\overline{\Omega})$. Prove that this minimizer u satisfies

$$\int_\Omega \nabla \cdot \left(\frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) \varphi(x)\, dx = 0$$

for all functions $\varphi \in C^\infty(\Omega)$ with compact support in Ω.

<u>Remark:</u> One can then conclude by the so-called *Fundamental Lemma of the Calculus of Variations* that u is a solution of the *minimal surface equation*

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) \quad = \quad 0 \qquad \text{in } \Omega.$$

<u>Hint:</u> Assuming that such a minimizer u exists consider the family of functions $u + t\varphi$, $t \in \mathbb{R}$, *for arbitrary* $\varphi \in C^\infty(\Omega)$ *with compact support. Which condition satisfies* $B(t) := J(u + t\varphi)$*?*

E1.3 **D'Alembert's formula**

Consider the one-dimensional wave equation

$$u_{tt} - u_{xx} = 0 \qquad \text{in } (0, \infty) \times \mathbb{R}. \tag{1.3}$$

(a) Show that for arbitrary functions $\phi, \psi \in C^2(\mathbb{R})$, the function

$$u(t, x) = \phi(x - t) + \psi(x + t)$$

is a solution of (1.3).

(b) In addition, let the solution satisfy the following *initial conditions*

$$\begin{aligned} u(0, x) &= f(x) \\ u_t(0, x) &= g(x) \end{aligned} \qquad x \in \mathbb{R}, \tag{1.4}$$

where $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$ are given. Use the ansatz in (a) to show that the solution of the problem is given by *D'Alembert's formula*

$$u(t, x) = \frac{1}{2}\Big(f(x + t) + f(x - t) + \int_{x-t}^{x+t} g(y)dy\Big).$$

Chapter 2

The Transport Equation

2.1 Motivation

Assume a chemical is dissolved in a fluid and flows at a constant velocity $c > 0$ along a horizontal thin pipe of fixed cross section in the positive x-direction. Let $u(t, x)$ denote the concentration of the substrate at time $t \geq 0$ and position $x \in \mathbb{R}$. The total amount of the chemical in the interval $[a, z] \subset \mathbb{R}$ is $M = \int_a^z u(t, x)dx$. At a later time $t + h$, the molecules have moved to the right by ch and therefore,

$$M = \int_a^z u(t, x)dx = \int_{a+ch}^{z+ch} u(t + h, x)dx.$$

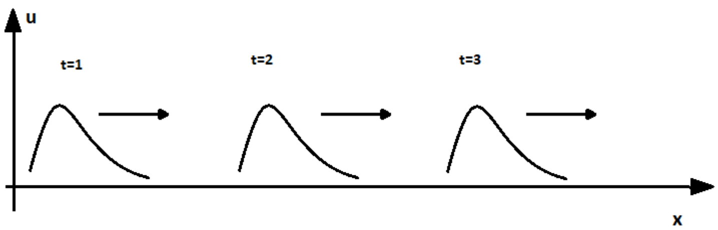
Assuming that u is smooth, then differentiating with respect to z we obtain

$$u(t, z) = u(t + h, z + ch).$$

Finally, differentiating with respect to h and setting $h = 0$, it follows that

$$0 = u_t(t, z) + cu_z(t, z),$$

which is a one-dimensinal *linear transport equation with constant coefficients*.



More generally, let $\Omega = (0, \infty) \times \mathbb{R}^3$, $t > 0$ denote time and $x \in \mathbb{R}^3$ the spatial position. In fluid dynamics, the *continuity equation* expresses the law of mass conservation. It is of the form

$$\rho_t + \nabla \cdot (\rho v) = 0 \qquad (0, \infty) \times \mathbb{R}^3,$$

where $\rho : \Omega \to \mathbb{R}^3$ denotes the density of the fluid, $v : \Omega \to \mathbb{R}^3$ the velocity field and ∇ the gradient with respect to x.

If we assume that the velocity of the fluid is given and constant $v \equiv \hat{v} \in \mathbb{R}^3$, the density ρ satisfies the linear transport equation with constant coefficients

$$\rho_t + \hat{v} \cdot \nabla \rho = 0.$$

It is one of the simplest PDEs and can be solved explicitly.

2.2 The homogeneous case

We consider the *linear transport equation with constant coefficients*,

$$u_t(t,x) + \sum_{i=1}^{n} b_i u_{x_i}(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^n,$$

where $b = (b_1, \dots, b_n)^T \in \mathbb{R}^n$ is a given, fixed vector. Typically, $x \in \mathbb{R}^n$ denotes a point in space and $t > 0$ the time. In compact notation, the equation can be written as

$$u_t + b \cdot \nabla u = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}^n, \tag{2.1}$$

where ∇u is the gradient of u with respect to $x \in \mathbb{R}^n$.

Note that if u is a classical solution of (2.1), then the left hand side of (2.1) is the directional derivative of u in the direction $\begin{pmatrix} 1 \\ b \end{pmatrix}$, and this directional derivative vanishes. In fact, for an arbitrary point $(t,x) \in (0,\infty) \times \mathbb{R}^n$ we define

$$z(s) := u(t+s, x+sb), \qquad s > -t.$$

The chain rule then implies that

$$\frac{d}{ds} z(s) = \frac{d}{ds} u(t+s, x+sb) = u_t(t+s, x+sb) + \nabla u(t+s, x+sb) \cdot b = 0,$$

where we used (2.1) in the last step. Hence,

$$z(s) = u(t+s, x+sb) \equiv \text{const.} \qquad \forall s > -t, \tag{2.2}$$

i.e. the value $u(t,x)$ is transported along the line

$$s \mapsto \begin{pmatrix} t \\ x \end{pmatrix} + s \begin{pmatrix} 1 \\ b \end{pmatrix}.$$

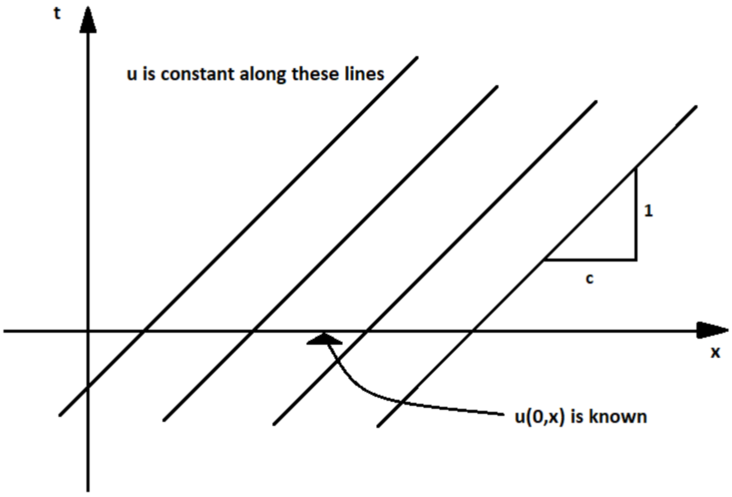
Thus, if $u \in C^1((0,\infty) \times \mathbb{R}^n) \cap C(\overline{(0,\infty) \times \mathbb{R}^n})$ satisfies in addition to (2.1) the *initial condition*

$$u(0,x) = g(x), \qquad x \in \mathbb{R}^n, \tag{2.3}$$

for a given function $g \in C^1(\mathbb{R}^n)$, then by (2.2) we have

$$u(t,x) = u(0, x - tb) = g(x - tb), \qquad t > 0, x \in \mathbb{R}^n. \tag{2.4}$$

The PDE (2.1) together with (2.3) is called an *initial value problem*.



Theorem 2.1. *Consider the linear transport equation with constant coefficients* (2.1)*. Then the following holds:*

(i) If u is a classical solution of (2.1)*, then*

$$u(t+s, x+sb) \equiv const., \qquad s > -t,$$

for all $(t,x) \in (0,\infty) \times \mathbb{R}^n$.

(ii) Let $g \in C^1(\mathbb{R}^n)$ be given. Then the initial value problem (2.1)*,* (2.3) *has a unique solution $u \in C^1((0,\infty) \times \mathbb{R}^n) \cap C(\overline{(0,\infty) \times \mathbb{R}^n})$, which is given by*

$$u(t,x) = u(0, x-tb) = g(x-tb),$$

for all $(t,x) \in (0,\infty) \times \mathbb{R}^n$.

Proof. (i) was already shown.

(ii): Uniqueness: If u is a classical solution, it satisfies (2.4), and this determines u uniquely. Existence: If g is $C^1(\mathbb{R}^n)$, then $u \in C^1((0,\infty) \times \mathbb{R}^n) \cap C(\overline{(0,\infty) \times \mathbb{R}^n})$. Moreover,

$$u_t(t,x) = \nabla g(x-tb) \cdot (-b),$$
$$\nabla u(t,x) = \nabla g(x-tb),$$

and hence, $u_t + b \cdot \nabla u = 0$. □

Remark 2.2. We were looking for classical solutions of the transport equation, and hence, by (2.4) we need to require that $g \in C^1(\mathbb{R}^n)$. If g is not of class $C^1(\mathbb{R}^n)$, a classical solution does not exist. However, one could still use the formula (2.4) to define a solution which satisfies the PDE *in a weak sense*. We will come back to the concept of *weak solutions* later.

2.3 The inhomogeneous case

More generally, we consider the inhomogeneous initial value problem

$$\begin{aligned} u_t + b \cdot \nabla u &= f \qquad \text{in } (0,\infty)\times\mathbb{R}^n, \\ u(0,\cdot) &= g \qquad \text{on } \mathbb{R}^n, \end{aligned} \tag{2.5}$$

where $g \in C^1(\mathbb{R}^n)$ and $f \in C^1([0,\infty)\times\mathbb{R}^n)$ are given. As before, the left-hand side of the PDE is the directional derivative of u in the direction $\begin{pmatrix}1\\ b\end{pmatrix}$. Hence, for an arbitrary point $(t,x) \in (0,\infty)\times\mathbb{R}^n$ the function

$$z(s) = u(t+s, x+sb), \qquad s > -t,$$

now satisfies

$$\frac{d}{ds}z(s) = u_t(t+s,x+sb) + \nabla u(t+s,x+sb)\cdot b = f(t+s,x+sb),$$

where the last equality holds by (2.5). Integrating the equation from $-t$ to 0 and using the initial condition yields

$$\begin{aligned} u(t,x) - g(x-tb) = z(0) - z(-t) &= \int_{-t}^{0} \frac{d}{ds}z(s)\,ds \\ &= \int_{-t}^{0} f(t+s,x+sb)\,ds = \int_0^t f(r, x+(r-t)b)\,dr. \end{aligned}$$

This yields the following result.

Theorem 2.3. *Let $b \in \mathbb{R}^n$, $f \in C^1([0,\infty)\times\mathbb{R}^n)$ and $g \in C^1(\mathbb{R}^n)$. Then the initial value problem (2.5) has a unique classical solution $u \in C^1((0,\infty)\times\mathbb{R}^n) \cap C(\overline{(0,\infty)\times\mathbb{R}^n})$, which is given by*

$$u(t,x) = g(x-tb) + \int_0^t f(s,x+(s-t)b)\,ds, \quad (t,x)\in(0,\infty)\times\mathbb{R}^n. \tag{2.6}$$

Proof. Uniqueness: We have shown that any classical solution u satisfies (2.6), and this determines u uniquely.
Existence: By assumption, $f \in C^1((0,\infty)\times\mathbb{R}^n)$ and $g \in C^1(\mathbb{R}^n)$, which implies that u defined by (2.6) is in the class $C^1((0,\infty)\times\mathbb{R}^n)\cap C(\overline{(0,\infty)\times\mathbb{R}^n})$. Moreover, u satisfies the initial condition and we have

$$\begin{aligned} u_t(t,x) &= \nabla g(x-tb)\cdot(-b) + f(t,x) - \int_0^t b\cdot\nabla_x f(s,x+(s-t)b)ds, \\ \nabla u(t,x) &= \nabla g(x-tb) + \int_0^t \nabla_x f(s,x+(s-t)b)ds. \end{aligned}$$

Therefore, u is a solution of the initial value problem (2.5). □

Note that in the proof we used the *Leibniz rule*: Assume that the functions $a: I\to\mathbb{R}$, $b: I\to\mathbb{R}$ and $f: I\times\mathbb{R}\to\mathbb{R}$ are continuously differentiable. Then,

$$\frac{\partial}{\partial t}\int_{a(t)}^{b(t)} f(t,s)ds = f(t,b(t))b'(t) - f(t,a(t))a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(t,s)ds.$$

Remark 2.4.
- We derived a solution formula for the transport equation by converting it into a family of ordinary differential equations. This technique to solve first order PDEs is called the *method of characteristics* and will be further discussed later in a more general context.
- To obtain classical solutions we require that f and g are continuously differentiable. However, the solution formula (2.6) also makes sense for non-differentiable (or even discontinuous) functions f and g, which would lead to *weak solutions*. The concept of weak solutions is, in fact, essential for a satisfying theory for PDEs. We will come back to it later.

2.4 Exercises

E2.1 **Transport equation**

Let $c \in \mathbb{R}$, $b \in \mathbb{R}^n$ be constant and $g \in C^1(\mathbb{R}^n)$ be given. Write down an explicit formula for a solution u of the initial value problem

$$\begin{aligned} u_t + b \cdot \nabla u + cu &= 0 \qquad \text{in } (0,\infty) \times \mathbb{R}^n, \\ u(0,\cdot) &= g \qquad \text{on } \{t = 0\} \times \mathbb{R}^n. \end{aligned}$$

<u>*Hint:*</u> *As in the lecture notes, transform the PDE into an ordinary differential equation and solve this equation with the given initial condition.*

Chapter 3

The Laplace and Poisson Equation

3.1 Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be open. In this chapter we consider the *Laplace equation*

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0 \qquad \text{in } \Omega, \tag{3.1}$$

and the *Poisson equation*

$$-\Delta u = f \qquad \text{in } \Omega, \tag{3.2}$$

where $f\colon \Omega \to \mathbb{R}$ is a given function and $u\colon \Omega \to \mathbb{R}$ is the unknown. They have many applications and typically model steady state phenomena.

Definition 3.1. Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C^2(\Omega)$. If u satisfies the Laplace equation (3.1), then u is called **harmonic** on Ω.

Moreover, u is called **subharmonic** if $-\Delta u \le 0$ on Ω, and **superharmonic** if $-\Delta u \ge 0$ on Ω.

Example 3.2. The real and imaginary part of an analytic function are harmonic.

Indeed, let the function $f : \Omega \to \mathbb{C}$, where $\Omega \subset \mathbb{C}$ is open, be analytic. Then, the real- and imaginary part of f,

$$u(x, y) = \mathrm{Re}(f(x + iy)), \qquad v(x, y) = \mathrm{Im}(f(x + iy)),$$

considered as functions $u, v : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$, are $C^\infty(\Omega)$. Moreover, they satisfy the Cauchy-Riemann differential equations

$$u_x = v_y, \qquad u_y = -v_x.$$

Thus, differentiating these equations, we have

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0 \qquad \text{and} \qquad v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0,$$

which shows that u and v satisfy (3.1).

Before we analyze the Laplace and Poisson equation we recall several facts from integration theory.

Definition 3.3. Let $\Omega \subset \mathbb{R}^n$ be open and bounded.

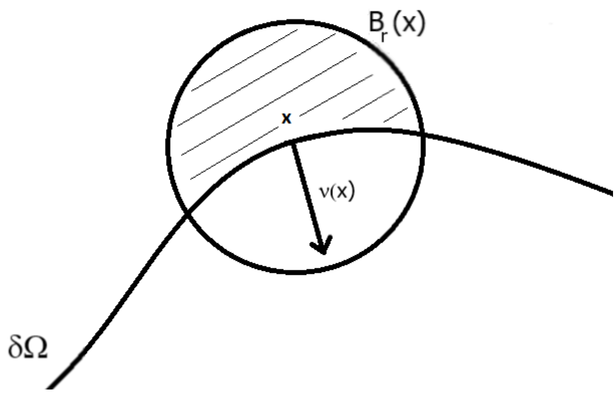
- We say that Ω has a C^k**-boundary**, if for every $x \in \partial\Omega$ there exists $r > 0$ and a function $\varphi \in C^k(\mathbb{R}^{n-1})$ such that (possibly after reordering the coordinates)

$$\Omega \cap B_r(x) = \{y \in B_r(x) \;:\; y_n > \varphi(y_1, \ldots, y_{n-1})\}.$$

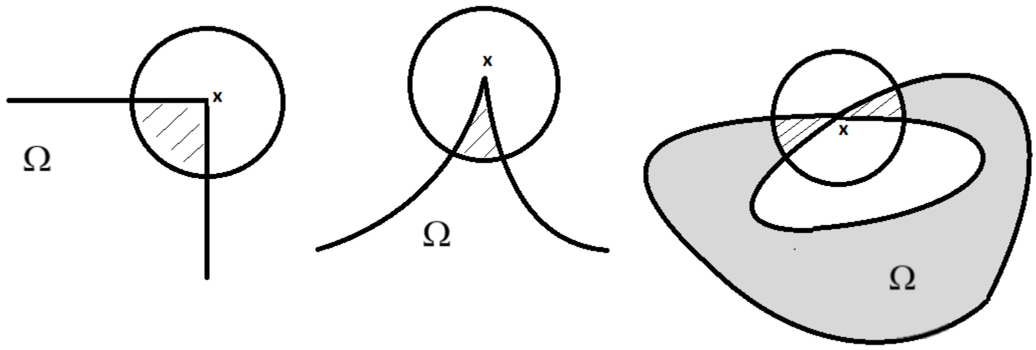
- If $\partial\Omega$ is of class C^1, we can define the **unit outer normal field** $\nu : \partial\Omega \to \mathbb{R}^n$, where $\nu(x)$, $|\nu(x)| = 1$, is the outward pointing unit normal vector at $x \in \partial\Omega$.

  The **normal derivative** of a function $u \in C^1(\overline{\Omega})$ is defined as

$$\frac{\partial u}{\partial \nu}(x) = \nu(x) \cdot \nabla u(x), \qquad x \in \partial\Omega.$$



Below are examples of domains that *do not possess* a C^1-boundary. The outward pointing unit normal vector in x cannot be defined.



We recall the *Gauß-Green theorem* and some direct consequences, a proof can be found in [5].

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary $\partial\Omega$. Then, for all $u \in C^1(\overline{\Omega})$ we have*

$$\int_\Omega u_{x_i}(x)dx = \int_{\partial\Omega} u(x)\nu_i(x)dS(x), \qquad i = 1, \ldots, n.$$

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary $\partial\Omega$. Then, the following properties hold:*

- *Integration by parts: For all $u, w \in C^1(\overline{\Omega})$ we have*

$$\int_\Omega u_{x_i} w = -\int_\Omega u w_{x_i} + \int_{\partial\Omega} u w \nu_i dS, \qquad i = 1, \ldots, n.$$

- *Green's formulas: For all $u, w \in C^2(\overline{\Omega})$ we have*

$$\int_\Omega \Delta u = \int_{\partial\Omega} \partial_\nu u dS,$$
$$\int_\Omega \nabla u \cdot \nabla w = -\int_\Omega u \Delta w + \int_{\partial\Omega} u \partial_\nu w dS,$$
$$\int_\Omega (u\Delta w - w\Delta u) = \int_{\partial\Omega} (u\partial_\nu w - w\partial_\nu u) dS.$$

Proof. See Problem E3.1. □

3.2 Motivation

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and suppose that $u\colon \Omega \to \mathbb{R}$ denotes the density or concentration of some quantity in equilibrium. Then, for an arbitrary open subset $V \subset \Omega$ with C^1-boundary the amount $\int_V u$ of the quantity contained in V does not change, i.e. the total flux through the boundary ∂V vanishes,

$$\int_{\partial V} F \cdot \nu \, dS = 0,$$

where $F\colon \Omega \to \mathbb{R}^n$ is the flux function. Therefore, by Theorem 3.4 we have

$$\int_V \operatorname{div} F = \int_{\partial V} F \cdot \nu \, dS = 0.$$

In many applications the flux function is proportional to the gradient of u but points in the opposite direction, i.e.

$$F(x) = -d\nabla u(x), \qquad x \in \Omega,$$

for some constant $d > 0$. For instance, if u denotes the concentration of a chemical substance, then particles move from regions of high concentrations to regions of low concentrations and this equation represents *Fick's law of diffusion*. Hence, we obtain

$$\operatorname{div} F = -d \operatorname{div}(\nabla u) = -d\Delta u,$$

and thus,

$$-\int_V d\Delta u = 0.$$

If $u \in C^2(\Omega)$, the integrand is continuous and since $V \subset \Omega$ was arbitrary it follows that $-d\Delta u = 0$ in Ω (see Problem E3.2). Therefore, u is a solution of the *Laplace equation*

$$\Delta u = 0 \qquad \text{in } \Omega.$$

In many cases a physical system has an additional source Q. The flux through the boundary ∂V then equals the amount generated by the source Q in V, i.e.

$$\int_{\partial V} F \cdot \nu \, dS = \int_V Q.$$

By the same arguments as above we conclude that

$$-d\Delta u = Q,$$

and hence, u satisfies the *Poisson equation*

$$-\Delta u = f \qquad \text{in } \Omega,$$

where $f = \frac{Q}{d}$.

The Poisson equation is used to model, e.g. the steady-state temperature in a solid (u is the temperature, f the heat source), the static deflection of a thin membrane in $\mathbb{R}^2$ (u is the deflection, f the pressure), electrostatics (u is the electrostatic potential, f the charge per unit volume) or Newtonian gravity (u is the gravitational potential, f the mass density).

3.3 Properties of harmonic functions

We first derive important properties of harmonic functions that have remarkable consequences for classical solutions of the Laplace and Poisson equation.

3.3.1 Mean value formulas

Let $\Omega \subset \mathbb{R}^n$ be open. Moreover, let $x \in \Omega$ and $r > 0$ be such that $\overline{B_r(x)} \subset \Omega$. For a function $u \in C(\Omega)$ we define the integral averages of u over balls $B_r(x)$ and spheres $\partial B_r(x)$,

$$\fint_{B_r(x)} u(y)\, dy := \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy,$$
$$\fint_{\partial B_r(x)} u(y)\, dS(y) := \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, dS(y),$$

where $|B_r(x)| = \int_{B_r(x)} 1 dy$ denotes the volume of $B_r(x)$ and $|\partial B_r(x)| = \int_{\partial B_r(x)} 1 dS(y)$ the surface area of the sphere $\partial B_r(x)$.

We will show that harmonic functions satisfy the *mean-value property*

$$u(x) = \fint_{B_r(x)} u(y)\, dy = \fint_{\partial B_r(x)} u(y)\, dS(y) \tag{3.3}$$

for all $x \in \Omega$ and $r > 0$ such that $\overline{B_r(x)} \subset \Omega$.

Recall that if $f \in C(\overline{B_r(x)})$ then using polar coordinates we have

$$\int_{B_r(x)} f(y)\, dy = \int_0^r \int_{\partial B_\rho(x)} f(y)\, dS(y)\, d\rho,$$

and by the transformation formula it follows that

$$\int_{\partial B_r(x)} f(y)\, dS(y) = r^{n-1} \int_{\partial B_1(0)} f(x + rz)\, dS(z).$$

In particular, this implies that

$$|\partial B_r(x)| = r^{n-1}|\partial B_1(0)| \qquad \text{and} \qquad |B_r(x)| = \frac{r}{n}|\partial B_r(x)| = \frac{r^n}{n}|\partial B_1(0)|. \tag{3.4}$$

We are now able to prove the mean-value-property.

Theorem 3.6 (Mean value formulas)**.** *Let $\Omega \subset \mathbb{R}^n$ be open.*

(a) If u is harmonic on Ω then u has the mean-value property (3.3).

(b) If u is subharmonic on Ω then u satisfies the inequalities

$$u(x) \le ⨍_{B_r(x)} u(y)\, dy, \tag{3.5}$$

$$u(x) \le ⨍_{\partial B_r(x)} u(y)\, dS(y), \tag{3.6}$$

for all $x \in \Omega$ and $r > 0$ such that $\overline{B_r(x)} \subset \Omega$.
If u is superharmonic on Ω then u satisfies these inequalities with a reversed sign, i.e. with "$\ge$" instead of "$\le$".

Proof. First, we observe that (a) immediately follows from (b). Indeed, if u is harmonic, then u is subharmonic and superharmonic. Therefore, the inequalities hold with "$\ge$" and "$\le$" and thus, equality must hold. This proves (3.3).

Moreover, assume that the inequalities hold for subharmonic functions and u is superharmonic. Then, $-u$ is subharmonic and thus, the inequalities for u hold with "$\ge$". Therefore, it suffices to prove (b) for subharmonic functions.

To this end let u be subharmonic, $x \in \Omega$ and $r > 0$ such that $\overline{B_r(x)} \subset \Omega$. We consider the function

$$\varphi(\rho) = ⨍_{\partial B_\rho(x)} u(y)\, dS(y), \qquad 0 < \rho \le r,$$

and prove that

$$\varphi'(\rho) \ge 0, \qquad \lim_{\rho \to 0} \varphi(\rho) = u(x). \tag{3.7}$$

Then,

$$u(x) = \lim_{\tilde\rho \to 0} \varphi(\tilde\rho) \le \varphi(\rho) \qquad \forall 0 < \rho \le r,$$

which is Inequality (3.6) for $\rho = r$. To show Inequality (3.5) we multiply the above inequality by $|\partial B_\rho(x)|$ and integrate from 0 to r,

$$|B_r(x)|u(x) = \int_0^r |\partial B_\rho(x)|u(x) d\rho \le \int_0^r \int_{\partial B_\rho(x)} u(y) dS(y) d\rho = \int_{B_r(x)} u(y) dy.$$

Dividing by $|B_r(x)|$ we obtain Inequality (3.5).

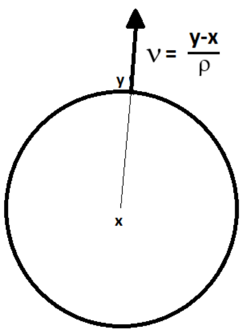
Hence, it remains to prove (3.7). We rewrite φ using the transformation formula and (3.4) as

$$\varphi(\rho) = \fint_{\partial B_\rho(x)} u(y) dS(y) = \fint_{\partial B_1(0)} u(x + \rho z)\, dS(z).$$

Differentiation now implies that

$$\begin{aligned}\varphi'(\rho) &= \fint_{\partial B_1(0)} \nabla u(x + \rho z) \cdot z\, dS(z) = \fint_{\partial B_\rho(x)} \nabla u(y) \cdot \frac{y - x}{\rho}\, dS(y) \\ &= \fint_{\partial B_\rho(x)} \frac{\partial u}{\partial \nu}(y)\, dS(y) = \frac{1}{|\partial B_\rho(x)|} \int_{\partial B_\rho(x)} \frac{\partial u}{\partial \nu}(y)\, dS(y),\end{aligned}$$

where we used that $\nu = \frac{y-x}{\rho}$ is the outer unit normal vector on $\partial B_\rho(x)$ at y.



Finally, we apply Green's formula (Theorem 3.5) and obtain

$$\varphi'(\rho) = \frac{1}{|\partial B_\rho(x)|} \int_{B_\rho(x)} \Delta u(y)\, dy \geq 0,$$

since u is subharmonic on Ω, which proves the first property in (3.7). To complete the proof we observe that

$$|\varphi(r) - u(x)| \leq \fint_{\partial B_r(x)} |u(y) - u(x)|\, dS(y) \leq \sup_{y \in \partial B_r(x)} |u(y) - u(x)| \to 0 \quad \text{as } r \to 0.$$

since u is continuous on $\overline{B_r(x)}$. □

The converse of the mean value property also holds.

Theorem 3.7. *Let $\Omega \subset \mathbb{R}^n$ be open. If $u \in C^2(\Omega)$ satisfies*

$$u(x) = \fint_{\partial B_r(x)} u(y)\, dS(y)$$

for all $x \in \Omega$ and $r > 0$ such that $\overline{B_r(x)} \subset \Omega$, then u is harmonic on Ω.

Proof. See Problem E3.7. □

3.3.2 Maximum principles and uniqueness for boundary value problems

An important consequence of the mean value property are the following maximum principles.

Theorem 3.8. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be subharmonic on Ω. Then, the following properties hold:*

(a) Maximum principle:

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x)$$

(b) Strong maximum principle: If Ω is connected and if there exists $x_0 \in \Omega$ such that

$$u(x_0) = \max_{x\in\overline{\Omega}} u(x),$$

then u is constant in Ω.

Proof. Since Ω is bounded, the sets $\overline{\Omega}$ and $\partial\Omega$ are compact. Thus, by the continuity of u, the maxima exist.

We observe that (a) is a consequence of (b). In fact, applying (b) on every connected component of Ω we conclude that

$$\max_{x\in\overline{\Omega}} u(x) \leq \max_{x\in\partial\Omega} u(x).$$

However, since $\partial\Omega \subset \overline{\Omega}$ it obviously holds that

$$\max_{x\in\partial\Omega} u(x) \leq \max_{x\in\overline{\Omega}} u(x),$$

which proves (a).

To show (b) let $x_0 \in \Omega$ be such that

$$M = u(x_0) = \max_{x\in\overline{\Omega}} u(x)$$

and let $A := \{x \in \Omega \ : \ u(x) = M\}$. Then, $A \subset \Omega$ is closed since it is the preimage of $\{M\}$ under the continuous mapping u, $A = u^{-1}(\{M\})$. On the other hand, if $x \in A$ then there exists $r > 0$ such that $\overline{B_r(x)} \subset \Omega$. By Theorem 3.6 we conclude that

$$M = u(x) \leq \fint_{B_r(x)} u(y)dy \leq M,$$

where we used that $u(y) \leq M$ in the last inequality. This enforces that $u \equiv M$ on $B_r(x)$ and proves that A is also open. Consequently, $A = \Omega$ since it is open and connected, which shows (b). □

We remark that a similar statement holds for superharmonic functions if the maxima are replaced minima. For harmonic functions we immediately obtain the following maximum principle.

Corollary 3.9 (Maximum principle for harmonic functions)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic on Ω. Then,*

$$\min_{y\in\partial\Omega} u(y) \leq u(x) \leq \max_{y\in\partial\Omega} u(y) \qquad \forall x \in \Omega.$$

Moreover, if Ω is connected then either strict inequalities hold or u is constant.

Proof. We observe that the functions u and $-u$ are both subharmonic. The statements are therefore direct consequences of Theorem 3.8. $\square$

An important application of the maximum principle is the uniqueness of solutions of the Dirichlet problem for Poisson's equation

$$-\Delta u = f \qquad \text{in } \Omega, \tag{3.8}$$
$$u = g \qquad \text{on } \partial\Omega, \tag{3.9}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded, and $g \in C(\partial\Omega)$ and $f \in C(\Omega)$ are given functions. The conditions in equation (3.9) are called *Dirichlet boundary conditions*.

