Chapter X. Tate modules, p-divisible groups, and the fundamental group.

If X is an abelian variety over the complex numbers, the associated analytic manifold can be described as a complex torus V/Λ, with V a $\mathbb{C}$-vector space and $\Lambda \subset V$ a lattice. Topologically this is a product of spheres, and the fundamental group can be identified with $\Lambda \cong \mathbb{Z}^{2g}$ (with $g = \dim(X)$). Many properties of X can be expressed in terms of this lattice, and in fact we see that Λ together with the complex structure on $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ completely determines X.

Over an arbitrary ground field, we can no longer naturally associate a lattice of rank $2g$ to a g-dimensional abelian variety (see ??), and we have to look for a substitute for Λ. The starting point is the remark that, over $\mathbb{C}$, the fundamental group is also the group of covering transformations of the universal covering of X, and its pro-finite completion classifies the finite coverings of X. Analytically, such finite coverings can be described as $V/\Lambda' \to V/\Lambda$ where $\Lambda' \subset \Lambda$ is a subgroup of finite index; the covering group is then Λ/Λ'. In particular, one finds that any finite covering is dominated by a covering of the form $[n]_X\colon X \to X$ (which corresponds to taking $\Lambda' = n\Lambda$), which has covering group isomorphic to the n-torsion subgroup $X[n] \subset X$. This leads to a description of the pro-finite completion of $\pi_1(X, 0)$ as the projective limit of the finite groups $X[n]$. (Cf. Cor. (10.37).)

Finite coverings of X, as well as torsion points of X, can be studied over arbitrary ground fields. Restricting to the ℓ-primary part, for a prime number ℓ, we are led to consider the so-called Tate-ℓ-module $T_\ell X$ of X, which is a good ℓ-adic analogue of the fundamental group, and which can be defined in elementary terms. These Tate modules turn out to be very useful, and will play an important role in the study of endomorphisms.

If the ground field has positive characteristic p then the Tate-p-module of X has a somewhat different structure than the $T_\ell X$ for $\ell \neq p$, and there is another object, called the p-divisible group, that contains finer information. This p-divisble group, denoted $X[p^\infty]$, will be introduced in the second paragraph.

In the second half of the chapter we give a brief introduction to Grothendieck's theory of the algebraic fundamental group. We then compute the (algebraic) π_1 of an abelian variety, and show that it can indeed be expressed—as already suggested above—in terms of Tate modules.

Throughout this chapter we work over a field k. We let k_s denote a separable closure of k and $\overline{k}$ an algebraic closure. The letter ℓ is reserved for a prime number different from $\mathrm{char}(k)$.

§ 1. *Tate-ℓ-modules.*

(10.1) Let X be a g-dimensional abelian variety over a field k. Let ℓ be a prime number different from $\mathrm{char}(k)$. As we have seen in (5.9) the group scheme $X[\ell^n]$ has rank ℓ^{2ng}, and since this is not divisible by $\mathrm{char}(k)$, Cor. (4.48) shows that $X[\ell^n]$ is étale-étale.

In (3.26) we have seen that a finite étale group scheme is fully described by its group of k_s-valued points equipped with its natural action of $\mathrm{Gal}(k_s/k)$. In the case of $X[\ell^n]$ this means we have to look at the group $X[\ell^n]\big(k_s\big)$ of ℓ^n-torsion points in $X(k_s)$, equipped with its natural

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Galois action.

Multiplication by ℓ on X induces a homomorphism of group schemes $\ell\colon X[\ell^{n+1}] \to X[\ell^n]$. Under the correspondence of (3.26) it corresponds to the homomorphism of abstract groups

$$\ell\colon X[\ell^{n+1}]\big(k_s\big) \to X[\ell^n]\big(k_s\big)\,, \tag{1}$$

which is $\mathrm{Gal}(k_s/k)$-equivariant. For varying n these maps make the collection $\big\{X[\ell^n]\big(k_s\big)\big\}_{n\in\mathbb{Z}_{\geqslant 0}}$ into a projective system of abelian groups with $\mathrm{Gal}(k_s/k)$-action.

(10.2) Definition. Let X be an abelian variety over a field k, and let ℓ be a prime number different from $\mathrm{char}(k)$. Then we define the *Tate-ℓ-module of X*, notation $T_\ell X$, to be the projective limit of the system $\big\{X[\ell^n]\big(k_s\big)\big\}_{n\in\mathbb{Z}_{\geqslant 0}}$ with respect to the transition maps (1). In other words,

$$T_\ell X := \lim\Big(\{0\} \xleftarrow{\ \ell\ } X[\ell]\big(k_s\big) \xleftarrow{\ \ell\ } X[\ell^2]\big(k_s\big) \xleftarrow{\ \ell\ } X[\ell^3]\big(k_s\big) \xleftarrow{\ \ell\ } \cdots\Big)\,.$$

If $\mathrm{char}(k) = p > 0$ then we define

$$T_{p,\text{ét}} X := \lim\Big(\{0\} \xleftarrow{\ p\ } X[p]\big(\overline{k}\big) \xleftarrow{\ p\ } X[p^2]\big(\overline{k}\big) \xleftarrow{\ p\ } X[p^3]\big(\overline{k}\big) \xleftarrow{\ p\ } \cdots\Big)\,.$$

In concrete terms this means that an element of $T_\ell X$ is a sequence $x = (0, x_1, x_2, \ldots)$ with $x_n \in X(k_s)$ an ℓ^n-torsion point, and with $\ell\cdot x_{n+1} = x_n$ for all n. The addition on $T_\ell X$ is done coordinatewise, and if we have an ℓ-adic number $a = (a_0, a_1, a_2, \ldots)$ with $a_i \in \mathbb{Z}/\ell^i\mathbb{Z}$ and $a_{i+1} \bmod \ell^i = a_i$, then $a\cdot x = (0, a_1x_1, a_2x_2, \ldots)$.

In practice we often simply call $T_\ell X$ the Tate module of X, especially when the choice of ℓ plays no particular role.

Note that for $\ell \neq \mathrm{char}(k)$ we get the same module $T_\ell X$ if in the definition we replace $X[\ell^n]\big(k_s\big)$ by $X[\ell^n]\big(\overline{k}\big)$; see Prop. (5.11). In fact, we prefer to state the definition using the separable closure k_s, as we usually want to consider $T_\ell X$ with its natural action of $\mathrm{Gal}(k_s/k)$; see below. For the definition of $T_{p,\text{ét}}X$, it does make a difference that we work with torsion points over $\overline{k}$ (and not k_s); see (5.24).

Though the definition of $T_{p,\text{ét}}X$ is perfectly analogous to that of $T_\ell X$, this "Tate-p-module" is not really a good analogue of the Tate-ℓ-modules. This is why we use a slightly different notation for it. See further the discussion in § 2.

(10.3) It follows from (5.11) that $T_\ell X$ is (non-canonically) isomorphic to

$$\lim\Big(\{0\} \xleftarrow{\ \ell\ } (\mathbb{Z}/\ell\mathbb{Z})^{2g} \xleftarrow{\ \ell\ } (\mathbb{Z}/\ell^2\mathbb{Z})^{2g} \xleftarrow{\ \ell\ } (\mathbb{Z}/\ell^3\mathbb{Z})^{2g} \xleftarrow{\ \ell\ } \cdots\Big) = \mathbb{Z}_\ell^{2g}\,.$$

In other words, $T_\ell X$ is a free $\mathbb{Z}_\ell$-module of rank $2g$. We also introduce

$$V_\ell X := T_\ell X \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell\,,$$

a $\mathbb{Q}_\ell$-vector space of dimension $2g$.

But T_ℓ is not just a $\mathbb{Z}_\ell$-module. We have a natural action of $\mathrm{Gal}(k_s/k)$ on the projective system $\big\{X[\ell^n]\big(k_s\big)\big\}$, and this gives rise to an integral ℓ-adic representation

$$\rho_\ell\colon \mathrm{Gal}(k_s/k) \to \mathrm{GL}(T_\ell X)\,.$$

We refer to Appendix ?? for some basic notions on ℓ-adic representations. If there is no risk of confusion we use the same notation ρ_ℓ for the ℓ-adic representation with values in $\mathrm{GL}(V_\ell X)$.

Note that we can find back the group scheme $X[\ell^n]$ from $T_\ell X$ with its Galois action, since $T_\ell X/\ell^n T_\ell X \cong X[\ell^n]\bigl(k_s\bigr)$. Therefore, knowing the Tate-ℓ-module with its action of $\mathrm{Gal}(k_s/k)$ is equivalent to knowing the full projective system of group schemes $X[\ell^n]$.

(10.4) The group $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ is the union of its subgroups $\ell^{-n}\mathbb{Z}_\ell/\mathbb{Z}_\ell$. Phrased differently, $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ is the inductive limit of the system $\{\mathbb{Z}/\ell^n\mathbb{Z}\}_{n\geqslant 0}$, where the transition maps are the homomorphisms $\mathbb{Z}/\ell^n\mathbb{Z} \hookrightarrow \mathbb{Z}/\ell^{n+1}\mathbb{Z}$ given by $(1 \bmod \ell^n) \mapsto (\ell \bmod \ell^{n+1})$.

