Models of Shimura varieties in mixed characteristics

by

Ben Moonen

Contents

Introduction

At the 1996 Durham symposium, a series of four lectures was given on Shimura varieties in mixed characteristics. The main goal of these lectures was to discuss some recent developments, and to familiarize the audience with some of the techniques involved. The present notes were written with the same goal in mind.

It should be mentioned right away that we intend to discuss only a small number of topics. The bulk of the paper is devoted to models of Shimura varieties over discrete valuation rings of mixed characteristics. Part of the discussion only deals with primes of residue characteristic p such that the group G in question is unramified at p, so that good reduction is expected. Even at such primes, however, many technical problems present themselves, to begin with the "right" definitions.

There is a rather large class of Shimura data—those called of pre-abelian type—for which the corresponding Shimura variety can be related, if maybe somewhat indirectly, to a moduli space of abelian varieties. At present, this seems the only available tool for constructing "good" integral models. Thus, if we restrict our attention to Shimura varieties of pre-abelian type, the construction of integral canonical models (defined in §3) divides itself into two parts:

Formal aspects. If, for instance, we have two Shimura data which are "closely related", then this should have consequences for the existence of integral canonical models. Loosely speaking, we would like to show that if one of the two associated Shimura varieties has an integral canonical model, then so does the other. Most of such "formal" results are discussed in §3.

Constructing models for Shimura varieties of Hodge type. By definition, these are the Shimura varieties that can be embedded into a Siegel modular variety. As we will see, the existence of an integral canonical model is essentially a problem about smoothness, which therefore can be studied using deformation theory. We are thus led to certain deformation problems for p-divisible groups. These can be dealt with using techniques of Faltings, which are the subject of §4. This is not to say that we can now easily prove the existence of integral canonical models. Faltings's results only apply under some assumptions, and in the situation where we want to use them, it is not at all clear that these are satisfied. To solve this, Adrian Vasiu has presented an ingenious, but technically complicated strategy. We will discuss this in §5. Unfortunately, it seems that Vasiu's program has not yet been brought to a successful end. We hope that our presentation of the material can help to clarify what technical points remain to be settled.

I have chosen to include quite a bit of "basic material" on Shimura varieties, which takes up sections 1 and 2. Most of this is a review of Deligne's papers [De1] and [De3]. I also included some examples and some references to fundamental work that was done later, such as the generalization of the theory to mixed Shimura varieties. The main strategy of [De3] is explained in some detail, since it will reappear in our study of integral models.

The only new result in the first two sections concerns the existence of canonical models for Shimura varieties which are not of abelian type. It was pointed out to me by J. Wildeshaus that the argument as it is found in the literature is not complete. We will discuss this in section 2, and we present an argument to complete the proof.

In §3 we take up the study of integral models of Shimura varieties. The first major problem here is to set up good definitions. We follow the pattern laid out by Milne in [Mi3], defining an integral canonical model as a smooth model which satisfies a certain Néron extension property. The main difficulty is to decide what class of "test schemes" to use. We explain why the class used by Milne in loc. cit. leads to unwanted results, and we propose to use a smaller class of test schemes which (at least for $p > 2$ and ramification $e < p - 1$) avoids this. Our definition differs from the one used by Vasiu in [Va2].

In the rest of §3 we prove a number of "formal" results about integral canonical models, and, inspired by Deligne's approach in [De3], we develop the notion of a connected Shimura variety in the p-adic setting. The main result of this section is Cor. 3.23. It says, roughly, that in order to prove the existence of integral canonical models for all Shimura varieties of pre-abelian type at primes of characteristic $p > 2$ where the group in question is unramified, it suffices to show that certain models obtained starting from an embedding into a Siegel modular variety, are formally smooth. As we will explain, there are finitely many primes that may cause additional problems if the group has simple factors of type A_ℓ. We give full proofs of most statements. Although the reader may find some details too cumbersome, we think that they are quite essential, and that only by going through all arguments we are able to detect some unexpected problems. Some of our results were also claimed in [Va2], but most proofs given here were obtained independently (see also remark 3.24).