A *classical solution* of the boundary value problem is a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ that satisfies (3.8), (3.9).

Theorem 3.10 (Uniqueness of solutions)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f \in C(\Omega)$ and $g \in C(\partial\Omega)$. Then, there exists at most one classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of the boundary value problem* (3.8), (3.9).

Proof. Assume that u and v are solutions of the boundary value problem, then their difference $w = u - v$ satisfies

$$-\Delta w = 0 \qquad \text{in } \Omega,$$
$$w = 0 \qquad \text{on } \partial\Omega.$$

Hence, by Corollary 3.9

$$0 = \min_{y \in \partial\Omega} w(y) \leq w(x) \leq \max_{y \in \partial\Omega} w(y) = 0 \qquad \forall x \in \Omega,$$

which implies that $w \equiv 0$ in Ω. $\square$

3.4 Fundamental solution

We aim at deriving explicit representation formulas for the solution of Poisson's equation. To this end, we first consider the Laplace equation in $\Omega = \mathbb{R}^n$ and construct a simple radial solution that we then use to build more complicated solutions.

To find explicit, special solutions of a PDE it is often useful to exploit symmetry properties of the equation. In fact, the Laplace operator is invariant under rotations (see Problem E3.4). This motivates to look for radial solutions of the Laplace equation (3.1) in $\Omega = \mathbb{R}^n$, i.e. solutions of the form

$$u(x) = v(r), \qquad r = |x|,$$

with a suitable function $v : [0, \infty) \to \mathbb{R}$. We observe that

$$r_{x_i}(x) = \frac{x_i}{|x|} = \frac{x_i}{r}, \qquad x \neq 0, \qquad i = 1, \ldots, n,$$

and hence, for the partial derivatives of u we obtain

$$u_{x_i}(x) = v'(r)\frac{x_i}{r}, \qquad u_{x_i x_i}(x) = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right).$$

This implies that

$$\Delta u(x) = v''(r) + \left(\frac{n}{r} - \frac{1}{r}\right) v'(r) = v''(r) + \frac{n-1}{r} v'(r).$$

Therefore, in this special case the PDE $\Delta u = 0$ for $x \neq 0$ is equivalent to the ODE

$$v''(r) + \frac{n-1}{r} v'(r) = 0, \qquad r > 0.$$

If $v' \neq 0$ then

$$\frac{d}{dr}(\ln |v'(r)|) = \frac{v''(r)}{v'(r)} = \frac{1-n}{r}$$

and thus,

$$\ln |v'(r)| = (1-n) \ln r + d = \ln r^{1-n} + d,$$

for some constant $d \in \mathbb{R}$. Consequently,

$$|v'(r)| = \frac{e^d}{r^{n-1}},$$

and we conclude that

$$v(r) = \begin{cases} b \ln r + c & \text{if } n = 2, \\ \frac{b}{(2-n) r^{n-2}} + c & \text{if } n \geq 3, \end{cases} \qquad r > 0,$$

for some constants $b, c \in \mathbb{R}$.

For the particular choice

$$b = -\frac{1}{|\partial B_1(0)|} = -\frac{1}{\omega_n}, \qquad c = 0,$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$, we obtain the so-called *fundamental solution of the Laplace equation*. The reason for choosing these particular constants will become apparent in the sequel.

Definition 3.11. The function $\Phi \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$,

$$\Phi(x) = \hat{\Phi}(|x|) = \begin{cases} -\frac{1}{2\pi} \ln |x|, & n = 2, \\ \frac{1}{(n-2)\omega_n} \frac{1}{|x|^{n-2}}, & n \geq 3, \end{cases} \tag{3.10}$$

is called the **fundamental solution of the Laplace equation**.

By construction, $\Delta \Phi = 0$ in $\mathbb{R}^n \setminus \{0\}$, but note that Φ has a singularity at the origin. Moreover, for $x \neq 0$ the partial derivatives of Φ are

$$\Phi_{x_i}(x) = -\frac{1}{\omega_n} \frac{x_i}{|x|^n}, \qquad \Phi_{x_i x_j}(x) = -\frac{1}{\omega_n} \left(\frac{\delta_{ij}}{|x|^n} - n \frac{x_i x_j}{|x|^{n+2}} \right), \tag{3.11}$$

$i, j = 1, \ldots, n$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Hence, $\Phi_{x_i x_j}$ is not integrable at the singularity $x = 0$ and it is precisely this property that allows us to construct solutions of the Dirichlet problem (3.8)-(3.9).

Remark 3.12. Recall that the function $x \mapsto |x|^{-s}$ is integrable over a ball $B_r(0), r > 0$, in $\mathbb{R}^n$ if $s < n$.

The function $x \mapsto \Phi(x)$ is harmonic for $x \neq 0$, and similarly, by shifting the origin, for any $y \in \mathbb{R}^n$ the function $x \mapsto \Phi(x - y)$ is harmonic for $x \neq y$. Moreover, taking a function $f : \mathbb{R}^n \to \mathbb{R}$, then $x \mapsto f(y)\Phi(x - y)$ is harmonic for every $y \in \mathbb{R}^n$, $x \neq y$, and thus, the same applies to the sum of finitely many such expressions. This might suggest that the *convolution*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y)\, dy$$

is a solution of the Laplace equation (3.1). However, this is wrong since $\Delta\Phi$ is not integrable near the singularity at $x = y$, and thus, interchanging differentiation and integration is not possible. In fact, the function u is not harmonic, but yields a solution of the Poisson equation in $\Omega = \mathbb{R}^n$ (see [4]).

We will consider the Poisson equation in bounded domains and use the fundamental solution Φ in (3.10) to construct a representation formula for the solution of the Dirichlet problem.

3.5 Green's function and representation formula

We now derive a representation formula for solutions of the boundary value problem (3.8)-(3.9)

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is open and bounded with C^1 boundary $\partial\Omega$, and $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ are continuous.

First, we prove an integral representation formula for arbitrary functions $u \in C^2(\overline{\Omega})$ that allows to express u in terms of Δu, $u|_{\partial\Omega}$ and $\partial_\nu u|_{\partial\Omega}$.

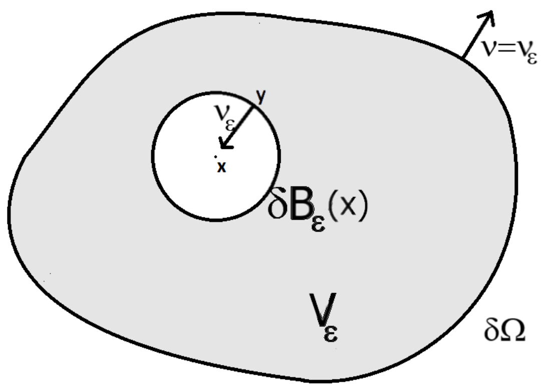
Proposition 3.13. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary $\partial\Omega$ and Φ be the fundamental solution in (3.10). Then, for any $u \in C^2(\overline{\Omega})$ we have

$$u(x) = \int_{\partial\Omega} \left(\Phi(y - x) \frac{\partial u}{\partial \nu}(y) - u(y) \frac{\partial \Phi}{\partial \nu_y}(y - x) \right) dS(y) - \int_{\Omega} \Phi(y - x) \Delta u(y)\, dy,$$

for all $x \in \Omega$, where $\frac{\partial \Phi}{\partial \nu_y} = \nu \cdot \nabla_y \Phi$ denotes the normal derivative with respect to y on $\partial\Omega$.

Proof. Let $x \in \Omega$ and $\varepsilon > 0$ be such that $\overline{B_\varepsilon(x)} \subset \Omega$. Moreover, let $V_\varepsilon = \Omega \setminus \overline{B_\varepsilon(x)}$ and ν^ε denote the outer normal field of V_ε. Then, for sufficiently small $\varepsilon > 0$ we have $\partial V_\varepsilon = \partial\Omega \dot\cup \partial B_\varepsilon(x)$ and

$$\nu^\varepsilon(y) = \nu(y), \quad y \in \partial\Omega, \qquad\qquad \nu^\varepsilon(y) = \frac{x - y}{\varepsilon}, \quad y \in \partial B_\varepsilon(x).$$



Applying Green's formula (Theorem 3.5) to u and $\Phi(\cdot - x)$ on V_ε we obtain

$$\begin{aligned}
&\int_{V_\varepsilon} u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u(y)\,dy \\
&= \int_{\partial V_\varepsilon} u(y)\frac{\partial \Phi}{\partial \nu^\varepsilon}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu^\varepsilon}(y)\,dS(y) \\
&= \int_{\partial\Omega} u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) \qquad (3.12)\\
&\quad + \underbrace{\int_{\partial B_\varepsilon(x)} u(y)\frac{x-y}{\varepsilon}\cdot\nabla\Phi(y-x)\,dS(y)}_{=:I_\varepsilon} - \underbrace{\int_{\partial B_\varepsilon(x)} \Phi(y-x)\frac{\partial u}{\partial \nu^\varepsilon}(y)\,dS(y)}_{=:J_\varepsilon}.
\end{aligned}$$

Note that for $y \in \partial B_\varepsilon(x)$ we have

$$\Phi(y-x) = \begin{cases} -\frac{1}{2\pi}\ln(\varepsilon), & n = 2, \\ \frac{1}{(n-2)\omega_n}\varepsilon^{2-n}, & n \geq 3, \end{cases}$$

and consequently,

$$\begin{aligned}
|J_\varepsilon| &\leq \omega_n \varepsilon^{n-1} \sup_{y\in\partial B_\varepsilon(x)} \left\{ \left| \frac{\partial u}{\partial \nu^\varepsilon}(y) \right| |\Phi(y-x)| \right\} \\
&\leq \sup_{y\in\partial B_\varepsilon(x)} \left\{ \left| \frac{\partial u}{\partial \nu^\varepsilon}(y) \right| \right\} \max\{\varepsilon, \varepsilon|\ln(\varepsilon)|\} \to 0 \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

Here, we used that ∇u is bounded, since $u \in C^2(\overline{\Omega})$.

To determine I_ε we use (3.11) and observe that

$$\frac{x-y}{\varepsilon}\cdot\nabla\Phi(y-x) = \frac{1}{\omega_n}\frac{(y-x)\cdot(y-x)}{\varepsilon|y-x|^n} = \frac{1}{\omega_n\varepsilon^{n-1}} = \frac{1}{|\partial B_\varepsilon(x)|} \qquad \forall y \in \partial B_\varepsilon(x).$$

Since u is continuous in x, this implies that

$$I_\varepsilon = \fint_{\partial B_\varepsilon(x)} u(y)\,dS(y) \to u(x) \qquad \text{as } \varepsilon \to 0.$$

Finally, we show that

$$\int_{V_\varepsilon} \Phi(y-x)\Delta u(y)dy \to \int_\Omega \Phi(y-x)\Delta u(y)dy \quad \text{as} \quad \varepsilon \to 0.$$

Indeed,

$$\int_{B_\varepsilon(x)} |\Phi(y-x)\Delta u(y)|\, dy \le \sup_{y\in B_\varepsilon(x)} \{|\Delta u(y)|\} \int_0^\varepsilon \int_{\partial B_\varepsilon(x)} |\hat{\Phi}(r)| dS(y) dr$$
$$\le \begin{cases} c\int_0^\varepsilon r|ln(r)|dr, & n=2 \\ c\int_0^\varepsilon rdr, & n\ge 3 \end{cases} \le \begin{cases} c\varepsilon^2|ln(\varepsilon)|dr, & n=2 \\ c\frac{\varepsilon}{2}, & n\ge 3 \end{cases} \to 0 \quad \text{as} \;\; \varepsilon \to 0,$$

where we used that $|\Delta u|$ is bounded, since $u \in C^2(\overline{\Omega})$.

Using this estimate as well as the limits for I_ε and J_ε the proposition follows by taking the limit in (3.12). □

An immediate consequence is the smoothness of harmonic functions.

Theorem 3.14. *Let $\Omega \subset \mathbb{R}^n$ be open and u be harmonic on Ω. Then, u satisfies $u \in C^\infty(\Omega)$.*

Proof. Let $x_0 \in \Omega$ and $r > 0$ be such that $\overline{B_r(x_0)} \subset \Omega$. Applying the representation formula in Proposition 3.13 to u and $B_r(x_0)$ we obtain

$$u(x) = \int_{\partial B_r(x_0)} (\Phi(y-x)\partial_\nu u(y) - u(y)\partial_\nu \Phi(y-x))\, dS(y) \qquad \forall x \in B_r(x_0).$$

The integrand and all partial derivatives with respect to x are continuous for $x \neq y$ and $\partial B_r(x_0)$ is compact. Therefore, we can differentiate the right hand side and interchange differentiation and integration. It follows that the right hand side is arbitrarily often continuously differentiable which proves the statement. □

The representation formula in Proposition 3.13 determines $u(x), x \in \Omega$, if Δu in Ω and $u, \frac{\partial u}{\partial \nu}$ on $\partial\Omega$ are known. If we apply the formula to solve the Dirichlet problem (3.8)-(3.9), the first two quantities are specified, but the normal derivative $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$ is unknown.

To eliminate this term, for fixed $x \in \Omega$, we introduce the *corrector function* w^x. The corrector function w^x is the solution (if it exists!) of the boundary value problem

$$\begin{aligned} \Delta w^x(y) &= 0, && y \in \Omega, \\ w^x(y) &= \Phi(y-x), && y \in \partial\Omega. \end{aligned} \tag{3.13}$$

Using Green's formula (Theorem 3.5) and the fact that $\Delta w^x = 0$ we obtain

$$\begin{aligned} -\int_\Omega w^x(y)\Delta u(y)\, dy &= \int_{\partial\Omega} u(y)\frac{\partial w^x}{\partial \nu}(y) - w^x(y)\frac{\partial u}{\partial \nu}(y)\, dS(y) \\ &= \int_{\partial\Omega} u(y)\frac{\partial w^x}{\partial \nu}(y) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\, dS(y), \end{aligned}$$

which implies that

$$0 = \int_\Omega w^x(y)\Delta u(y)\, dy + \int_{\partial\Omega} u(y)\frac{\partial w^x}{\partial \nu}(y) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\, dS(y). \tag{3.14}$$

Adding this equation to the representation formula in Proposition 3.13 we can eliminate the term involving the normal derivative $\partial_\nu u$. Hence, we obtain a representation formula for solutions of the Dirichlet problem for Poisson's equation. This motivates the definition of *Green's function*.

Definition 3.15. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary. Then, the function G defined by

$$G(x, y) = \Phi(y - x) - w^x(y), \qquad x, y \in \Omega, x \neq y,$$

where Φ is the fundamental solution and $w^x \in C^2(\overline{\Omega})$ the solution of (3.13), is called **Green's function** for Ω.

Adding the representation formula in Proposition 3.13 and the equation (3.14) we obtain

$$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu}(x, y)\, dS(y) - \int_\Omega G(x, y)\Delta u(y)\, dy, \qquad x \in \Omega.$$

This formula holds for arbitrary functions $u \in C^2(\overline{\Omega})$. In particular, if u is a classical solution of the Dirichlet problem (and if Green's function exists), it yields the desired representation formula.

Theorem 3.16. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary $\partial\Omega$ and assume that Green's function G for Ω exists. Moreover, let $f \in C(\Omega)$ and $g \in C(\partial\Omega)$. Then, a classical solution $u \in C^2(\overline{\Omega})$ of the Dirichlet problem* (3.8)-(3.9) *satisfies*

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x, y)\, dS(y) + \int_\Omega f(y)G(x, y)\, dy \qquad \forall x \in \Omega.$$

Proof. The representation formula immediately follows from Proposition 3.13 and the above computations. □

Remark 3.17. One can show that Green's function G is symmetric, i.e.,

$$G(y, x) = G(x, y) \qquad \forall x, y \in \Omega, x \neq y$$

(see Problem E3.15).

The explicit construction of Green's function for a given Ω can be difficult, or may not even be possible. It requires to solve the auxiliary Dirichlet problem (3.13), and this can be complicated, or even impossible. However, Green's function can be computed for geometrically simple domains Ω which we will illustrate for the ball $B_r(0) \subset \mathbb{R}^n$.

3.6 Green's function and existence result for the ball

Consider the Dirichlet problem (3.8)-(3.9) for $\Omega = B_r(0)$. To determine Green's function for a given $x \in B_r(0)$ we need to find the solution w^x of the auxiliary problem

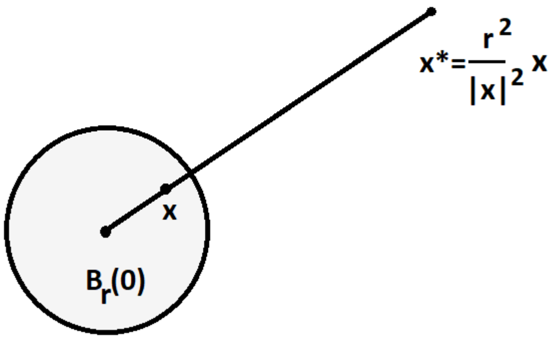
$$\begin{aligned} \Delta w^x &= 0 && \text{in } B_r(0), \\ w^x &= \Phi(\cdot - x) && \text{on } \partial B_r(0). \end{aligned}$$

The idea is to use the fundamental solution $\Phi(\cdot - x)$, which is harmonic in $\mathbb{R}^n \setminus \{x\}$, and to reflect the singularity outside of the sphere. Since Φ is radially symmetric, we make the following ansatz

$$w^x(y) = \Phi\left(\frac{|y-x|}{|y-x^*|}(y-x^*)\right), \qquad y \in \partial B_r(0),$$

and aim to find a suitable $x^* \notin B_r(0)$ such that $\frac{|y-x|}{|y-x^*|}$ is independent of $y \in \partial B_r(0)$. This can be achieved by *inversion on the sphere*,

$$x \mapsto x^* = \frac{r^2}{|x|^2}x, \qquad x \in B_r(0) \setminus \{0\}.$$



In fact, then we have

$$\frac{|y-x|^2}{|y-x^*|^2} = \frac{r^2 - 2x\cdot y + |x|^2}{r^2 - 2\frac{r^2}{|x|^2}x\cdot y + \frac{r^4}{|x|^2}} = \frac{|x|^2}{r^2}, \qquad y \in \partial B_r(0).$$

This leads to $w^x(y) = \Phi\left(\frac{|x|}{r}(y-x^*)\right)$, $y \in \partial B_r(0)$, and extending w^x to all $y \neq x$ yields

$$w^x(y) = \Phi\left(\frac{|x|}{r}(y-x^*)\right) = \begin{cases} \Phi\left(\frac{|x|}{r}y - \frac{r}{|x|}x\right), & y \neq x, x \neq 0, \\ \hat{\Phi}(r), & x = 0. \end{cases}$$

Certainly, $w^x \in C^2(\overline{B_r(0)})$, w^x is harmonic on $B_r(0)$ and by construction, it satisfies

$$w^x(y) = \hat{\Phi}\left(\frac{|x|}{r}|y-x^*|\right) = \hat{\Phi}(|y-x|) = \Phi(y-x), \qquad y \in \partial B_r(0).$$

Thus, w^x is the desired corrector function, and we obtain **Green's function for the ball**,

$$G(x,y) = \Phi(y-x) - \begin{cases} \Phi\left(\frac{|x|}{r}y - \frac{r}{|x|}x\right), & x \neq 0, x \neq y, \\ \hat{\Phi}(r), & x = 0. \end{cases}$$

We remark that in this case the symmetry of G can be directly verified.

To obtain an explicit representation formula for the solution of Dirichlet's problem we compute

$$\begin{aligned} G_{y_i}(x,y) &= \Phi_{y_i}(y-x) - \Phi_{y_i}\left(\frac{|x|}{r}y - \frac{r}{|x|}x\right)\frac{|x|}{r} \\ &= -\frac{1}{\omega_n}\frac{y_i - x_i}{|y-x|^n} + \frac{1}{\omega_n}\frac{\frac{|x|^2}{r^2}y_i - x_i}{|y-\tilde{x}|^n} = \frac{1}{\omega_n}\frac{y_i(\frac{|x|^2}{r^2} - 1)}{|y-x|^n}. \end{aligned}$$

Since $\nu(y) = \frac{y}{r}$ this implies that for $y \in \partial B_r(0)$,

$$\frac{\partial G}{\partial \nu}(x, y) = \nu(y) \cdot \nabla G(x, y) = \frac{1}{\omega_n} \frac{|x|^2 - r^2}{r|x - y|^n}.$$

Hence, we expect that a solution of the Dirichlet problem (3.8)-(3.9) for $\Omega = B_r(0)$ is given by the representation formula

$$u(x) = \frac{r^2 - |x|^2}{r\omega_n} \int_{\partial B_r(0)} \frac{g(y)}{|x - y|^n} \, dS(y) + \int_{B_r(0)} f(y) G(x, y) \, dy$$

In the spacial case that $f \equiv 0$ we obtain the **Poisson formula**,

$$u(x) = \frac{r^2 - |x|^2}{r\omega_n} \int_{\partial B_r(0)} \frac{g(y)}{|x - y|^n} \, dS(y), \tag{3.15}$$

for solutions of the *Dirichlet problem for Laplace's equation* in $\Omega = B_r(0)$,

$$\begin{aligned} \Delta u &= 0 && \text{in } B_r(0), \\ u &= g && \text{on } \partial B_r(0). \end{aligned}$$

So far, we have shown that classical solutions $u \in C^2(\overline{B_r(0)})$ of the Dirichlet problem for Laplace's equation on the ball $B_r(0)$ satisfy Poisson's formula. Finally, we show that the representation formula actually provides a solution if $g \in C(\partial B_r(0))$.

Theorem 3.18 (Existence for the ball)**.** *Suppose that* $g \in C(\partial B_r(0))$*, then Poisson's formula* (3.15) *defines the unique classical solution* $u \in C^2(B_r(0)) \cap C(\overline{B_r(0)})$ *of the Dirichlet problem for Laplace's equation in* $\Omega = B_r(0)$*. Moreover,* $u \in C^\infty(B_r(0))$*.*

Proof. Poisson's formula is a special case of Green's representation formula in Theorem 3.16 for $\Omega = B_r(0)$ and $f = 0$, namely

$$u(x) = -\int_{\partial B_r(0)} g(y)\nu(y) \cdot \nabla G(x, y) dS(y).$$

The integrand and all its partial derivatives with respect to x are continuous on $B_r(0) \times \partial B_r(0)$. Since $\partial B_r(0)$ is compact, the derivatives of u can be obtained by interchanging differentiation and integration and therefore, the integral defines a function in $C^\infty(B_r(x))$.

Next, we show that u is harmonic on $B_r(0)$. In fact, Green's function G is harmonic with respect to the second variable and symmetric for $x \neq y$ (cf. Remark 3.17). Hence, it is also harmonic with respect to the first variable, $\Delta_x G(x, y) = 0 = \Delta_y G(x, y)$, for all $(x, y) \in B_r(0) \times \partial B_r(0)$. We conclude that

$$\Delta u(x) = -\int_{\partial B_r(0)} g(y)\nu(y) \cdot \nabla \Delta_x G(x, y) dS(y) = 0, \qquad x \in B_r(0).$$

To conclude the proof it remains to show that $u \in C(\overline{B_r(0)})$ and $u|_{\partial B_r(0)} = g$. This follows from the continuity of g, an ε-δ-argument and by estimating the integrals involved (see Problem E3.16). □

Another region with a simple geometry for which we can construct a Green's function is the half space

$$\Omega = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\},$$

see Problem E.3.18.

3.7 Energy methods

So far, we used the mean value property and explicit representation formulas to derive the existence, uniqueness and properties of solutions of the Laplace and Poisson equation. Now, we apply a different approach, so-called *energy methods* that are based on L^2-norms of solutions and its derivatives. These methods foreshadow techniques that are used to study weak solutions of PDEs.

Definition 3.19. Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C(\Omega; \mathbb{R}^m), m \in \mathbb{N}$. For $1 \le p < \infty$ we defined the $\mathbf{L^p}$**-norm** of u by

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|_p^p \right)^{\frac{1}{p}},$$

where $|y|_p := (|y_1|^p + \cdots + |y_m|^p)^{\frac{1}{p}}$ for a vector $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$.

Moreover, we call a function $u \in C(\Omega; \mathbb{R}^m)$ **integrable** if $\|u\|_{L^1(\Omega)} < \infty$.

3.7.1 Uniqueness

In Theorem 3.10 we already proved uniqueness for solutions of the Dirichlet problem (3.8)-(3.9) based on the maximum principle. We now present an alternative proof using energy methods.

Theorem 3.20. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary $\partial\Omega$. Then, for every $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ there exists at most one solution $u \in C^2(\overline{\Omega})$ of the boundary value problem* (3.8)-(3.9),

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

Proof. Suppose that v is another solution. Then, the difference $w = u - v$ satisfies $\Delta w = 0$ in Ω and $w|_{\partial\Omega} = 0$. Hence, multiplying the PDE by w and integrating over Ω it follows that

$$0 = -\int_\Omega w\Delta w = \int_\Omega |\nabla w|^2 = \|\nabla w\|^2_{L^2(\Omega)},$$

where we used integration by parts. Since ∇w is continuous, we conclude that $\nabla w \equiv 0$ in Ω (see Problem E3.2). Therefore, w must be constant, and since $w|_{\partial\Omega} = 0$, this implies that $w \equiv 0$, i.e. $u \equiv v$. □

3.7.2 Dirichlet's principle

The Poisson equation describes, e.g. steady state deflections of a thin membrane or steady state distributions of a chemical substrate. It is therefore natural that the solution of the Dirichlet problem (3.8)-(3.9) corresponds to a minimum of some *energy functional*. In fact, we will see that the solution can be characterized as a minimizer of an appropriate functional.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary $\partial\Omega$. For given $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ consider the energy functional

$$J(w) := \int_\Omega \left(\frac{1}{2} |\nabla w|^2 - wf \right),$$

for w belonging to the *admissible set*

$$\mathcal{A} = \{w \in C^2(\overline{\Omega}) \;:\; w = g \text{ on } \partial\Omega\}.$$

Theorem 3.21 (Dirichlet principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary $\partial\Omega$, $f \in C(\Omega)$ and $g \in C(\partial\Omega)$.*

Assume that $u \in C^2(\overline{\Omega})$ is a solution of the boundary value problem (3.8)-(3.9)*,*

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega, \end{aligned}$$

then

$$J(u) = \min_{w \in \mathcal{A}} J(w). \tag{3.16}$$

Conversely, if $u \in \mathcal{A}$ satisfies (3.16)*, then u is a solution of the Dirichlet problem* (3.8)-(3.9).

Proof. (i) Let $u \in C^2(\overline{\Omega})$ be a solution of (3.8)-(3.9). Then, $u \in \mathcal{A}$. Moreover, if $w \in \mathcal{A}$, then multiplying the PDE by $(u - w)$ and integrating over Ω we obtain

$$0 = \int_\Omega (-\Delta u - f)(u - w) = \int_\Omega (|\nabla u|^2 - \nabla u \cdot \nabla w - fu + fw).$$

Note that no boundary term occurs since $u, w \in \mathcal{A}$, which implies that $(u - w)|_{\partial\Omega} = 0$. By the Cauchy–Schwarz inequality it follows that

$$|\nabla u \cdot \nabla w| \leq |\nabla u||\nabla w| \leq \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla w|^2,$$

where we used the inequality $a^2 + b^2 - 2ab = (a - b)^2 \geq 0, \forall a, b \in \mathbb{R}$, in the second step. Using this estimate in the equality above leads to

$$0 \geq \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - uf\right) - \int_\Omega \left(\frac{1}{2}|\nabla w|^2 - wf\right),$$

i.e. $J(w) \geq J(u)$ for all $w \in \mathcal{A}$.

(ii) Conversely, let $u \in \mathcal{A}$ satisfy (3.16). For arbitrary $v \in C_c^\infty(\Omega)$, where

$$C_c^\infty(\Omega) = \{u \in C^\infty(\Omega) \;:\; \text{supp}(u) \text{ is compact in } \Omega\},$$

consider the function

$$j(s) := J(u + sv), \qquad s \in \mathbb{R}.$$

Then, since $u + sv \in \mathcal{A}$, $j : \mathbb{R} \to \mathbb{R}$ is well-defined and has a minimum in $s = 0$. Moreover, j is continuously differentiable and

$$j'(s) = \int_\Omega ((\nabla u + s\nabla v) \cdot \nabla v - fv),$$

since Ω is bounded and the integrand of J and its partial derivatives with respect to s are continuous for $s \in \mathbb{R}, x \in \overline{\Omega}$. Therefore,

$$0 = j'(0) = \int_\Omega (\nabla u \cdot \nabla v - vf) = \int_\Omega (-\Delta u - f)v,$$

where we used integration by parts in the last step. Since $v \in C_c^\infty(\Omega)$ was arbitrary and $(-\Delta u - f) \in C(\Omega)$, it follows that $-\Delta u = f$ in Ω by the Fundamental Lemma of the Calculus of Variations (see Lemma 3.22). The boundary conditions (3.9) are satisfied, since $u \in \mathcal{A}$. □

If the data f and g or the boundary $\partial\Omega$ are less regular, it is not guaranteed that $\mathcal{A} \neq \emptyset$ or that J attains a minimum in $\mathcal{A}$. It is therefore desirable to enlarge the admissible set $\mathcal{A}$ by considering less regular classes of functions in order to ensure the existence of a minimizer. This minimizer is a natural candidate for a weak solution of Poisson's equation.

Lemma 3.22 (Fundamental Lemma of the Calculus of Variations)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. If a function $u \in C(\overline{\Omega})$ satisfies*

$$\int_\Omega u(x)\eta(x)dx = 0 \qquad \forall \eta \in C_c^\infty(\Omega),$$

then $u \equiv 0$ in $\overline{\Omega}$.

Proof. By contradiction, we assume that $u \not\equiv 0$ in $\overline{\Omega}$. Then, there exists $x_0 \in \Omega$ such that $u(x_0) \neq 0$. Since u is continuous, there exists $\delta > 0$ such that $B_\delta(x_0) \subset \Omega$ and

$$u(x) > \frac{1}{2}u(x_0) > 0 \qquad \text{or} \qquad u(x) < \frac{1}{2}u(x_0) < 0 \qquad \forall x \in B_\delta(x_0).$$

We now choose a function $\psi \in C_c^\infty(\Omega)$ with

$$\text{supp}(\psi) \subset B_\delta(x_0), \qquad \psi(x_0) > 0, \qquad \psi \geq 0 \quad \text{in } B_\delta(x_0).$$

It then follows that

$$\int_\Omega u(x)\psi(x)dx = \int_{B_\delta(x_0)} u(x)\psi(x)dx \neq 0,$$

which is a contraction. □

Lemma 3.22 can be shown for a larger class of functions. In particular, the assumption that u is continuous can be weakened, but the proof is more involved (see [2]).