The definition of the Tate-ℓ-module may be reformulated by saying that

$$T_\ell X = \mathrm{Hom}\bigl(\mathbb{Q}_\ell/\mathbb{Z}_\ell, X(k_s)\bigr)\,,$$

where we take homomorphisms of abstract groups. Indeed,

$$\mathrm{Hom}\bigl(\mathbb{Q}_\ell/\mathbb{Z}_\ell, X(k_s)\bigr) = \varprojlim_n \mathrm{Hom}\bigl(\mathbb{Z}/\ell^n\mathbb{Z}, X(k_s)\bigr) = \varprojlim_n X[\ell^n]\bigl(k_s\bigr)\,,$$

where in the last term the transition maps are given by multiplication by ℓ. Concretely, if $(0, x_1, x_2, \ldots)$ with $x_n \in X[\ell^n](k_s)$ is an element of $T_\ell X$ then the corresponding homomorphism $\mathbb{Q}_\ell/\mathbb{Z}_\ell \to X(k_s)$ sends the class of ℓ^{-n} to x_n. In this description the $\mathrm{Gal}(k_s/k)$-action on $T_\ell X$ is induced by the Galois action on $X(k_s)$.

(10.5) A homomorphism $f\colon X \to Y$ gives rise to a $\mathbb{Z}_\ell$-linear, $\mathrm{Gal}(k_s/k)$-equivariant map $T_\ell f\colon T_\ell X \to T_\ell Y$. It sends a point $(0, x_1, x_2, \ldots)$ of $T_\ell X$ to the point $\bigl(0, f(x_1), f(x_2), \ldots\bigr)$ of $T_\ell Y$.

Suppose f is an isogeny with kernel $N \subset X$. Applying $\mathrm{Hom}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, -)$ to the exact sequence $0 \longrightarrow N(k_s) \longrightarrow X(k_s) \longrightarrow Y(k_s) \longrightarrow 0$ we obtain an exact sequence

$$\begin{aligned} 0 \longrightarrow T_\ell X \xrightarrow{T_\ell f} T_\ell Y \longrightarrow &\mathrm{Ext}^1\bigl(\mathbb{Q}_\ell/\mathbb{Z}_\ell, N(k_s)\bigr) \\ &\longrightarrow \mathrm{Ext}^1\bigl(\mathbb{Q}_\ell/\mathbb{Z}_\ell, X(k_s)\bigr) \longrightarrow \mathrm{Ext}^1\bigl(\mathbb{Q}_\ell/\mathbb{Z}_\ell, Y(k_s)\bigr)\,, \end{aligned} \tag{2}$$

where the Ext terms are computed in the category Ab of abelian groups.

Let us first try to understand the term $\mathrm{Ext}^1\bigl(\mathbb{Q}_\ell/\mathbb{Z}_\ell, N(k_s)\bigr)$. We use that if A and B are abelian groups, multiplication by an integer n on $\mathrm{Ext}^1(A, B)$ equals the map induced by $[n]_A$ (multiplication by n on A), and also equals the map induced by $[n]_B$.

Write $N = N_\ell \times N^\ell$ with N^ℓ a group scheme of order prime to ℓ and N_ℓ a group scheme of ℓ-power order. If m is the order of N^ℓ then multiplication by m kills $N^\ell(k_s)$ but is a bijection on $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. Hence

$$\begin{aligned} \mathrm{Ext}^1\bigl(\mathbb{Q}_\ell/\mathbb{Z}_\ell, N(k_s)\bigr) &= \mathrm{Ext}^1\bigl(\mathbb{Q}_\ell/\mathbb{Z}_\ell, N_\ell(k_s)\bigr) \times \mathrm{Ext}^1\bigl(\mathbb{Q}_\ell/\mathbb{Z}_\ell, N^\ell(k_s)\bigr) \\ &= \mathrm{Ext}^1\bigl(\mathbb{Q}_\ell/\mathbb{Z}_\ell, N_\ell(k_s)\bigr)\,. \end{aligned}$$

Next consider the long exact sequence

$$\begin{aligned} \cdots \longrightarrow \mathrm{Hom}\bigl(\mathbb{Q}_\ell, N_\ell(k_s)\bigr) \longrightarrow &\mathrm{Hom}\bigl(\mathbb{Z}_\ell, N_\ell(k_s)\bigr) \\ &\longrightarrow \mathrm{Ext}^1\bigl(\mathbb{Q}_\ell/\mathbb{Z}_\ell, N_\ell(k_s)\bigr) \longrightarrow \mathrm{Ext}^1\bigl(\mathbb{Q}_\ell, N_\ell(k_s)\bigr) \longrightarrow \cdots\,. \end{aligned} \tag{3}$$

For a sufficiently big $N_\ell(k_s)$ is killed by ℓ^a, so multiplication by ℓ^a induces the zero map on all terms in (3). On the other hand, multiplication by ℓ^a is a bijection on $\mathbb{Q}_\ell$ and therefore induces an bijection on the terms $\mathrm{Ext}^i\big(\mathbb{Q}_\ell, N_\ell(k_s)\big)$. Hence the terms $\mathrm{Hom}\big(\mathbb{Q}_\ell, N_\ell(k_s)\big)$ and $\mathrm{Ext}^1\big(\mathbb{Q}_\ell, N_\ell(k_s)\big)$ vanish, and the conclusion is that

$$\mathrm{Ext}^1\big(\mathbb{Q}_\ell/\mathbb{Z}_\ell, N(k_s)\big) \cong \mathrm{Hom}\big(\mathbb{Z}_\ell, N_\ell(k_s)\big) \cong N_\ell(k_s)\,. \tag{4}$$

Write $E^1(f)\colon \mathrm{Ext}^1\big(\mathbb{Q}_\ell/\mathbb{Z}_\ell, X(k_s)\big) \to \mathrm{Ext}^1\big(\mathbb{Q}_\ell/\mathbb{Z}_\ell, Y(k_s)\big)$ for the map induced by f. We claim it is injective. If the ground field k is perfect, so that $k_s = \overline{k}$, then we know from Cor. (5.10) that $X(k_s)$ is a divisible group, and is therefore an injective object in the category of abelian groups. Hence in this case $\mathrm{Ext}^1\big(\mathbb{Q}_\ell/\mathbb{Z}_\ell, X(k_s)\big) = 0$. In the general case, we first choose an isogeny $g\colon Y \to X$ such that $g \circ f = [n]_X$ for some positive integer n. Then $E(g\circ f)$ is multiplication by n on $\mathrm{Ext}^1\big(\mathbb{Q}_\ell/\mathbb{Z}_\ell, X(k_s)\big)$. Now write $n = \ell^m \cdot n'$ with $\ell \nmid n'$. Multiplication by n' is a bijection on $\mathbb{Q}_\ell/\mathbb{Z}_\ell$; so it suffices to show that $E^1(\ell^m)$ is injective. But if we take $f = \ell^m$ then the sequence (2) becomes

$$\begin{aligned} 0 \longrightarrow T_\ell X \xrightarrow{\ell^m} T_\ell X \xrightarrow{\delta} \mathrm{Ext}^1\big(\mathbb{Q}_\ell/\mathbb{Z}_\ell, X[\ell^m](k_s)\big) \\ \longrightarrow \mathrm{Ext}^1\big(\mathbb{Q}_\ell/\mathbb{Z}_\ell, X(k_s)\big) \xrightarrow{E^1(\ell^m)} \mathrm{Ext}^1\big(\mathbb{Q}_\ell/\mathbb{Z}_\ell, X(k_s)\big)\,, \end{aligned}$$

and it follows from (4) that δ is surjective. This proves our claim.

Finally we remark that the maps in (2) are equivariant for the natural Galois actions on all terms. To summarize, we have the following conclusion.

(10.6) Proposition. *Let $f\colon X \to Y$ be an isogeny of abelian varieties over a field k, with kernel N. If ℓ is a prime number with $\ell \neq \mathrm{char}(k)$ then we have an exact sequence of $\mathbb{Z}_\ell\big[\mathrm{Gal}(k_s/k)\big]$-modules*

$$0 \longrightarrow T_\ell X \xrightarrow{T_\ell f} T_\ell Y \longrightarrow N_\ell(k_s) \longrightarrow 0$$

where $N_\ell(k_s)$ is the ℓ-Sylow subgroup of $N(k_s)$.

(10.7) Corollary. *If $f\colon X \to Y$ is an isogeny then for all $\ell \neq \mathrm{char}(k)$ the induced map $V_\ell f\colon V_\ell X \to V_\ell Y$ is an isomorphism.*

(10.8) The construction of the Tate module makes sense for arbitrary group varieties. Thus, if G is a group variety over k and $\ell \neq \mathrm{char}(k)$ then we can form

$$T_\ell G := \lim\Big(\{0\} \xleftarrow{\ \ell\ } G[\ell](k_s) \xleftarrow{\ \ell\ } G[\ell^2](k_s) \xleftarrow{\ \ell\ } G[\ell^3](k_s) \xleftarrow{\ \ell\ } \cdots\Big)\,.$$

In some cases the result is not very interesting. For instance, $T_\ell\mathbb{G}_a = 0$. But the Tate module of the multiplicative group $\mathbb{G}_m$ is a fundamental object; so much so that it has a special notation: we write

$$\mathbb{Z}_\ell(1) := T_\ell\mathbb{G}_m = \lim\Big(\{1\} \xleftarrow{(\ )^\ell} \mu_\ell(k_s) \xleftarrow{(\ )^\ell} \mu_{\ell^2}(k_s) \xleftarrow{(\ )^\ell} \mu_{\ell^3}(k_s) \xleftarrow{(\ )^\ell} \cdots\Big)\,.$$

(In this case we of course use multiplicative notation.) As a $\mathbb{Z}_\ell$-module, $\mathbb{Z}_\ell(1)$ is free of rank 1. The action of $\mathrm{Gal}(k_s/k)$ is therefore given by a character

$$\chi_\ell\colon \mathrm{Gal}(k_s/k) \to \mathbb{Z}_\ell^{\,*} = \mathrm{GL}\big(\mathbb{Z}_\ell(1)\big)\,,$$

called the ℓ-adic cyclotomic character.