In §4 we study deformation theory of p-divisible groups with given Tate classes. The main results are based on a series of remarks in Faltings's paper [Fa3], of which we provide detailed proofs.

In §5 we attempt to follow Vasiu's paper [Va2]. Our main goal here is to explain Vasiu's strategy, and to explain which technical problems remain to be solved. This section consists of two parts. Up until Thm. 5.8.3, we prove most statements in detail. This leads to a result about the existence of integral canonical models under a certain additional hypothesis (5.6.1). After that we indicate a number of statements that should allow to remove this hypothesis. It is in this part of Vasiu's work that, to our understanding, further work needs to be done before the main result (see 5.9.6) can be accepted as a solid theorem.[1]

The last section contains a hodgepodge of questions and results, due to

[1]After completing our manuscript we received new versions of Vasiu's work (A. Vasiu, *Integral canonical models of Shimura varieties of Preabelian type*, third version, July 15, 1997, UC at Berkeley, and *Ibid.*, December 1997, UC at Berkeley.) We have not yet had the opportunity to study this work in detail, and we therefore cannot say whether it can take away all doubts we have about the arguments in [Va2]. We strongly recommend the interested reader to consult Vasiu's original papers.

various people. We will try to give references in the main text. The main topic here is no longer the existence of integral canonical models per se. Instead, we discuss some results about the local structure of (examples of) such models, in relation to conjectures of Coleman and Oort.

There are some interesting related topics for which we did not find place in this article. Among the casualties are the recent work [RZ] of Rapoport and Zink, examples of bad reduction (see, e.g., [R2]), the Newton polygon stratification of A_g in characteristic p (for an overview, see [Oo1], [Oo2]) and the study of isocrystals with additional structure as in [Ko1], [Ko3], [RR].

Acknowledgements. In preparing this paper I benefited a lot from discussions with Y. André, D. Blasius, C. Deninger, B. Edixhoven, O. Gabber, J. de Jong, G. Kings, E. Landvogt, F. Oort, A. Vasiu, A. Werner and J. Wildeshaus. I thank them all cordially. Also I wish to thank the referee for several useful comments.

Notations. *Superscripts and subscripts:* ^{0} denotes connected components for the Zariski topology, $^+$ connected components for other (usually analytic) topologies. A superscript $^-$ (as in $G(\mathbb{Q})_+^-$ for example) denotes the closure of a subset of a topological space. If G is an algebraic group then $^{\mathrm{ad}}$ (adjoint group), $^{\mathrm{ab}}$ (maximal abelian quotient), $^{\mathrm{der}}$ (derived group) have the usual meaning, $G(\mathbb{R})_+$ denotes the pre image of $G^{\mathrm{ad}}(\mathbb{R})^+$ under the adjoint map, and in case G is defined over $\mathbb{Q}$ we write $G(\mathbb{Q})_+$ for the intersection of $G(\mathbb{Q})$ and $G(\mathbb{R})_+$ inside $G(\mathbb{R})$. For fields, $^{\mathrm{ab}}$ denotes the maximal abelian extension. A superscript ^{p} usually denotes a structure "away from p"; a subscript $_p$ something "at p".

If (X, λ) is a g-dimensional principally polarized abelian scheme over a basis S then λ gives rise to a Weil pairing $e^\lambda\colon\ X[n] \times X[n] \to \mu_{n,S}$. Write $\psi_n\colon\ (\mathbb{Z}/n\mathbb{Z})^{2g} \times (\mathbb{Z}/n\mathbb{Z})^{2g} \to (\mathbb{Z}/n\mathbb{Z})$ for the standard symplectic form. By a Jacobi level n structure on (X, λ) we mean an isomorphism $\eta\colon\ X[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})_S$ such that there exists an isomorphism $\alpha\colon\ (\mathbb{Z}/n\mathbb{Z})_S \xrightarrow{\sim} \mu_{n,S}$ with $\alpha \circ \psi_n \circ (\eta \times \eta) = e^\lambda$. We write $\mathsf{A}_{g,1,n}$ for the (coarse) moduli scheme over $\mathrm{Spec}(\mathbb{Z}[1/n])$ of principally polarized, g-dimensional abelian varieties with a Jacobi level n structure. If $n \geq 3$ then it is a fine moduli scheme.

Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m},\mathbb{C}}$. We write $\mu\colon\ \mathbb{G}_{\mathrm{m},\mathbb{C}} \to \mathbb{S}_\mathbb{C}$ for the cocharacter which on complex points is given by $\mathbb{C}^* \ni z \mapsto (z, 1) \in \mathbb{C}^* \times \mathbb{C}^* \cong (\mathbb{C} \otimes_\mathbb{R} \mathbb{C})^*$. The natural inclusion $w\colon\ \mathbb{G}_{\mathrm{m},\mathbb{R}} \to \mathbb{S}$ is called the weight cocharacter.

We write $\mathbb{A}_f$ for the ring of finite adèles of $\mathbb{Q}$ and $\mathbb{A}_L$ for the ring of (full) adèles of a number field L. We refer to [Pi], 0.6, for an explanation of when a subgroup $K \subseteq G(\mathbb{A}_f)$ (where G is an algebraic group over $\mathbb{Q}$) is called neat, and for some basic properties concerning this notion.

Abbreviations: H.S. for Hodge structure, V.H.S. for variation of Hodge structure, d.v.r. for discrete valuation ring, p.p.a.v. for principally polarized abelian variety, i.c.m. for integral canonical model (see 3.3), a.t.s. for admissible test scheme (see 3.5), e.e.p. for extended extension property (see 3.20).

§1 Shimura varieties

1.1 Recall ([De2]) that a pure Hodge structure of weight n with underlying $\mathbb{Q}$-vector space V is given by a homomorphism of algebraic groups $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$ such that the weight cocharacter $h \circ w\colon \mathbb{G}_{\mathrm{m}} \to \mathrm{GL}(V)_{\mathbb{R}}$ maps z to $z^{-n} \cdot \mathrm{id}_V$. The Tate twist $\mathbb{Q}(1)$ corresponds to the norm character $\mathrm{Nm}\colon \mathbb{S} \to \mathbb{G}_{\mathrm{m},\mathbb{R}}$. An element $v \in V$ is called a Hodge class (in the strict sense) if v is purely of type $(0,0)$ in the Hodge decomposition $V_{\mathbb{C}} = \oplus V^{p,q}$. In other words: the Hodge classes are the *rational* classes $v \in V$ which, as elements of $V_{\mathbb{R}}$, are invariant under the action of $\mathbb{S}$ given by h.

The Mumford-Tate group $\mathrm{MT}(V)$ of V is defined as the smallest algebraic subgroup of $\mathrm{GL}(V) \times \mathbb{G}_{\mathrm{m}}$ which is defined over $\mathbb{Q}$ and such that $h \times \mathrm{Nm}\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}} \times \mathbb{G}_{\mathrm{m},\mathbb{R}}$ factors through $\mathrm{MT}(V)_{\mathbb{R}}$. In Tannakian language $\mathrm{MT}(V)$ is the automorphism group of the forgetful fibre functor $\langle V, \mathbb{Q}(1)\rangle^{\otimes} \to \mathsf{Vec}_{\mathbb{Q}}$, where $\langle V, \mathbb{Q}(1)\rangle^{\otimes} \subset \mathsf{Hdg}_{\mathbb{Q}}$ is the Tannakian subcategory generated by V and $\mathbb{Q}(1)$. Concretely, this means that for every tensor space

$$V(r_1, r_2; s) := V^{\otimes r_1} \otimes (V^*)^{\otimes r_2} \otimes \mathbb{Q}(s)\,,$$

the Hodge classes in $V(r_1, r_2; s)$ are precisely the invariants under the natural action of $\mathrm{MT}(V)$.