3.8 Exercises

E3.1 **Consequences of the Gauss–Green Theorem**

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary $\partial\Omega$ and let $\nu \colon \partial\Omega \to \mathbb{R}^n$ denote the outward pointing unit normal vector field of $\partial\Omega$. We recall the *Gauss–Green Theorem*: If $u \in C^1(\overline{\Omega})$ then

$$\int_\Omega u_{x_i} = \int_{\partial\Omega} u\nu_i \, dS, \qquad i = 1, \ldots, n.$$

Prove the following integration formulas:

(a) *Integration by parts*
Let $u, v \in C^1(\overline{\Omega})$. Then,

$$\int_\Omega u_{x_i} v = -\int_\Omega u v_{x_i} + \int_{\partial\Omega} uv\nu_i dS, \qquad i = 1, \ldots, n.$$

(b) *Green's formulas*
Let $u, v \in C^2(\overline{\Omega})$. Then,

$$\text{(i)} \qquad \int_\Omega \Delta u = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, dS,$$

$$\text{(ii)} \qquad \int_\Omega \nabla u \cdot \nabla v = -\int_\Omega u\,\Delta v + \int_{\partial\Omega} u \frac{\partial v}{\partial \nu}\, dS,$$

$$\text{(iii)} \qquad \int_\Omega (u\Delta v - v\Delta u) = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS.$$

E3.2 **Averages**

Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C(\Omega)$. Moreover, let $x \in \Omega$ and $r > 0$ be such that $\overline{B_r(x)} \subset \Omega$.

(a) Show that

$$\lim_{r \to 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy = u(x) \qquad \forall x \in \Omega.$$

(b) Prove that

$$\int_{B_r(x)} u(y)dy = 0 \qquad \forall B_r(x) \subset \Omega$$

implies that $u \equiv 0$ in Ω.

E3.3 **Harmonic functions**

Let $V \subset \mathbb{R}^2 \setminus \{0\}$ and $W \subset \mathbb{R}^3 \setminus \{0\}$ be open. Which of the following functions are harmonic, subharmonic, or superharmonic?

(i) $u\colon V \to \mathbb{R}, \quad u(x,y) = \ln \sqrt{x^2+y^2}$,

(ii) $v\colon W \to \mathbb{R}, \quad v(x,y,z) = \ln \sqrt{x^2+y^2+z^2}$,

(iii) $w\colon W \to \mathbb{R}, \quad w(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$.

E3.4 **Invariance of the Laplacian**

Let $u : \mathbb{R}^n \to \mathbb{R}$ be an harmonic function and $A \in \mathbb{R}^{n\times n}$ be an orthogonal matrix (i.e. $AA^T = \mathrm{Id}$). Show that $v : \mathbb{R}^n \to \mathbb{R}$, $v(x) = u(Ax)$, is also an harmonic function.

Remark: Note that this implies that the Laplacian is invariant under rotations.

E3.5 **Neumann problem for the Poisson equation** Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary $\partial\Omega$. Consider the Poisson equation with *Neumann boundary conditions*

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= g && \text{on } \partial\Omega, \end{aligned}$$

where $f \in C(\overline{\Omega})$ and $g \in C(\partial\Omega)$ are given.

Show that if a classical solution $u \in C^2(\overline{\Omega})$ of the problem exists then g and f must satisfy

$$\int_\Omega f(x)dx + \int_{\partial\Omega} g(x)dS(x) = 0.$$

E3.6 **Averages**

Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C(\Omega)$. Show that the following statements are equivalent:

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)dy = u(x) \qquad \forall x \in \Omega, r > 0 \text{ s.t. } \overline{B_r(x)} \subset \Omega$$
$$\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)dS(y) = u(x) \qquad \forall x \in \Omega, r > 0 \text{ s.t. } \overline{B_r(x)} \subset \Omega$$

E3.7 **Converse of the mean value property**

Prove the converse of the mean-value property (see Theorem 3.6):

Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C^2(\Omega)$ satisfy

$$u(x) = \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)dS(y) \qquad \forall x \in \Omega \text{ s.t. } \overline{B_r(x)} \subset \Omega.$$

Then, u is harmonic on Ω.

E3.8 **Mean value formulas**

Let $n \geq 3$ and $u \in C^2(\overline{\Omega})$ be a solution of the boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } B_r(0), \\ u &= g \quad \text{on } \partial B_r(0). \end{aligned}$$

Modify the proof of the mean value formulas to show that

$$u(0) = \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} g(x)\,dS(x) + \frac{1}{(n-2)|\partial B_1(0)|}\int_{B_r(0)}\left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f(x)\,dx.$$

Hint: *Consider the function φ used in the proof of Theorem 3.6 and first show that*

$$⨍_{\partial B_r(0)} g(x)\,dS(x) - ⨍_{\partial B_\varepsilon(0)} u(x)\,dS(x) = -\frac{1}{|\partial B_1(0)|}\int_\varepsilon^r \frac{1}{\rho^{n-1}}\int_{B_\rho(0)} f(x)\,dxd\rho.$$

Then, use integration by parts to evaluate the integral on the right and side and take the limit $\varepsilon \to 0$.

E3.9 **Subharmonic functions**

Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C^3(\Omega)$. Prove that $v = |\nabla u|^2$ is subharmonic if u is harmonic.

E3.10 **Fundamental solution.**

(a) Let $r > 0$ and consider the ball $B_r(0) \subset \mathbb{R}^n$. Show that the integral

$$\int_{B_r(0)} \frac{1}{|x|^s} dx$$

is finite if and only if $s < n$.

(b) Derive the following estimates for the derivatives for the fundamental solution $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ of the Laplace equation,

$$|D\Phi(x)| \leq \frac{c}{|x|^{n-1}}, \qquad |D^2\Phi(x)| \leq \frac{c}{|x|^n},$$

for some constant $c > 0$.

(c) Is the fundamental solution Φ integrable near the singularity, i.e. is the integral $\int_{B_r(0)} \Phi$ finite? What about the partial derivatives of first order and the Laplacian of Φ?

E3.11 **Bound for the derivatives**

Let $\Omega \subset \mathbb{R}^n$ be open and u be harmonic on Ω. Show that

$$|u_{x_i}(\bar{x})| \leq \frac{n}{r} \sup_{y \in \partial B_r(\bar{x})} |u(y)|$$

for every $\bar{x} \in \Omega$ and $r > 0$ such that $\overline{B_r(\bar{x})} \subset \Omega$.

Hint: Use the mean-value property.

E3.12 **Maxima and minima**

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Prove that:

(a) If $x \in \Omega$ is a local maximum of u then $\Delta u(x) \leq 0$.

(b) Let u be a solution of the boundary value problem

$$\begin{aligned} \Delta u &= u^3 - u && \text{in } \Omega, \\ u &= \frac{1}{2} && \text{on } \partial\Omega. \end{aligned}$$

Show that $-1 \leq u \leq 1$ throughout Ω.

E3.13 **Maximum principle I**

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $v \in C^2(\Omega) \cap C(\overline{\Omega})$ are solutions of the following system of semilinear equations

$$\begin{aligned} \Delta u &= -u^2 - v^2 - 2uv && \text{in } \Omega, \\ \Delta v &= -v^2 && \text{in } \Omega, \\ u|_{\partial\Omega} &= v|_{\partial\Omega} = c && \text{on } \partial\Omega, \end{aligned}$$

where the constant $c > 0$.

(a) Show that the solutions u and v are non-negative, i.e. $u, v \geq 0$ in Ω.

(b) Consider their difference $w = u - v$ prove that u and v satisfy $u \geq v$ in Ω.

E3.14 **Maximum principle II**

Use separation of variables to find a nonzero solution for the Dirichlet problem in the strip,

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 \ : \ 0 < y < \pi\}$. What does this example tell us about the maximum principle?

Hint: *Assume that the solution is of the form $u(x, y) = X(x)Y(y)$ and solve the resulting ODEs for X and Y.*

E3.15 **Symmetry of Green's function**

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary $\partial\Omega$. Prove that if G is a Green's function on Ω, then

$$G(y, x) = G(x, y),$$

for all $x, y \in \Omega$, $x \neq y$.

You only need to prove the statement for $n \geq 3$, the case $n = 2$ can be shown similarly.

Hint: *For fixed $x, y \in \Omega$, $x \neq y$, consider*

$$v(z) = G(x, z), \qquad w(z) = G(y, z), \qquad z \in \Omega,$$

and show that $w(x) = v(y)$.

E3.16 **Existence result for the ball (Theorem 3.18)**

Let $g \in C(\partial B_r(0))$. Show that the function u given by Poisson's formula,

$$u(x) = \frac{r^2 - |x|^2}{r\omega_n} \int_{\partial B_r(0)} \frac{g(y)}{|y - x|^n} dS(y),$$

satisfies $u|_{\partial B_r(0)} = g$.

Hint: *First, conclude using the representation formula in Theorem 3.13 that*

$$\frac{r^2 - |x|^2}{r\omega_n} \int_{\partial B_r(0)} \frac{1}{|y - x|^n} dS(y) = 1.$$

Then, use Poisson's formula to show that

$$\lim_{x \to \hat{x}} |u(x) - g(\hat{x})| = 0,$$

if $\hat{x} \in \partial B_r(0)$.

E3.17 **Green's function for the half space**

Let Φ be the fundamental solution of the Laplace equation. For the half space

$$\Omega = \{x = (x_1, x_2, \ldots, x_n) \ : \ x_n > 0\}$$

let $x^* := (x_1, x_2, \ldots, -x_n)$ be the reflection of x on the plane $\partial\Omega$.

(i) Show that $G(x, y) = \Phi(y - x) - w^x(y)$ is the Green's function for the Laplace equation on Ω, where $w^x(y) = \Phi(y - x^*)$.

(ii) Find an integral representation for a solution $u \in C^2(\overline{\Omega})$ of

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

For this problem you can use results shown in the lecture notes for bounded domains without justifying their validity in unbounded domains.

E3.18 **Harnack's inequality**

Use Poisson's formula for the ball to prove that

$$r^{n-2}\frac{r - |x|}{(r + |x|)^{n-1}}u(0) \le u(x) \le r^{n-2}\frac{r + |x|}{(r - |x|)^{n-1}}u(0),$$

whenever u is positive, continuous in $\overline{B_r(0)}$, and harmonic in $B_r(0)$.

E3.19 **Energy estimates**

(a) Let $\Omega \subset \mathbb{R}^n$ be open and $u, v : \Omega \to \mathbb{R}$ be functions such that u^2 and v^2 are integrable over Ω. Show that for arbitrary $\varepsilon > 0$ the following inequality holds:

$$\|uv\|_{L^1(\Omega)} \;\le\; \frac{1}{2}\left(\frac{1}{\varepsilon}\|u\|^2_{L^2(\Omega)} + \varepsilon\|v\|^2_{L^2(\Omega)}\right).$$

(b) Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary $\partial\Omega$ and $f \in C(\overline{\Omega})$. Moreover, suppose that $u \in C^2(\overline{\Omega})$ is a solution of the boundary value problem

$$\begin{aligned} -\Delta u + \lambda u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

with some constant $\lambda > 0$. Use the inequality in (a) to show the estimate

$$\|\,|\nabla u|\,\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} \le \frac{1}{2\lambda}\|f\|^2_{L^2(\Omega)}.$$

E3.20 **Maximum principle III**

Let $R > 0$ and $w \in C^2(B_R(0)) \cap C(\overline{B_R(0)})$ be such that $f := \Delta w$ is bounded on $B_R(0)$. Show that

$$w \le R^2 \sup(f_-)/(2d) + \max_{\partial B_R(0)}(w_+),$$

where $a_+ = \max\{a, 0\}$ and $a_- = \max\{-a, 0\}$. Also prove that

$$|w| \le R^2 \sup|f|/(2d) + \max_{\partial B_R(0)}|w|.$$

Hint: Observe that the function $v(x) = R^2 - |x|^2$ satisfies $\Delta v = -2d$ and consider $u = \pm w - v\sup(f_\mp)/(2d)$.

Chapter 4

The Heat Equation

In this chapter, we consider the *heat equation*

$$u_t(t,x) - \Delta u(t,x) = 0, \qquad (t,x) \in \Omega, \tag{4.1}$$

and the *inhomogeneous heat equation*

$$u_t(t,x) - \Delta u(t,x) = f(t,x), \qquad (t,x) \in \Omega, \tag{4.2}$$

where $\Omega = (0,\infty) \times U$ and $U \subset \mathbb{R}^n$, $n \geq 1$, is open. Moreover, $f\colon [0,\infty) \times U \to \mathbb{R}$ is given and $u\colon [0,\infty) \times \overline{U} \to \mathbb{R}$ is the unknown. Here, $t > 0$ denotes time, $x \in U$ a point in space and $\Delta = \Delta_x$ is the Laplacian with respect to the space variable x.

4.1 Motivation

Typically, the heat equation (or *diffusion equation*) describes the time evolution of some quantity such as heat or a chemical concentration. Let $U \subset \mathbb{R}^n$ be open and $V \subset U$ be an arbitrary open and bounded subset with C^1-boundary. Moreover, we assume that $u(t,x)$ is the density of a physical quantity at time $t \geq 0$ at the point $x \in \overline{U}$. Then, the rate of change of the physical quantity within V equals the negative flux through the boundary ∂V, i.e.

$$\frac{d}{dt}\int_V u(t,x)\,dx = -\int_{\partial V} F(t,x)\cdot \nu(x)\,dS(x),$$

where $F\colon [0,\infty) \times U \to \mathbb{R}^n$ is the flux function. By the Gauß-Green theorem (Theorem 3.4), it follows that

$$\int_{\partial V} F(t,x)\cdot \nu(x)\,dS(x) = \int_V \operatorname{div} F(t,x)\,dx,$$

which implies that

$$\int_V u_t = -\int_V \operatorname{div} F.$$

In many cases, the flux function F is proportional to the (spatial) gradient of u, but points in the opposite direction (since particles flow from regions of higher to regions of lower concentration),

$$F = -a\nabla u,$$

for some constant $a > 0$. Consequently, we have

$$\int_V u_t = \int_V a\Delta u,$$

and since $V \subset U$ was arbitrary it follows that

$$u_t - a\Delta u = 0 \qquad \text{in } (0,\infty)\times U,$$

if $u \in C^2((0,\infty)\times U)$ (see Problem E3.2). For $a = 1$ we obtain the heat equation.

If, in addition, the physical quantity is generated by a source Q, then we obtain the inhomogeneous heat equation

$$u_t - a\Delta u = Q \qquad \text{in } (0,\infty)\times U$$

(cf. the derivation of Laplace's equation).

4.2 Fundamental solution

As we noticed in the case of the Laplace equation, an important step in studying a PDE is often to find a specific special solution (called *fundamental solution*) of the equation that allows to derive representation formulas for solutions.

To construct a fundamental solution we consider the heat equation (4.1) in $\Omega = (0,\infty)\times\mathbb{R}^n$ and exploit particular properties of the differential operator. If u is a solution, then for every $\lambda \in \mathbb{R}$ the function $u_\lambda(t,x) = (\lambda^2 t, \lambda x)$ also solves the heat equation (see Problem E.4.1). Together with the rotational invariance of the Laplace operator, this scaling invariance suggests to look for solutions of the form $u(t,x) = v\big(\frac{|x|}{\sqrt{t}}\big)$. Although this ansatz would lead to the solution, it turns out to be quicker to seek for solutions of the form

$$u(t,x) = t^\alpha v\left(\frac{|x|}{\sqrt{t}}\right) = t^\alpha v\left(\frac{r}{\sqrt{t}}\right), \tag{4.3}$$

for some $\alpha \in \mathbb{R}$ and a suitable function $v\colon [0,\infty) \to \mathbb{R}$, where $r = |x|$. We compute the partial derivatives,

$$\begin{aligned}
u_t(t,x) &= \alpha t^{\alpha-1} v\left(\frac{r}{\sqrt{t}}\right) - t^{\alpha-1}\frac{r}{2\sqrt{t}} v'\left(\frac{r}{\sqrt{t}}\right),\\
u_{x_i}(t,x) &= t^\alpha \frac{x_i}{r\sqrt{t}} v'\left(\frac{r}{\sqrt{t}}\right), && i = 1,\dots,n,\\
u_{x_i x_i}(t,x) &= t^{\alpha-1}\frac{x_i^2}{r^2} v''\left(\frac{r}{\sqrt{t}}\right) + \frac{t^\alpha}{\sqrt{t}} v'\left(\frac{r}{\sqrt{t}}\right)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right), && i = 1,\dots,n,
\end{aligned}$$

and hence, inserting the ansatz (4.3) into the heat equation leads to

$$\begin{aligned}
0 &= u_t(t,x) - \Delta u(t,x)\\
&= t^{\alpha-1}\left(\alpha v\left(\frac{r}{\sqrt{t}}\right) - v'\left(\frac{r}{\sqrt{t}}\right)\left(\frac{r}{2\sqrt{t}} + \frac{(n-1)t}{r\sqrt{t}}\right) - v''\left(\frac{r}{\sqrt{t}}\right)\right).
\end{aligned}$$

Denoting $s = \frac{r}{\sqrt{t}}$ and dividing the equation by $t^{\alpha-1}$ we obtain

$$\alpha v(s) - \left(\frac{s}{2} + \frac{n-1}{s}\right) v'(s) - v''(s) = 0.$$

Moreover, choosing $\alpha = -\frac{n}{2}$ we can rewrite this ODE as

$$\left(s^{n-1} v'(s)\right)' + \frac{1}{2} \left(s^n v(s)\right)' = 0,$$

and consequently,

$$\frac{1}{2} s^n v(s) + s^{n-1} v'(s) = c,$$

for some $c \in \mathbb{R}$. If we assume that $v(s) \to 0, v'(s) \to 0$ sufficiently fast as $s \to \infty$, then $c = 0$. We obtain

$$v'(s) = -\frac{s}{2} v(s),$$

which implies that

$$v(s) = be^{-\frac{s^2}{4}},$$

for some constant $b \in \mathbb{R}$. Recalling that $\alpha = -\frac{n}{2}$ and the ansatz (4.3), it follows that

$$u(t,x) = \frac{b}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, \qquad t > 0, x \in \mathbb{R}^n.$$

For the particular choice of the constant $b = \frac{1}{(4\pi)^{\frac{n}{2}}}$, the function $u(t,\cdot)$ is the density of the n-dimensional normal distribution $\mathcal{N}(0, 2t\mathrm{Id})$, and we obtain the fundamental solution of the heat equation.

Definition 4.1. The function $\Phi \colon (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^n \to \mathbb{R}$, defined by

$$\Phi(t,x) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & t > 0, x \in \mathbb{R}^n, \\ 0, & t < 0, x \in \mathbb{R}^n, \end{cases} \tag{4.4}$$

is called the **fundamental solution of the heat equation** (or *heat kernel*).

Lemma 4.2. *The fundamental solution* (4.4) *satisfies* $\Phi(t,\cdot) > 0$ *and*

$$\int_{\mathbb{R}^n} \Phi(t,x)\, dx = 1 \qquad \text{for all } t > 0.$$

Proof. The first statement is clear. To show the second one we observe that

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(t,x)\, dx &= \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\, dx \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{2}}\, dz = \prod_{i=1}^n \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z_i^2}{2}}\, dz_i}_{=1} = 1, \end{aligned} \qquad \square$$

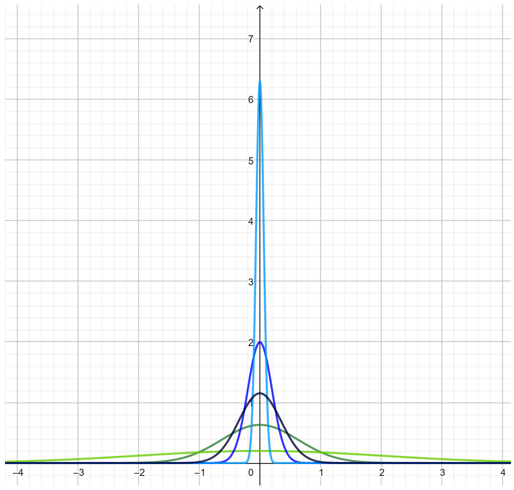
see Example A.7.

Lemma 4.3. *Let Φ be the fundamental solution in* (4.4). *For every compact interval* $[t_1, t_2] \subset (0, \infty)$ *and* $\alpha \in \mathbb{N}_0^{n+1}$ *there exists an integrable function* F_α *with*

$$|D^\alpha_{(t,x)}\Phi(t, x)| \le F_\alpha(x) \qquad \text{for all } (t, x) \in [t_1, t_2] \times \mathbb{R}^n.$$

Proof. See Problem E4.2. □

Below, the one-dimensional heat kernel is plotted for different time instances.



4.3 Initial value problems

We now use the fundamental solution to construct solutions of initial value problems for the heat equation in $(0, \infty) \times \mathbb{R}^n$.

4.3.1 Homogeneous case

Consider the initial value problem

$$\begin{aligned} u_t - \Delta u &= 0 \qquad \text{in } (0, \infty) \times \mathbb{R}^n, \\ u(0, \cdot) &= g \qquad \text{on } \mathbb{R}^n, \end{aligned} \tag{4.5}$$

where the given function $g \in C(\mathbb{R}^n)$ is bounded, i.e.

$$\|g\|_{L^\infty} := \sup_{x \in \mathbb{R}^n} |g(x)| < \infty.$$

Definition 4.4. A function $u \in C(\overline{(0, \infty) \times \mathbb{R}^n}) \cap C^{1,2}((0, \infty) \times \mathbb{R}^n)$ that satisfies (4.5) is called a **classical solution**, where

$$C^{1,2}((0, \infty) \times \mathbb{R}^n) := \Big\{ v \in C^1((0, \infty) \times \mathbb{R}^n)) \,:\, D_x^2 v \text{ exists and } D_x^2 v \in C((0, \infty) \times \mathbb{R}^n; \mathbb{R}^{n \times n}) \Big\}.$$

Note that Φ solves the heat equation away from the singularity in $t = 0$, and so does the function $(t, x) \mapsto \Phi(t, x - y)$ for every fixed $y \in \mathbb{R}^n$. This motivates that the *convolution*

$$u(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y) g(y)\, dy \tag{4.6}$$

is a solution of the heat equation as well. In fact, we will show that (4.6) indeed yields a classical solution of the initial value problem (4.5). Since we are integrating over the whole space $\mathbb{R}^n$ we need to be more careful when justifying that we can interchange differentiation and integration.

Theorem 4.5. *Let $g \in C(\mathbb{R}^n)$ be a bounded function. Then, the function u defined by* (4.6) *satisfies $u \in C^\infty((0,\infty) \times \mathbb{R}^n) \cap C(\overline{(0,\infty) \times \mathbb{R}^n})$. Moreover, u is a classical solution of* (4.5) *and*

$$\|u(t,\cdot)\|_{L^\infty} \leq \|g\|_{L^\infty} \qquad \forall t \geq 0.$$

Proof. Since g is bounded, it follows from the properties of the fundamental solution that the integrand $h(t,x,y) := \Phi(t, x-y)g(y)$, $(t,x,y) \in (0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n$ satisfies:

- For every fixed y, the function $h(\cdot,\cdot,y)$ is in $C^\infty((0,\infty) \times \mathbb{R}^n)$.
- For every fixed (t,x) the function $h(t,x,\cdot)$ is integrable on $\mathbb{R}^n$.
- For every compact set $I \times K \subset (0,\infty) \times \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^{n+1}$ we have

  $$|D^\alpha_{(t,x)} h(t,x,y)| \leq \|g\|_{L^\infty} \sup_{x \in K} F_\alpha(x-y) =: G_\alpha(y), \qquad (t,x,y) \in I \times K \times \mathbb{R}^n,$$

  by Lemma 4.3, and the function G_α is integrable.

Hence, by Theorems A.6 and A.5 (see also [5], Theorem 2 §11), we conclude that $u \in C^\infty((0,\infty) \times \mathbb{R}^n)$, and the derivatives can be computed by differentiation under the integral sign. Therefore, we obtain

$$u_t(t,x) - \Delta u(t,x) = \int_{\mathbb{R}^n} \underbrace{(\Phi_t - \Delta\Phi)(t, x-y)}_{=0}\, g(y)\, dy = 0,$$

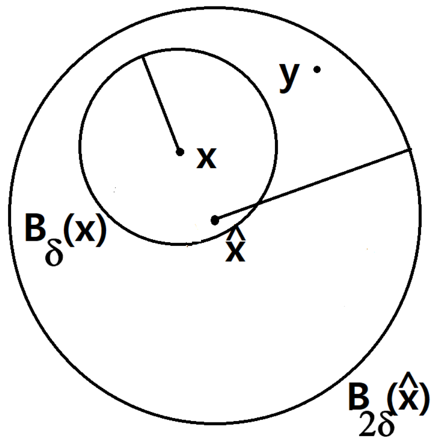
which shows that u satisfies the heat equation.

It remains to show that u fulfills the initial data, $u(0,\cdot) = g$, i.e. for $\hat{x} \in \mathbb{R}^n$ we have

$$u(t,x) \to g(\hat{x}) \qquad \text{as } (t,x) \to (0,\hat{x}).$$

Let $\hat{x} \in \mathbb{R}^n$ and $\varepsilon > 0$. Since g is continuous, there exists $\delta > 0$ such that

$$|g(y) - g(\hat{x})| < \varepsilon \qquad \forall |y - \hat{x}| < 2\delta. \tag{4.7}$$



Therefore, if $x \in \mathbb{R}^n$ with $|x - \hat{x}| < \delta$, then by Lemma 4.2 we have

$$\begin{aligned}|u(t,x) - g(\hat{x})| &= \left| \int_{\mathbb{R}^n} \Phi(t, x-y)(g(y) - g(\hat{x}))\, dy \right| \\ &\leq \underbrace{\int_{B_\delta(x)} \Phi(t,x-y)|g(y) - g(\hat{x})|\, dy}_{=:I} + \underbrace{\int_{\mathbb{R}^n \setminus B_\delta(x)} \Phi(t,x-y)|g(y) - g(\hat{x})|\, dy}_{=:J}.\end{aligned}$$

By (4.7) and since $B_\delta(x) \subset B_{2\delta}(\hat{x})$, it follows that

$$I \leq \varepsilon \int_{\mathbb{R}^n} \Phi(t, x-y)\, dy = \varepsilon.$$

For the second integral we obtain

$$\begin{aligned} J &\leq 2\|g\|_{L^\infty} \int_{\mathbb{R}^n \setminus B_\delta(x)} \Phi(t,x-y)\, dy \leq \frac{c}{t^{\frac{n}{2}}} \int_{\mathbb{R}^n \setminus B_\delta(x)} e^{-\frac{|x-y|^2}{4t}}\, dy \\ &= \frac{c\omega_n}{t^{\frac{n}{2}}} \int_\delta^\infty r^{n-1} e^{-\frac{r^2}{4t}}\, dr = c\omega_n \int_{\frac{\delta}{\sqrt{t}}}^\infty s^{n-1} e^{-\frac{s^2}{4}}\, ds \to 0 \quad \text{as } t \to 0, \end{aligned}$$

for some $c > 0$, where we used the change of variables $s = \frac{r}{\sqrt{t}}$ in the last step. Consequently, $|u(t,x) - g(\hat{x})| \leq 2\varepsilon$ for all $x \in B_\delta(\hat{x})$ and $t > 0$ sufficiently small, which shows that $u(0, \cdot) = g$.

The last statement of the theorem is a direct consequence of Lemma 4.2. □

Remark 4.6. In view of Theorem 4.5, the fundamental solution Φ formally satisfies the initial value problem

$$\begin{aligned} \Phi_t - \Delta\Phi &= 0 \qquad \text{in } (0,\infty) \times \mathbb{R}^n, \\ \Phi(0,\cdot) &= \delta_0 \qquad \text{on } \mathbb{R}^n, \end{aligned}$$

where δ_0 is the *Dirac measure* on $\mathbb{R}^n$ centered at $x = 0$. The Dirac measure (or Dirac distribution) is not a function in the usual sense. Formally, it has the properties

$$\int_{\mathbb{R}^n} \delta_0(x) dx = 1, \qquad \int_{\mathbb{R}^n} \delta_0(x)\varphi(x) dx = \varphi(0) \quad \forall \varphi \in C^\infty(\mathbb{R}^n),$$

but a rigorous definition requires the theory of distributions.

Remark 4.7 (*Infinite speed of progagation*). Let g be as in Theorem 4.5. Moreover, we assume that $g \geq 0$ and $g \not\equiv 0$. Then, the solution of (4.5) satisfies

$$u(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy > 0 \qquad \forall t > 0,\ x \in \mathbb{R}^n.$$

Due to this observation we say that the heat equation forces an *infinite speed of propagation of disturbances*. In the context of heat conduction, that means that if the initial temperature is nonnegative and positive somewhere in $\mathbb{R}^n$, then at any later time (no matter how short the time interval) the temperature is strictly positive everywhere. This is a characteristic property of the heat equation. As we will later see, the wave equation in contrast supports a *finite speed of propagation*.

Furthermore, we observe that the heat equation has an immediate *smoothing effect*. Even if the initial data g is only continuous, the solution is infinitely times continuously differentiable for all $(t,x) \in (0,\infty) \times \mathbb{R}^n$.

4.3.2 Inhomogeneous case

We now consider the inhomogeneous initial value problem

$$\begin{aligned} u_t - \Delta u &= f \qquad \text{in } (0,\infty) \times \mathbb{R}^n, \\ u(0,\cdot) &= 0 \qquad \text{on } \mathbb{R}^n, \end{aligned} \tag{4.8}$$

where, for simplicity, we assume that $f \in C_c^{1,2}((0,\infty) \times \mathbb{R}^n)$, i.e. $f \in C^{1,2}((0,\infty) \times \mathbb{R}^n)$ and $\operatorname{supp}(f) \subset (0,\infty) \times \mathbb{R}^n$ is compact.