As discussed in Appendix ??, if T is any ℓ-adic representation of $\mathrm{Gal}(k_s/k)$ then we define $T(n)$, called "T twisted by n", to be

$$\begin{cases} T \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)^{\otimes n} & \text{if } n \geqslant 0, \\ T \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(-1)^{\otimes -n} & \text{if } n \leqslant 0, \end{cases}$$

where $\mathbb{Z}_\ell(-1) := \mathbb{Z}_\ell(1)^\vee$ and $\mathbb{Z}_\ell(1)^{\otimes 0}$ is defined to be $\mathbb{Z}_\ell$ with trivial Galois action. Concretely, if ρ is the Galois action on T then $T(n)$ is isomorphic to T as a $\mathbb{Z}_\ell$-module, but with $\sigma \in \mathrm{Gal}(k_s/k)$ acting via $\chi_\ell(\sigma)^n \cdot \rho(\sigma)$.

(10.9) Proposition. *We have a canonical isomorphism*

$$T_\ell X^t \cong (T_\ell X)^\vee(1)\,.$$

Proof. By Thm. (7.5) we have $X^t[\ell^n] \cong X[\ell^n]^D$, and therefore

$$X^t[\ell^n](k_s) \cong \mathrm{Hom}\Big(X[\ell^n](k_s), k_s^*\Big) = \mathrm{Hom}\Big(X[\ell^n](k_s), \mu_{\ell^n}(k_s)\Big)$$

as groups with Galois action. Now take projective limits. □

§ 2. *The p-divisible group.*

If $\mathrm{char}(k) = p > 0$ then the "Tate-p-module" $T_{p,\text{ét}}X$ is in many respects not the right object to consider. For instance, whereas $T_\ell X$ (for $\ell \neq \mathrm{char}(k)$, as always) has rank $2g$ over $\mathbb{Z}_\ell$, independent of ℓ, the rank of the module $T_{p,\text{ét}}X$ equals the p-rank of X, and as we know this is an integer with $0 \leqslant f(X) \leqslant g$. In particular, $T_{p,\text{ét}}X$ could be zero.

We have seen that the Tate-ℓ-module captures the full system of group *schemes* $X[\ell^n]$. That this system can be encoded into a single $\mathbb{Z}_\ell$-module with Galois action is due to the fact that $X[\ell^n]$ is étale for every n. So we should really consider the full system of group schemes $X[p^n]$. It turns out that it is most convenient to put these into an inductive system, and in this way we arrive at the p-divisible group of an abelian variety.

Let us now first give the definition of a p-divisible group in a general setting.

(10.10) Definition. Let S be a base scheme. A p-divisible group over S, also called a Barsotti-Tate group over S, is an inductive system

$$\big\{G_n\,;\, i_n\colon G_n \to G_{n+1}\big\}_{n\in\mathbb{N}}\,, \quad \text{in other words:} \quad G_1 \xrightarrow{i_1} G_2 \xrightarrow{i_2} G_3 \xrightarrow{i_3} \cdots\,,$$

where:

(i) each G_n is a commutative finite locally free S-group scheme, killed by p^n, and flat when viewed as a sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules;

(ii) each $i_n\colon G_n \to G_{n+1}$ is a homomorphism of S-group schemes, inducing an isomorphism $G_n \xrightarrow{\sim} G_{n+1}[p^n]$.

Homomorphisms of p-divisible groups are defined to be the homomorphisms of inductive systems of group schemes.

The flatness condition in (i) of the definition can be rephrased in more elementary terms, as in the following lemma.

(10.11) Lemma. *Let S be a scheme. Let p be a prime number. If H is an fppf sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules on S then the following are equivalent:*
(i) *H is flat as a sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules;*
(ii) $\mathrm{Ker}(p^i) = \mathrm{Im}(p^{n-i})$ *for all $i \in \{0, 1, \ldots, n\}$.*

Proof. We closely follow Messing [1], Chap. I, § 1. For (i) ⇒ (ii), start with the exact sequence

$$\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{p^{n-i}} \mathbb{Z}/p^n\mathbb{Z} \xrightarrow{p^i} \mathbb{Z}/p^n\mathbb{Z}\,.$$

If H is flat over $\mathbb{Z}/p^n\mathbb{Z}$ then $-\otimes H$ gives an exact sequence

$$H \xrightarrow{p^{n-i}} H \xrightarrow{p^i} H \tag{5}$$

and we see that (ii) holds.

For the proof of (ii) ⇒ (i) we use some results of Bourbaki [2]. These results are stated in the context of modules over rings, but they carry over (with the same proofs) to the setting of sheaves.

We use the flatness criterion, loc. cit., Chap. III, § 5, Thm. 1 together with ibid., Prop. 1. This tells us that H is flat over $\mathbb{Z}/p^n\mathbb{Z}$ if and only if the following two conditions hold:
(a) H/pH is flat as a sheaf of $\mathbb{F}_p$-modules;
(b) $\mathrm{Tor}_1^{\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{Z}/p^i\mathbb{Z}, H) = 0$ for all $i \geqslant 0$.
But (a) is trivially true, as $\mathbb{F}_p$ is a field. To see that (b) holds, start with

$$0 \longrightarrow \mathbb{Z}/p^{n-i}\mathbb{Z} \xrightarrow{p^i} \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathbb{Z}/p^i\mathbb{Z} \longrightarrow 0\,.$$

This gives a long exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{Z}/p^i\mathbb{Z}, H) \longrightarrow H/p^{n-i}H \xrightarrow{p^i} H \longrightarrow H/p^iH \longrightarrow 0\,.$$

But assumption (ii), equivalent to the exactness of (5), says precisely that $p^i\colon H/p^{n-i}H \to H$ is injective. □

(10.12) Let $\{G_n; i_n\}$ be a p-divisible group over S. If m and n are natural numbers then the composition

$$i_{m,n}\colon G_m \xrightarrow{i_m} G_{m+1} \xrightarrow{i_{m+1}} \cdots \xrightarrow{i_{m+n-1}} G_{m+n}$$

gives an identification $G_m \xrightarrow{\sim} G_{m+n}[p^m]$. Hence we may view G_m as a subgroup scheme of G_{m+n}.

On the other hand, since G_{m+n} is killed by p^{m+n} the map $[p^m]\colon G_{m+n} \to G_{m+n}$ factors through $G_{m+n}[p^n] = G_n$. If there is no risk of confusion we simply write $p^m\colon G_{m+n} \to G_n$ for the induced homomorphism. By Lemma (10.11) this last map is an epimorphism. Hence the sequence

$$0 \longrightarrow G_m \xrightarrow{i_{m,n}} G_{m+n} \xrightarrow{p^m} G_n \longrightarrow 0 \tag{6}$$

is exact.

(10.13) Given a p-divisible group as in the above definition, we may consider the G_n as fppf sheaves on S and form the limit

$$G := \varinjlim_n G_n ,$$

in the category of fppf sheaves of abelian groups. We can recover G_n from G by $G_n = G[p^n]$.

If $\{G_n\}$ and $\{H_n\}$ are two p-divisible groups and we form $G := \varinjlim G_n$ and $H := \varinjlim H_n$, then the homomorphisms from $\{G_n\}$ to $\{H_n\}$ are just the homomorphisms from G to H as fppf sheaves. In other words, by passing from the inductive system $\{G_n\}$ to the limit G we can identify the category of p-divisible groups over S with a full subcategory of the category of fppf sheaves in abelian groups over S.

An fppf sheaf G is (or "comes from") a p-divisible group if and only if it satisfies the following conditions:

(i) G is p-divisible in the sense that $[p]_G\colon G \to G$ is an epimorphism;
(ii) G is p-torsion, meaning that $G = \varinjlim_n G[p^n]$;
(iii) the subsheaves $G[p^n]$ are representable by finite locally free S-group schemes.

To go back from a sheaf G satisfying these conditions to a p-divisible group as defined in (10.10), take $G_n := G[p^n]$, and let $i_n\colon G_n \to G_{n+1}$ be the natural inclusion. It follows from (i) that $[p^n]_G$ is an epimorphism for all n, and this implies that for all m and n we have an exact sequence as in (6). By Lemma (10.11), we conclude that each G_n is flat as a sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules; hence the system $\{G_n; i_n\}_n$ is a p-divisible group. As a further simplification, it can be shown that it suffices to require (iii) for $G[p]$; see Messing [1], Chap. I, § 1.