In more classical language one would define a Hodge class to be a rational class $v \in V$ which is purely of type $(n/2, n/2)$ in the Hodge decomposition. Clearly there are in general more Hodge classes in this sense than in the "strict" sense, but the difference is only a matter of weights. If we define the Hodge group $\mathrm{Hg}(V)$ (sometimes called the special Mumford-Tate group) to be the kernel of the second projection map $\mathrm{MT}(V) \twoheadrightarrow \mathbb{G}_{\mathrm{m}}$, then the Hodge classes (in the more general sense) of a tensor space $V^{\otimes r_1} \otimes (V^*)^{\otimes r_2}$ are precisely the invariants of $\mathrm{Hg}(V)$. All in all, the Hodge group contains essentially the same information as the Mumford-Tate group, except that it does not keep track of the weight.

The main principle that we want to stress here is the following: if $h\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$ defines a Hodge structure on the $\mathbb{Q}$-vector space V, and if we are given tensors $t_1, \ldots, t_k$ in spaces of the form $V(r_1, r_2; s)$, then there is an algebraic group $G \subseteq \mathrm{GL}(V)$ (depending on the classes t_i) such that

$$t_1, \ldots, t_k \quad \text{are Hodge classes} \quad \Longleftrightarrow \quad h \quad \text{factors through } G_{\mathbb{R}}\,.$$

1.2 To illustrate the usage of Mumford-Tate groups, let us discuss some examples pertaining to Hodge classes on abelian varieties. There are at least two reasons why abelian varieties are special:

(i) Riemann's theorem tells us that there is an equivalence of categories

$$\{\text{complex abelian varieties}\} \xrightarrow{\text{eq.}} \{\text{polarizable } \mathbb{Z}\text{-H.S. of type } (0,1)+(1,0)\}\,,$$

sending X to $\mathrm{H}^1(X, \mathbb{Z})$. (This should really be done covariantly, using H_1; as we shall later always work with cohomology we phrase everything in terms of H^1.) This result has some important variants, in that polarized abelian varieties are in equivalence with polarized $\mathbb{Z}$-H.S. of type $(0,1) + (1,0)$, abelian varieties up to isogeny correspond to polarizable $\mathbb{Q}$-H.S. of type $(0,1)+(1,0)$, and if S is a smooth variety over $\mathbb{C}$ then abelian schemes over S correspond to polarizable $\mathbb{Z}$-V.H.S. of type $(0,1)+(1,0)$ over S. (See [De2], section 4.4.) Furthermore, all cohomology of X and of its powers X^m, can be expressed directly in terms of $\mathrm{H}^1(X, \mathbb{Z})$: we have natural isomorphisms of Hodge structures

$$\mathrm{H}^k(X^m, \mathbb{Z}) \cong \bigwedge^k \big(\oplus^m\, \mathrm{H}^1(X, \mathbb{Z})\big)\,.$$

(ii) Let $V := \mathrm{H}^1(X, \mathbb{Q})$, and write $\mathrm{Hg}(X) := \mathrm{Hg}(V)$. Choose a polarization of X. The corresponding Riemann form φ is a Hodge class in $\mathrm{Hom}\big(V^{\otimes 2}, \mathbb{Q}(-1)\big) = V(0, 2; -1)$, hence it is invariant under $\mathrm{Hg}(X)$. This means that $\mathrm{Hg}(X)$ is contained in the symplectic group $\mathrm{Sp}(V, \varphi)$. Next we remark that, because of the above equivalence of categories,

$$\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q} =: \mathrm{End}^0(X) \cong \{\text{Hodge classes in } \mathrm{End}(V)\,\} = \mathrm{End}(V)^{\mathrm{Hg}(X)}\,.$$

We conclude that $\mathrm{Hg}(X)$ is contained in the centralizer of $\mathrm{End}^0(X)$ inside $\mathrm{Sp}(V, \varphi)$, and that the commutant of $\mathrm{Hg}(X)$ in $\mathrm{End}(V)$ equals $\mathrm{End}^0(X)$. These observations become even more useful if we remark that for abelian varieties of a given dimension, the Albert classification (see [Mu2], section 21) gives a finite list of possible types for the endomorphism algebra $\mathrm{End}^0(X)$. When combined with other properties of the Hodge group, knowing $\mathrm{End}^0(X)$ is in some cases sufficient to determine $\mathrm{Hg}(X)$ and its action on V. This then enables us—at least in principle—to determine the Hodge ring of all powers of X. In general, however, the endomorphism algebra does not determine the Hodge group.