Note that by Theorem 4.5, for fixed $s > 0$ the function

$$u(t,x;s) := \int_{\mathbb{R}^n} \Phi(t-s, x-y) f(s,y)\, dy, \qquad t > s,\ x \in \mathbb{R}^n,$$

solves the initial value problem

$$\begin{aligned} u_t(\,\cdot\,,\,\cdot\,;s) - \Delta u(\,\cdot\,,\,\cdot\,;s) &= 0 \qquad \text{in } (s,\infty) \times \mathbb{R}^n, \\ u(s,\cdot\,;s) &= f(s,\cdot\,) \qquad \text{on } \mathbb{R}^n. \end{aligned} \tag{4.9}$$

This is a homogeneous initial value problem of the form (4.5) with starting time $t = s$ and initial data $g = f(s,\cdot\,)$. To build a solution of the inhomogeneous problem (4.8) we apply *Duhamel's principle*. Namely, integrating $u(t,x;s)$ from $s = 0$ to $s = t$ leads to

$$u(t,x) = \int_0^t u(x,t;s)\, ds = \int_0^t \int_{\mathbb{R}^n} \Phi(t-s, x-y) f(s,y)\, dy\, ds, \tag{4.10}$$

for $t > 0$, $x \in \mathbb{R}^n$. The formal computation

$$(u_t - \Delta u)(t,x) = \underbrace{u(t,x;t)}_{=f(t,x)} + \int_0^t \big(\underbrace{u_t(t,x;s) - \Delta_x u(t,x;s)}_{=0}\big)\, ds = f(t,x)$$

indicates that the formula (4.10) indeed yields a solution. Due to the singularity of Φ, however, this formal calculation requires rigorous justification.

Theorem 4.8. *Let $f \in C_c^{1,2}((0,\infty)\times\mathbb{R}^n)$. Then, the function u defined by* (4.10) *satisfies $u \in C^{1,2}((0,\infty)\times\mathbb{R}^n)\cap C(\overline{(0,\infty)\times\mathbb{R}^n})$, $u_t - \Delta u = f$ in $(0,\infty)\times\mathbb{R}^n$ and for every $\hat{x}\in\mathbb{R}^n$*

$$u(t,x)\to 0 \qquad \text{as } (t,x)\to(0,\hat{x}),$$

i.e. u is a classical solution of the initial value problem (4.8).

Proof. First, we apply a change of variables and rewrite u as

$$u(t,x) = \int_0^t \int_{\mathbb{R}^n} \Phi(s,y) f(t-s,x-y)\,dy\,ds.$$

Since f has compact support, we can extend f by zero to a function $f \in C_c^{1,2}(\mathbb{R}^{n+1})$. Similarly as in the proof of Theorem 4.5, we can conclude that for any $\tau > 0$ the function

$$\tilde{u}(t,x) = \int_0^\tau \int_{\mathbb{R}^n} \Phi(s,y) f(t-s,x-y)\,dy\,ds$$

is in $C^{1,2}((0,\infty)\times\mathbb{R}^n)$ and its derivatives can be obtained by differentiation under the integral sign. Moreover, we observe that $f(t-s,\cdot) = 0$ if $|t-s| < \delta$ for sufficiently small $\delta > 0$, which implies that $u(t,\cdot) = \tilde{u}(t,\cdot)$ for $|t-\tau| < \delta$. Consequently, for $0 < \varepsilon < t$ we obtain

$$\begin{aligned} u_t(t,x) - \Delta u(t,x) &= \int_0^t \int_{\mathbb{R}^n} \Phi(s,y)(f_t - \Delta f)(t-s,x-y)\,dy\,ds \\ &= \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(s,y)(f_t - \Delta f)(t-s,x-y)\,dy\,ds \\ &\quad + \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(s,y)(-f_s - \Delta_y f)(t-s,x-y)\,dy\,ds =: I_\varepsilon + J_\varepsilon. \end{aligned}$$

For the first integral we have

$$|I_\varepsilon| \le \varepsilon \|f_t - \Delta f\|_{L^\infty} \max_{0<s<\varepsilon} \int_{\mathbb{R}^n} \Phi(s,y)\,dy = \varepsilon\|f_t - \Delta f\|_{L^\infty} \to 0 \quad \text{as } \varepsilon \to 0.$$

To estimate the second integral we apply integration by parts and use that f has compact support,

$$\begin{aligned} J_\varepsilon = &\int_\varepsilon^t \int_{\mathbb{R}^n} \underbrace{(\Phi_s - \Delta\Phi)(s,y)}_{=0} f(t-s,x-y)\,dy\,ds \\ &- \int_{\mathbb{R}^n} \big((\Phi(t,y)\underbrace{f(0,x-y)}_{=0} - \Phi(\varepsilon,y) f(t-\varepsilon,x-y)\big)\,dy \\ = &\int_{\mathbb{R}^n} \Phi(\varepsilon,y) f(t-\varepsilon,x-y))\,dy \to f(t,x) \qquad \text{as } \varepsilon \to 0. \end{aligned}$$

The last limit follows from Theorem 4.5 and the assumption $f \in C_c^{1,2}((0,\infty)\times\mathbb{R}^n)$ (see Problem E4.4).

Finally, we observe that

$$|u(t,x)| \le t\|f\|_{L^\infty} \max_{0<s<t} \int_{\mathbb{R}^n} \Phi(s,y)dy \le t\|f\|_{L^\infty} \to 0 \quad \text{as } t \to 0,$$

which shows that u satisfies the initial data.

□

Adding the solutions of (4.5) and (4.8) we obtain a solution for general inhomogeneous initial value problems.

Corollary 4.9. *Let $g \in C(\mathbb{R}^n)$ be bounded and $f \in C_c^{1,2}((0,\infty) \times \mathbb{R}^n)$. Then*

$$u(t,x) = \int_{\mathbb{R}^n} \Phi(t, x-y) g(y)\, dy + \int_0^t \int_{\mathbb{R}^n} \Phi(t-s, x-y) f(s,y)\, dy\, ds$$

is a classical solution of the initial value problem

$$\begin{aligned} u_t - \Delta u &= f \qquad \text{in } (0,\infty) \times \mathbb{R}^n, \\ u(0,\cdot) &= g \qquad \text{on } \mathbb{R}^n. \end{aligned}$$

Proof. This immediately follows from Theorem 4.5 and Theorem 4.4. □

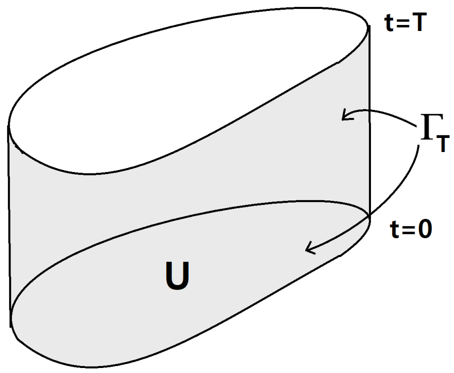
4.4 Maximum principles

We prove maximum principles for classical solutions of the heat equation. First, we consider bounded, open sets $U \subset \mathbb{R}^n$ and later the case $U = \mathbb{R}^n$. For $T > 0$ we define the *parabolic cylinder*

$$U_T := (0, T] \times U,$$

and the *parabolic boundary*

$$\Gamma_T := \overline{U_T} \setminus U_T = ([0,T] \times \partial U) \cup (\{0\} \times U).$$



Theorem 4.10. *Let $U \subset \mathbb{R}^n$ be open and bounded. Assume that $u \in C^{1,2}(U_T) \cap C(\overline{U_T})$ satisfies $u_t - \Delta u \leq 0$ in U_T. Then, the following statements hold:*

(i) Weak maximum principle:

$$\max_{(t,x) \in \overline{U_T}} u(t,x) = \max_{(t,x) \in \Gamma_T} u(t,x).$$

(ii) Strong maximum principle: If U is also connected and if there exists a point $(t_0, x_0) \in U_T$ with

$$u(t_0, x_0) = \max_{(t,x) \in \overline{U_T}} u(t,x),$$

then u is constant on $\overline{U_{t_0}}$.

Remark 4.11. Similar statements hold replacing u by $-u$ and the maxima by minima.

Note that if u is a solution of the heat equation and if u attains a maximum (or minimum) at an interior point $(x_0, t_0) \in U_T$, then u is constant *at all earlier times* $t \le t_0$ provided that the boundary and initial conditions are constant. However, the solution may change for $t > t_0$ if the boundary conditions alter at a later time $t > t_0$.

Proof. We only prove the weak maximum principle (i). The proof of the strong maximum requires a mean value formula for solutions of the heat equation (see, e.g. [4]).

Let L denote the differential operator $Lu := u_t - \Delta u$. First, suppose that $Lu < 0$ in U_T. We assume that u assumes a maximum in a point $(t_0, x_0) \in (0, T) \times U$. Then, $u_t(t_0, x_0) = 0$ and the Hessian $D^2u(t_0, x_0)$ is negative semidefinite. In particular, we have $u_{x_i x_i}(t_0, x_0) \le 0$ and therefore $Lu(t_0, x_0) = (u_t - \Delta u)(t_0, x_0) \ge 0$ which contradicts our assumption. Hence, we conclude that

$$\max_{(t,x)\in\overline{U_T}} u(t, x) = \max_{(t,x)\in\partial U_T} u(t, x).$$

Next, we show that the same holds true if $Lu \le 0$ in U_T. To this end consider the perturbed function $u_\varepsilon(t, x) = u(t, x) + \varepsilon e^{x_1}$ for $\varepsilon > 0$. We observe that

$$Lu_\varepsilon(t, x) = Lu(t, x) - \varepsilon e^{x_1} < 0, \qquad (t, x) \in U_T.$$

As we have shown above, this implies that

$$\max_{(t,x)\in\overline{U_T}} u_\varepsilon(t, x) = \max_{(t,x)\in\partial U_T} u_\varepsilon(t, x).$$

Taking the limit $\varepsilon \to 0$ yields the result for u.

It remains to show that u cannot attain a maximum in a point (T, x_0) with $x_0 \in U$. As before, we first assume that $Lu < 0$ in U_T. Suppose that u assumes a maximum in (T, x_0), $x_0 \in \Omega$. Then, $D^2u(T, x_0)$ is negative semidefinite and thus $-\Delta u(T, x_0) \ge 0$. We conclude that

$$0 > (u_t - \Delta u)(T, x_0) \ge u_t(T, x_0).$$

However, $u_t(T, x_0) < 0$ is a contradiction to the original assumption that $u(T, x_0)$ is a maximum.

Finally, the general case $Lu \le 0$ follows by considering the perturbed function $u_\varepsilon(t, x) = u(t, x) + \varepsilon e^{-t}$ for $\varepsilon > 0$. We obtain

$$Lu_\varepsilon(t, x) = Lu(t, x) - \varepsilon e^{-t} < 0, \qquad (t, x) \in U_T,$$

and hence u_ε cannot attain a maximum on $\{T\} \times U$. Taking the limit $\varepsilon \to 0$ the result remains valid for u. □

In order to derive maximum principles for the *unbounded domain* $U = \mathbb{R}^n$ we need an additional growth condition for the solutions.

Theorem 4.12. *Let $u \in C^{1,2}((0, T] \times \mathbb{R}^n) \cap C([0, T] \times \mathbb{R}^n)$ be a classical solution of the initial value problem* (4.5)*,*

$$\begin{aligned} u_t - \Delta u &= 0 \qquad \text{in } (0, T] \times \mathbb{R}^n, \\ u(0, \cdot) &= g \qquad \text{on } \mathbb{R}^n, \end{aligned}$$

where $g \in C(\mathbb{R}^n)$ is bounded, and assume that u satisfies the growth estimate

$$u(t,x) \le Ae^{a|x|^2}, \qquad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^n,$$

for some constants $a, A > 0$. Then,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} u(t,x) = \sup_{x\in\mathbb{R}^n} g(x).$$

Proof. First, we assume that $4aT < 1$. Then, there exists $\varepsilon > 0$ such that

$$4a(T+\varepsilon) < 1. \tag{4.11}$$

For fixed $y \in \mathbb{R}^n$ and $\delta > 0$ we consider the function

$$u_\delta(t,x) := u(t,x) - \frac{\delta}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{|x-y|^2}{4(T+\varepsilon-t)}}.$$

Note that we can rewrite u_δ as

$$u_\delta(t,x) = u(t,x) - \delta(4\pi)^{\frac{n}{2}}\Phi(T+\varepsilon-t, i(x-y)),$$

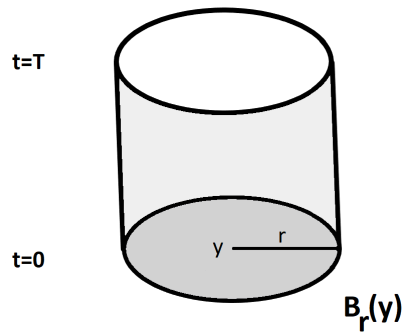
if we consider the fundamental solution Φ on $(0,\infty)\times\mathbb{C}^n$. It is easy to verify that the function $(t,x) \mapsto \Phi(T+\varepsilon-t, i(x-y))$ satisfies the heat equation on $(0,T]\times\mathbb{R}^n$. Consequently, we have

$$(u_\delta)_t - \Delta u_\delta = 0 \qquad \text{on } (0,T]\times\mathbb{R}^n,$$

i.e. u_δ solves the heat equation.

Let $U = B_r(y)$, for any $r > 0$. Then, Theorem 4.10 implies that

$$\max_{(t,x)\in\overline{U_T}} u_\delta(t,x) = \max_{(t,x)\in\Gamma_T} u_\delta(t,x).$$



For arbitrary $x \in U$ we have that

$$u_\delta(0,x) \le u(0,x) = g(x),$$

i.e. $u_\delta \le g$ on the subset $\{0\}\times U$. On the set $[0,T]\times\partial B_r(y)$ we have

$$u_\delta(t,x) = u(t,x) - \frac{\delta}{(T+\varepsilon-t)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\varepsilon-t)}} \le Ae^{a(|y|+r)^2} - \frac{\delta}{(T+\varepsilon)^{\frac{n}{2}}} e^{\frac{r^2}{4(T+\varepsilon)}}.$$

Due to (4.11) it follows that $\frac{1}{4(T+\varepsilon)} = a + \gamma$ for some $\gamma > 0$, and thus

$$u_\delta(t,x) \le Ae^{a(|y|+r)^2} - \delta(4(a+\gamma))^{\frac{n}{2}} e^{(a+\gamma)r^2} \le \sup_{x\in\mathbb{R}^n} g(x)$$

if we choose r large enough. Thus, we conclude that

$$u_\delta(t,x) \le \sup_{y\in\mathbb{R}^n} g(y) \qquad \text{for all } (t,x) \in [0,T]\times\mathbb{R}^n,$$

and taking the limit $\delta \to 0$, the same remains true for u.

Finally, if $4aT > 1$ we apply the result iteratively on subsequent time intervals of length $\frac{1}{8a}$. □

4.5 Uniqueness

A direct consequence of the maximum principle is the uniqueness of solutions. However, a uniqueness result in $\mathbb{R}^n$ does not hold without additional growth assumptions on the solutions such as in Theorem 4.12. In fact, one can show that there exist *infinitely many* solutions of the initial value problem

$$\begin{aligned} u_t - \Delta u &= 0 \qquad \text{in } (0,T]\times\mathbb{R}^n, \\ u(0,\cdot) &= 0 \qquad \text{on } \mathbb{R}^n, \end{aligned}$$

e.g. see [6], Chapter 7. Each of the solutions grows very rapidly, except for the trivial solution $u \equiv 0$, which is the only physical solution.

Theorem 4.13. *(i) Let $U \subset \mathbb{R}^n$ be open, bounded and connected. The initial-boundary value problem*

$$\begin{aligned} u_t - \Delta u &= f \qquad \text{in } U_T, \\ u &= g \qquad \text{on } \Gamma_T, \end{aligned}$$

where f and g are continuous functions, has at most one classical solution $u \in C^{1,2}(U_T) \cap C(\overline{U_T})$.

(ii) The initial value problem

$$\begin{aligned} u_t - \Delta u &= f \qquad \text{in } (0,T]\times\mathbb{R}^n, \\ u(0,\cdot) &= g \qquad \text{on } \mathbb{R}^n, \end{aligned}$$

where f and g are continuous functions, has at most one classical solution $u \in C^{1,2}((0,T]\times\mathbb{R}^n) \cap C([0,T]\times\mathbb{R}^n)$ satisfying the growth condition

$$|u(t,x)| \le Ae^{a|x|^2}, \qquad (t,x) \in [0,T]\times\mathbb{R}^n,$$

for some constants $a, A > 0$.

Proof. (i) Let u and v be two classical solutions of the initial-boundary value problem. Then, their difference $w = u - v$ satisfies

$$w_t - \Delta w = 0 \quad \text{in } U_T, \qquad w = 0 \quad \text{on } \Gamma_T.$$

The weak maximum principle in Theorem 4.10 applied to w and $-w$ implies that $w \le 0$ and $w \ge 0$ in $\overline{U_T}$ and consequently, $w \equiv 0$.

(ii) The statement can be shown similarly, see Problem E4.8. □

4.6 Energy methods

As for the Laplace and the Poisson equation, we now provide an alternative uniqueness proof for solutions of initial-boundary value problems that is based on energy methods.

Let $U \subset \mathbb{R}^n$ be open and bounded. We consider the heat equation

$$\begin{aligned} u_t - \Delta u &= f \qquad \text{in } U_T, \\ u(0,\cdot) &= g \qquad \text{on } U, \end{aligned} \tag{4.12}$$

with either homogeneous *Dirichlet boundary conditions*

$$u = 0 \qquad \text{on } [0,T] \times \partial U, \tag{4.13}$$

or homogeneous *Neumann boundary conditions*

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } [0,T] \times \partial U, \tag{4.14}$$

where $\frac{\partial u}{\partial \nu}$ denotes the normal derivative of u.

Theorem 4.14. *Let $U \subset \mathbb{R}^n$ be open and bounded with C^1-boundary ∂U. Suppose that $f \in C(\overline{U_T})$ and $g \in C(\overline{U})$. Then, every solution $u \in C^{1,2}(\overline{U_T})$ of the initial-boundary value problem* (4.12)-(4.13) *or* (4.12)-(4.14) *satisfies the energy estimate*

$$\|u(t,\cdot)\|_{L^2(U)}^2 + 2\|\nabla u\|_{L^2(U_t)}^2 \le e^t \left(\|g\|_{L^2(U)}^2 + \|f\|_{L^2(U_T)}^2 \right),$$

for all $t \in (0,T]$.

For the proof of theorem we need Gronwall's lemma.

Lemma 4.15 (Gronwall's inequality)**.** *Let $v \colon [0,T] \to [0,\infty)$ be an integrable function that satisfies*

$$v(t) \le a + b \int_0^t v(s)\, ds \qquad \forall t \in [0,T], \tag{4.15}$$

for some constants $a, b \ge 0$. Then, we have

$$v(t) \le a e^{bt} \le a(1 + bte^{bt}) \qquad \forall t \in [0,T].$$

Proof. Consider the function $h(t) := b \int_0^t v(s) ds$. Then, (4.15) is equivalent to

$$w(t) := v(t) - h(t) \le a \qquad \forall t \in [0,T].$$

We observe that

$$h'(t) = bv(t) = bw(t) + bh(t),$$

which is a linear ODE with constant coefficients. Its solution satisfies

$$h(t) = h(0)e^{bt} + \int_0^t e^{b(t-s)} bw(s) ds \le ab \int_0^t e^{b(t-s)} ds = a(e^{bt} - 1),$$

where we used that $h(0) = 0$ and $w(t) \le a$. Therefore, it follows that

$$v(t) \le a + h(t) \le a + a(e^{bt} - 1) = ae^{bt},$$

which proves the lemma. □

We now prove Theorem 4.14.

Proof of Theorem 4.14. Let $t \in (0, T]$. We multiply the heat equation (4.12) by $2u$ and integrate over U_t,

$$\int_0^t \int_U 2u(s,x)u_s(s,x) - 2u(s,x)\Delta u(s,x)\,dx\,ds = \int_0^t \int_U 2u(s,x)f(s,x)\,dx\,ds.$$

Observing that $2uu_s = (u^2)_s$ it follows that

$$\int_0^t \int_U 2u(s,x)u_s(s,x)\,dx\,ds = \int_U u^2(t,x)\,dx - \int_U g^2(x)\,dx.$$

Consequently, using integration by parts (see Theorem 3.5) we obtain

$$\|u(t,\cdot)\|^2_{L^2(U)} - \|g\|^2_{L^2(U)} - \int_0^t \int_{\partial U} \underbrace{2u(s,x)\frac{\partial u}{\partial \nu}(s,x)}_{=0 \text{ by (4.13) or (4.14)}}\,dS(x)\,ds + \int_0^t \int_U 2\nabla u(s,x)\cdot \nabla u(s,x)\,dx\,ds$$

$$= \int_0^t \int_U 2u(s,x)f(s,x)\,dx\,ds.$$

Using the estimate $2uf \le u^2 + f^2$ are rearranging the terms, we finally obtain

$$\|u(t,\cdot)\|^2_{L^2(U)} + 2\|\nabla u\|^2_{L^2(U_t)} \le \|g\|^2_{L^2(U)} + \|f\|^2_{L^2(U_T)} + \int_0^t \|u(s,\cdot)\|^2_{L^2(U)}\,ds.$$

By Lemma 4.15 applied to $v(t) = \|u(t,\cdot)\|^2_{L^2(U)} + 2\|\nabla u\|^2_{L^2(U_t)}$, it follows that

$$\|u(t,\cdot)\|^2_{L^2(U)} + 2\|\nabla u\|^2_{L^2(U_t)} \le e^t(\|g\|^2_{L^2(U)} + \|f\|^2_{L^2(U_T)})$$

for all $t \in [0, T]$. □

An immediate consequence is the uniqueness of initial-boundary value problems.

Corollary 4.16. *Let $U \subset \mathbb{R}^n$ be open and bounded. Moreover, let f and g be as in Theorem 4.14 and $h \in C([0,T] \times \partial U)$. Then, the initial-boundary value problem*

$$\begin{aligned} u_t - \Delta u &= f && \text{in } U_T,\\ u &= h && \text{on } [0,T]\times\partial U,\\ u(0,\cdot) &= g && \text{on } U \end{aligned}$$

has at most one classical solution $u \in C^{1,2}(\overline{U_T})$.

Proof. Let u and v be two solutions. Then, their difference $w = u - v$ satisfies the initial-boundary value problem (4.12)–(4.13) with $g \equiv 0$ and $f \equiv 0$. Thus, the energy estimate in Theorem 4.14 implies that

$$\|w(s,\cdot)\|_{L^2(U)} = 0, \qquad s \in (0, T],$$

and consequently, $w \equiv 0$. □

4.7 Exercises

E4.1 **Scalings**

Suppose that $u \in C^\infty((0,\infty)\times\mathbb{R}^n)$ solves the heat equation

$$u_t - \Delta u = 0 \qquad \text{in } (0,\infty)\times\mathbb{R}^n.$$

(a) Show that for every $\lambda \in \mathbb{R}$ the function $u_\lambda(t,x) := u(\lambda^2 t, \lambda x), (t,x) \in (0,\infty)\times\mathbb{R}^n$, also solves the heat equation.

(b) Show that the function $v(t,x) := x\cdot\nabla u(t,x) + 2t\,u_t(t,x), (t,x) \in (0,\infty)\times\mathbb{R}^n$, solves the heat equation as well.

Hint: You can deduce it from (a), or verify it by direct computation.

E4.2 **Derivatives of the fundamental solution**

Show that, for every compact interval $[t_0,t_1] \subset (0,\infty)$ and any $\alpha \in \mathbb{N}_0^{n+1}$, there exists an integrable function F_α with

$$|D^\alpha_{(t,x)}\Phi(t,x)| \le F_\alpha(x) \qquad \forall (t,x) \in [t_0,t_1]\times\mathbb{R}^n.$$

Hint: First show that for all $t \in [t_0,t_1]$ we have

$$|D^\alpha_x \Phi(t,x)| \le C(t_0)(1+|x|^{|\alpha|})e^{-\frac{|x|^2}{4t_1}},$$

for some constant $C(t_0) > 0$ depending on t_0. Then, show that the right hand side is integrable on $\mathbb{R}^n$. The integrable bounds for $|D^\alpha_{(t,x)}\Phi|$ can be deduced from the bounds for the x-derivatives.

E4.3 **Fourier's method and superposition principle**

Can you find an explicit solution of the following problem? Consider the one-dimensional heat equation

$$\begin{aligned} u_t(t,x) &= u_{xx}(t,x) && (t,x) \in (0,\infty)\times(0,1),\\ u(t,0) &= u(t,1) = 0 && t \ge 0,\\ u(0,x) &= f(x) && x \in [0,1], \end{aligned}$$

where $f(x) = \sum_{k=1}^n c_k \sin(k\pi x)$ and $c_1,\dots,c_n \in \mathbb{R}$.

Hint: First consider and solve the auxiliary problems

$$\begin{aligned} \frac{\partial}{\partial t}u_k(t,x) &= \frac{\partial^2}{\partial x^2}u_k(t,x),\\ u_k(t,0) &= u_k(t,1) = 0,\\ u_k(0,x) &= \sin(k\pi x). \end{aligned}$$

To this end use the method of separation of variables, i.e. assume that the solution is of the form $u_k(t,x) = g_k(x)h_k(t)$ and solve the resulting ODEs for g_k and h_k.

E4.4 **Inhomogeneous heat equation**

Let $\tau > 0$, $f \in C_c^{1,2}((0,\infty)\times\mathbb{R}^n)$ and Φ be the fundamental solution of the heat equation.

(a) Show that the function

$$\tilde{u}(t,x) = \int_0^\tau \int_{\mathbb{R}^n} \Phi(s,y) f(t-s, x-y)\,dyds$$

satisfies $\tilde{u} \in C^{1,2}((0,\infty)\times\mathbb{R}^n)$ and

$$\tilde{u}_t(t,x) - \Delta\tilde{u}(t,x) = \int_0^\tau \int_{\mathbb{R}^n} \Phi(s,y)(f_t - \Delta f)(t-s, x-y)\,dyds.$$

(b) Show that

$$\int_{\mathbb{R}^n} \Phi(\varepsilon, y) f(t-\varepsilon, x-y)\,dy \to f(t,x) \qquad \text{as } \varepsilon \to 0.$$

<u>Remark:</u> *These properties are used in the proof of Theorem 4.8.*

E4.5 **Heat equation on the half line**

Let $f \in C_c^{1,2}((0,\infty)\times(0,\infty))$ and consider the initial-boundary value problem

$$\begin{aligned} u_t(t,x) &= u_{xx}(t,x) + f(t,x), && t>0,\ x>0,\\ u(t,0) &= 0, && t>0,\\ u(0,x) &= 0, && x>0. \end{aligned} \tag{4.16}$$

(a) Show that for every solution v of the initial value problem

$$\begin{aligned} v_t(t,x) &= v_{xx}(t,x) + f(t,x), && t>0,\ x\in\mathbb{R},\\ v(0,x) &= 0, && x\in\mathbb{R}, \end{aligned}$$

the function $u = v + \tilde{v}$ is a solution of (4.16), where $\tilde{v}$ is the function obtained by odd reflection of v, i.e. $\tilde{v}(t,x) = -v(t,-x)$.

(b) Use (a) to show that the solution of (4.16) can be written as

$$u(t,x) = \int_0^t \int_0^\infty f(s,y)(\Phi(t-s, x-y) - \Phi(t-s, x+y))\,dyds,$$

where Φ is the fundamental solution of the heat equation.

E4.6 **Product Ansatz for the heat equation**

Let $u_j \in C^2((0,\infty)\times\mathbb{R})$ for $j = 1,2,\ldots,n$. Assume that u_j solves the one-dimensional heat equation $(u_j)_t - (u_j)_{xx} = 0$. Show that

$$u(t,x_1,x_2,\ldots,x_n) := \prod_{j=1}^n u_j(t,x_j), \qquad x\in\mathbb{R}^n, t>0,$$

is a solution for the heat equation in $\mathbb{R}^n$.

E4.7 **Periodic temperature fluctuations**

(a) Find $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{R}^n$ such that

$$u(t,x) := e^{i(\lambda t + \xi \cdot x)}, \qquad x \in \mathbb{R}^n, t > 0,$$

is a (bounded) solution for the heat equation.

(b) Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_1 > 0\}$ be the upper half space and let $\lambda > 0$. Use the ansatz

$$u(t,x) := e^{i(\lambda t + \xi_1 x_1)}, \qquad x \in \mathbb{R}^n_+, t > 0,$$

to find a bounded solution for the boundary value problem

$$\begin{aligned} v_t(t,x) - \Delta_x v(t,x) &= 0, && x \in \mathbb{R}^n_+, t > 0, \\ v(t,0,x_2,\dots,x_n) &= \cos(\lambda t), && x_2,\dots,x_n \in \mathbb{R}, \end{aligned}$$

where $v = \mathrm{Re}(u)$ is the real part of u, and $\xi_1 \in \mathbb{C}$ is to be determined.

(c) What is the behavior of the temperature distribution v in the set

$$\mathbb{R}_d = \{x \in \mathbb{R}^n \mid x_1 = d\},$$

for $d > 0$? In which distance d from the boundary R_0 of $\mathbb{R}^n_+$ did the amplitude decrease to half of the amplitude on the boundary?

E4.8 **Uniqueness in $\mathbb{R}^n$**

Let $f \in C^{1,2}((0,T] \times \mathbb{R}^n) \cap C([0,T] \times \mathbb{R}^n)$ and $g \in C(\mathbb{R}^n)$. Show that the initial value problem

$$\begin{aligned} u_t - \Delta u &= f && \text{in } (0,T] \times \mathbb{R}^n, \\ u(0,\cdot) &= g && \text{on } \mathbb{R}^n, \end{aligned}$$

has at most one classical solution satisfying the growth condition

$$|u(t,x)| \le A e^{a|x|^2}, \qquad (t,x) \in [0,T] \times \mathbb{R}^n,$$

for some constants $a, A > 0$.

E4.9 **Comparison principle**

Let $U \subset \mathbb{R}^n$ be open and bounded with C^1-boundary ∂U. Assume that $u_1, u_2 \in C^{1,2}(U_T) \cap C(\overline{U_T})$ are solutions of the (semilinear) initial-boundary value problems

$$\begin{aligned} \partial_t u_i(t,x) - \Delta u_i(t,x) &= f(t,x,u_i(t,x)), && (t,x) \in U_T, \\ u_i(t,x) &= g_i(t,x), && (t,x) \in \Gamma_T, \end{aligned}$$

where $f_i \in C(U_T \times \mathbb{R})$ and $g_i \in C(\Gamma_T)$, $i = 1,2$. Show that if

$$\begin{aligned} f(t,x,u_1) &\le f(t,x,u_2) && \text{in } U_T \times \mathbb{R}, \\ g_1(t,x) &\le g_2(t,x) && \text{on } \Gamma_T, \end{aligned}$$

then, the solutions satisfy $u_1 \le u_2$ in U_T.

E4.10 **Maximum principle**

Let $U \subset \mathbb{R}^n$ be open and bounded and assume that $u \in C^{1,2}(U_T)$. Consider the partial differential operator

$$Lu := u_t - \Delta u + b \cdot \nabla u + cu \qquad \text{on } U_T,$$

with $b \in \mathbb{R}^n$ and $c \in C(\overline{U_T})$. Show that if $Lu \leq 0$ and $c \geq 0$ on U_T then

$$\max_{\overline{U_T}} u \leq \max_{\Gamma_T} u^+,$$

where $u^+ = \max\{u, 0\}$ is the positive part of u.