We can go one step further by remarking that, as a consequence of (ii), a p-divisible group G has a natural structure of a sheaf in $\mathbb{Z}_p$-modules. More concretely, suppose we have a p-adic number $a = (a_1, a_2, \ldots)$ with $a_i \in \mathbb{Z}/p^i\mathbb{Z}$. Then a acts on G_n as multiplication by a_n; this gives a well-defined $\mathbb{Z}_p$-module structure on the limit G because the diagrams

$$\begin{tikzcd} G_n \arrow[r, "a_n\cdot"] \arrow[d, "i_n"'] & G_n \arrow[d, "i_n"] \\ G_{n+1} \arrow[r, "a_{n+1}\cdot"'] & G_{n+1} \end{tikzcd}$$

are commutative. Homomorphisms of p-divisible groups are automatically $\mathbb{Z}_p$-linear. In particular, $\mathrm{Hom}(G, H)$ has a natural structure of a $\mathbb{Z}_p$-module.

(10.14) Remark. The name "p-divisible group" refers to condition (i) in (10.13). But we see that the requirement for an fppf sheaf G to be a p-divisble group in the sense of Def. (10.10) is stronger than only this condition. Thus, strictly speaking the terminology "p-divisible group" is not correct. This is one of the reasons that some prefer the terminology "Barsotti-Tate group", after two of the pioneers in this area.

(10.15) If $G = \varinjlim G_n$ is a p-divisible group over a connected base scheme S then, by definition, the group scheme G_1 is locally free and killed by p. It follows that the rank of G_1 equals p^h for some integer h. (Use Exercise (4.4).) This integer $h = h(G)$ is called the *height* of G. It readily follows from (6) and Lemma (4.46) that G_n, which is again locally free, has rank p^{nh}.

Over an arbitrary basis S, we define the height of a p-divisible group G as the locally constant function $|S| \to \mathbb{Z}_{\geqslant 0}$ given by $s \mapsto h\big(G(s)\big)$.

(10.16) Definition. Let X be an abelian variety over a field k. Let p be a prime number. Then we define the *p-divisible group of X*, notation $X[p^\infty]$, to be the inductive system

$$\big\{X[p^n]\big\}_{n\geqslant 0}$$

with respect to the natural inclusion homomorphisms $X[p^n] \hookrightarrow X[p^{n+1}]$.

Note that $X[p^\infty]$ has height $2g$, where $g = \dim(X)$.

(10.17) A homomorphism $f\colon X \to Y$ of abelian varieties over k induces a homomorphism $f[p^\infty]\colon X[p^\infty] \to Y[p^\infty]$ of p-divisible groups.

If we take $f = [n]_X$ for some integer n then the induced endomorphism of $X[p^\infty]$ is multiplication by n, which for $n \neq 0$ is surjective (as a homomorphism of fppf sheaves). Using Prop. (5.12) it follows that if f is an isogeny then $f[p^\infty]$ is an epimorphism of fppf sheaves. Hence if f is an isogeny with kernel N we find an exact sequence of fppf sheaves

$$0 \longrightarrow N_p \longrightarrow X[p^\infty] \xrightarrow{f[p^\infty]} Y[p^\infty] \longrightarrow 0\,,$$

where we write $N = N_p \times N^p$ with N_p of p-power order and N^p a group scheme of order prime to p.

(10.18) Let us return to the general context of a p-divisible group G over a base scheme S. Applying Cartier duality to (6) gives an exact sequence

$$0 \longrightarrow G_n^D \longrightarrow G_{m+n}^D \longrightarrow G_m^D \longrightarrow 0\,.$$

In particular, taking $m = 1$ this gives homomorphisms $\iota_n\colon G_n^D \to G_{n+1}^D$. The inductive system $\big\{G_n^D\,;\,\iota_n\big\}$ is again a p-divisible group; it is called the Serre dual of G.

A homomorphism $f\colon G \to H$ induces a dual homomorphism $f^D\colon H^D \to G^D$; in this way $G \mapsto G^D$ gives a contravariant functor from the category of p-divisible groups over S to itself. The collection of isomorphisms $(G_n^D)^D \xrightarrow{\sim} G_n$ give a canonical isomorphism $(G^D)^D \xrightarrow{\sim} G$.

It is immediate from the defnitions that the Serre-dual of G has the same height as G.

(10.19) Proposition. *If X/k is an abelian variety then we have a canonical isomorphism*

$$X^t[p^\infty] \cong X[p^\infty]^D\,.$$

Proof. Immediate from Thm. (7.5) and the definition of the Serre dual. $\square$

(10.20) Like the construction of a Tate module, the definition of a p-divisible group also makes sense for certain other commutative group varieties. Beyond abelian varieties, the main example of interest is the p-divisible group $\mathbb{G}_m[p^\infty]$ associated to $\mathbb{G}_m$. By definition, $\mathbb{G}_m[p^\infty]$ is the inductive system of group schemes μ_{p^n} with respect to the natural inclusions $\mu_{p^n} \hookrightarrow \mu_{p^{n+1}}$. If we work over a field k and view $\mathbb{G}_m[p^\infty]$ as an fppf sheaf on $\mathrm{Spec}(k)$ then we have

$$\mathbb{G}_m[p^\infty]\big(R\big) = \big\{x \in R^* \;\big|\; x^{p^n} = 1 \text{ for some } n \geqslant 0\big\}\,,$$

for any k-algebra R. The height of $\mathbb{G}_m[p^\infty]$ is 1.

The Serre-dual of $\mathbb{G}_m[p^\infty]$ is the p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$, i.e., the inductive limit of constant group schemes $\mathbb{Z}/p^n\mathbb{Z}$ with respect to the inclusion maps $\mathbb{Z}/p^n\mathbb{Z} \xrightarrow{\sim} p\mathbb{Z}/p^{n+1}\mathbb{Z} \subset \mathbb{Z}/p^{n+1}\mathbb{Z}$.

(10.21) As we have seen in Prop. (4.45), a finite commutative group scheme over a field k is, in a canonical way, an extension of an étale group scheme by a local group scheme. An immediate consequence of this is that any p-divisible group $G = \varinjlim G_n$ over k is an extension

$$1 \longrightarrow G_{\mathrm{loc}} \longrightarrow G \longrightarrow G_{\text{ét}} \longrightarrow 1 \tag{7}$$

of the "ind-étale" p-divisible group $G_{\text{ét}} = \varinjlim G_{n,\text{ét}}$ by the "ind-local" p-divisible group $G_{\mathrm{loc}} = \varinjlim G_{n,\mathrm{loc}}$. To simplify terminology, the prefix "ind-" is often omitted; e.g., G is called an étale p-divisible group if $G \xrightarrow{\sim} G_{\text{ét}}$.

If k is perfect then the sequence (7) splits. See Exercise (10.1).

Combining the above with the Serre-duality functor $G \mapsto G^D$ of (10.18), we can further decompose G_{loc} as an extension of a local-local p-divisble group by a local-étale one. Here we extend the terminology introduced in (4.42) in an obvious way to p-divisible groups. Similarly, $G_{\text{ét}}$ is an extension of an étale-local p-divisible group by an étale-étale one.

(10.22) If G is a p-divisible group over k, viewed as an fppf sheaf, then we define its Tate-p-module by $T_pG := \mathrm{Hom}\big(\mathbb{Q}_p/\mathbb{Z}_p, G(\overline{k})\big)$. Concretely, we take the limit of the projective system

$$G_1(k_s) \xleftarrow{\pi_{1,1}} G_2(k_s) \xleftarrow{\pi_{1,2}} G_3(k_s) \xleftarrow{\pi_{1,3}} \cdots .$$

As usual, T_pG is a $\mathbb{Z}_p$-module that comes equipped with a continuous action of $\mathrm{Gal}(k_s/k)$.

It is clear from the definitions that T_pG only sees the étale part of G, i.e., the canonical map $T_pG \to T_pG_{\text{ét}}$ is an isomorphism. It follows that T_pG is a free $\mathbb{Z}_p$-module of rank $h(G_{\text{ét}})$.

If $p \neq \mathrm{char}(k)$ then clearly the Tate module of $X[p^\infty]$ is the same as the Tate-p-module of X as defined in (10.2). The Tate module of $\mathbb{G}_m[p^\infty]$ is $\mathbb{Z}_p(1)$.

(10.23) Thus far we have not made any assumptions on the prime p in relation to the characteristic of the ground field k. But if $p \neq \mathrm{char}(k)$ then it follows from Prop. (4.47) that any p-divisible group G over k is étale-étale. More precisely, G_{k_s} is non-canonically isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^{h(G)}$. In this case it is an easy exercise to show that the functor $G \mapsto T_pG$ gives an equivalence from the category of p-divisible groups over k to the category of free $\mathbb{Z}_p$-modules of finite rank equipped with a continuous action of $\mathrm{Gal}(k_s/k)$. This functor is compatible with duality, in the sense that $T_p(G^D)$ is canonically isomorphic to $(T_pG)^\vee(1)$.

In sum, for $p \neq \mathrm{char}(k)$, a p-divisible group carries the same information as the corresponding Tate module, and we typically work with the latter. (To stress that $p \neq \mathrm{char}(k)$ we shall use the letter ℓ rather than p.) By contrast, if $\mathrm{char}(k) = p > 0$ then a p-divisible group in general contains finer information than the associated Tate module.

(10.24) To conclude this general section on p-divisible groups, let us discuss the relation with formal groups. (??TO BE COMPLETED??)