Example 1. Main references: [Ri], [Haz], [Ku], [Se2], [Ch]. Let X be a simple abelian variety of dimension 1 or of prime dimension. Then the Hodge group is equal to the centralizer of $\mathrm{End}^0(X)$ in $\mathrm{Sp}(V,\varphi)$. (This does not depend on the choice of the polarization.) The Hodge ring of every power of X is generated by divisor classes; in particular, the Hodge conjecture is true for all powers of X.

Example 2. Main references: [We], [MZ2]. Suppose k is an imaginary quadratic field, acting on X by endomorphisms. If σ and τ are the two complex embeddings of k, then $\mathrm{H}^0(X,\Omega^1_X) = V^{1,0}$ is a module over $k \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}^{(\sigma)} \times \mathbb{C}^{(\tau)}$, hence it decomposes as $V^{1,0} = V^{1,0}(\sigma) \oplus V^{1,0}(\tau)$. Suppose that the dimensions $n_\sigma = \dim V^{1,0}(\sigma)$ and $n_\tau = \dim V^{1,0}(\tau)$ are equal. This implies that $\dim(X)$ is even, say $\dim(X) = 2n$. The 1-dimensional k-vector space

$$W_k := \bigwedge_k^{2n} V$$

can be identified with a subspace of $\wedge^{2n}_{\mathbb{Q}} V = \mathrm{H}^{2n}(X,\mathbb{Q})$. Moreover, the condition that $n_\sigma = n_\tau$ implies that $W_k \subset \mathrm{H}^{2n}(X,\mathbb{Q})$ consists of Hodge classes. This construction was first studied by Weil in [We]; we call W_k the space of Weil classes with respect to k. Weil showed that for a generic abelian variety X with an action of k (subject to the condition $n_\sigma = n_\tau$), the non-zero classes in W_k are exceptional, i.e., they do not lie in the $\mathbb{Q}$-subalgebra $\mathcal{D}^\bullet(X) \subset \oplus \mathrm{H}^{2i}(X,\mathbb{Q})$ generated by the divisor classes.

The construction of Weil classes works in much greater generality. They play a role in Deligne's proof of "Hodge = absolute Hodge" for abelian varieties. In [MZ2] the space W_F of Weil classes w.r.t. the action of an arbitrary field $F \hookrightarrow \mathrm{End}^0(X)$ is studied. We find here criteria, purely in terms of F, $\mathrm{End}^0(X)$ and the action of F on the tangent space $V^{1,0}$, of when W_F contains Hodge classes, and of when these Hodge classes are exceptional.

Example 3. Main references: [Mu1], [MZ1], [Ta]. Let X be an abelian fourfold with $\mathrm{End}^0(X) = \mathbb{Q}$. Then either $\mathrm{Hg}(X)$ is the full symplectic group $\mathrm{Sp}_{8,\mathbb{Q}}$, in which case the Hodge ring of every power of X is generated by divisor classes, or $\mathrm{Hg}(X)$ is isogenous to a $\mathbb{Q}$-form of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$, where the representation V is the tensor product of the standard 2-dimensional representations of the three factors. Both possibilities occur. In the latter case, the Hodge ring of X is generated by divisor classes, but for X^2 this is no longer true: $\mathrm{H}^4(X^2, \mathbb{Q})$ contains exceptional classes. These are not of the same kind as in example 2, i.e., they are not Weil classes with respect to the action of a field on X^2.

In case X is defined over a number field, we have "the same" two possibilities for the image of the Galois group acting on the Tate module. In particular, knowing $\mathrm{End}^0(X)$ here is not sufficient to prove the Mumford-Tate conjecture. Known by many as "the Mumford example", this is actually the lowest dimensional case where the Mumford-Tate conjecture for abelian varieties remains, at present, completely open. Mumford's example can be generalized to abelian varieties of dimension 4^k, see [Ta].