Hint: *Use ideas applied in the proof of the maximum principle for the heat equation. First, consider the case that* $Lu > 0$*, and then extend the result for the case* $Lu \geq 0$.

E4.11 **Energy methods**

Let $U \subset \mathbb{R}^n$ be open and bounded with C^1-boundary, where $f \in C(U_T)$ and $g \in C(U)$. Use the energy method to prove uniqueness of classical solutions $u \in C^{1,2}(\overline{(0,\infty) \times U})$ of the following initial-boundary value problem,

$$\begin{aligned} u_t(t,x) - \Delta u(t,x) + c^2 u(t,x) &= f(t,x), && t > 0, x \in U, \\ u(0,x) &= g(x), && x \in U, \\ au(t,x) + \frac{\partial u}{\partial \nu}(t,x) &= 0, && t > 0, x \in \partial U, \end{aligned}$$

where $c \in \mathbb{R}, a > 0$ and $\frac{\partial u}{\partial \nu}$ denotes the normal derivative of u.

Hint: *Let* u, v *be two solutions and consider their difference* $w = u - v$*. Multiply the resulting PDE for* w *by* w *and integrate over* U.

Chapter 5

The Wave Equation

In this chapter we analyze the *wave equation*

$$u_{tt} - \Delta u = 0 \qquad \text{in } (0, \infty) \times \mathbb{R}^n, \tag{5.1}$$

and the *inhomogeneous wave equation*

$$u_{tt} - \Delta u = f \qquad \text{in } (0, \infty) \times \mathbb{R}^n, \tag{5.2}$$

where $f\colon [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is given and $u\colon [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is the unknown. As for the heat equation, $\Delta u = \Delta_x u$ is the Laplacian with respect to the spatial variable $x \in U$ and $t \geq 0$ denotes time.

5.1 Motivation

The wave equation is a simplified model for a vibrating string ($n = 1$), a membrane ($n = 2$) or an elastic solid ($n = 3$). In these cases, the solution $u(t, x)$ denotes the displacement in a point $x \in \mathbb{R}^n$ at time $t > 0$.

Let $V \subset \mathbb{R}^n$ be any open set with C^1-boundary. If the mass density is taken to be unity, the acceleration within V is given by

$$\frac{d^2}{dt^2} \int_V u(t, x)\, dx = \int_V u_{tt}(t, x)\, dx,$$

and the net contact force is

$$-\int_{\partial V} F(t, x) \cdot \nu(x)\, dS(x),$$

where F is the force acting through the boundary ∂V on V. By Newton's law the net force equals mass times acceleration and hence,

$$\int_V u_{tt}(t, x)\, dx = -\int_{\partial V} F(t, x) \cdot \nu(x)\, dS(x).$$

The Gauß–Green Theorem (Theorem 3.4) now implies that

$$\int_V u_{tt}(t, x)\, dx = -\int_V \operatorname{div}_x F(t, x)\, dx.$$

For elastic bodies, F is a function of the displacement gradient ∇u, and for small displacements the linearization

$$F(t,x) \approx -a\nabla u(t,x),$$

for some $a > 0$, is often appropriate. This leads to the integral equation

$$\int_V u_{tt}(t,x)\,dx = \int_V a\Delta u(t,x)\,dx,$$

and if $u \in C^2((0,\infty)\times V)$, it follows that

$$u_{tt} - a\Delta u = 0 \qquad \text{in } (0,\infty)\times\mathbb{R}^n,$$

since V was arbitrary (see Problem E3.2). If there is an additional volume force Q acting, we would obtain the inhomogeneous wave equation

$$u_{tt} - a\Delta u = Q \qquad \text{in } (0,\infty)\times\mathbb{R}^n.$$

In both cases, rescaling of the time variable leads to (5.1) and (5.2).

The physical interpretation of the wave equation suggests that we specify two initial conditions, the *initial displacement* $u(0,\cdot)$ and the *initial velocity* $u_t(0,\cdot)$. First, we analyze the initial value problem for the homogeneous wave equation,

$$\begin{aligned} u_{tt} - \Delta u &= 0 && \text{in } (0,\infty)\times\mathbb{R}^n, \\ u(0,\cdot) = u_0, \quad u_t(0,\cdot) &= u_1 && \text{on } \mathbb{R}^n, \end{aligned} \tag{5.3}$$

where $u_0 \in C^2(\mathbb{R}^n)$ and $u_1 \in C^1(\mathbb{R}^n)$.

Definition 5.1. A function $u \in C^2([0,\infty)\times\mathbb{R}^n)$ that satisfies the IVP (5.3) is called a **classical solution**.

5.2 D'Alembert's formula (1D)

In this section, we consider the one-dimensional case, $n = 1$. Let u be a classical solution of the IVP (5.3). Note that we can write the wave equation as

$$0 = u_{tt} - u_{xx} = (\partial_t - \partial_x)(\partial_t + \partial_x)u.$$

Thus, if we define $v := u_t + u_x$, then v satisfies the initial value problem for a linear homogeneous transport equation,

$$\begin{aligned} v_t(t,x) - v_x(t,x) &= 0, && (t,x) \in (0,\infty)\times\mathbb{R}, \\ v(0,x) &= u_1(x) + u_0'(x), && x \in \mathbb{R}. \end{aligned}$$

By Theorem 2.1, the unique solution is given by

$$v(t,x) = u_1(x+t) + u_0'(x+t),$$

and hence, u satisfies the inhomogeneous transport equation

$$\begin{aligned} u_t(t,x) + u_x(t,x) &= v(t,x) = u_1(x+t) + u_0'(x+t), && (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(0,x) &= u_0, && x \in \mathbb{R}. \end{aligned}$$

By Theorem 2.3, the unique solution of this initial value problem is

$$\begin{aligned} u(t,x) &= u_0(x-t) + \int_0^t u_1(x+(s-t)+s) + u_0'(x+(s-t)+s)ds \\ &= u_0(x-t) + \frac{1}{2}(u_0(x+t) - u_0(x-t)) + \int_0^t u_1(x+2s-t)ds. \end{aligned}$$

Finally, the substitution $y = x + 2s - t$ yields **D'Alembert's formula**

$$u(t,x) = \frac{1}{2}(u_0(x-t) + u_0(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)dy. \tag{5.4}$$

Remark 5.2. Another way to solve the wave equation is via *characteristic coordinates*,

$$\xi = x + t, \qquad \eta = x - t.$$

Then, the wave equation takes the form

$$u_{\xi\eta} = 0,$$

and the solutions of this equation are of the form

$$u(t,x) = \phi(\xi) + \psi(\eta) = \phi(x+t) + \psi(x-t).$$

Using the initial conditions we obtain d'Alembert's formula (see Problem E1.3).

Conversely, (5.4) can be rewritten in the form $u(t,x) = \phi(x+t) + \psi(x-t)$ with suitable functions $\phi, \psi \in C^2(\mathbb{R})$, and it is easy to see that functions of this form satisfy the wave equation. This general form shows the simple geometry of the wave equation. The solution is a combination of two waves, $\phi(x+t)$ is traveling to the right with speed 1 and $\psi(x-t)$ is traveling to the left with speed 1.

Theorem 5.3. *Let* $u_0 \in C^2(\mathbb{R})$ *and* $u_1 \in C^1(\mathbb{R})$. *Then,* d'Alembert's formula (5.4) *defines the unique classical solution* $u \in C^2([0,\infty)\times\mathbb{R})$ *of the initial value problem for the wave equation* (5.3) *in the case* $n = 1$.

Proof. The regularity of u follows immediately from the regularity of u_0 and u_1. That the function u defined by d'Alembert's formula satisfies the wave equation and attains the initial values in (5.3) can be verified by direct calculation.

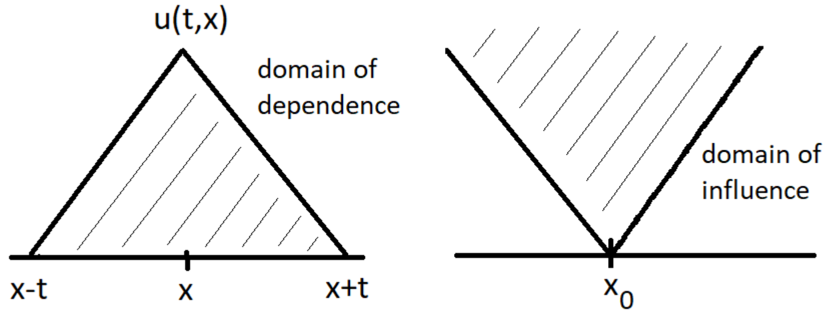
Furthermore, the above derivation shows that any classical solution u of the initial value problem (5.3) satisfies (5.4), and the formula defines u uniquely. □

Remark 5.4.

- We observe that the smoothness of the solution of the wave equation depends on the smoothness of the initial conditions. This is essentially different from the heat and Laplace equation, whose solutions are infinitely times continuously differentiable.

- While the heat equation forces an infinite speed of propagation of disturbances, the wave equation has a *finite speed of propagation* of information. In fact, let $x \in \mathbb{R}$ and $t > 0$. Then, $u(t, x)$ is uniquely determined by the values of u_1 in the interval $[x-t, x+t]$ and by the values of u_0 at the endpoints of this interval.

  Conversely, if $y \in \mathbb{R}$, the values $u_0(y)$ and $u_1(y)$ influence the value of the solution $u(t, x)$ for those values of x and t such that $y - t \leq x \leq y + t$. In particular, if $x_0 \in \mathbb{R}$, $r > 0$ and u_0 and u_1 vanish in the interval $|x - x_0| \leq r$ then $u(t, x) = 0$ for all t and x such that $|x - x_0| \leq r - t$.



5.3 Spherical means

We aim to find solution formulas for the wave equation in higher space dimensions $n \geq 2$. This can be done by studying *spherical means* of the solution, i.e. averages over certain spheres. Namely, for $x \in \mathbb{R}^n$, $t > 0$ and $r > 0$ we define

$$U(x; t, r) := \fint_{\partial B_r(x)} u(t, y)\, dS(y).$$

It turns out that these averages as functions of t and r satisfy a PDE that, for odd space dimensions, can be converted into a one-dimensional wave equation. Using d'Alembert formula and transforming back then leads to a solution formula for the wave equation in higher odd dimensions.

Using the transformation formula, we can rewrite U as

$$U(x; t, r) = \fint_{\partial B_r(0)} u(t, x + y)\, dS(y) = \frac{1}{\omega_n} \int_{\partial B_1(0)} u(t, x + ry)\, dS(y),$$

and by setting

$$U(x; t, r) := U(x; t, -r)$$

we get an extension for all $r \in \mathbb{R}$. Note that whenever u is of class C^k the extension $U(x; \cdot)$ is C^k as well. Similarly, we define the (extended) spherical means for the initial data u_0 and u_1 by

$$U_0(x; r) := \frac{1}{\omega_n} \int_{\partial B_1(0)} u_0(x + ry)\, dS(y),$$
$$U_1(x; r) := \frac{1}{\omega_n} \int_{\partial B_1(0)} u_1(x + ry)\, dS(y).$$

We remark that the solution u can be recovered from U by taking the limit $r \to 0$,

$$u(t,x) = \lim_{r\to 0} U(x;t,r) = U(x;t,0).$$

As in the proof of Theorem 3.6 (mean value formulas), we conclude that for $r > 0$

$$U_r(x;t,r) = \frac{1}{|\partial B_r(x)|}\int_{B_r(x)} \Delta u(t,y)\,dy = \frac{1}{|\partial B_r(0)|}\int_{B_r(0)} \Delta_x u(t,x+y)\,dy.$$

Since $|\partial B_r(0)| = \left(\frac{r}{\rho}\right)^{n-1}|\partial B_\rho(0)|$, the right hand side can be expressed in terms of U, namely,

$$\begin{aligned} U_r(x;t,r) &= \int_0^r \frac{\left(\frac{r}{\rho}\right)^{1-n}}{|\partial B_\rho(0)|}\int_{\partial B_\rho(0)} \Delta_x u(t,x+y)\,dS(y)\,d\rho \\ &= r^{1-n}\Delta_x \int_0^r \rho^{n-1} U(x;t,\rho)\,d\rho. \end{aligned}$$

Multiplying the equation by r^{n-1} and taking the derivative with respect to r we obtain

$$\partial_r(r^{n-1}U_r) = r^{n-1}\Delta_x U.$$

Thus, for $r > 0$ the spherical means satisfy *Darboux's equation*

$$U_{rr} + \frac{n-1}{r}U_r = \Delta_x U.$$

On the other hand, u is a solution of the wave equation $u_{tt} - \Delta u = 0$, and hence, we obtain

$$\begin{aligned} \Delta_x U(x;t,r) &= \Delta_x \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} u(t,x+y)\,dS(y) \\ &= \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} \Delta_x u(t,x+y)\,dS(y) \\ &= \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} u_{tt}(t,x+y)\,dS(y) = U_{tt}(x;t,r). \end{aligned}$$

That is, U satisfies the initial value problem

$$\begin{aligned} U_{tt}(x;t,r) - U_{rr}(x;t,r) - \frac{n-1}{r}U_r(x;t,r) &= 0 && \text{in } (0,\infty)\times(0,\infty), \\ U(x;0,\cdot) = U_0(x,\cdot), \quad U_t(x;0,\cdot) &= U_1(x,\cdot) && \text{on } (0,\infty). \end{aligned}$$

By definition, our extension of U is an even function in r and hence,

$$\begin{aligned} U_r(x;t,-r) &= -U_r(x;t,r), \\ U_{rr}(x;t,-r) &= U_{rr}(x;t,r). \end{aligned}$$

Therefore, the PDE for U holds for all $r \neq 0$, and we obtain the following result.

Theorem 5.5. *Let u be a classical solution of the initial value problem* (5.3)*. Then for all $x \in \mathbb{R}^n$ the spherical means $U(x;\cdot,\cdot)$ satisfy the* Euler–Poisson–Darboux equation

$$U_{tt}(x;t,r) - U_{rr}(x;t,r) - \frac{n-1}{r}U_r(x;t,r) = 0, \qquad t \in (0,\infty),\ r \in \mathbb{R},$$

and the initial data

$$U(x;0,r) = U_0(x;r), \qquad U_t(x;0,r) = U_1(x;r), \qquad r \in \mathbb{R}.$$

Proof. That the spherical means satisfy the Euler–Poisson–Darboux equation was shown above. Moreover, we note that for $x \in \mathbb{R}^n$ and $t \geq 0$ the function $h(r) := U(x;t,r)$ satisfies $h \in C^2(\mathbb{R})$. Since h is even, the derivative h' is odd, and hence, $h(0) = 0$. By L'Hospital's rule we conclude that

$$\lim_{r\to 0}\frac{h'(r)}{r}$$

exists and therefore, the Euler–Poisson–Darboux equation can be considered for all $r \in \mathbb{R}$. □

5.4 Kirchhoff's formula (3D)

For $n = 3$ the Euler–Poisson–Darboux equation is

$$U_{tt}(x;t,r) - U_{rr}(x;t,r) - \frac{2}{r}U_r(x;t,r) = 0, \qquad t \in (0,\infty), r \in \mathbb{R}. \tag{5.5}$$

This PDE can be transformed into a one-dimensional wave equation which can be solved by d'Alembert's formula.

Theorem 5.6. *Let u be a classical solution of the initial value problem* (5.3) *in three dimensions, $n = 3$. Then, for all $x \in \mathbb{R}^3$ the function*

$$\tilde{U}(x;t,r) := rU(x;t,r), \qquad t > 0, r \in \mathbb{R},$$

where U denotes the spherical mean, is a classical solution of the initial value problem for the one-dimensional wave equation

$$\begin{aligned} \tilde{U}_{tt}(x;t,r) - \tilde{U}_{rr}(x;t,r) &= 0, && t \in (0,\infty), r \in \mathbb{R}, \\ \tilde{U}(x;0,r) = rU_0(x;r), \quad \tilde{U}_t(x;0,r) &= rU_1(x;r), && r \in \mathbb{R}. \end{aligned} \tag{5.6}$$

Moreover, if $u_0 \in C^3(\mathbb{R}^3)$ and $u_1 \in C^2(\mathbb{R}^3)$, then u is given by ***Kirchhoff's formula***

$$u(t,x) = \tilde{U}_r(x;t,0) = \frac{1}{4\pi t^2}\int_{\partial B_t(x)} (u_0(y) + \nabla u_0(y)\cdot(y-x) + tu_1(y))\, dS(y), \tag{5.7}$$

$t \geq 0, x \in \mathbb{R}^3$.

Proof. Differentiation implies that

$$\tilde{U}_{tt} = rU_{tt}, \quad \tilde{U}_r = rU_r + U, \quad \tilde{U}_{rr} = 2U_r + rU_{rr}.$$

Hence by the Euler–Poisson–Darboux equation (5.5) we conclude that

$$\tilde{U}_{tt} - \tilde{U}_{rr} = r\left(U_{tt} - \frac{2}{r}U_r - U_{rr}\right) = 0 \qquad \text{in } (0,\infty)\times\mathbb{R}.$$

We observe that $\tilde{U} \in C^2([0,\infty)\times\mathbb{R})$ and the initial conditions follow directly from the definition of $\tilde{U}$ and from those for U. Hence, $\tilde{U}$ is a classical solution of the initial value problem (5.6) and by d'Alembert's formula, we obtain

$$\tilde{U}(x;t,r) = \frac{1}{2}((r-t)U_0(x;r-t) + (r+t)U_0(x;,r+t)) + \frac{1}{2}\int_{r-t}^{r+t} yU_1(x;y)\,dy.$$

Moreover, taking the derivative with respect to r we observe that

$$\tilde{U}_r(x;t,0) = 0U_r(x;t,0) + U(x;t,0) = u(t,x)$$

and thus,

$$\begin{aligned} u(t,x) &= \tilde{U}_r(x;t,0) \\ &= \frac{1}{2}\left(U_0(x;-t) + U_0(x;t) - t(U_0'(x;-t) - U_0'(x;t)) + t(U_1(x;t) + U_1(x;-t))\right). \end{aligned}$$

To differentiate the last term we used the Leibniz rule. Since U_0 and U_1 are even in r, it follows that

$$u(t,x) = U_0(x;t) + tU_0'(x;t) + tU_1(x;t).$$

Finally, we observe that

$$\begin{aligned} U_0'(x;t) &= \frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)} \nabla u_0(x+ty)\cdot y\,dS(y) \\ &= \frac{1}{|\partial B_t(x)|}\int_{\partial B_t(x)} \nabla u_0(z)\cdot\frac{z-x}{t}\,dS(z), \end{aligned}$$

and since $|\partial B_t(x)| = 4\pi t^2$, we obtain Kirchhoff's formula

$$u(t,x) = \frac{1}{4\pi t^2}\int_{\partial B_t(x)} (u_0(y) + \nabla u_0(y)\cdot(y-x) + tu_1(y))\,dS(y). \qquad \square$$

Remark 5.7. This method can be generalized for arbitrary odd space dimensions. In fact, for $n = 2k+1, k \in \mathbb{N}$, the Euler–Poisson–Darboux equation in Theorem 5.5 can be reduced to the wave equation in one space dimension via the function

$$\tilde{U}(x;t,r) := \left(\tfrac{1}{r}\partial_r\right)^{k-1}(r^{2k-1}U(x;t,r)).$$

For details, see e.g. [4].

We have shown that every classical solution of (5.3) satisfies Kirchhoff's formula. We now prove the existence of classical solutions of the initial value problem by verifying that Kirchhoff's formula indeed provides a classical solution.

Theorem 5.8. *Let $n = 2k+1$, $k \in \mathbb{N}$, and $u_0 \in C^{k+2}(\mathbb{R}^n)$, $u_1 \in C^{k+1}(\mathbb{R}^n)$. Then the initial value problem for the wave equation* (5.3) *has a unique classical solution $u \in C^2([0,\infty)\times\mathbb{R}^n)$.*

For $n = 3$ it is given by Kirchhoff's formula (5.7). *Moreover, $u(t,x)$, $t > 0$ and $x \in \mathbb{R}^3$, only depends on the initial data on $\partial B_t(x)$.*

Proof. We prove the theorem only for $n = 3$, for the general case, see [4].

First, we show that Kirchhoff's formula provides a classical solution. We can rewrite the formula as

$$\begin{aligned} u(t,x) &= \frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)} u_0(x+ty) + t\nabla u_0(x+ty)\cdot y + tu_1(x+ty)\,dS(y) \\ &= \frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)} \frac{d}{dt}(tu_0(x+ty)) + tu_1(x+ty)\,dS(y). \end{aligned} \tag{5.8}$$

Due to the regularity of the initial data, the integrand is twice continuously differentiable with respect to t, x and y and $\partial B_1(0)$ is compact. Therefore, we can interchange differentiation and integration and the right hand side is in $C^2([0,\infty)\times\mathbb{R}^n)$.

Let us consider the case $u_0 \equiv 0$ first. Then, (5.7) can be written as

$$u(t,x) = \frac{t}{|\partial B_1(0)|}\int_{\partial B_1(0)} u_1(x+ty)\,dS(y) = tU_1(x;t).$$

One the one hand, this implies that

$$\Delta u(t,x) = \frac{t}{|\partial B_t(0)|}\int_{\partial B_t(0)} \Delta u_1(x+y)\,dy.$$

On the other hand, differentiating with respect to t we obtain

$$u_t = (tU_1)_t = U_1 + t(U_1)_t, \qquad u_{tt} = (tU_1)_{tt} = 2(U_1)_t + t(U_1)_{tt}.$$

As previously observed for the spherical means (see the proof of Theorem 3.6 and the derivation of the Euler-Poisson-Darboux equation), it follows that

$$(U_1)_t(x;t) = \frac{1}{|\partial B_t(0)|}\int_{B_t(0)} \Delta u_1(x+y)\,dy.$$

Moreover, note that $|\partial B_t(0)| = t^2|\partial B_1(0)|$ which implies that $\frac{d}{dt}\frac{1}{|\partial B_t(0)|} = -\frac{2}{t|\partial B_t(0)|}$. Consequently, we have

$$(U_1)_{tt}(x;t) = -\frac{2}{t}(U_1)_t(x;t) + \frac{1}{|\partial B_t(0)|}\int_{\partial B_t(0)} \Delta u_1(x+y)\,dS(y),$$

and we conclude that

$$u_{tt} = 2(U_1)_t + t(U_1)_{tt} = \frac{t}{|\partial B_t(0)|}\int_{\partial B_t(0)} \Delta u_1(x+y)\,dS(y) = \Delta u(t,x).$$

For the general case that $u_0 \not\equiv 0$ we note that also the term involving u_0 in Kirchhoff's formula satisfies the wave equation. Indeed, if we replace u_1 by u_0 in the arguments above, it follows that the function

$$v(t,x) = \frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)} tu_0(x+ty)\,dS(y) = tU_0(x;t)$$

solves the wave equation as well. Moreover, v is three times continuously differentiable and therefore, v_t also satisfies the wave equation. Finally, we observe that

$$v_t(t,x) = \frac{1}{|\partial B_1(0)|} \int_{\partial B_1(0)} \frac{d}{dt}(tu_0(x+ty))\, dS(y)$$

which is the first term in the formula (5.8).

It remains to show that the function u given by (5.7) satisfies the initial conditions. To this end we observe that by (5.8), $u(0,x) = u_0(x)$ and

$$u_t(0,x) = (U_1 + t(U_1)_t)(0,x) = u_1(x).$$

The statement concerning the dependence of the solution on the initial data directly follows from Kirchhoff's formula (5.7). □

Remark 5.9. If we compare Kirchhoff's formula with d'Alembert's formula we observe that the latter does not involve derivatives of the initial data. Hence, the solution of the wave equation at $t > 0$ for $n > 1$ may be less regular than the initial data.

5.5 Poisson's formula (2D)

For even dimensions a reduction of the Euler–Poisson–Darboux equation to a one-dimensional wave equation is not possible. Instead, we consider the initial value problem for $n = 2$ as a problem in three space dimensions. More precisely, assuming that $u \in C^2([0,\infty)\times\mathbb{R}^2)$ is a solution of (5.3) for $n = 2$ let

$$\bar{u}(t,x,x_3) := u(t,x), \qquad (t,x,x_3) \in [0,\infty)\times\mathbb{R}^2\times\mathbb{R}.$$

Then, $\bar{u}$ satisfies the initial value problem

$$\begin{aligned} \bar{u}_{tt} - \Delta\bar{u} &= 0 && \text{in } (0,\infty)\times\mathbb{R}^3, \\ \bar{u}(0,\cdot) = \bar{u}_0, \quad \bar{u}_t(0,\cdot) &= \bar{u}_1 && \text{on } \mathbb{R}^3, \end{aligned}$$

where $\bar{u}_0(x,x_3) := u_0(x)$ and $\bar{u}_1(x,x_3) := u_0(x)$, $x \in \mathbb{R}^2, x_3 \in \mathbb{R}$. We can now apply Kirchhoff's formula (5.7) and obtain

$$u(t,x) = \frac{1}{4\pi t^2} \int_{\partial B_t^3(x)} (\bar{u}_0(y) + \nabla\bar{u}_0(y)\cdot(y-x) + t\bar{u}_1(y))\, dS(y).$$

We observe that the integrand does not depend on the third space variable which allows us to simplify the formula.

To distinguish dimensions we denote by $B_r^k(x)$ a ball in $\mathbb{R}^k$ with radius $r > 0$ around x. Let w be a continuous function that is independent of y_3. Then, using the parametrization $(y_1, y_2, \pm\gamma(y_1,y_2))$ with $\gamma(y_1,y_2) = \sqrt{r^2 - y_1^2 - y_2^2}$ for $\partial B_r^3(0)$, it follows that

$$\begin{aligned} \int_{\partial B_r^3(0)} w(y)\, dS(y,y_3) &= 2\int_{B_r^2(0)} w(y)\sqrt{1+|\nabla\gamma(y)|^2}\, dy \\ &= 2\int_{B_r^2(0)} \frac{r}{\sqrt{r^2-|y|^2}}\, w(y)\, dy. \end{aligned}$$

Thus, for initial data $\bar{u}_0$ and $\bar{u}_1$ Kirchhoff's formula yields **Poisson's formula**

$$\begin{aligned} u(t,x) &= \frac{1}{4\pi t^2}\int_{\partial B_t^3(x)} (\bar{u}_0(y) + \nabla \bar{u}_0(y)\cdot(y-x) + t\bar{u}_1(y))\,dS(y) \\ &= \frac{1}{2\pi t}\int_{B_t(x)} \frac{u_0(y) + \nabla u_0(y)\cdot(y-x) + tu_1(y)}{\sqrt{t^2 - |y-x|^2}}\,dy, \end{aligned} \tag{5.9}$$

for the solution of the initial value problem (5.3) for $n = 2$.

This approach to first consider and solve the problem in dimension $n = 3$ is called the *method of descent*.

Remark 5.10. For general even space dimensions $n = 2k, k \in \mathbb{N}$, the method of descent can be applied by considering the solution $\bar{u}(t, x, x_{n+1}) = u(t, x)$ of the $(n+1)$-dimensional wave equation.

We now show that Poisson's formula provides a classical solution of the initial value problem (5.3) for $n = 2$, if the initial data is sufficiently regular.

Theorem 5.11. *Let $n = 2k$, $k \in \mathbb{N}$, and $u_0 \in C^{k+2}(\mathbb{R}^n)$, $u_1 \in C^{k+1}(\mathbb{R}^n)$. Then, the initial value problem for the wave equation* (5.3) *has a classical solution $u \in C^2([0,\infty)\times\mathbb{R}^n)$.*

For $n = 2$ it is given by Poisson's formula (5.9)*. Moreover, $u(t,x)$, $t > 0$, $x \in \mathbb{R}^2$, only depends on the initial data in $\overline{B_t(x)}$.*

Proof. We only prove the theorem for $n = 2$. A proof of the general case can be found in [4].

Poisson's formula is Kirchhoff's formula for initial data that are independent of the third space variable x_3. Therefore, by Theorem 5.8 it provides a classical solution for the initial value problem (5.3) for $n = 2$.

The statement concerning the dependence of solutions on initial data directly follows from Poisson's formula (5.9). □

The set of points in $[0,\infty)\times\mathbb{R}^n$ that determine the value $u(t,x)$, $t > 0, x \in \mathbb{R}^n$, is called the *domain of dependence* and forms a cone. If we change the initial conditions outside of this region, the value $u(t,x)$ will not change. Similarly, for $x_0 \in \mathbb{R}^n$ the *domain of influence* consists of all points in $[0,\infty)\times\mathbb{R}^n$ that are influenced by the values of the initial data in x_0. This set forms an inverted cone.

Remark 5.12. Theorem 5.8 and Theorem 5.11 reveal two important properties of the wave equation.

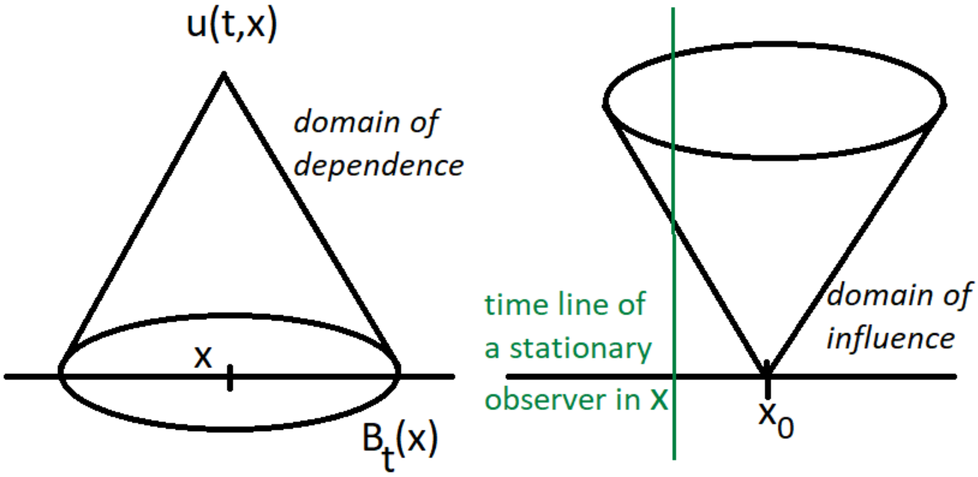
- For $n \geq 2$ the solution can be less regular than the initial data. This is caused by a *focusing effect*, i.e. irregularities in u_0 may focus at a later time and cause u to be less regular. This is essentially different from solutions of the heat equation that are C^∞ for $t > 0$ if the initial data is bounded and continuous.