§ 3. *The algebraic fundamental group—generalities.*

In Topology one defines the fundamental group $\pi_1(X,x)$ of a space X with base point $x \in X$ as the group of homotopy classes (rel. $\{0,1\}$) of paths $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1) = x$. Now suppose we want to define the fundamental group of an algebraic variety over an arbitrary field. Working with the Zariski topology does not give reasonable answers—for instance, any two algebraic curves over the same field are homeomorphic as topological spaces! Further, the above topological definition via paths does not have an obvious "algebraic" analogue that works well. (In fact, an algebraic analogue of homotopy theory was developed only in the 1990's; see Morel and Voevodsky [1].)

Assuming that X is locally connected and locally simply connected, an alternative description of $\pi_1(X,x)$ is that it is the automorphism group of the universal covering $\tilde{X} \to X$. See for instance Massey [1] or Rotman [1]. In this description the fundamental group becomes the group which classifies topological coverings of X. This is similar to Galois theory of fields, and it was one of Grothendieck's fundamental insights that it is possible to develop an abstract Galois theory of which both are special instances. Using finite étale morphisms as coverings, this theory also applies to algebraic schemes and gives rise to a notion of an algebraic fundamental group.

We shall now recall the definition of the algebraic fundamental group $\pi_1(X,\bar{x})$, and some basic properties. For further introduction we refer to SGA1. On a more advanced level, but very readable, is Deligne [4], § 10. We shall write π_1 for the algebraic fundamental group and use the notation π_1^{top} for the fundamental group in the classical setting of topological spaces.

(10.25) Definition. Let X be a scheme. By an *étale covering* of X we mean a finite étale morphism $Y \to X$. (Do not confuse this with the notion of a covering for the étale topology.) We write $\mathsf{FEt}_{/X} \subset \mathsf{Sch}_{/X}$ for the full subcategory of such étale coverings. Note that the morphisms in $\mathsf{FEt}_{/X}$ are automatically again étale coverings. We say that an étale covering $f\colon Y \to X$ *dominates* the étale covering $g\colon Z \to X$ if there exists a morphism $h\colon Y \to Z$ with $f = g \circ h$.

Fix an algebraically closed field Ω and a geometric point $\bar{x}\colon \mathrm{Spec}(\Omega) \to X$. We define a functor

$$F_{\bar{x}}\colon \mathsf{FEt}_{/X} \to \mathsf{Sets}$$

by $F_{\bar{x}}(f\colon Y \to X) = \bigl\{y \in Y(\Omega) \bigm| f(y) = x\bigr\}$. In other words, $F_{\bar{x}}$ associates to an étale covering of X the set of geometric points lying over $\bar{x}$.

(10.26) Definition. (Grothendieck) Assume X to be locally noetherian and connected. Then the algebraic fundamental group $\pi_1(X,\bar{x})$ is defined to be the automorphism group of the functor $F_{\bar{x}}$.

(10.27) Example. Suppose $X = \mathrm{Spec}(k)$ is the spectrum of a field. The geometric point $\bar{x}$ corresponds to an embedding $\sigma\colon k \hookrightarrow \Omega$. An étale covering of X is a finite disjoint union of schemes $\mathrm{Spec}(L)$, where $k \subset L$ is a finite separable field extension. For such a scheme $Y = \mathrm{Spec}(L)$ we have

$$F_{\bar{x}}(Y) = \bigl\{\text{embeddings } \tau\colon L \hookrightarrow \Omega \text{ with } \tau_{|k} = \sigma\bigr\}\,.$$

Write k_s for the separable closure of k inside Ω. Clearly, every element of $\mathrm{Gal}(k_s/k)$ gives an automorphism of the functor $F_{\bar{x}}$. Conversely, if $\alpha \in \mathrm{Aut}(F_{\bar{x}})$ and $\xi \in k_s$ then the inclusion $k(\xi) \subset \Omega$ gives an Ω-valued point of $\mathrm{Spec}\bigl(k(\xi)\bigr)$ lying above $\bar{x}$, in other words, an element $i \in F_{\bar{x}}\bigl(\mathrm{Spec}(k(\xi))\bigr)$. Then $\alpha(i)$ is another embedding of $k(\xi)$ into Ω that extends σ. Sending ξ

to its image under $\alpha(i)$ defines an element of $\mathrm{Gal}(k_s/k)$. These two constructions are inverse to each other, so we find a canonical isomorphism of pro-finite groups

$$\pi_1\bigl(\mathrm{Spec}(k),\bar{x}\bigr) \cong \mathrm{Gal}(k_s/k)\,.$$

Notice that the elements of $\pi_1\bigl(\mathrm{Spec}(k),\bar{x}\bigr)$ do not directly appear as automorphisms of the field k_s. Rather, if $\alpha \in \pi_1\bigl(\mathrm{Spec}(k),\bar{x}\bigr)$ corresponds to $\beta \in \mathrm{Gal}(k_s/k)$ then α describes the effect that β has on all embeddings $L \hookrightarrow k_s$ (= the geometric points lying over $\bar{x}$ in the covering $\mathrm{Spec}(L) \to X$). So, to phrase it in a more topological way, the point here is that an automorphism of the "universal covering" of X is completely determined by its effect on the points in the fibre over the base point $\bar{x}$.

(10.28) Theorem. (Grothendieck) *Assume X to be locally noetherian and connected. Then $\pi_1 = \pi_1(X,\bar{x})$ is a pro-finite group, and $F_{\bar{x}}$ induces an equivalence of categories*

$$\mathsf{FEt}_{/X} \xrightarrow{\text{eq}} \begin{pmatrix}\text{finite}\\ \pi_1\text{-sets}\end{pmatrix}\,,$$

where the right hand side denotes the category of finite sets with a continuous action of $\pi_1(X,\bar{x})$.

For the proof of this theorem we refer to SGA1, in particular Exp. V. Note that in case $X = \mathrm{Spec}(k)$ we have already seen this result in (3.25).

From now on, whenever we consider an algebraic fundamental group, it is assumed that the scheme in question is locally noetherian and connected.

We shall briefly review some basic properties of the fundamental group. Proofs may be found in SGA1. Note that some of the results discussed below are ingredients of the proof of Thm. (10.28), rather than being consequences of it.

(10.29) *Dependence on the choice of a base point.* Suppose we have two geometric points $\bar{x}_1\colon \mathrm{Spec}(\Omega_1) \to X$ and $\bar{x}_2\colon \mathrm{Spec}(\Omega_2) \to X$; here the (algebraically closed) fields Ω_1 and Ω_2 may be different, and may even have different characteristics. The theorem implies that there is an equivalence of categories

$$\begin{pmatrix}\text{finite}\\ \pi_1(X,\bar{x}_1)\text{-sets}\end{pmatrix} \xrightarrow{\text{eq}} \begin{pmatrix}\text{finite}\\ \pi_1(X,\bar{x}_2)\text{-sets}\end{pmatrix}\,. \tag{8}$$

Notice that this equivalence is not canonical, as it depends on the choice of a quasi-inverse of the equivalence $F_{\bar{x}_2}$. Now it is not difficult to show that the equivalence in (8) is induced by an isomorphism of topological groups $\pi_1(X,\bar{x}_1) \xrightarrow{\sim} \pi_1(X,\bar{x}_2)$. Hence up to isomorphism the fundamental group of the (connected!) scheme X does not depend on the chosen base point.

As in topology, a more elegant way to express that the fundamental group does not depend on the chosen base point is to work with the fundamental groupoid. See SGA1, Exp. V, sect. 5 or Deligne [4], § 10.

(10.30) *Functoriality.* Let $f\colon Y \to X$ be a morphism between connected, locally noetherian schemes. Let $\bar{y}$ be a geometric point of Y, and write $\bar{x} = f(\bar{y})$. Associating to an étale covering $X' \to X$ its pull-back $Y' := (X' \times_X Y) \to Y$ gives a functor $f^*\colon \mathsf{FEt}_X \to \mathsf{FEt}_Y$, and $F_{\bar{x}} = F_{\bar{y}} \circ f^*$. In particular, every automorphism of the functor $F_{\bar{y}}$ induces an automorphism of $F_{\bar{x}}$, and this gives a canonical homomorphism

$$f_*\colon \pi_1(Y,\bar{y}) \to \pi_1(X,\bar{x})\,.$$

If $g\colon Z \to Y$ is a second morphism then $(f\circ g)_* = f_*\circ g_*$.

If $f\colon Y \to X$ is an étale covering (still with X and Y connected and locally noetherian), one shows that f_* gives an isomorphism

$$\pi_1(Y,\bar{y}) \xrightarrow{\sim} \mathrm{Stab}(\bar{y}) \subset \pi_1(X,\bar{x})\,,$$

where $\mathrm{Stab}(\bar{y})$ is the stabilizer of the point $\bar{y} \in f^{-1}(\bar{x})$ under the natural action of $\pi_1(X,\bar{x})$ on $f^{-1}(\bar{x})$. Indeed, if $g\colon Z \to Y$ is an étale covering of Y, then $f\circ g$ is an étale covering of X and $g^{-1}(\bar{y}) \subset (f\circ g)^{-1}(\bar{x})$. If $\sigma \in \mathrm{Stab}(\bar{y}) \subset \pi_1(X,\bar{x})$ then its natural action on $(f\circ g)^{-1}(\bar{x})$ preserves the subset $g^{-1}(\bar{y})$; hence σ induces an automorphism of the functor $F_{\bar{y}}$. This gives a homomorphism $\mathrm{Stab}(\bar{y}) \to \pi_1(Y,\bar{y})$ inverse to f_*.