1.3 Guided by the considerations in 1.1, we can make sense of the problem to study Hodge structures with "a given collection of Hodge classes". How one translates this in purely group-theoretical terms is explained with great clarity in [De3], especially section 1.1. Here we summarize the most important points.

Fix an algebraic group $G_\mathbb{R}$ over $\mathbb{R}$, and consider the space $\mathrm{Hom}(\mathbb{S}, G_\mathbb{R})$ of homomorphisms of algebraic groups $h\colon \mathbb{S} \to G_\mathbb{R}$. Its connected components are the $G(\mathbb{R})^+$-conjugacy classes. Given one such component X^+, and fixing a representation $\rho_\mathbb{R}\colon G_\mathbb{R} \to \mathrm{GL}(V_\mathbb{R})$, we obtain a family of $\mathbb{R}$-Hodge structures on $V_\mathbb{R}$, parametrized by X^+. From an algebro-geometric point of view, the natural conditions to impose on this family are:

(a) the weight decomposition $V_\mathbb{R} = \oplus_{n\in\mathbb{Z}} V_\mathbb{R}^n$ does not depend on $h \in X^+$,

(b) there is a complex structure on X^+ such that the family of Hodge structures on each $V_\mathbb{R}^n$ is a polarizable $\mathbb{R}$-V.H.S. over X^+.

Now an important fact is that (a) and (b) can be expressed directly in terms of $G_\mathbb{R}$ and X^+, and that, at least for faithful representations, they do not depend on $\rho_\mathbb{R}$. If (a) and (b) are satisfied for some (equivalently: every) faithful representation $\rho_\mathbb{R}$, then the complex structure in (b) is unique and X^+ is a hermitian symmetric domain. (For all this, see [De3], 1.13–17.) By adding a $\mathbb{Q}$-structure on $G_\mathbb{R}$, one is led to the following definition.

1.4 Definition. A Shimura datum is a pair (G, X) consisting of a connected reductive group G over $\mathbb{Q}$, and a $G(\mathbb{R})$-conjugacy class $X \subset \mathrm{Hom}(\mathbb{S}, G_\mathbb{R})$, such that for all (equivalently: for some) $h \in X$,

(i) the Hodge structure on $\mathrm{Lie}(G)$ defined by $\mathrm{Ad}\circ h$ is of type $(-1,1)+(0,0)+(1,-1)$,

(ii) the involution $\mathrm{Inn}\big(h(i)\big)$ is a Cartan involution of $G^{\mathrm{ad}}_{\mathbb{R}}$,

(iii) the adjoint group G^{ad} does not have factors defined over $\mathbb{Q}$ onto which h has a trivial projection.

In this definition we have followed [De3], section 2.1. Pink has suggested (cf. [Pi]) to allow not only $G(\mathbb{R})$-conjugacy classes $X \subset \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ but also finite coverings of such. We will not use this generalization in this paper.

There are some other conditions that sometimes play a role. For instance, condition (i) implies that the weight cocharacter $h\circ w\colon\ \mathbb{G}_{\mathrm{m},\mathbb{C}} \to G_{\mathbb{C}}$ (for which we sometimes simply write w) does not depend on $h \in X$, and one could require that it is defined over $\mathbb{Q}$. It turns out, however, that the theory works well without this assumption, and that there are rather natural examples where it is not satisfied.

1.5 Let (G, X) be a Shimura datum, and let K be a compact open subgroup of $G(\mathbb{A}_f)$. We set

$$\mathit{Sh}_K(G,X)_{\mathbb{C}} = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K\,,$$

where $G(\mathbb{Q})$ acts diagonally on $X \times \big(G(\mathbb{A}_f)/K\big)$. If $X^+ \subseteq X$ is a connected component, and if $g_1, \ldots, g_m$ are representatives in $G(\mathbb{A}_f)$ for the finite set $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$, then we can rewrite $\mathit{Sh}_K(G,X)_{\mathbb{C}}$ as a disjoint sum