- The wave equation exhibits a *finite speed of propagation*, i.e. the value $u(t,x)$ depends only on the values of u_0 and u_1 in the set $\overline{B_t(x)}$. In contrast, the solution of the heat equation depends on the initial data u_0 in the whole space $\mathbb{R}^n$.

Huygen's principle: In dimension $n = 3$, by Kirchhoff's formula, the initial data in a given point $x \in \mathbb{R}^3$ only affect the solution on the boundary $\{(t,y) \ : \ t > 0, |y-x| = t\}$ of the cone

$C = \{(t, y) \ : \ t > 0, |y - x| < t\}$. That is, a disturbance in the point x propagates along a sharp wave front.

On the other hand, in dimension $n = 2$, by Poission's formula, the initial data in a given point $x \in \mathbb{R}^n$ affect the solution in the whole set $\overline{C}$. That is, a disturbance in the point x continues to have an effect even after the leading edge of the wavefront has passed.



5.6 Inhomogeneous initial value problems

We now consider the initial value problem for the inhomogeneous wave equation with vanishing initial data,

$$\begin{aligned} u_{tt} - \Delta u = f \qquad & \text{in } (0, \infty) \times \mathbb{R}^n, \\ u(0, \cdot) = 0 \quad u_t(0, \cdot) = 0 \qquad & \text{on } \mathbb{R}^n, \end{aligned} \tag{5.10}$$

where $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is given.

As previously done for the heat equation we apply *Duhamel's principle* and define

$$u(t, x) = \int_0^t \hat{u}(t, x; s) \, ds, \tag{5.11}$$

where $\hat{u}(t, x; s)$, $s \geq 0$, is the solution of the initial value problem

$$\begin{aligned} \hat{u}_{tt}(\cdot, \cdot; s) - \Delta \hat{u}(\cdot, \cdot; s) = 0 \qquad & \text{in } (s, \infty) \times \mathbb{R}^n, \\ \hat{u}(s, \cdot; s) = 0, \quad \hat{u}_t(s, \cdot; s) = f(s, \cdot) \qquad & \text{on } \mathbb{R}^n, \end{aligned} \tag{5.12}$$

given by Theorem 5.8 and Theorem 5.11.

For $a \in \mathbb{R}$ we denote by $\lfloor a \rfloor$ the greatest integer less than or equal to a.

Theorem 5.13. *Let $n \geq 1$ and $f \in C^{\lfloor \frac{n}{2} \rfloor + 1}([0, \infty) \times \mathbb{R}^n)$. Then u defined by* (5.11) *is a classical solution of the initial value problem* (5.10).

Proof. By Theorem 5.8 and Theorem 5.11 it follows that for $s \geq 0$ the solution $\hat{u}(\cdot,\cdot;s)$ of the initial value problem (5.12) is in $C^2([s,\infty)\times\mathbb{R}^n)$. Consequently, the function u defined by (5.11) satisfies $u \in C^2([0,\infty)\times\mathbb{R}^n)$. Furthermore, computing the derivatives we observe that

$$\begin{aligned}
u_t(t,x) &= \int_0^t \hat{u}_t(t,x;s)\,ds + \hat{u}(t,x;t) = \int_0^t \hat{u}_t(t,x;s)\,ds,\\
u_{tt}(t,x) &= \int_0^t \hat{u}_{tt}(t,x;s)\,ds + \hat{u}_t(t,x;t) = \int_0^t \hat{u}_{tt}(t,x;s)\,ds + f(t,x),\\
\Delta u(t,x) &= \int_0^t \Delta\hat{u}(t,x;s)\,ds.
\end{aligned}$$

Therefore, since $\hat{u}(\cdot,\cdot;s)$ satisfies the wave equation, it follows that

$$u_{tt}(t,x) - \Delta u(t,x) = \int_0^t \hat{u}_{tt}(t,x;s) - \Delta\hat{u}(t,x;s)\,ds + f(t,x) = f(t,x).$$

Moreover, we observe that $u(0,x) = 0$ and $u_t(0,x) = 0$ for $x \in \mathbb{R}^n$, which shows that the initial conditions are satisfied as well. □

An explicit representation formula for the dimensions $n = 1,2,3$ can be derived from (5.11) (see Exercises).

Corollary 5.14. *Let $u_0 \in C^{\lfloor\frac{n}{2}\rfloor+2}(\mathbb{R}^n)$, $u_1 \in C^{\lfloor\frac{n}{2}\rfloor+1}(\mathbb{R}^n)$ and $f \in C^{\lfloor\frac{n}{2}\rfloor+1}([0,\infty)\times\mathbb{R}^n)$. Then, there exists a classical solution of the general inhomogeneous problem*

$$\begin{aligned}
u_{tt} - \Delta u &= f \qquad \text{in } (0,\infty)\times\mathbb{R}^n,\\
u(0,\cdot) = u_0, \quad u_t(0,\cdot) &= u_1 \qquad \text{on } \mathbb{R}^n.
\end{aligned}$$

Proof. The solution can be obtained by adding the solutions of the initial value problems (5.11) and (5.3). □

5.7 Energy methods

The explicit solution formulas for the wave equation show that with increasing space dimension n higher and higher regularity assumptions are required for the initial data u_0 and u_1 in order to obtain a classical solution. Energy "norms" are an alternative to measure the size and regularity of solutions. In this section we use energy methods to prove the uniqueness and to examine the domain of dependence of solutions.

We derive an energy inequality for solutions of the initial value problem

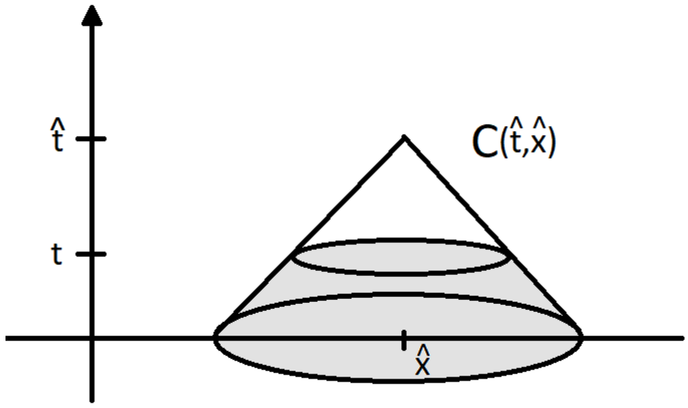
$$\begin{aligned}
u_{tt} - \Delta u &= f \qquad \text{in } (0,\infty)\times\mathbb{R}^n,\\
u(0,\cdot) = u_0, \quad u_t(0,\cdot) &= u_1 \qquad \text{on } \mathbb{R}^n.
\end{aligned} \tag{5.13}$$

To this end for a given point $(\hat{t},\hat{x}) \in (0,\infty)\times\mathbb{R}^n$ we define the *backwards wave cone* (domain of dependence) by

$$C(\hat{t},\hat{x}) = \{(s,x) \,:\, 0 \leq s \leq \hat{t},\ |x-\hat{x}| \leq \hat{t}-s\},$$

and for $t \in (0,\hat{t})$ the *cone sections* by

$$C(t;\hat{t},\hat{x}) = \{(s,x) \,:\, 0 \leq s \leq t,\ |x-\hat{x}| \leq \hat{t}-s\}.$$



Theorem 5.15. *Let $u_0 \in C^2(\mathbb{R}^n)$, $u_1 \in C^1(\mathbb{R}^n)$ and $f \in C([0,\infty)\times\mathbb{R}^n)$. Then, any classical solution $u \in C^2([0,\infty)\times\mathbb{R}^n)$ of* (5.13) *satisfies for all $(\hat{t},\hat{x}) \in (0,\infty)\times\mathbb{R}^n$ and $\varepsilon > 0$ the energy estimate*

$$\begin{aligned}\|u_t(t,\cdot)\|^2_{L^2(B_{\hat{t}-t}(\hat{x}))} &+ \|\nabla u(t,\cdot)\|^2_{L^2(B_{\hat{t}-t}(\hat{x}))} \\ &\le e^{\varepsilon t}\left(\|u_1\|^2_{L^2(B_{\hat{t}}(\hat{x}))} + \|\nabla u_0\|^2_{L^2(B_{\hat{t}}(\hat{x}))} + \frac{1}{\varepsilon}\|f\|^2_{L^2(C(\hat{t},\hat{x}))}\right),\end{aligned}$$

where $t \in (0,\hat{t})$. If $f \equiv 0$, then $\varepsilon = 0$ is allowed.

Proof. Let $\hat{t} > 0, \hat{x} \in \mathbb{R}^n$ and $\varepsilon > 0$. To shorten notations we write $B_t = B_{\hat{t}-t}(\hat{x})$. Defining the energy

$$e(t) = \int_{B_t} \left(u_t^2(t,\cdot) + |\nabla u(t,\cdot)|^2\right), \qquad t > 0,$$

we observe that

$$e'(t) = \int_{B_t} (2u_t(t,\cdot)u_{tt}(t,\cdot) + 2\nabla u_t(t,\cdot)\cdot\nabla u(t,\cdot)) - \int_{\partial B_t}\left(u_t^2(t,\cdot) + |\nabla u(t,\cdot)|^2\right) dS,$$

since $\int_{B_t} g(y)dy = \int_0^{\hat{t}-t}\int_{\partial B_\rho(\hat{x})} g(y)dS(y)d\rho$ for a continuous function g. Using integration by parts it follows that

$$\begin{aligned}\int_{B_t}(2u_tu_{tt} + 2\nabla u_t\cdot\nabla u) &= \int_{B_t} 2u_t(u_{tt} - \Delta u) + \int_{\partial B_t} 2u_t\nabla u\cdot\nu\, dS \\ &= \int_{B_t} 2u_t f + \int_{\partial B_t} 2u_t\nabla u\cdot\nu\, dS,\end{aligned}$$

where we used that u is a solution of the inhomogeneous wave equation (5.13). We further estimate the right hand side using the inequalities $2ab \le a^2 + b^2$ and $2ab \le \varepsilon a^2 + \frac{b^2}{\varepsilon}$, $a, b \in \mathbb{R}$,

$$\int_{B_t} 2u_t f + \int_{\partial B_t} 2u_t\nabla u\cdot\nu\, dS \le \varepsilon\|u_t\|^2_{L^2(B_t)} + \frac{1}{\varepsilon}\|f\|^2_{L^2(B_t)} + \int_{\partial B_t}\left(u_t^2 + |\nabla u|^2\right) dS.$$

Consequently, this estimate implies that

$$e'(t) \le \varepsilon\|u_t(t,\cdot)\|^2_{L^2(B_t)} + \frac{1}{\varepsilon}\|f(t,\cdot)\|^2_{L^2(B_t)} \le \varepsilon e(t) + \frac{1}{\varepsilon}\|f\|^2_{L^2(B_t)},$$

and integrating the inequality from 0 to t we obtain

$$e(t) \leq e(0) + \frac{1}{\varepsilon}\|f\|^2_{L^2(C(t;\hat{t},\hat{x}))} + \varepsilon \int_0^t e(s)ds.$$

Finally, the energy inequality follows by Gronwall's lemma applied to the function e. □

An immediate consequence of the energy estimate in Theorem 5.15 is the uniqueness of solutions.

Corollary 5.16. *Let u_0, u_1 and f be as in Theorem 5.15. Then, there exists at most one classical solution of the initial value problem* (5.13)*.*

Proof. Let u and v be two classical solution. Their difference $w = u - v$ satisfies the initial value problem (5.13) with $u_0 = u_1 = f \equiv 0$. Hence, the energy inequality in Theorem 5.15 implies that $w_t \equiv \nabla w \equiv 0$ in $C(\hat{t}, \hat{x})$ for all $\hat{t} > 0$ and $\hat{x} \in \mathbb{R}^n$. Since $w(0, \cdot) \equiv 0$, we conclude that $w \equiv 0$. □

The energy estimate in Theorem 5.15 also provides an alternative proof for the finite speed of propagation for solutions of the homogeneous wave equation.

Corollary 5.17. *Let $u_0 \in C^2(\mathbb{R}^n)$, $u_1 \in C^1(\mathbb{R}^n)$ and u be a classical solution of the homogeneous initial value problem* (5.3)*.*

If $\hat{t} > 0, \hat{x} \in \mathbb{R}^n$ and $u_0 \equiv u_1 \equiv 0$ on $B_{\hat{t}}(\hat{x})$, then $u \equiv 0$ within the cone $C(\hat{t}, \hat{x})$.

Proof. By the energy inequality for $f \equiv 0$ and $u_0 \equiv u_1 \equiv 0$ in Theorem 5.15 it follows that $e(t) = 0$ for all $0 \leq t \leq \hat{t}$. Hence, we conclude that $u_t \equiv \nabla u \equiv 0$ on $C(\hat{t}, \hat{x})$, which implies that $u \equiv 0$ on $C(\hat{t}, \hat{x})$ as $u(0, \cdot) \equiv 0$. □

We notice that any disturbance originating outside of $B_{\hat{t}}(\hat{x})$ has no effect on the solution within the cone $C(\hat{t}, \hat{x})$, and consequently, has a finite speed of propagation. We had already observed this property based on the representation formulas for solutions in dimensions $n = 1, 2, 3$. Energy methods provide a much simpler proof and do not require the knowledge of explicit solution formulas.

5.8 Exercises

E5.1 **An initial value problem**

Find the solution of the initial value problem

$$\begin{aligned} u_{xx} - 3u_{xt} - 4u_{tt} &= 0 && \text{in } (0,\infty) \times \mathbb{R}, \\ u(0,x) = x^2, \quad u_t(0,x) &= e^x, && x \in \mathbb{R}. \end{aligned}$$

Hint: *Factor the partial differential operator as done in the lecture for the one-dimensional wave equation.*

E5.2 **Spherical waves**

A *spherical wave* is a solution of the three-dimensional wave equation of the form $u(t, r)$, where $r = |x|$ is the distance to the origin. In spherical coordinates the wave equation takes the form

$$u_{tt} = u_{rr} + \frac{2}{r}u_r \qquad \text{in } (0,\infty) \times (0,\infty).$$

(a) Use the change of variables $v = ru$ and show that v satisfies $v_{tt} = v_{rr}$. Note that you can consider the equation for $t > 0$ and $r \in \mathbb{R}$.

(b) Find a solution of the spherical wave equation with the initial conditions $u(0, r) = u_0(r)$, $u_t(0, r) = u_1(r)$, assuming that u_0 and u_1 are even functions of r. To this end, first use d'Alembert's formula to solve the wave equation for v.

E5.3 **Equipartition of the energy**

Let $u \in C^2([0, \infty) \times \mathbb{R})$ solve the one-dimensional wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } (0, \infty) \times \mathbb{R}, \\ u(0, \cdot) &= u_0 && \text{on } \{0\} \times \mathbb{R}, \\ u_t(0, \cdot) &= u_1 && \text{on } \{0\} \times \mathbb{R}. \end{aligned}$$

Suppose that $u_0 \in C^2(\mathbb{R}), u_1 \in C^1(\mathbb{R})$ have compact support. The *kinetic energy* is

$$k(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(t, x)\, dx$$

and the *potential energy* is

$$p(t) = \frac{1}{2} \int_{-\infty}^{\infty} u_x^2(t, x)\, dx.$$

Prove the following:

(a) $k(t) + p(t)$ is constant, that is, the total energy is conserved.

(b) $k(t) = p(t)$ for all large enough times t.

Hint: By d'Alembert's formula

$$\begin{aligned} u_t(t, x) &= \phi'(x + t) - \psi'(x - t), \\ u_x(t, x) &= \phi'(x + t) + \psi'(x - t), \end{aligned}$$

where $\phi' = \frac{1}{2}(u_0' + u_1)$ *and* $\psi' = \frac{1}{2}(u_0' - u_1)$.

E5.4 **Wave equation in 1D**

Consider the initial value problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } (0, \infty) \times \mathbb{R}, \\ u = u_0, \quad u_t &= u_1 && \text{on } \{0\} \times \mathbb{R}, \end{aligned} \tag{5.14}$$

where $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$.

(a) Verify that the function given by d'Alembert's formula is a classical solution of the initial value problem (5.14).

(b) Consider the inhomogeneous initial value problem

$$\begin{aligned} u_{tt} - u_{xx} &= f && \text{in } (0, \infty) \times \mathbb{R}, \\ u = 0, \quad u_t &= 0 && \text{on } \{0\} \times \mathbb{R}, \end{aligned} \tag{5.15}$$

where $f \in C^2([0,\infty)\times\mathbb{R})$.

A solution can be found by *Duhamel's principle*: Show that the unique solution of the initial-value problem (5.15) is given by

$$u(t,x) = \int_0^t \hat{u}(t,x;s)ds,$$

where $\hat{u}$ solves

$$\begin{aligned} \hat{u}_{tt}(t,x;s) - \hat{u}_{xx}(t,x;s) &= 0, && t > s, x \in \mathbb{R}, \\ \hat{u}(s,x;s) = 0, \quad \hat{u}_t(s,x;s) &= f(s,x), && x \in \mathbb{R}, \end{aligned}$$

for $s > 0$.

E5.5 **Duhamel's principle** ($n = 3$)

Use Duhamel's principle together with Kirchhoff's formula to get an explicit formula for the solution for the equation

$$\begin{aligned} u_{tt} - \Delta u &= f && \text{in } (0,\infty)\times\mathbb{R}^3, \\ u(0,\cdot) = 0, \quad u_t(0,\cdot) &= 0 && \text{on } \mathbb{R}^3, \end{aligned}$$

To obtain a classical solution, which regularity do you need to require for f?

E5.6 **Propagation of singularities**

Use d'Alembert's formula to express the solution of the initial value problem for the wave equation

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } (0,\infty)\times\mathbb{R}, \\ u(0,\cdot) = u_0, \quad u_t(0,\cdot) &= u_1 && \text{on } \mathbb{R}, \end{aligned}$$

where

(a) $u_1(x) = 0$ and

$$u_0(x) = \begin{cases} 1 & \text{if } -1 \le x \le 1, \\ 0 & \text{else.} \end{cases}$$

(b) $u_0(x) = 0$ and

$$u_1(x) = \begin{cases} 1 & \text{if } -1 \le x \le 1, \\ 0 & \text{else.} \end{cases}$$

Plot $u(t_0, x)$ for $t_0 = 0, \frac{1}{2}, 1, 2$, $x \in \mathbb{R}$.

In case (a), where are the discontinuities of u in space-time $(t,x) \in (0,\infty)\times\mathbb{R}$?

In case (b), is u discontinuous? Is the derivative u_x discontinuous, and if so, where?

Remark: Note that the function u is not a classical solution of the initial value problem.

E5.7 **Wave equation in $\mathbb{R}^3$**

Let u be a solution of the initial value problem

$$\begin{aligned} u_{tt} - \Delta u &= 0 && \text{in } (0,\infty)\times\mathbb{R}^3,\\ u(0,\cdot) = u_0, \qquad u_t(0,\cdot) &= u_1 && \text{on } \mathbb{R}^3. \end{aligned}$$

We assume that the support of the functions $u_0 \in C^3(\mathbb{R})$ and $u_1 \in C^2(\mathbb{R})$ is compact. Use Kirchhoff's formula to show that u satisfies

$$|u(t,x)| \le \frac{c}{t}, \qquad \text{for all } x \in \mathbb{R}^3, t > 0,$$

for some constant $c > 0$.

E5.8 **Energy and momentum density**

For a solution $u(t,x)$ of the one-dimensional wave equation $u_{tt} = u_{xx}$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$. Assume that $u \in C^3((0,\infty)\times\mathbb{R})$.

(a) Show that $\frac{\partial e}{\partial t} = \frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial t} = \frac{\partial e}{\partial x}$.

(b) Show that both, $e(t,x)$ and $p(t,x)$ also satisfy the wave equation.

E5.9 **Separation of variables**

Consider the initial-boundary value problem for the wave equation in a bounded one-dimensional interval,

$$\begin{aligned} u_{tt}(t,x) - u_{xx}(t,x) &= 0 && t > 0, x \in (0,2\pi),\\ u(t,0) = u(t,2\pi) &= 0 && t \ge 0,\\ u(0,x) = u_0(x), \quad u_t(0,x) &= u_1(x) && x \in (0,2\pi), \end{aligned}$$

where $u_0 = \sum_{k=1}^n \frac{a_k}{\sqrt{\pi}} \sin(\frac{kx}{2})$ and $u_1 \sum_{k=1}^n \frac{b_k}{\sqrt{\pi}} \sin(\frac{kx}{2})$ or some constants $a_k, b_k \in \mathbb{R}$.

Show that the general solution can be written as

$$u(t,x) = \sum_{k=1}^n \left(a_k \cos(\frac{kt}{2}) + b_k \frac{\sin(\frac{kt}{2})}{\frac{k}{2}} \right) \frac{1}{\sqrt{\pi}} \sin(\frac{kx}{2}).$$

To this end use the method of *separation of variables*, i.e. assume that the solution is of the form $u(t,x) = g(x)h(t)$ and solve the resulting ODEs for g and h.

E5.10 **Wave equation in bounded domains**

Let $U \subset \mathbb{R}^n$ be open and bounded with C^1 boundary ∂U. For $T > 0$ we define

$$U_T := U \times (0,T], \qquad \Gamma_T := \overline{U_T} \setminus U_T.$$

Consider the initial boundary value problem

$$\begin{aligned} u_{tt} - \Delta u &= f && \text{in } U_T,\\ u &= g && \text{on } \Gamma_T,\\ u_t &= h && \text{on } \{t = 0\} \times U, \end{aligned} \tag{5.16}$$

where the functions f, g and h are twice continuously differentiable. Use energy methods to prove that there exists at most one classical solution $u \in C^2(\overline{U_T})$ of (5.16).

<u>Hint:</u> For $w \in C^2(\overline{U_T})$ consider the energy

$$e(t) := \int_U \underbrace{w_t^2(t,x)}_{E_{kin}} + \underbrace{|\nabla w(t,x)|^2}_{E_{pot}} \, dx, \qquad t \in [0,T].$$

E5.11 **Damped vibrating string**

Let u describe the displacement from equilibrium of a flexible, elastic, infinite string. If significant air resistance $r > 0$ is present, the vibrating string is modeled by a wave equation with an additional term proportional to the speed u_t,

$$u_{tt} - u_{xx} + r u_t = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}.$$

Let u denote a solution of the *damped wave equation* and assume that there exists $R > 0$ such that u vanishes for $|x| \geq R$. Show that the energy

$$e(t) = \int_{-\infty}^{\infty} u_t^2(t,x) + u_x^2(t,x) dx$$

is preserved if $r = 0$ and *decreases* in time if $r > 0$.

E5.12 **Maximum principle**

Recall the maximum principle for the heat equation (Theorem 4.10). Construct a counterexample to show that such a result does not hold for the wave equation.

Chapter 6

Nonlinear First Order PDEs

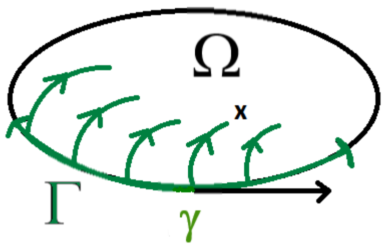
We analyze general nonlinear PDEs of first order,

$$F(\nabla u(x), u(x), x) = 0, \qquad x \in \Omega, \tag{6.1}$$

$$u(x) = g(x), \qquad x \in \Gamma, \tag{6.2}$$

where $\Omega \subset \mathbb{R}^n$ is open and $\Gamma \subset \partial\Omega$ is a C^1-hypersurface in $\mathbb{R}^n$ (see Appendix B). Moreover, the functions $F \in C^1(\mathbb{R}^n \times \mathbb{R} \times \overline{\Omega})$ and $g \in C^1(\Gamma)$ are given, and $u : \overline{\Omega} \to \mathbb{R}$ is the unknown.

To solve the boundary value problem, we apply the *method of characteristics*, that transforms the PDE (6.1) into an appropriate system of ODEs. The idea is that for a given $x \in \Omega$ we aim to find a curve that connects x with a point $\gamma \in \Gamma$ and along which we can calculate u.



We had applied this method to solve the linear transport equation in Chapter 2 and generalize it now for nonlinear equations.

6.1 The method of characteristics

Let $\Omega \subset \mathbb{R}^2$ be open and $\Gamma \subset \partial\Omega$ be a regular C^1 curve in $\mathbb{R}^2$. We consider the linear boundary value problem

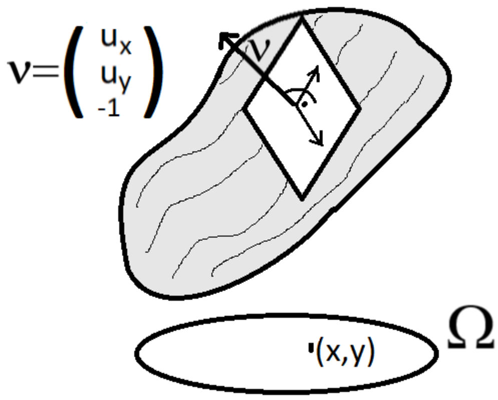
$$a(x, y) \cdot \nabla u(x, y) + b(x, y)u(x, y) = f(x, y), \qquad (x, y) \in \Omega, \tag{6.3}$$

$$u(x, y) = g(x, y), \qquad (x, y) \in \Gamma, \tag{6.4}$$

where $a \in C^1(\overline{\Omega}; \mathbb{R}^2)$, $b \in C^1(\overline{\Omega})$, $f \in C^1(\overline{\Omega})$ and $g \in C^1(\Gamma)$. The PDE (6.3) can be written as

$$\begin{pmatrix} u_x \\ u_y \\ -1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ -bu + f \end{pmatrix} = 0 \qquad \text{in } \Omega.$$

If $u \in C^1(\overline{\Omega})$ is a solution, the graph of u is a two-dimensional surface in $\mathbb{R}^3$, and since $\begin{pmatrix} u_x \\ u_y \\ -1 \end{pmatrix}$ is normal to the surface, the vector $\begin{pmatrix} a_1(x,y) \\ a_2(x,y) \\ -b(x,y)u(x,y) + f(x,y) \end{pmatrix}$, $(x,y) \in \Omega$, lies in the tangent plane to the graph of u in $(x, y, u(x,y))$, see the figure below.



Consequently, the system of first order ODEs,

$$\begin{aligned} x'(s) &= a_1(x(s), y(s)), \\ y'(s) &= a_2(x(s), y(s)), \qquad\qquad s > 0, \\ z'(s) &= -b(x(s), y(s))z(s) + f(x(s), y(s)), \end{aligned} \tag{6.5}$$

where $z(s) = u(x(s), y(s))$, define spatial curves lying on the graph of the solution u. These are the *characteristic equations* and the solutions of these ODEs are called *characteristic curves*. We require that the initial data lies on Γ, and since each curve emanates from a different point $\gamma = (\gamma_1, \gamma_2) \in \Gamma$ we indicate this dependency by writing $x_\gamma, y_\gamma, z_\gamma$, and hence, the initial conditions are

$$x_\gamma(0) = \gamma_1, \qquad y_\gamma(0) = \gamma_2, \qquad z_\gamma(0) = g(\gamma), \qquad \gamma \in \Gamma.$$

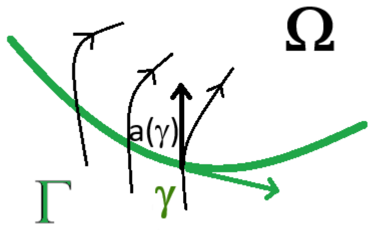
The idea of the method of characteristics is that solving the system of ODEs (6.5), we can reconstruct the solution $u(x,y)$ of the original PDE for

$$(x,y) \in \{(x_\gamma(t), y_\gamma(t)) : \gamma \in \Gamma, t \in [0,T)\} =: W,$$

for some $T > 0$. This is suitable, if W is a “large set”, i.e. an open neighborhood of Γ in $\overline{\Omega}$. Certainly, a necessary condition is that the vector $a(\gamma)$ is not tangential to Γ in γ, i.e.

$$a(\gamma) \notin \mathcal{T}_\gamma\Gamma, \qquad \gamma \in \Gamma,$$

where $\mathcal{T}_\gamma\Gamma$ denotes the tangent space of Γ in γ.



We notice that the first two equations in (6.5) are independent of the last equation, and the last equation is a linear ODE for z that we can explicitly solve if x and y are known. This will be different for quasi-linear and fully non-linear PDEs.

Example 6.1. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$ and $\Gamma = \{(x, y) \in \mathbb{R}^2 : x > 0, y = 0\} \subset \partial\Omega$. We consider the boundary value problem

$$\begin{aligned} xu_y(x, y) - yu_x(x, y) &= u(x, y), && (x, y) \in \Omega, \\ u(x, 0) &= g(x), && x > 0. \end{aligned}$$

Hence, using the previous notation we have $a = \begin{pmatrix} -y \\ x \end{pmatrix}$, $b = -1$ and $f = 0$, and the system of characteristic equations is

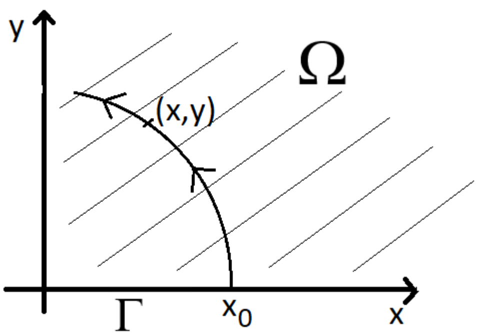
$$\begin{aligned} x'(s) &= -y(s), \\ y'(s) &= x(s), \\ z'(s) &= z(s), \end{aligned}$$

with the initial conditions

$$x(0) = x_0 > 0, \qquad y(0) = 0, \qquad z(0) = g(x_0).$$

We observe that $x'' = -x$, $y'' = -y$ and $(x^2 + y^2)' = 0$. Therefore, the solutions are

$$\begin{aligned} x(s) &= x_0 \cos s, \\ y(s) &= x_0 \sin s, \\ z(s) &= g(x_0)e^s. \end{aligned}$$



We observe that all vectors that are tangential to the curve Γ are of the form $v = \begin{pmatrix} c \\ 0 \end{pmatrix}$, $c \in \mathbb{R} \setminus \{0\}$. Moreover, if $\gamma = (x_0, 0) \in \Gamma$ then $a(\gamma) = \begin{pmatrix} 0 \\ x_0 \end{pmatrix}$. Hence, $a(\gamma)$ is not tangential to Γ and the condition $a(\gamma) \notin \mathcal{T}_\gamma \Gamma$ for all $\gamma \in \Gamma$ is satisfied.