Conversely, if $H \subset \pi := \pi_1(X,\bar{x})$ is an open subgroup (equivalently, a subgroup of finite index) then π/H is a finite set with a natural action of π by left multiplication, so by Thm. (10.28) there exists an étale covering $f_H\colon Y_H \to X$ such that we have an isomorphism $\gamma\colon f_H^{-1}(\bar{x}) \xrightarrow{\sim} \pi/H$ as π-sets. Since the π-action on π/H is transitive, Y_H is connected. If we let $\bar{y} \in f_H^{-1}(\bar{x})$ be the geometric point with $\gamma(\bar{y}) = (1 \bmod H)$ then $\mathrm{Stab}(\bar{y}) = H$ as subgroups of π, and the pair $(Y,\bar{y})$ is uniquely determined up to isomorphism over X. In this way we obtain a bijective correspondence between pairs $(Y,\bar{y})$ up to X-isomorphism (with connected Y) and open subgroups of $\pi_1(X,\bar{x})$. As a variant, we may forget the choice of a geometric point $\bar{y}$ above $\bar{x}$; then we get a bijective correspondence between connected étale coverings $Y \to X$ up to isomorphism over X and conjugacy classes of open subgroups of $\pi_1(X,\bar{x})$.

(10.31) *Geometric and arithmetic fundamental group.* Let X be a geometrically connected scheme of finite type over a field k. Let k_s be a separable closure of k and write $\bar{X} := X \times_k k_s$. Choose a geometric point $\bar{x}$ of $\bar{X}$, and write $\bar{x}'$ for its image in X. Then there is an exact sequence

$$1 \longrightarrow \pi_1(\bar{X},\bar{x}) \xrightarrow{p_*} \pi_1(X,\bar{x}') \xrightarrow{s_*} \mathrm{Gal}(k_s/k) \longrightarrow 1\,, \qquad (9)$$

where the homomorphisms are induced by the projection $p\colon \bar{X} \to X$ and the structural morphism $s\colon X \to \mathrm{Spec}(k)$, and where we use the isomorphism of (10.27). If $\bar{x}\colon \mathrm{Spec}(\Omega) \to X$ factors through a k-rational point $x\colon \mathrm{Spec}(k) \to X$ then $x_*\colon \mathrm{Gal}(k_s/k) \to \pi_1(X,\bar{x})$ is a section of s_*.

The group $\pi_1(\bar{X},\bar{x})$ is referred to as the *geometric fundamental group* of X. The "full" fundamental group $\pi_1(X,\bar{x})$ is occasionally called the *arithmetic fundamental group.* If $\mathrm{char}(k) = 0$ or if X is proper over k then $\pi_1(\bar{X},\bar{x})$ does not change under extension of scalars to a bigger separably closed field. More precisely, if L is a separably closed field containing k_s such that $\bar{x}$ lifts to a geometric point $\tilde{x}$ of $X \times_k L$, then the natural map $\gamma\colon \pi_1^{\mathrm{alg}}(X\times_k L,\tilde{x}) \to \pi_1^{\mathrm{alg}}(X\times_k k_s,\bar{x})$ is an isomorphism. Note however that for $\mathrm{char}(k) > 0$ and X not proper, γ need not be an isomorphism; see SGA1, Exp. X, Sect. 1.

Writing $\mathrm{Out}(\pi) := \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ for the group of *outer automorphisms* of a group π, the exact sequence (9) gives rise to a homomorphism

$$\mathrm{Gal}(k_s/k) \to \mathrm{Out}\bigl(\pi_1(\bar{X},\bar{x})\bigr)\,.$$

In the special case that $\bar{x}$ factors through $x\colon \mathrm{Spec}(k) \to X$, this naturally lifts (via the section x_*) to a homomorphism

$$\mathrm{Gal}(k_s/k) \to \mathrm{Aut}\bigl(\pi_1(\bar{X},\bar{x})\bigr)\,.$$

(10.32) *Comparison with the topological fundamental group.* Let X be a variety over $\mathbb{C}$. Choose a base point $x \in X(\mathbb{C})$. Let us write $\pi_1^{\mathrm{top}}(X,x)$ for the usual fundamental group of $X(\mathbb{C})$ with its analytic topology. If $Y \to X$ is an étale covering then the induced map on points $Y(\mathbb{C}) \to X(\mathbb{C})$ is a finite topological covering (taking the analytic topology on both sides). Since $\pi_1^{\mathrm{top}}(X,x)$ naturally acts on the fibre of $Y(\mathbb{C})$ over x, we obtain a homomorphism $\pi_1^{\mathrm{top}}(X,x) \to \pi_1(X,x)$. It can be shown that this map induces an isomorphism

$$\left[\pi_1^{\mathrm{top}}(X,x)\right]^{\wedge} \xrightarrow{\sim} \pi_1^{\mathrm{alg}}(X,x)\,,$$

where the left hand side denotes the pro-finite completion of π_1^{top}, that is, the projective limit of all its finite quotients. The geometric content of this statement is that every finite topological covering of X can be realised as an algebraic variety which is finite étale over X, and this algebraic structure is unique up to isomorphism over X.

Note that $\pi_1^{\mathrm{top}}(X,x)$ may not be residually finite, i.e., it may happen that the natural homomorphism $\pi_1^{\mathrm{top}} \to [\pi_1^{\mathrm{top}}]^{\wedge}$ is not injective. (For examples, see Toledo [1].) Geometrically this means that the natural map from the universal covering $\tilde{X}$ of X (in the context of topological spaces) to the "algebraic universal covering" $\tilde{X}^{\mathrm{alg}}$, obtained as the projective limit of all finite étale coverings of X, is not injective.

(10.33) *Galois coverings.* As before, let X be a connected, locally noetherian scheme. Fix a geometric base point $\bar{x} \in X(\Omega)$. If we claim that the theory of the fundamental group can be viewed as an abstract Galois theory, one may expect that certain étale coverings $Y \to X$ play the role of Galois extensions.

Consider an étale covering $f\colon Y \to X$ with Y connected. Choose a base point $\bar{y} \in Y(\Omega)$ above $\bar{x}$. For simplicity of notation, write $\pi := \pi_1(X,\bar{x})$, and let $H \subset \pi$ be the stabilizer of $\bar{y}$. As discussed in (10.30), we have an isomorphism $f_*\colon \pi_1(Y,\bar{y}) \xrightarrow{\sim} H$, and we get an identification $\pi/H \xrightarrow{\sim} F_{\bar{x}}(Y)$ of finite sets with π-action. Write $N := N_\pi(H) \subset G$ for the normaliser of H.

Let $G := \mathrm{Aut}(Y/X)$ be the group of automorphisms of Y over X. Note that Y is affine over X (as $Y \to X$ is finite), so any G-equivalence class in $|Y|$ is contained in an affine subset, and there exists a quotient of Y by G.

By Theorem (10.28), G maps isomorphically to the automorphism group of $F_{\bar{x}}(Y)$ as a π-set. Using the above description we readily find that the latter group is isomorphic to $(N/H)^{\mathrm{opp}}$, the opposite group of N/H. Indeed, if $a \in N/H$ then $\varphi_a\colon \pi/H \to \pi/H$ given by $gH \mapsto gaH$ is a well-defined automorphism of π-sets, any automorphism is of this form, and $\varphi_b \circ \varphi_a = \varphi_{ab}$.

We conclude that G is finite and that its natural action on $F_{\bar{x}}(Y)$ is faithful. As this holds for any choice of the base point $\bar{x}$, it follows that G acts freely on Y. Hence the morphism $Y \to X$ factors as a composition of two étale coverings $Y \to (G\backslash Y) \to X$. From the given description of G we then see that the following conditions are equivalent:

(i) the group G acts transitively on $F_{\bar{x}}(Y)$;
(ii) the group G acts simply transitively on $F_{\bar{x}}(Y)$;
(iii) the natural map $\bar{f}\colon G\backslash Y \to X$ is an isomorphism, i.e., X is the quotient of Y under G;
(iv) the subgroup $H = f_*\pi_1(Y,\bar{y}) \subset \pi_1(X,\bar{x})$ is normal.

If these conditions are satisfied we say that $f\colon Y \to X$ is a *Galois covering* with group G, and we have an exact sequence of groups

$$1 \longrightarrow \pi_1(Y,\bar{y}) \longrightarrow \pi_1(X,\bar{x}) \longrightarrow \mathrm{Aut}(Y/X)^{\mathrm{opp}} \longrightarrow 1\,.$$

(Caution: we here only consider étale coverings. The terminology "Galois covering" is also used in the context of ramified coverings.)

Using condition (ii), it readily follows from Theorem (10.28) that every étale covering $Z \to X$ with connected Z is dominated by a Galois covering.

Suppose we have étale coverings $g\colon Z \to Y$ and $f\colon Y \to X$, where all three schemes are connected and locally noetherian. Suppose $h := f \circ g\colon Z \to X$ is a Galois covering. Then g is a Galois covering, too. Further, f is Galois if and only if $\mathrm{Aut}(Z/Y) \subset \mathrm{Aut}(Z/X)$ is a normal subgroup, and if this holds then we have a short exact sequence

$$1 \longrightarrow \mathrm{Aut}(Z/Y) \longrightarrow \mathrm{Aut}(Z/X) \longrightarrow \mathrm{Aut}(Y/X) \longrightarrow 1\,.$$

§ 4. *The fundamental group of an abelian variety.*

We now specialize to the case of an abelian variety. The key result of this paragraph is a theorem of Lang and Serre which says that, for an abelian variety X, the finite étale coverings $f\colon Y \to X$ with a rational point $e_Y \in f^{-1}(e_X)$ are precisely the separable isogenies with target X.