$$\mathit{Sh}_K(G,X)_{\mathbb{C}} = \coprod_{i=1,\ldots,m} \Gamma_i\backslash X^+\,,$$

where Γ_i is the image of $G(\mathbb{Q})_+ \cap g_i K g_i^{-1}$ inside $G^{\mathrm{ad}}(\mathbb{Q})^+$, which is an arithmetic subgroup. By the results of Baily and Borel in [BB], the quotients $\Gamma_i\backslash X^+$ have a natural structure of a quasi-projective algebraic variety. For compact open subgroups $K_1 \subseteq K_2$, the natural map

$$\mathit{Sh}(K_1,K_2)\colon\ \mathit{Sh}_{K_1}(G,X)_{\mathbb{C}} \longrightarrow \mathit{Sh}_{K_2}(G,X)_{\mathbb{C}}$$

is algebraic. We thus obtain a projective system of (generally non-connected) algebraic varieties $\mathit{Sh}_K(G,X)_{\mathbb{C}}$, indexed by the compact open subgroups $K \subset G(\mathbb{A}_f)$. This system, or its limit

$$\mathit{Sh}(G,X)_{\mathbb{C}} = \varprojlim_K \mathit{Sh}_K(G,X)_{\mathbb{C}}\,,$$

(which exists as a scheme, since the transition maps are finite) is called the Shimura variety defined by the datum (G, X).

1.6 We will briefly recall some basic definitions and results. For further discussion of these topics, see [De1], [De3], [Mi2].

1.6.1 The group $G(\mathbb{A}_f)$ acts continuously on $Sh(G,X)_{\mathbb{C}}$ from the right. The continuity here means that the action of an element $g \in G(\mathbb{A}_f)$ is obtained as the limit of a system of isomorphism $\cdot g\colon Sh_K(G,X)_{\mathbb{C}} \xrightarrow{\sim} Sh_{g^{-1}Kg}(G,X)_{\mathbb{C}}$, see [De3], 2.1.4 and 2.7, or [Mi2], II.2 and II.10. On "finite levels", the $G(\mathbb{A}_f)$-action gives rise to Hecke correspondences: for compact open subgroups K_1, $K_2 \subset G(\mathbb{A}_f)$ and $g \in G(\mathbb{A}_f)$, set $K' = K_1 \cap gK_2g^{-1}$; then the Hecke correspondence $\mathcal{T}_g$ from $Sh_{K_1}(G,X)_{\mathbb{C}}$ to $Sh_{K_2}(G,X)_{\mathbb{C}}$ is given by

$$\mathcal{T}_g\colon \quad Sh_{K_1}(G,X)_{\mathbb{C}} \xleftarrow{Sh(K',K_1)} Sh_{K'}(G,X)_{\mathbb{C}} \xrightarrow{\cdot g} Sh_{K_2}(G,X)_{\mathbb{C}}\,.$$

1.6.2 A morphism of Shimura data $f\colon (G_1,X_1) \to (G_2,X_2)$ is given by a homomorphism of algebraic groups $f\colon G_1 \to G_2$ defined over $\mathbb{Q}$ which induces a map from X_1 to X_2. Such a morphism induces a morphism of schemes

$$Sh(f)\colon Sh(G_1,X_1)_{\mathbb{C}} \longrightarrow Sh(G_2,X_2)_{\mathbb{C}}\,.$$

If $f\colon G_1 \to G_2$ is a closed immersion then so is $Sh(f)$. (See [De1], 1.14–15.)

1.6.3 Let (G,X) be a Shimura datum. Associated to $h \in X$, we have a cocharacter $h \circ \mu\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to G_{\mathbb{C}}$, whose $G(\mathbb{C})$-conjugacy class is independent of $h \in X$. The reflex field $E(G,X) \subset \mathbb{C}$ is defined as the field of definition of this conjugacy class. It is a finite extension of $\mathbb{Q}$. If $f\colon (G_1,X_1) \to (G_2,X_2)$ is a morphism of Shimura data, then $E(G_2,X_2) \subseteq E(G_1,X_1) \subset \mathbb{C}$.