For a given $(x, y) \in \Omega$ there exist $s \geq 0$, $x_0 > 0$ such that $(x(s), y(s)) = (x_0 \cos s, x_0 \sin s)$. Indeed, $x_0 = \sqrt{x^2 + y^2}$ and $s = \arctan \frac{x}{y}$. Hence, we obtain

$$u(x, y) = u(x(s), y(s)) = z(s) = g(x_0)e^s = g\Big(\sqrt{x^2 + y^2}\Big)e^{\arctan(\frac{y}{x})}.$$

6.2 Quasilinear equations

Before we consider general nonlinear equations we analyze boundary value problems for quasilinear equations,

$$a(x, u(x)) \cdot \nabla u(x) = b(x, u(x)), \qquad x \in \Omega, \tag{6.6}$$

$$u(x) = g(x), \qquad x \in \Gamma, \tag{6.7}$$

where $\Omega \subset \mathbb{R}^n$ is open, $\Gamma \subset \partial\Omega$ is a C^1-hypersurface in $\mathbb{R}^n$, $a \in C^1(\overline{\Omega} \times \mathbb{R}; \mathbb{R}^n)$, $b \in C^1(\overline{\Omega} \times \mathbb{R})$ and $g \in C^1(\Gamma)$.

As we observed for linear problems, we can rewrite the PDE (6.6) as

$$\begin{pmatrix} a(x, u(x)) \\ b(x, u(x)) \end{pmatrix} \cdot \begin{pmatrix} \nabla u(x) \\ -1 \end{pmatrix} = 0, \qquad x \in \Omega.$$

If $u \in C^1(\overline{\Omega})$, then the graph of u is a hypersurface in $\mathbb{R}^{n+1}$. Moreover, since $\begin{pmatrix} \nabla u(x) \\ -1 \end{pmatrix}$, $x \in \Omega$, is normal to the graph of u in $(x, u(x))$, we observe that u is a solution of (6.6) if and only if the vector $\begin{pmatrix} a(x, u(x)) \\ b(x, u(x)) \end{pmatrix}$ is in the tangent space of the graph of u in $(x, u(x))$, for all $x \in \Omega$. This motivates the following definitions.

Definition 6.2. The **characteristic equations** for the quasilinear PDE (6.6) are

$$\begin{aligned} x'(s) &= a(x(s), z(s)), \\ z'(s) &= b(x(s), z(s)), \end{aligned} \tag{6.8}$$

with the initial conditions

$$x(0) = \gamma, \qquad z(0) = g(\gamma), \qquad \gamma \in \Gamma.$$

Moreover, the boundary data (6.7) is **non-characteristic** if

$$a(\gamma, g(\gamma)) \notin \mathcal{T}_\gamma \Gamma \qquad \forall \gamma \in \Gamma.$$

The solutions of the characteristic equations are curves that lie on the graph of the solution u of (6.6). In particular, $z(s) = u(x(s))$, determines the values of the solution along the curve $x(s)$. Different from linear and semilinear equations, the system of ODEs for x is not decoupled from the ODE for z.

Example 6.3. Let $\Omega = \{(x, y) \in \mathbb{R}^2 \ : \ y > 0\}$ and $\Gamma = \{(x, y) \in \mathbb{R}^2 \ : \ y = 0\} = \partial\Omega$. We consider the semilinear boundary value problem

$$\begin{aligned} u_x(x, y) + u_y(x, y) &= u^2(x, y) && (x, y) \in \Omega, \\ u(x, 0) &= g(x) && x \in \mathbb{R}. \end{aligned}$$

Hence, we have $a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $b = z^2$, and the system of characteristic equations is

$$\begin{aligned} x'(s) &= 1, \\ y'(s) &= 1, \\ z'(s) &= z^2(s), \end{aligned}$$

with the initial conditions

$$x(0) = x_0 \in \mathbb{R}, \quad y(0) = 0, \qquad z(0) = g(x_0).$$

We obtain $x(s) = x_0 + s$ and $y(s) = s$, for $s \geq 0$. The last equation implies that $\frac{dz}{z^2} = ds$ and hence, $-\frac{1}{z} = s - \frac{1}{z(0)}$, which leads to

$$z(s) = \frac{1}{-s + \frac{1}{z(0)}} = \frac{z(0)}{1 - sz(0)} = \frac{g(x_0)}{1 - sg(x_0)}, \quad s \geq 0,$$

as long as the denominator is nonzero.

We observe that the vectors that are tangential to Γ are of the form $\begin{pmatrix} c \\ 0 \end{pmatrix}$, $c \neq 0$. Since $a_2 = 1 \neq 0$, the boundary data is non-charateristic. For given $(x, y) \in \Omega$ there exist $s \geq 0$ and $x_0 \in \mathbb{R}$ such that $(x, y) = (x(s), y(s)) = (x_0 + s, s)$. Namely, $x_0 = x - y$ and $s = y$, and hence we obtain

$$u(x, y) = u(x(s), y(s)) = z(s) = \frac{g(x_0)}{1 - sg(x_0)} = \frac{g(x - y)}{1 - yg(x - y)},$$

if $1 - yg(x - y) \neq 0$.

The system of characteristic equations and ODE theory can be used to prove the local existence and uniqueness of solutions for the boundary value problem (6.6)–(6.7).

Theorem 6.4. *If the boundary data is non-characteristic, then there exists a neighborhood U of Γ in $\overline{\Omega}$ such that there exists a unique solution $u \in C^1(U)$ of the boundary value problem* (6.6)–(6.7) *in U.*

Proof. The theorem is a special case of the general result for nonlinear equations (Theorem 6.12). The proof is given in the following section. □

6.3 Fully nonlinear equations

We now consider boundary value problems for fully nonlinear equations (6.1)–(6.2),

$$\begin{aligned} F(\nabla u(x), u(x), x) &= 0, && x \in \Omega, \\ u(x) &= g(x), && x \in \Gamma, \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is open and $\Gamma \subset \partial\Omega$ is a C^1-hypersurface. Moreover, we assume that $F \in C^1(\mathbb{R}^n \times \mathbb{R} \times \overline{\Omega})$ and $g \in C^1(\Gamma)$.

From now on, we will use the notation

$$(p, z, x) \mapsto F(p, z, x), \qquad p \in \mathbb{R}^n, z \in \mathbb{R}, x \in \overline{\Omega},$$

i.e. p substitutes the gradient of u and z substitutes u.

6.3.1 Characteristic equations

We search for suitable characteristics. Suppose that $u \in C^2(\Omega)$ is a solution of (6.1) and $x : I \to \Omega$ is a curve in Ω, where $I \subset \mathbb{R}$ is an interval. Let

$$z(s) := u(x(s)), \tag{6.9}$$
$$p(s) := \nabla u(x(s)), \tag{6.10}$$

$s \in I$. We aim to derive a system of ODEs for x, z and p that allows to compute the solution u. Differentiating $p_i = u_{x_i}(x)$ we obtain

$$p_i'(s) = \sum_{j=1}^{n} u_{x_i x_j}(x(s)) x_j'(s), \qquad i = 1, \ldots, n. \tag{6.11}$$

To eliminate the second order derivatives of u we differentiate the PDE (6.1) with respect to x_i,

$$\sum_{j=1}^{n} F_{p_j}(\nabla u, u, \cdot) u_{x_j x_i} + F_z(\nabla u, u, \cdot) u_{x_i} + F_{x_i}(\nabla u, u, \cdot) = 0. \tag{6.12}$$

Consequently, we set

$$x_j'(s) = F_{p_j}(p(s), z(s), x(s)), \qquad j = 1, \ldots, n, \tag{6.13}$$

and inserting $x = x(s)$ and (6.9)–(6.10) in (6.12) we obtain

$$\sum_{j=1}^{n} F_{p_j}(p(s), z(s), x(s)) u_{x_i x_j}(x(s)) + F_z(p(s), z(s), x(s)) p_i(s) + F_{x_i}(p(s), z(s), x(s)) = 0.$$

Finally, using this relation and (6.13) in the ODE (6.11) it follows that

$$p_i'(s) = -F_{x_i}(p(s), z(s), x(s)) - F_z(p(s), z(s), x(s)) p_i(s).$$

Moreover, by (6.9) and (6.10) together with (6.13) we obtain

$$z'(s) = \sum_{j=1}^{n} u_{x_j}(x(s)) x_j'(s) = \sum_{j=1}^{n} p_j(s) F_{p_j}(p(s), z(s), x(s)).$$

Hence, for the PDE (6.1) we obtain the *system of characteristic equations*

$$x'(s) = \nabla_p F(p(s), z(s), x(s)), \tag{6.14}$$
$$z'(s) = \nabla_p F(p(s), z(s), x(s)) \cdot p(s), \tag{6.15}$$
$$p'(s) = -\nabla_x F(p(s), z(s), x(s)) - F_z(p(s), z(s), x(s))p(s). \tag{6.16}$$

This systems consists of $2n+1$ first order ODEs. The functions p, z and x are called *characteristics*. The curve x is the projection of the full characterstics (p, z, x) onto the set $\Omega \subset \mathbb{R}^n$ and is therefore sometimes called the *projected characteristic*.

The following theorem summarizes our observations.

Theorem 6.5. *Let $\Omega \subset \mathbb{R}^n$ be open and $u \in C^2(\Omega)$ be a solution of* (6.1) *in Ω. If x is a solution of* (6.14)*, where $z = u \circ x$ and $p = \nabla u \circ x$, then z solves* (6.15) *and p solves* (6.16) *for those s such that $x(s) \in \Omega$.*

Remark 6.6.
- In order to solve the system of characteristic equations (6.14)–(6.16) we still need to specify suitable initial values.
- If u is a C^2-solution of (6.1), then (6.14)–(6.16) is an explicit system of ODEs for x, $z = u \circ x$ and $p = \nabla u \circ x$. The key step in the derivation was to set $x' = \nabla_p F$ such that the second order derivatives of u dropped out. This avoids introducing ODEs for the derivatives of second and higher order of u.
- In case of quasi-linear equations, the ODE for p is not required and the characteristic equations reduce to a system of ODEs for x and z.

Example 6.7. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and $\Gamma = \{(x, y) \in \mathbb{R}^2 : x = 0\} = \partial\Omega$. We consider the fully nonlinear boundary value problem

$$\begin{aligned} u_x(x, y)u_y(x, y) &= u(x, y) & (x, y) &\in \Omega, \\ u(0, y) &= y^2, & y &\in \mathbb{R}. \end{aligned}$$

Thus, $F(p, z, x) = p_1 p_2 - z$, and the system of characteristic equations (6.14)–(6.16) is

$$\begin{aligned} x'(s) &= p_2(s), \\ y'(s) &= p_1(s), \\ z'(s) &= 2p_1(s)p_2(s), \\ p_1'(s) &= p_1(s), \\ p_2'(s) &= p_2(s). \end{aligned}$$

The solutions of the first two equations are

$$p_1(s) = p_1(0)e^s, \qquad p_2(s) = p_2(0)e^s,$$

and inserting these functions in the system of ODEs we obtain

$$\begin{aligned} x(s) &= p_2(0)(e^s - 1), \\ y(s) &= y_0 + p_1(0)(e^s - 1), \\ z(s) &= y_0^2 + p_1(0)p_2(0)(e^{2s} - 1), \end{aligned}$$

where we used the initial values $x(0) = 0$, $y(0) = y_0 \in \mathbb{R}$ and $z(0) = y_0^2$.

We still need to determine suitable initial values for p. Since $u(x, y) = y^2$ on Γ, we have $p_2(0) = u_y(0, y) = 2y_0$. Furthermore, the PDE $u_x u_y = u$ implies that $p_1(0)p_2(0) = z(0) = u_0^2$, and consequently, $p_2(0) = \frac{y_0}{2}$.

Finally, for a given $(x, y) \in \Omega$, $x \neq 4y$, there exist $s \geq 0$ and $y_0 \in \mathbb{R}$ such that

$$(x, y) = (x(s), y(s)) = \left(2y_0(e^s - 1), \frac{y_0}{2}(e^s + 1)\right).$$

In fact, we find

$$y_0 = \frac{4y - x}{4}, \quad e^s = \frac{x + 4y}{4y - x},$$

and therefore, the solution of the boundary value problem is given by

$$u(x, y) = u(x(s), y(s)) = z(s) = (y_0)^2 e^{2s} = \frac{(x + 4y)^2}{16}.$$

This example illustrates that we need to specify suitable initial data for the system of characteristics (6.14)–(6.16), which we will do in the next subsection.

6.3.2 Boundary data

From now on we make the simplifying assumption that

$$\Gamma \subset \{x \in \mathbb{R}^n \ : \ x_n = 0\} \tag{6.17}$$

and $\Gamma \subset \mathbb{R}^{n-1}$ is open. We will comment on the general case in Section 6.3.4.

We look for suitable initial conditions

$$p(0) = p_0, \quad z(0) = z_0, \quad x(0) = x_0,$$

for the system of characteristic equations (6.14)–(6.16) that allow to construct a solution of the boundary value problem (6.1)–(6.2). Since $x(0) = x_0 \in \Gamma$, a necessary condition is that

$$z_0 = z(0) = u(x(0)) = u(x_0) = g(x_0). \tag{6.18}$$

To determine $p(0) = p_0$ we note that $u(x_1, \ldots, x_{n-1}, 0) = g(x_1, \ldots, x_{n-1})$ near x_0, and differentiating with respect to x_i we obtain

$$p_i(0) = u_{x_i}(x_0) = g_{x_i}(x_0), \qquad i = 1, \ldots, n - 1.$$

In addition, the PDE (6.1) should hold and hence, we obtain the following n equations that determine $p_0 = ((p_0)_1, \ldots, (p_0)_n)$,

$$\begin{aligned} (p_0)_i &= g_{x_i}(x_0), \qquad i = 1, \ldots, n - 1, \\ F(p_0, z_0, x_0) &= 0. \end{aligned} \tag{6.19}$$

Definition 6.8. The conditions (6.18)–(6.19) are called **compatibility conditions**, and a triple $(p_0, z_0, x_0) \in \mathbb{R}^{2n+1}$ satisfying (6.18)–(6.19) is called **admissible**.

We remark that z_0 is uniquely determined by g and the choice of $x_0 \in \Gamma$, but a vector p_0 fulfilling (6.19) may not exist or may not be unique.

Now suppose that (p_0, z_0, x_0) is an admissible triple. Then $x(0) = x_0$, $z(0) = z_0$ and $p(0) = p_0$ are possible initial values for the system of characteristics (6.14)–(6.16). We need to solve this system also for nearby initial values $y \in \Gamma$, and consequently, must ensure that the compatibility conditions remain valid. Hence, we also want to solve (6.14)–(6.16) with initial values

$$p(0) = q(y), \quad z(0) = g(y), \quad x(0) = y,$$

where $y \in \Gamma$ is close to x_0 and $q = (q_1, \dots, q_n)$ is a function such that

$$q(x_0) = p_0 \tag{6.20}$$

and $(q(y), g(y), y)$ is admissible, i.e.

$$\begin{aligned} q_i(y) &= g_{x_i}(y), \qquad i = 1, \dots, n-1, \\ F(q(y), g(y), y) &= 0. \end{aligned} \tag{6.21}$$

Lemma 6.9. *Let $F \in C^2(\mathbb{R}^n \times \mathbb{R} \times \overline{\Omega})$ and (p_0, z_0, x_0) be an admissible triple. If*

$$F_{p_n}(p_0, z_0, x_0) \neq 0, \tag{6.22}$$

then there exists a unique C^2-solution q of (6.21) *for all $y \in \Gamma$ near x_0.*

Proof. Consider the function $G \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where

$$\begin{aligned} G_i(p, y) &= p_i - g_{x_i}(y), \qquad i = 1, \dots, n-1, \\ G_n(p, y) &= F(p, g(y), y). \end{aligned}$$

Since (p_0, z_0, x_0) is an admissible triple, $G(p_0, x_0) = 0$. Moreover, we obtain

$$D_p G(p_0, x_0) = \begin{pmatrix} 1 & 0 & & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ F_{p_1}(p_0, z_0, x_0) & \dots & \dots & \dots & F_{p_n}(p_0, z_0, x_0) \end{pmatrix},$$

and consequently, $\det D_p G(p_0, x_0) = F_{p_n}(p_0, z_0, x_0) \neq 0$ by assumption (6.22). The Implicit Function Theorem now implies that the equation

$$G(p, y) = 0$$

is uniquely solvable for y near x_0, i.e. there exists a function q such that $p = q(y)$. Moreover, q is twice continuously differentiable if $F \in C^2 C^2(\mathbb{R}^n \times \mathbb{R} \times \overline{\Omega})$. □

Definition 6.10. An admissible triple (p_0, z_0, x_0) that satisfies (6.22) is called **non-characteristic**.

6.3.3 Local solution

We want to solve the system of characteristic equations (6.14)–(6.16) for non-characteristic initial data (p_0, z_0, x_0), where $x_0 \in \Gamma \subset \{x \in \mathbb{R}^n : x_n = 0\}$, and values $y \in \Gamma$ close to x_0. By Lemma 6.9, there exists a function q such that $p_0 = q(x_0)$ and $(q(y), g(y), y)$ is admissible for all $y \in \Gamma$ close to x_0. Let $y = (y_1, \dots, y_{n-1}) \in \Gamma$ (to simplify notations we omit here the zero in $y = (y_1, \dots, y_{n-1}, 0)$) and let

$$\begin{aligned} p(s) &= p(y, s) = p(y_1, \dots, y_{n-1}, s), \\ z(s) &= z(y, s) = z(y_1, \dots, y_{n-1}, s), \\ x(s) &= x(y, s) = x(y_1, \dots, y_{n-1}, s), \end{aligned} \tag{6.23}$$

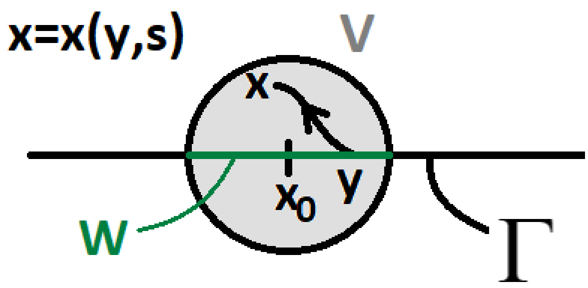
denote the solutions of the characteristic equations with initial data

$$p(0) = q(y), \quad z(0) = g(y), \quad x(0) = y. \tag{6.24}$$

Lemma 6.11 (Local invertibility)**.** *Let (p_0, z_0, x_0) be admissible, $F \in C^3(\mathbb{R}^n \times \mathbb{R} \times \overline{\Omega})$ and let $F_{p_n}(p_0, z_0, x_0) \neq 0$. Then, there exists an open interval I containing 0, a neighborhood W of x_0 in $\Gamma \subset \mathbb{R}^{n-1}$ and a neighborhood V of x_0 in $\mathbb{R}^n$ such that for $x \in V$ there is a unique $(y, s) \in W \times I$ with*

$$x = x(y, s).$$

Moreover, the map $x \mapsto (y, s)$ is $C^2(V; W \times I)$.



Proof. By Lemma 6.9 and since $F \in C^3(\mathbb{R}^n \times \mathbb{R} \times \overline{\Omega})$, the function q is C^2 and the solutions p, z, x of the characteristic equations in (6.23) as well. We observe that

$$x(x_0, 0) = x_0,$$

and hence, the claim follows from the Inverse Function Theorem if $\det Dx(x_0, 0) \neq 0$. By (6.23) and (6.24) we have

$$x(y, 0) = y,$$

and thus, we conclude that

$$\frac{\partial x_j}{\partial y_i}(x_0, 0) = \begin{cases} \delta_{ij} & j = 1, \dots, n-1, \\ 0 & j = n, \end{cases}$$

for $i = 1, \dots, n-1$. The characteristic equations (6.14) furthermore imply that

$$\frac{\partial x_j}{\partial s}(x_0, 0) = F_{p_j}(p_0, z_0, x_0).$$

Therefore, we obtain

$$Dx(x_0, 0) = \begin{pmatrix} 1 & \cdots & 0 & F_{p_1}(p_0, z_0, x_0) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & \vdots \\ 0 & \dots & 0 & F_{p_n}(p_0, z_0, x_0) \end{pmatrix},$$

and consequently,

$$\det Dx(x_0, 0) = F_{p_n}(p_0, z_0, x_0) \neq 0. \qquad \square$$

By Lemma 6.11, for every $x \in V$ there exist unique solution $y = y(x)$ and $s = s(x)$ of the equation

$$x = x(y, s).$$

Finally, we define

$$\begin{aligned} u(x) &:= z(y(x), s(x)), \\ p(x) &:= p(y(x), s(x)), \end{aligned} \tag{6.25}$$

for $x \in V$ and show that u is indeed a (local) solution of the PDE (6.1).

Theorem 6.12. *Let $F \in C^3(\mathbb{R}^n \times \mathbb{R} \times \overline{\Omega})$, $g \in C^2(\Gamma)$ and (p_0, z_0, x_0) be admissible. Moreover, we assume that $F_{p_n}(p_0, z_0, x_0) \neq 0$. Then, u defined in (6.25) is a C^2-function and the unique solution of the initial value problem*

$$\begin{aligned} F(\nabla u(x), u(x), x) &= 0, & x &\in V, \\ u(x) &= g(x), & x &\in \Gamma, \end{aligned}$$

with $\nabla u(x_0) = p_0$.

Proof. Let I and W be as in Lemma 6.11, $y \in W$ and let

$$p(s) = p(y, s), \quad z(s) = z(y, s), \quad x(s) = x(y, s)$$

denote the solutions of the characteristic equations (6.23) with initial data (6.24).

Step 1. First, we show that F vanishes along the characteristic curves, i.e.

$$f(y, s) := F(p(y, s), z(y, s), x(y, s)) = 0, \qquad s \in I. \tag{6.26}$$

Indeed,

$$f(y, 0) = F(p(y, 0), z(y, 0), x(y, 0)) \overset{(6.24)}{=} F(q(y), g(y), y) \overset{(6.21)}{=} 0,$$

and

$$\begin{aligned}\frac{\partial f}{\partial s} &= \nabla_p F(p,z,x)\cdot p' + F_z(p,z,x)z' + \nabla_x F(p,z,x)\cdot x' \\ &\overset{(6.14)\text{–}(6.16)}{=} \nabla_p F(p,z,x)\cdot(-\nabla_x F(p,z,x) - F_z(p,z,x)p) \\ &\qquad + F_z(p,z,x)p\cdot F_p(p,z,x) + \nabla_x F(p,z,x)\cdot \nabla_p F(p,z,x) = 0,\end{aligned}$$

which implies (6.26)

Step 2. By (6.25) and Lemma 6.11 we conclude that

$$F(p(x), u(x), x) = 0, \qquad x \in V,$$

and therefore, it remains to show that $p(x) = \nabla u(x)$, $x \in V$. To this end we first prove that

$$\frac{\partial z}{\partial s}(y,s) = p(y,s)\cdot\frac{\partial x}{\partial s}(y,s), \tag{6.27}$$

$$\frac{\partial z}{\partial y_i}(y,s) = p(y,s)\cdot\frac{\partial x}{\partial y_i}(y,s), \qquad i = 1,\dots,n-1, \tag{6.28}$$

for $y \in W$ and $s \in I$. The first condition follows immediately from the system of characteristics, more precisely, from (6.15) and (6.14). To show the second equation let

$$r_i(s) := \frac{\partial z}{\partial y_i}(y,s) - p(y,s)\cdot\frac{\partial x}{\partial y_i}(y,s), \qquad i = 1,\dots,n-1.$$

Then, $r_i(0) = g_{x_i}(y) - q_i(y) = 0$ by the compatibility conditions (6.20)–(6.21). Moreover, differentiating r_i we obtain

$$\begin{aligned} r_i' &= \frac{\partial}{\partial y_i}\frac{\partial z}{\partial s} - \frac{\partial p}{\partial s}\cdot\frac{\partial x}{\partial y_i} - p\cdot\frac{\partial}{\partial y_i}\frac{\partial x}{\partial s} \\ &\overset{(6.27)}{=} \frac{\partial p}{\partial y_i}\cdot\frac{\partial x}{\partial s} - \frac{\partial p}{\partial s}\cdot\frac{\partial x}{\partial y_i} \\ &\overset{(6.14),(6.16)}{=} \frac{\partial p}{\partial y_i}\cdot\nabla_p F(p,z,x) - (-\nabla_x F(p,z,x) - pF_z(p,z,x))\cdot\frac{\partial x}{\partial y_i} \\ &= \frac{\partial}{\partial y_i}F(p,z,x) - F_z(p,z,x)\frac{\partial z}{\partial y_i} + F_z(p,z,x)p\cdot\frac{\partial x}{\partial y_i} \\ &\overset{\text{Step 1}}{=} -F_z(p,z,x)\left(\frac{\partial z}{\partial y_i} - p\cdot\frac{\partial x}{\partial y_i}\right) = -F_z(p,z,x)r_i. \end{aligned}$$

Consequently, r_i satisfies a linear homogeneous ODE with initial data $r_i(0) = 0$, and we conclude that $r_i \equiv 0$ in I.

Step 3. We now show that $p(x) = \nabla u(x)$, $x \in V$, using the formulas (6.27) and (6.28). Indeed, for $j = 1, \ldots, n$, we obtain

$$\begin{aligned}
\frac{\partial u}{\partial x_j} &\overset{(6.25)}{=} \sum_{i=1}^{n-1} \frac{\partial z}{\partial y_i}\frac{\partial y_i}{\partial x_j} + \frac{\partial z}{\partial s}\frac{\partial s}{\partial x_j} \\
&\overset{(6.27),(6.28)}{=} \sum_{i=1}^{n-1}\left(p\cdot\frac{\partial x}{\partial y_i}\right)\frac{\partial y_i}{\partial x_j} + \left(p\cdot\frac{\partial x}{\partial s}\right)\frac{\partial s}{\partial x_j} \\
&= p\cdot\left(\sum_{i=1}^{n-1}\frac{\partial x}{\partial y_j}\frac{\partial y_i}{\partial x_j} + \frac{\partial x}{\partial s}\frac{\partial s}{\partial x_j}\right) = \sum_{k=1}^{n} p_k \underbrace{\frac{\partial x_k}{\partial x_j}}_{=\delta_{kj}} = p_j.
\end{aligned}$$

Step 4. Uniqueness. Two different solutions u and v with $\nabla u(x_0) = p_0$ would lead via the relation

$$z(y,s) = u(x(y,s)), \qquad \tilde{z}(y,s) = v(x(y,s)),$$

to two different solutions of the initial value problem for the characteristic equations which is impossible by Lemma 6.11. In fact, for every $x \in V$ there exists a unique $y \in W$ and $s \in I$ such that $x(y,s) = x$. □

Remark 6.13. In the quasilinear case, the compatibility conditions are a system of *n linear* alregbraic equations, and the non-characteristic condition ensures that there exists a unique solution.

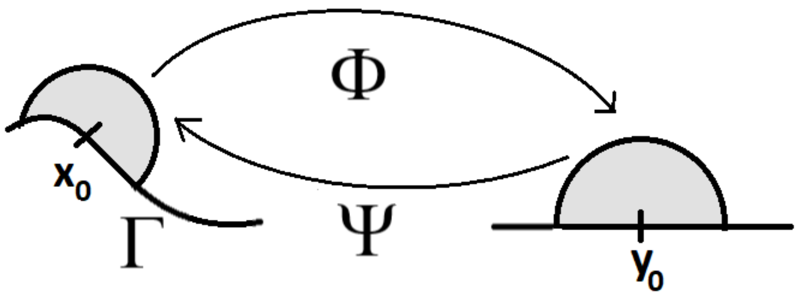
For fully nonlinear equations the compatibility conditions are a system of *nonlinear* equations and the non-characteristic condition is not sufficient to ensure that a solution exists.

6.3.4 Straightening the boundary

Theorems 6.4 and 6.12 can be extended for more general domains $\Omega \subset \mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $\partial\Omega$ be of class C^1, i.e. for every $x_0 \in \partial\Omega$ there is an $r > 0$ and a C^1-function $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that (possibly after relabeling and reorienting the coordinate axes)

$$\Omega \cap B_r(x_0) = \{x \in B_r(x_0) \,:\, x_n > \varphi(x_1, \ldots, x_{n-1})\}.$$

Via a change of coordinates near x_0 we can *flatten out the boundary* $\partial\Omega$ near x_0.



In fact, let $x_0 \in \partial\Omega$ and consider r and φ as above. We define

$$\begin{aligned}
\Phi^i(x) &:= x_i, && \text{for } i = 1, \ldots, n-1, \\
\Phi^n(x) &:= x_n - \varphi(x_1, \ldots, x_{n-1}),
\end{aligned}$$

and write $y = \Phi(x)$. Similarly, we define

$$\Psi^i(y) := y_i, \qquad \text{for } i = 1, \dots, n-1,$$
$$\Psi^n(y) := y_n + \varphi(y_1, \dots, y_{n-1}),$$

and write $x = \Psi(y)$. Then, $\Psi = \Phi^{-1}$ and the mapping $x \to \Phi(x) = y$ straightens out $\partial\Omega$ near x_0.

We reformulate the initial value problem (6.1)–(6.2) accordingly. Suppose that $u\colon \Omega \to \mathbb{R}$ is a solution and define $V := \Phi(\Omega)$ and $v(y) := u(\Psi(y))$ for $y \in V$. Then,

$$u(x) = v(\Phi(x)), \qquad x \in \Omega,$$

and

$$\nabla u(x) = \nabla v(\underbrace{\Phi(x)}_{=y})D\Phi(x).$$

Consequently, we obtain

$$0 = F(\nabla u(x), u(x), x) = F(\nabla v(y)D\Phi(\Psi(y)), v(\Psi(y)), \Psi(y)),$$

and this equation is of the form

$$G(\nabla v(y), v(y), y) = 0, \qquad y \in V.$$

Moreover, with $h(y) := g(\Psi(y))$, in the new coordinates the initial value problem (6.1)–(6.2) takes the form

$$\begin{aligned} G(\nabla v, v, \cdot) &= 0 \qquad \text{in } V, \\ v &= h \qquad \text{on } \Phi(\Gamma). \end{aligned}$$

This system is of the same form as the original one, but with a "flat" boundary near x_0. This shows that our simplifying assumption (6.17), $\Gamma \subset \{x \in \mathbb{R}^n \;:\; x_n = 0\}$, is not restrictive, and the theory can be extended to more general C^1-hypersurfaces Γ.

6.4 Exercises

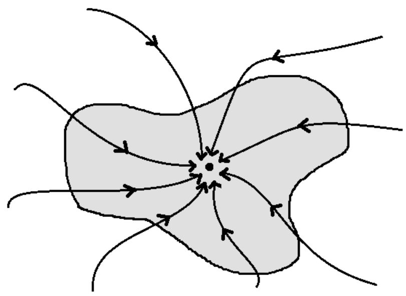
E6.1 **Solvability of first order PDEs**

(a) Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in C^1(\overline{\Omega})$ be a solution of the quasilinear PDE

$$a(x, u(x)) \cdot \nabla u(x) = 0, \qquad x \in \Omega, \tag{6.29}$$

where $a \in C^1(\overline{\Omega} \times \mathbb{R}; \mathbb{R}^n)$.
Show that the solution u is constant along every characteristic curve.