(10.34) Proposition. *Let X be a complete variety over a field k. Suppose given a point $e \in X(k)$ and a k-morphism $m\colon X \times X \to X$ such that $m(x,e) = x = m(e,x)$ for all $x \in X$. Then X is an abelian variety with group law m and origin e.*

Proof. Let $g := \dim(X)$, and write $x \cdot y$ for $m(x,y)$. Consider the morphism $\tau\colon X \times X \to X \times X$ given by $\tau(x,y) = \bigl(x \cdot y, y\bigr)$. (If the proposition is true then τ is the universal right translation.) We have $\tau^{-1}(e,e) = \{(e,e)\}$, so the image of τ has dimension $2g$. (We use a standard result on the dimension of the fibres of a morphism; see HAG, Chap. II, Exercise 3.22.) As $X \times X$ is complete and irreducible, it follows that τ is surjective.

We reduce the problem to the case that k is algebraically closed. Namely, suppose m induces a group structure on $X(\overline{k})$, with origin e. Then for every $x \in X(\overline{k})$ the translation $\tau_x\colon y \mapsto x \cdot y$ is an automorphism of $X_{\overline{k}}$ as a variety, and by the argument of Prop. (1.5) it follows that X is non-singular. It also follows that τ induces a bijection on $\overline{k}$-valued points. Hence τ gives a purely inseparable extension on function fields. On the other hand, by looking at the restrictions of τ to $\{e\} \times X$ and $X \times \{e\}$ we see that the tangent map of τ at (e,e) is an isomorphism. The conclusion is that τ is an isomorphism. Now define $i\colon X \to X$ by $i(y) = p_1\bigl(\tau^{-1}(e,y)\bigr)$. Using that X is geometrically reduced and that we know the group axioms to hold on $X(\overline{k})$, it follows that m, i and e define the structure of an abelian variety on X. Hence to complete the proof of the proposition, we may assume that $k = \overline{k}$ and it suffices to prove that m gives a group structure on $X(k)$.

Consider the closed subscheme $\Gamma \subset X \times X$ given by $\Gamma := \bigl\{(x,y) \bigm| x \cdot y = e\bigr\}$. Then $\Gamma = \tau^{-1}\bigl(\{e\} \times X\bigr)$, so the surjectivity of τ implies that the second projection $p_2\colon \Gamma \to X$ is surjective. Let $\Gamma_1 \subset \Gamma$ be an irreducible component with $p_2(\Gamma_1) = X$. Notice that Γ_1 is complete, and that $\dim(\Gamma_1) \geqslant g$. Further note that $p_1^{-1}(e) \cap \Gamma = \bigl\{(e,e)\bigr\} = p_2^{-1}(e) \cap \Gamma$; this implies that $(e,e) \in \Gamma_1$. Again by comparing dimensions it follows that $p_1\colon \Gamma_1 \to X$ is surjective, too.

Define $f\colon \Gamma_1 \times X \times X \to X$ by $f\bigl((x,y),z,w\bigr) = x \cdot \bigl((y \cdot z) \cdot w\bigr)$. We have $f\bigl(\Gamma_1 \times \{e\} \times \{e\}\bigr) = \{e\}$. Applying the rigidity lemma we find

$$x \cdot \bigl((y \cdot z) \cdot w\bigr) = z \cdot w \qquad \text{for all } (x,y) \in \Gamma_1 \text{ and } z,\, w \in X. \tag{10}$$

As a particular case, taking $w = e$, we have

$$x \cdot (y \cdot z) = z \qquad \text{for all } (x, y) \in \Gamma_1 \text{ and } z \in X. \tag{11}$$

Now fix $y \in X(k)$. Choose any $x \in X(k)$ with $(x, y) \in \Gamma_1$, and any $z \in X(k)$ with $(y, z) \in \Gamma_1$. (Such x and z exist, as we have shown the two projections $p_i \colon \Gamma_1 \to X$ to be surjective.) Then (11) gives $x = x \cdot (y \cdot z) = z \cdot e = z$. The conclusion is that y has a unique left and right inverse in $X(k)$. Finally, multiplying (10) from the left by $y = x^{-1}$, and using (11) gives

$$y \cdot (z \cdot w) = y \cdot \bigl(x \cdot \bigl((y \cdot z) \cdot w\bigr)\bigr) = (y \cdot z) \cdot w\,,$$

which shows that the group law on $X(k)$ is associative. $\square$

(10.35) Lemma. *Let Z be a k-variety, let Y be an integral k-scheme of finite type, and let $f \colon Y \to Z$ be a smooth proper morphism of k-schemes. If there exists a section $s \colon Z \to Y$ of f then all fibres of f are irreducible.*

Proof. As the fibres of f are non-singular, it suffices to show that they are connected. Write $Z' := \mathrm{Spec}(f_* O_Y)$, and consider the Stein factorization

$$f = g \circ f' \colon Y \xrightarrow{f'} Z' \xrightarrow{g} Z\,.$$

By Zariski's connectedness theorem (EGA III, Thm. 4.3.1) the morphism f' has connected fibres. The composition $f' \circ s$ is a proper section of g, hence it induces an isomorphism of Z with a closed subscheme of Z'. As g is finite and Z' is integral, it follows that g is an isomorphism. $\square$

(10.36) Theorem. (Lang-Serre) *Let X be an abelian variety over a field k. Let Y be a k-variety and $e_Y \in Y(k)$. If $f \colon Y \to X$ is an étale covering with $f(e_Y) = e_X$ then Y has the structure of an abelian variety such that f is a separable isogeny.*

Proof. With Proposition (10.34) at our disposal, the main point of the proof is to construct the group law $m_Y \colon Y \times Y \to Y$. Let $\Gamma_X \subset X \times X \times X$ be the graph of the multiplication on X, and write $\Gamma'_Y \subset Y \times Y \times Y$ for the pull-back of Γ_X via $f \times f \times f$. Let $\Gamma_Y \subset \Gamma'_Y$ be the connected component containing the point (e_Y, e_Y, e_Y), and if $I \subseteq \{1, 2, 3\}$ write q_I for the restriction of the projection $p_I \colon Y^3 \to Y^I$ to Γ_Y. We want to show that the projection $q_{12} \colon \Gamma_Y \to Y \times Y$ is an isomorphism—if this is true then we can define the desired group law by taking $m_Y := q_3 \circ q_{12}^{-1} \colon Y \times Y \to Y$. Note that q_{12} has a section s_1 over $\{e_Y\} \times Y$ and a section s_2 over $Y \times \{e_Y\}$, given on points by $s_1(e_Y, y) = (e_Y, y, y)$ and $s_2(y, e_Y) = (y, e_Y, y)$. This readily implies that the proposed group law m_Y satisfies the conditions of (10.34).

By construction we have a commutative diagram

$$\begin{tikzcd} \Gamma_Y \arrow[r] \arrow[d, "q_{12}"'] & \Gamma_X \arrow[d, "p_{12}"] \\ Y \times Y \arrow[r, "f \times f"] & X \times X \end{tikzcd}\quad,$$

in which both the upper arrow $\Gamma_Y \to \Gamma_X$ and the morphism $f \times f$ are étale coverings, and the right hand arrow $p_{12} \colon \Gamma_X \to X \times X$ is an isomorphism. Hence $q_{12} \colon \Gamma_Y \to Y \times Y$ is an étale covering, too.

The projection $q_2\colon \Gamma_Y \to Y$ is a smooth proper morphism, being the composition of q_{12} and $p_2\colon Y\times Y \to Y$. As s_1 gives a section of q_2 we conclude from the above lemma that all fibres of q_2 are irreducible. In particular, $Z := q_2^{-1}(e_Y) = q_{12}^{-1}\bigl(Y\times\{e_Y\}\bigr)$ is irreducible. Further, q_{12} restricts to an étale covering $r\colon Z \to Y = Y\times\{e_Y\}$ of the same degree. But s_2 gives a section of r. Hence r is an isomorphism. It follows that the étale covering q_{12} has degree 1 and is therefore an isomorphism. $\square$

(10.37) Corollary. *Let X be an abelian variety over a field k. Let Ω be an algebraically closed field containing k, and regard $0 = e_X$ as an Ω-valued point of X. Write k_s for the separable closure of k inside Ω. Then there are canonical isomorphisms*

$$\pi_1^{\mathrm{alg}}(X_{k_s},0) \cong \varprojlim_n X[n](k_s) \cong \begin{cases} \prod_\ell T_\ell X & \text{if } \mathrm{char}(k)=0,\\ T_{p,\text{ét}}X \times \prod_{\ell\neq p} T_\ell X & \text{if } \mathrm{char}(k)=p>0,\end{cases}$$

where the projective limit runs over all maps $X[nm](k_s)\to X[n](k_s)$ given by $P\mapsto m\cdot P$, and where ℓ runs over the prime numbers. In particular, $\pi_1^{\mathrm{alg}}(X_{k_s},0)$ is abelian. Further there is a canonical isomorphism

$$\pi_1^{\mathrm{alg}}(X,0) \cong \pi_1^{\mathrm{alg}}(X_{k_s},0) \rtimes \mathrm{Gal}(k_s/k)\,,$$

where $\mathrm{Gal}(k_s/k)$ acts on $\pi_1^{\mathrm{alg}}(X_{k_s},0)$ through its natural action on the groups $X[n](k_s)$.

Proof. For the proof of the first assertion we may assume that $k=k_s$. Write $\pi := \pi_1^{\mathrm{alg}}(X_{k_s},0)$. We have $\pi = \varprojlim(\pi/H)$ where H runs over the open subgroups of π. By (10.30), each H corresponds to an étale covering $f_H\colon Y_H\to X$ together with the choice of a point $e_H\in Y_H(\Omega)$ above 0, the pair (Y_H,e_H) being unique up to isomorphism over X. By the Lang-Serre theorem, we have the structure of an abelian variety on Y_H with origin e_H such that f_H is a separable isogeny. Further, it is clear that a separable isogeny $f\colon Y\to X$ is a Galois covering (in the sense of (10.33)) with group $\mathrm{Ker}(f)\bigl(k\bigr)$. (Recall that we assume $k=k_s$.) By what was explained in (10.33) we find that $\pi/H \cong \mathrm{Ker}(f_H)\bigl(k\bigr)^{\mathrm{opp}} = \mathrm{Ker}(f_H)\bigl(k\bigr)$, for any open subgroup $H\subset\pi$.

Let $\mathscr{I}$ be the set of isomorphism classes of separable isogenies $f\colon Y\to X$, where we call $f\colon Y\to X$ and $f'\colon Y'\to X$ isomorphic if there is an isomorphism of abelian varieties $\alpha\colon Y\xrightarrow{\sim} Y'$ with $f'\circ\alpha = f$. We partially order $\mathscr{I}$ by dominance; so $f\geqslant f'$ if there is a homomorphism of abelian varieties $\alpha\colon Y\to Y'$ with $f'\circ\alpha=f$. If $f\geqslant f'$ then we get a homomorphism $\mathrm{Ker}(f)\twoheadrightarrow\mathrm{Ker}(f')$, independent of the choice of α. In this way we have a projective system of finite groups $\bigl\{\mathrm{Ker}(f)\bigl(k\bigr)\bigr\}_{f\in\mathscr{I}}$, and the conclusion of the above discussion is that

$$\pi \xrightarrow{\sim} \varprojlim_{f\in\mathscr{I}} \mathrm{Ker}(f)\bigl(k\bigr) \tag{12}$$

as pro-finite groups.

If n is a positive integer then $[n]=[n]_X$ factors as $X\xrightarrow{f} X/X[n]_{\mathrm{loc}} \xrightarrow{g} X$ where f is purely inseparable and g is separable. Of course, if $\mathrm{char}(k)=0$ or $\mathrm{char}(k)=p>0$ and $p\nmid n$ then f is the identity and $g=[n]$. For the purpose of this discussion, write $g=[n]_{\mathrm{sep}}$. The Galois group of $[n]_{\mathrm{sep}}$ is $X[n]\bigl(k\bigr)$. Let $\mathscr{I}'\subset\mathscr{I}$ be the subset of all isogenies $[n]_{\mathrm{sep}}$ for $n\in\mathbb{Z}_{\geqslant 1}$. Then $\mathscr{I}'$ is cofinal in $\mathscr{I}$; indeed, if $f\colon Y\to X$ is any separable isogeny, say of degree d, then by Prop. (5.12) there is an isogeny $g\colon X\to Y$ with $[d]_X = f\circ g$, and then it follows from Cor. (5.8) that $[d]_{\mathrm{sep}}$ dominates f. Hence we may restrict the limit in (12) to the terms $f\in\mathscr{I}'$; this gives the desired isomorphism

$$\pi \xrightarrow{\sim} \varprojlim_{f\in\mathscr{I}'} \mathrm{Ker}(f)\bigl(k\bigr) = \varprojlim_n X[n]\bigl(k\bigr)\,,$$

where n runs over the set $\mathbb{Z}_{\geqslant 1}$, partially ordered by divisibility.

The last assertion of the theorem (now again over an arbitrary ground field) follows by using what was explained in (10.31), noting that $0 \in X(\Omega)$ factors through a k-rational point. $\square$

(10.38) As an application of this theorem, let us now discuss how the ℓ-adic cohomology of an abelian variety can be described in terms of its Tate-ℓ-module.

First let X be any complete variety over a field k, say with $\dim(X) = g$. Let k_s be a separable closure of k, and let ℓ be a prime number different from $\mathrm{char}(k)$. The ℓ-adic cohomology $H^\bullet(X_{k_s}, \mathbb{Z}_\ell) = \oplus_{i=0}^{2g} H^i(X_{k_s}, \mathbb{Z}_\ell)$ is a graded-commutative $\mathbb{Z}_\ell$-algebra of finite type that comes equipped with a continuous action of $\mathrm{Gal}(k_s/k)$. If $\bar{x} \in X(k_s)$ then the first ℓ-adic cohomology and the fundamental group of X_{k_s} are related by

$$H^1(X_{k_s}, \mathbb{Z}_\ell) \cong \mathrm{Hom}_{\mathrm{cont}}\bigl(\pi_1(X_{k_s}, \bar{x}), \mathbb{Z}_\ell\bigr)\,, \tag{13}$$

where the right hand side is the group of continuous homomorphisms $\pi_1(X_{k_s}, \bar{x}) \to \mathbb{Z}_\ell$. The homomorphism $\mathrm{Gal}(k_s/k) \to \mathrm{Out}\bigl(\pi_1(X_{k_s}, \bar{x})\bigr)$ of (10.31) induces a homomorphism $\mathrm{Gal}(k_s/k) \to \mathrm{Aut}\bigl(\pi_1(X_{k_s}, \bar{x})^{\mathrm{ab}}\bigr)$, and this gives a continuous Galois action on $\mathrm{Hom}_{\mathrm{cont}}\bigl(\pi_1(X_{k_s}, \bar{x}), \mathbb{Z}_\ell\bigr) = \mathrm{Hom}_{\mathrm{cont}}\bigl(\pi_1(X_{k_s}, \bar{x})^{\mathrm{ab}}, \mathbb{Z}_\ell\bigr)$. The isomorphism (13) is equivariant for the Galois actions on the two sides.

Now we specialize this to the case where X is an abelian variety. As we shall prove later, $H^\bullet(X_{k_s}, \mathbb{Z}_\ell)$ is then the exterior algebra on $H^1(X_{k_s}, \mathbb{Z}_\ell)$; see Cor. (13.32). Admitting this, we find the following result.

(10.39) Corollary. *Let X be an abelian variety over a field k, let $k \subset k_s$ be a separable algebraic closure, and let ℓ be a prime number with $\ell \neq \mathrm{char}(k)$. Then we have*

$$H^1(X_{k_s}, \mathbb{Z}_\ell) \cong (T_\ell X)^\vee := \mathrm{Hom}(T_\ell X, \mathbb{Z}_\ell)$$

as $\mathbb{Z}_\ell$-modules with continuous action of $\mathrm{Gal}(k_s/k)$. Further we have an isomorphism of graded-commutative $\mathbb{Z}_\ell$-algebras with continuous $\mathrm{Gal}(k_s/k)$-action

$$H^\bullet(X_{k_s}, \mathbb{Z}_\ell) \cong \wedge^\bullet\left[(T_\ell X)^\vee\right].$$

Exercises.

(10.1) Let G be a p-divisible group over a perfect field k. Show that for every n the square

$$\begin{array}{ccc} G_{n,\mathrm{red}} & \longrightarrow & G_{n+1,\mathrm{red}} \\ \downarrow & & \downarrow \\ G_n & \xrightarrow{i_n} & G_{n+1} \end{array}$$

is Cartesian. Conclude that the exact sequence (7) splits.

(10.2) Let K be a field, $K \subset K_s$ a separable algebraic closure. Let $K \subset L$ be a finite extension inside K_s.

(i) Let H be a finite étale group scheme over L, and consider $G := \mathrm{Res}_{L/K}(H)$, the K-group scheme obtained by Weil restriction of scalars from L to K. By definition of the Weil restriction, G represents the functor $\mathsf{Sch}_{/K}^{\mathrm{opp}} \to \mathsf{Gr}$ given by $T \mapsto H(T_L)$. Show that G is again a finite étale group scheme.

(ii) Assume (for simplicity) that H is commutative. Write $\Gamma_L := \mathrm{Gal}(K_s/L) \subset \Gamma_K := \mathrm{Gal}(K_s/K)$. Show that $G(K_s) \cong \mathrm{Ind}_{\Gamma_L}^{\Gamma_K} H(K_s)$ as representations of $\mathrm{Gal}(K_s/K)$.

(iii) Let X be an abelian variety over L, and write $Y := \mathrm{Res}_{L/K}(X)$, which is an abelian variety over K of dimension $\dim(X) \cdot [L : K]$. If ℓ is a prime number different from $\mathrm{char}(K)$, show that $T_\ell(Y) \cong \mathrm{Ind}_{\Gamma_L}^{\Gamma_K} T_\ell(X)$ as $\mathbb{Z}_\ell\big[\mathrm{Gal}(K_s/K)\big]$-modules.