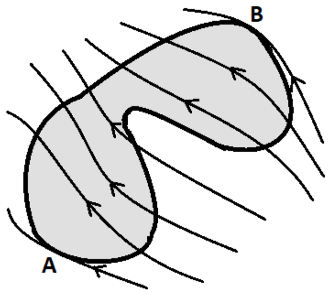
(b) Assume that Ω is open and bounded with C^1-boundary $\partial\Omega$ and $a(x, u(x)) = a(x)$, i.e. the equation (6.29) is linear. Moreover, suppose that the trajectories of the characteristic curves are as in the figure below,



i.e. a vanishes only in one point $x_0 \in \Omega$, $a(x) \neq 0$ in $\Omega \setminus \{x_0\}$ and $a(x) \cdot \nu(x) < 0$ for $x \in \partial\Omega$. Does there exist a solution $u \in C^1(\overline{\Omega})$ of the boundary value problem

$$\begin{aligned} a(x) \cdot \nabla u(x) &= 0, & x &\in \Omega, \\ u(x) &= g(x), & x &\in \partial\Omega \ ? \end{aligned} \tag{6.30}$$

(c) Now, assume that the trajectories look as follows:



Can then a solution $u \in C^1(\overline{\Omega})$ of the boundary value problem (6.30) exist?

E6.2 **Semilinear equation**

Consider the semilinear boundary value problem

$$\begin{aligned} x^2 u_x(x,y) - y^2 u_y(x,y) &= u^2(x,y), & (x,y) &\in \mathbb{R}^2, \\ u &= 1 & &\text{on } \Gamma, \end{aligned}$$

where $\Gamma = \{(x,y) \in \mathbb{R}^2 \mid x > 0, y = 2x\}$.
Verify that the boundary value problem is locally solvable and use the method of characteristics to find an explicit solution.

E6.3 **Method of characteristics**

Use the method of characteristics to find a solution of the following boundary value problems (the shape of the boundary and the boundary values are determined by the second equation):

(a)

$$\begin{aligned} (y + u(x,y))u_x(x,y) + yu_y(x,y) &= x - y \\ u(x,1) &= 1 + x \end{aligned}$$

(b)

$$\begin{aligned} uu_x + u_y &= 1 \\ u(x,x) &= \frac{1}{2}x \end{aligned}$$

(c)

$$\begin{aligned} xu_x(x,y,z) + 2yu_y(x,y,z) + u_z(x,y,z) &= 3u(x,y,z) \\ u(x,y,0) &= g(x,y) \end{aligned}$$

Are the PDEs linear, semilinear or quasilinear? Do there exist unique local solutions of the boundary value problems? Do the solutions exist globally?

E6.4 **Quasilinear equation**

Consider the initial value problem

$$\begin{aligned} 2u(x,y)u_x(x,y) - yu_y(x,y) &= 2x, \\ u(1+s^2, s) &= 1 - s^2, \qquad\qquad s \in \mathbb{R}. \end{aligned}$$

Argue that there exists a unique solution in a sufficiently small neighborhood of $\{(1+s^2, s) | s \in \mathbb{R}\} \subset \mathbb{R}^2$ and compute the solution explicitly.

E6.5 **Burger's equation**

Consider the inviscid Burgers' equation

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= 0 \qquad\qquad \text{in } (0,\infty) \times \mathbb{R}, \\ u(0,\cdot) &= g_\varepsilon \qquad\qquad \text{in } \mathbb{R}, \end{aligned}$$

where the initial data $g_\varepsilon \in C^\infty(\mathbb{R}), \varepsilon > 0$, is smooth and such that

$$g_\epsilon(x) = \begin{cases} 1 & x \leq 0, \\ 1-x & [\varepsilon, 1-\varepsilon], \\ 0 & x \geq 1. \end{cases}$$

(a) Check whether the initial data is non-characteristic and solve the characteristic equations. *Hint: Recall Problem 1, Sheet 12.*

(b) Sketch the characteristic curves in the x-t–plane. For simplicity restrict yourself to the limiting case $\varepsilon \to 0$. Pay special attention to what happens at the lines $\{x = 1\}$ and $\{t = 1\}$.

(c) Use part (b) to deduce a formula for the solution for times $0 \leq t < 1$ in the limit $\varepsilon \to 0$. Which peculiarity occurs when t approaches 1?

E6.6 **Fully nonlinear equation**

Use the method of characteristics to solve the initial value problem

$$u + \frac{1}{2}(u_x)^2 + u_y = 0 \qquad \text{in } \mathbb{R} \times (-\infty, 0),$$
$$u(x, 0) = -x^2, \qquad x \in \mathbb{R},$$

where $x \in \mathbb{R}, y \leq 0$. Verify that the boundary data is non-characteristic and use the compatibility conditions.

E6.7 **Conservation laws**

Consider in $\Omega = (0, \infty) \times \mathbb{R}^n$ the equation

$$G(Du, u_t, u, x, t) = u_t + \operatorname{div} F(u) = u_t + F'(u) \cdot \nabla u = 0$$

with the initial condition

$$u = g \qquad \text{on } \Gamma = \{0\} \times \mathbb{R}^n.$$

(a) Show that the non-characteristic condition is satisfied on Γ.

(b) Compute the system of characteristics and show that the projected characteristic is a straight line along with the solution is constant.

(c) Derive an *implicit* formula for the solution u.

E6.8 **Fully nonlinear equation**

Consider the initial value problem

$$\begin{aligned} u_t u_x &= 0 \qquad \text{in } (0, \infty) \times \mathbb{R}, \\ u(0, x) &= 1, \qquad x \in \mathbb{R}, \end{aligned} \tag{6.31}$$

where $t \in (0, \infty)$.

(a) Does the local existence theorem apply? Explicitly check the non-characteristic and compatibility conditions.

(b) Find all solutions of the initial value problem (6.31).

Hint: Prove by contradiction that $u_x \equiv 0$. To this end consider a shift of the initial surface $\Gamma = \{0\} \times \mathbb{R}$ and solve the corresponding characteristic equations.

E6.9 **Eikonal equation**

The *eikonal equation* arises in problems of wave propagation and provides the foundation of geometrical optics. In $\mathbb{R}^2$ it takes the form

$$u_x^2 + u_y^2 = n^2 \qquad \text{in } \mathbb{R}^2,$$

where level sets of the solution correspond to wave fronts and n is the refraction index of the medium.

Find the solution of the eikonal equation for a medium with constant refraction index $n = n_0 \in \mathbb{R}$ and initial condition $u(x, 2x) = 1$.

Appendix A

Integration Theory in $\mathbb{R}^n$

In this appendix we summarize basic facts from integration theory that are used throughout the course. For further details and proofs we refer to [3] and [5].

If we write $U \subset \mathbb{R}^n$ then either $U = \mathbb{R}^n$ or $U \subsetneq \mathbb{R}^n$. Similarly, if $U, V \subset \mathbb{R}^n$, we write $U \subset V$ if $U = V$ or $U \subsetneq V$. We denote by $\overset{\circ}{U}$ the interior of U, by $\overline{U}$ the closure of U and by ∂U the boundary of U.

A.1 Riemann integrability

We call D an *n-dimensional rectangle* parallel to the coordinate axis if it is a compact subset of the form

$$D = \left\{ x \in \mathbb{R}^n : a_j \leq x_j \leq b_j,\ j = 1, \ldots, n \right\},$$

where $a_j \leq b_j$, $a_j, b_j \in \mathbb{R}$. The *n-dimensional volume* of D is

$$|D| = \prod_{j=1}^{n} (b_j - a_j).$$

Note that $|D| = 0$ if there exists j such that $a_j = b_j$.

A *partition of D* is a finite collection $\mathcal{D} = \{D_i : i \in I\}$, I an index set, of n-dimensional rectangles D_i such that

$$D = \bigcup_{i \in I} D_i, \qquad D_i \cap D_j = \emptyset \quad \text{or} \quad |D_i \cap D_j| = 0 \quad \text{if} \quad i \neq j.$$

Throughout this section we assume that D is an n-dimensional rectangle and $f : D \to \mathbb{R}$ is a bounded function.

Definition A.1. For every partition $\mathcal{D} = \{D_i : i \in I\}$ of D we define the **lower sum** and **upper sum** of f by

$$\overline{S}(f, \mathcal{D}) = \sum_{i \in I} \sup_{x \in D_i} f(x)|D_i|, \qquad \underline{S}(f, \mathcal{D}) = \inf_{i \in I} \sup_{x \in D_i} f(x)|D_i|.$$

The function f is **(Riemann-)integrable over** D if

$$\sup_{\mathcal{D} \text{ partition of } D} \underline{S}(f, \mathcal{D}) = \inf_{\mathcal{D} \text{ partition of } D} \overline{S}(f, \mathcal{D}).$$

The common value is called the **integral of** f **over** D and denoted by

$$\int_D f(x)dx.$$

We recall a special case of *Fubini's Theorem* that allows to interchange the order of integration.

Theorem A.2. *Let $f : \mathbb{R}^{n+m} \to \mathbb{R}$ be a continuous function with compact support. Then, for every $y \in \mathbb{R}^n$, the integral $\int_{\mathbb{R}^m} f(y,z)dz$ is well-defined and Riemann integrable on $\mathbb{R}^n$, and for every $z \in \mathbb{R}^m$, the integral $\int_{\mathbb{R}^n} f(y,z)dy$ is well-defined and Riemann integrable on $\mathbb{R}^m$. Moreover, we have*

$$\int_{\mathbb{R}^{n+m}} f(x)dx = \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} f(y,z)dydz = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} f(y,z)dzdy.$$

For $A \subset \mathbb{R}^n$ we denote by χ_A the *characteristic function* of A, i.e. $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$.

Definition A.3. A bounded subset $A \subset D$ is called **(Jordan-)measurable** if χ_A is integrable over D. Then its **(Jordan-)volume** is given by

$$|A| := \int_D \chi_A(x)dx = \int_A dx.$$

We call a function $f : D \to \mathbb{R}$ **(Riemann-)integrable over** A if $f\chi_A$ is integrable and write

$$\int_A f(x)dx := \int_D f(x)\chi_A(x)dx.$$

Definition A.4. Let $U \subset \mathbb{R}^n$ be open. We call a function $f : U \to \mathbb{R}$ **absolutely (Riemann-) integrable in** U if for every $x \in U$ there exists a rectangle $D \subset U$ such that $x \in \mathring{D}$ and f is (Riemann-) integrable over D, and

$$\sup_{K \in \mathcal{J}(U)} \int_K |f(x)|dx < \infty,$$

where $\mathcal{J}(U)$ denotes the set of compact and Jordan measurable subsets of U.

The following theorem provides a criterion for the absolute integrability of a continuous function over an open set.

Theorem A.5. *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be continuous. Suppose that $(K_j)_{j\in\mathbb{N}}$ is a family of subset such that K_j is compact and Jordan measurable, $K_j \subset \mathring{K}_{j+1}$ for all $j \in \mathbb{N}$ and $\bigcup_{j\in\mathbb{N}} K_j = U$. Then the following statements are equivalent:*

(i) The function f is absolutely Riemann integrable over U.

(ii) $\left(\int_{K_j} |f(x)|dx\right)_{j\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}$.

If one of the statements holds then $\left(\int_{K_j} |f(x)|dx\right)_{j\in\mathbb{N}}$ is monotonically nondecreasing and

$$\lim_{j\to\infty}\left(\int_{K_j} f(x)dx\right)_{j\in\mathbb{N}} = \int_U f(x)dx.$$

A.2 Interchanging differentiation and integration

Theorem A.6. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open and $f : U \times V \to \mathbb{R}$ be a function with the following properties.*

(a) For every $x \in U$, the function $t \mapsto f(x,t)$ is absolutely Riemann integrable over V.

(b) The total derivative $D_1 f$ with respect to the variable $x \in U$ exists and, for every $x \in U$, the mapping $t \mapsto D_1 f(x,t)$ is absolutely Riemann integrable over V (here, integrability is meant component-wise).

(c) There exists a function $g : V \to [0,\infty)$ that is bounded on V and absolutely Riemann integrable over V, such that $\|D_1 f(x,t)\| \leq g(t)$ for all $(x,t) \in U \times V$.

Then the function $F : U \to \mathbb{R}$, defined by $F(x) = \int_V f(x,t)dt$, is differentiable and

$$D_1 F(x) = \int_V D_1 f(x,t)dt, \qquad x \in U.$$

Example A.7. The function $f(x) = e^{-|x|^2}$ is continuous on $\mathbb{R}^2$. Let $r > 0$. Using polar coordinates we observe that

$$\int_{B_r(0)} = \int_{-\pi}^{\pi}\int_0^r re^{-r^2} dr d\varphi = \pi(1 - e^{-r^2}) \leq \pi.$$

Let $K \subset \mathbb{R}^2$ be a compact, Jordan measurable set. Then, there exists $r > 0$ such that $K \subset B_r(0)$. Moreover, since f is positive, we have

$$\int_K f(x)dx \leq \int_{B_r(0)} f(x)dx \leq \pi,$$

which shows that f is absolutely integrable over $\mathbb{R}^2$.

By Theorem A.2 we conclude that

$$\left(\int_{-r}^{r} e^{-s^2} ds\right)^2 = \left(\int_{-r}^{r} e^{-x_1^2} dx_1\right)\left(\int_{-r}^{r} e^{-x_2^2} dx_2\right) = \int_{C_r} e^{-(x_1^2+x_2^2)} dx,$$

where $C_r = [-r,r] \times [-r,r]$. We observe that $B_r(0) \subset C_r \subset B_{r\sqrt{2}}(0)$ and therefore,

$$\pi(1 - e^{-r^2}) \leq \int_{C_r} f(x)dx \leq \pi(1 - e^{-2r^2}).$$

Finally, by Theorem A.5 we conclude that

$$\left(\int_{\mathbb{R}} e^{-s^2} ds\right)^2 = \lim_{r\to\infty} \int_{C_r} f(x)dx = \pi,$$

which shows that

$$\int_{\mathbb{R}} e^{-s^2} ds = \sqrt{\pi}.$$

A.3 Change of variables

Let U and V be open subsets of $\mathbb{R}^n$. Recall that a function $\Psi : V \to U$ is a C^1-*diffeomorphism* if it is bijective and if Ψ and Ψ^{-1} are continuously differentiable.

Theorem A.8. *Let U and V be open subsets of $\mathbb{R}^n$ and $\Psi : V \to U$ be a C^1-diffeomorphism. Let $f : U \to \mathbb{R}$ be a bounded function with compact support. Then f is integrable over U if and only if the function $y \mapsto (f \circ \Psi)(y)|detD\Psi(y)|$ is integrable over V. In this case we have*

$$\int_{\Psi(V)} f(x)dx = \int_U f(x)dx = \int_V (f \circ \Psi)(y)|detD\Psi(y)|dy.$$

An important special case of this theorem are *polar and spherical coordinates*.

Polar coordinates

Let $U = \mathbb{R}^2 \setminus ([0,\infty) \times \{0\})$ and $V = [0,\infty) \times (0, 2\pi)$. Then $\Psi_2 : V \to U$, defined by

$$\Psi_2(r,\phi) = (r\cos\phi, r\sin\phi), \qquad (r,\phi) \in V,$$

is a C^1-diffeomorphism. We observe that

$$\det D\Psi_2(r,\phi) = \det\begin{pmatrix}\cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi\end{pmatrix} = r > 0.$$

Hence, for a continuous function f with compact support in U we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} f(x)dx &= \int_V (f \circ \Psi_2)(y)|\det D\Psi_2(y)|dy = \int_0^\infty r \int_{-\pi}^{\pi} f(r\cos\phi, r\sin\phi)d\phi\, dr \\ &= \int_{-\pi}^{\pi}\int_0^\infty rf(r\cos\phi, r\sin\phi)dr\, d\phi.\end{aligned}$$

Spherical coordinates

Let $U = \mathbb{R}^3 \setminus \{x \in \mathbb{R}^3 : x_1 > 0, x_2 = 0\}$ and $V = [0,\infty) \times (0,2\pi) \times (0,\pi)$. Then $\Psi_3 : V \to U$, defined by

$$\Psi_3(r,\phi,\theta) = (r\cos\phi\sin\theta, r\sin\phi\sin\theta, r\cos\theta), \qquad (r,\phi,\theta) \in V,$$

is a C^1-diffeomorphism. To shorten notations we write $\Psi_3 = \widetilde{\Psi} \circ \psi$, where

$$\begin{aligned}\psi(r,\phi,\theta) &= (r\sin\theta, \phi, r\cos\theta) = (\rho,\phi,z), \\ \widetilde{\Psi}(\rho,\phi,z) &= (\Psi_2(\rho,\phi), z).\end{aligned}$$

The chain rule then implies that $D\Psi_3 = D\widetilde{\Psi} \circ D\psi$ and consequently,

$$|\det D\Psi_3(r,\phi,\theta)| = |\det D\widetilde{\Psi}(\rho,\phi,z)| \cdot |\det D\psi(r,\phi,\theta)| = \rho r = r^2\sin\theta.$$

We can generalize this to arbitrary dimensions $n \in \mathbb{N}$. Let $U = \mathbb{R}^n \setminus \{x \in \mathbb{R}^3 : x_1 \geq 0, x_2 = 0\}$ and $V = (0,\infty) \times (0,2\pi) \times (0,\pi)^{n-2}$. Then $\Psi_n : V \to U$, defined by

$$\Psi_n(r,\phi,\theta_1,\ldots,\theta_{n-2}) = x, \tag{A.1}$$

where

$$\begin{aligned}
x_1 &= r\cos\phi\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-2},\\
x_2 &= r\sin\phi\sin\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-2},\\
x_3 &= r\cos\theta_1\sin\theta_2\sin\theta_3\cdots\sin\theta_{n-2},\\
x_4 &= r\cos\theta_2\sin\theta_3\cdots\sin\theta_{n-2},\\
&\vdots\\
x_{n-1} &= r\cos\theta_{n-3}\sin\theta_{n-2},\\
x_n &= r\cos\theta_{n-2},
\end{aligned}$$

is a C^1-diffeomorphism. Moreover, by induction one can show that

$$|\det D\Psi_n(r,\phi,\theta_1,\ldots,\theta_{n-2})| = r^{n-1}\sin\theta_1(\sin\theta_2)^2\cdots(\sin\theta_{n-2})^{n-2}. \tag{A.2}$$

A.4 Surface integrals

We shortly discuss integration over k-dimensional surfaces in $\mathbb{R}^n$. Recall that the *Gram determinant* $G(v_1,\ldots,v_k)$ of m vectors $v_1,\ldots,v_k \in \mathbb{R}^n$ is defined as

$$G(v_1,\ldots,v_k) = \det\begin{pmatrix} v_1\cdot v_1 & \cdots & v_1\cdot v_k\\ \vdots & & \vdots\\ v_k\cdot v_1 & \cdots & v_k\cdot v_k\end{pmatrix},$$

where $\cdot$ denotes the inner product in $\mathbb{R}^n$. Moreover, the volume of a parallelepiped spanned by $v_1,\ldots,v_k$ equals $\sqrt{G(v_1,\ldots,v_k)}$.

Let now $U\subset\mathbb{R}^m$ be open and $\varphi\in C^1(U;\mathbb{R}^n)$ be an *immersion*, i.e.

$$\operatorname{rank}(D\varphi(x)) = m \qquad \forall x\in U.$$

This condition is equivalent to the linear independence of the vectors $D_1\varphi(x),\ldots,D_m\varphi(x)$ and hence, these vectors span an m-dimensional subspace in $\mathbb{R}^n$. Therefore, $M := \varphi(U)$ is called an m-dimensional *surface* in $\mathbb{R}^n$.

The area of S is given by

$$|M| = \int_U \sqrt{G(D\varphi(x))}dx.$$

The intuition is that the surface consists of infinitely many spanned parallelepipeds spanned by the vectors $D_1\varphi(x),\ldots,D_m\varphi(x)$ and we integrate over all of them to determine the area. For a continuous function $f:\mathbb{R}^n\to\mathbb{R}$ the *integral of f over M* is defined as

$$\int_M f(x)dS(x) := \int_U f(\varphi(x))\sqrt{G(D\varphi(x))}dx.$$

Formally, one often writes $dS(x) = \sqrt{G(D\varphi(x))}dx$, and $dS(x)$ is called the *m-dimensional surface element.*

Spheres and balls

As an example, we calculate the volume of the unit ball $B_1(0)$ in $\mathbb{R}^n$ and the surface area of the unit sphere $\partial B_1(0)$ in $\mathbb{R}^n$. We first observe that if $m = n$ and $A := (v_1, \dots, v_n)$ for given vectors $v_1, \dots, v_n \in \mathbb{R}^n$, then

$$G(v_1, \dots, v_n) = \det(A^2) = (\det(A))^2.$$

We can parametrize the unit sphere in $\mathbb{R}^n$ using (A.1), $\varphi := \Psi_n\big|_{r=1} : (0, 2\pi) \times (0, \pi)^{n-2} \to \mathbb{R}^n$,

$$y = (\phi, \theta_1, \dots, \theta_{n-2}) \mapsto \Psi_n(1, \phi, \theta_1, \dots, \theta_{n-2}).$$

First, we observe that (A.1) implies that

$$\partial_r \Psi_n \cdot \partial_r \Psi_n = 1, \qquad \partial_r \Psi_n \cdot \partial_\phi \Psi_n = 0 = \partial_r \Psi_n \cdot \partial_{\theta_i} \Psi_n = 1,\ i = 1, \dots, n-2,$$

and hence, it follows that

$$G(D\Psi_n)\big|_{r=1} = \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & D\varphi & \\ 0 & & & \end{pmatrix} = \det(D\varphi) = G(D\varphi).$$

Consequently, using (A.2) we obtain

$$\begin{aligned} dS(y) &= \sqrt{G(D\varphi(y))}dy = \sqrt{G(D\Psi_n)}\big|_{r=1} = \sqrt{(\det D\Psi_n)^2}\big|_{r=1} = |\det D\Psi_n|\,\big|_{r=1} \\ &= \sin\theta_1 (\sin\theta_2)^2 \cdots (\sin\theta_{n-2})^{n-2} d\phi d\theta_1 \dots d\theta_{n-2}. \end{aligned}$$

We can now compute the *surface area of the unit ball*,

$$|\partial B_1(0)| = \int_{\partial B_1(0)} dS(y) = \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} \sin\theta_1 (\sin\theta_2)^2 \cdots (\sin\theta_{n-2})^{n-2} d\phi d\theta_1 \dots d\theta_{n-2} = \omega_n,$$

and one can show that

$$\omega_n = n \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)},$$

where Γ denotes the *Gamma-function*.

As a consequence, we can calculate the *volume of the unit ball*. To this end we observe that $\overset{\circ}{B}_1(0) = \Psi_n((0,1) \times (0, 2\pi) \times (0, \pi)^{n-2}) =: \Psi_n(U)$ and hence, the change of variables formula implies that

$$\begin{aligned} |B_1(0)| = \overset{\circ}{B}_1(0) &= \int_{\overset{\circ}{B}_1(0)} dx = \int_U |\det D\Psi_n(x)| dx \\ &= \int_0^1 \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} r^{n-1} \sin\theta_1 (\sin\theta_2)^2 \cdots (\sin\theta_{n-2})^{n-2} d\phi d\theta_1 \dots d\theta_{n-2} \\ &= \int_0^1 r^{n-1} \int_{\partial B_1(0)} dS(y) dr = \frac{r^n}{n} \omega_n \Big|_0^1 = \frac{\omega_n}{n}. \end{aligned}$$

Finally, we aim to integrate continuous functions over general balls $B_r(x)$ and spheres $\partial B_r(x)$ for some $x \in \mathbb{R}^n$ and $r > 0$. For the sphere $\partial B_r(x)$ we use the parametrization

$$y = (\phi, \theta_1, \ldots, \theta_{n-2}) \mapsto x + r\Psi_n(1, \phi, \theta_1, \ldots .\theta_{n-2})$$

Analogously as above, we obtain

$$dS(y) = |\text{det}D\Psi_n|\Big|_r = r^{n-1} \sin\theta_1 (\sin\theta_2)^2 \cdots (\sin\theta_{n-2})^{n-2} d\phi d\theta_1 \ldots d\theta_{n-2},$$

and we conclude that

$$\begin{aligned}
&\int_{\partial B_r(x)} u(y) dS(y) \\
&= \int_0^{\pi} \cdots \int_0^{\pi} \int_0^{2\pi} u(x + r\Psi_n(1, \phi, \theta_1, \ldots, \theta_{n-2})) r^{n-1} \sin\theta_1 (\sin\theta_2)^2 \cdots (\sin\theta_{n-2})^{n-2} d\phi d\theta_1 \ldots d\theta_{n-2} \\
&= \int_{\partial B_1(0)} u(x + ry) dS(y).
\end{aligned}$$

Similarly, for a general ball we write $\mathring{B}_r(x) = x + \Psi_n\left((0, r) \times (0, 2\pi) \times (0, \pi)^{n-2}\right) =: \Psi(U)$, where $\Psi(r, \phi, \theta_1, \ldots, \theta_{n-2}) = x + \Psi_n(r, \phi, \theta_1, \ldots, \theta_{n-2})$. Hence, we obtain

$$\begin{aligned}
\int_{B_r(x)} u(x) dx &= \int_U u(\Psi(r, \phi, \theta_1, \ldots, \theta_{n-2})) |\text{det}D\Psi| ds d\phi d\theta_1 \ldots d\theta_{n-2} \\
&= \int_0^r s^{n-1} \int_{\partial B_1(0)} u(x + sy) dS(y) ds = \int_0^r \int_{\partial B_s(x)} u(y) dS(y) ds.
\end{aligned}$$

If we now choose $u \equiv 1$, we obtain the surface area of spheres and the volume of balls in $\mathbb{R}^n$ with radius $r > 0$,

$$\begin{aligned}
|\partial B_r(x)| &= r^{n-1} |\partial B_1(0)| = r^{n-1} \omega_n, \\
|B_r(x)| &= \int_0^r s^{n-1} \omega_n ds = \frac{r^n \omega_n}{n}.
\end{aligned}$$

A.5 Integral theorems and integral formulas

In this section we recall *Gauß' divergence theorem* and several of its consequences that are frequently used throughout the course.

Theorem A.9. *Let $U \subset \mathbb{R}^n$ be open and bounded with C^1-boundary ∂U and lying on one side of ∂U. Moreover, let $\nu : \partial U \to \mathbb{R}^n$ be the outer unit normal vector and $F \in C^1(\overline{U}; \mathbb{R})$. Then we have*

$$\int_U divF(x) dx = \int_{\partial U} F(x) \cdot \nu dS(x),$$

where $\cdot$ denotes the inner product.

Applying the divergence theorem to a function F of the form $F = (0, \dots, 0, u, 0 \dots, 0)$ where the i-th component of F is given by a function $u \in C^1(\overline{U}; \mathbb{R})$ we obtain the *Gauß-Green theorem*

$$\int_U u_{x_i}(x)dx = \int_{\partial U} u(x)\nu_i(x)dS(x).$$

Moreover, applying this formula with u replaced by the product uv of two functions $u, v \in C^1(\overline{U}; \mathbb{R})$ we obtain the *integration by parts formula*

$$\int_U u_{x_i}(x)v(x)dx = -\int_U u(x)v_{x_i}(x)dx + \int_{\partial U} u(x)v(x)\nu_i(x)dS(x).$$

Appendix B

Submanifolds and tangent spaces

In this section we recall the notions of submanifolds in $\mathbb{R}^n$, hypersurfaces and tangent spaces that are used in Chapter 6. For further details and proofs we refer to [3] and [5].

Definition B.1. Let $\emptyset \neq V \subset \mathbb{R}^n$ be a subset, $k \in \mathbb{N} \cup \{\infty\}$ and $0 \leq d \leq n$. Then V is a C^k **submanifold in** $\mathbb{R}^n$ **of dimension** d, if for every $x \in V$ there exists an open neighborhood U of x in $\mathbb{R}^n$ such that $V \cap U$ is the graph of a C^k mapping $\varphi : W \to \mathbb{R}^{n-d}$, where $W \subset \mathbb{R}^d$ is open, i.e.

$$V \cap U = \{(w, \varphi(w)) \in \mathbb{R}^n : w \in W\}.$$

If $d = 1$ we call V a C^k **curve**, if $d = 2$, we call it a C^k **surface**, and if $d = n - 1$ then we call V a C^k **hypersurface** in $\mathbb{R}^n$.

Definition B.2. Let $\emptyset \neq W \subset \mathbb{R}^d$ be open, $d \leq n$ and $k \in \mathbb{N} \cup \{\infty\}$. Then a C^k mapping, $\psi : D \to \mathbb{R}^n$, $k \in \mathbb{N}$, is called C^k **immersion** if

$$\text{rank} D\psi(w) = k \qquad \forall w \in W.$$

The following theorem provides a characterization of submanifolds in $\mathbb{R}^n$.

Theorem B.3. *A subset $\emptyset \neq V \subset \mathbb{R}^n$ is a C^k submanifold in $\mathbb{R}^n$ of dimension d if and only if for every $v \in V$ there exists a relatively open neighborhood $U \subset V$ of v, an open subset $W \subset \mathbb{R}^d$ and a C^k immersion $\psi : W \to \mathbb{R}^d$ such that $\psi : D \to U$ is a homeomorphism. In this case, $\psi^{-1} : \psi(D) \to D$ is called a* ***local chart of*** V.

Next, we recall the notion of tangent space. Let $I \subset \mathbb{R}$ be an interval and $\gamma : I \to \mathbb{R}^n$ be a differentiable mapping. Then, γ is called a *differentiable curve in* $\mathbb{R}^n$ and the vector $\gamma'(t) \in \mathbb{R}^n$ is the *tangent vector* of γ at the point $\gamma(t), t \in I$.

Definition B.4. Let V be a C^1 submanifold of $\mathbb{R}^n$ of dimension d and $v \in V$. A vector $w \in \mathbb{R}^n$ is called **tangential vector** of V in v if there exists a differentiable curve $\gamma : I \to \mathbb{R}^n$, $I \subset \mathbb{R}$, and $t_0 \in I$ such that

$$\gamma(I) \subset V \quad \forall t \in I, \qquad \gamma(t_0) = v, \qquad \gamma'(t_0) = w.$$

The set of all tangential vectors of V at a point $v \in V$ is the **tangent space** $\mathcal{T}_v V$ of V at v.

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