1.6.4 A point $h \in X$ is called a special point if there is a torus $T \subseteq G$, defined over $\mathbb{Q}$, such that $h\colon \mathbb{S} \to G_{\mathbb{R}}$ factors through $T_{\mathbb{R}}$. In this case $(T,\{h\})$ is a Shimura datum, and the inclusion $T \hookrightarrow G$ gives a morphism $(T,\{h\}) \to (G,X)$. A point $x \in Sh_K(G,X)_{\mathbb{C}}$ is called a special point if it is of the form $x = [h,gK]$ with h special. (This does not depend on the choice of the representative (h,gK) for x.) Here we follow [De3]; the definition in [De1], 3.15, is more restrictive.

1.6.5 Consider a triplet $(G^{\mathrm{ad}}, G', X^+)$ consisting of an adjoint group G^{ad} over $\mathbb{Q}$, a covering G' of G^{ad}, and a $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class $X^+ \subset \mathrm{Hom}(\mathbb{S}, G^{\mathrm{ad}}_{\mathbb{R}})$ such that the conditions (i), (ii) and (iii) in 1.4 are satisfied. Let $\tau(G')$ be the linear topology on $G^{\mathrm{ad}}(\mathbb{Q})$ for which the images in $G^{\mathrm{ad}}(\mathbb{Q})$ of the congruence subgroups in $G'(\mathbb{Q})$ form a fundamental system of neighbourhoods of the identity. The connected Shimura variety $Sh^0(G^{\mathrm{ad}}, G', X^+)_{\mathbb{C}}$ is defined as the projective system

$$Sh^0(G^{\mathrm{ad}}, G', X^+)_{\mathbb{C}} = \varprojlim_{\Gamma} \Gamma\backslash X^+ ,$$

where Γ runs through the arithmetic subgroups of $G^{\mathrm{ad}}(\mathbb{Q})$ which are open in $\tau(G')$. It comes equipped with an action of the completion $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ of $G^{\mathrm{ad}}(\mathbb{Q})^+$ for the topology $\tau(G')$.

Given a Shimura datum (G, X) and a connected component $X^+ \subseteq X$, we obtain a triplet $(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)$ as above. The associated connected Shimura variety $Sh^0(G^{\mathrm{ad}}, G^{\mathrm{der}}, X^+)_{\mathbb{C}}$ is the connected component of $Sh(G, X)_{\mathbb{C}}$ containing the image of $X^+ \times \{e\} \subset X \times G(\mathbb{A}_f)$. In particular, we see that this component only depends on G^{ad}, G^{der} and $X^+ \subset X$. In the sequel, when working with connected Shimura varieties, we will usually omit G^{ad} from the notation. For lack of better terminology, we will refer to a pair (G', X^+) as above as "a pair defining a connected Shimura variety".

1.6.6 Let G be a reductive group over a number field L. Write $\rho\colon \widetilde{G} \to G^{\mathrm{der}}$ for the universal covering (in the sense of algebraic groups) of its derived group. By [De1], Prop. 2.2 and [De3], Cor. 2.0.8, $G(L) \cdot \rho\widetilde{G}(\mathbb{A}_L)$ is a closed subgroup of $G(\mathbb{A}_L)$ with abelian quotient $\pi(G) := G(\mathbb{A}_L)/G(L) \cdot \rho\widetilde{G}(\mathbb{A}_L)$. (Note: $\mathbb{A}_L$ is the ring of full adèles of L.) Consequently, the set of connected components $\pi_0\pi(G)$ is also an abelian group.

Now let (G, X) be a Shimura datum. If $K \subset G(\mathbb{A}_f)$ is a compact open subgroup then $Sh_K(G, X)_{\mathbb{C}}$ is a scheme of finite type over $\mathbb{C}$. For K getting smaller, its number of connected components generally increases. Deligne proves in [De3], 2.1.3 that

$$\pi_0\bigl(Sh_K(G, X)_{\mathbb{C}}\bigr) \cong G(\mathbb{A}_f)/G(\mathbb{Q})_+ \cdot K \cong \overline{\pi}_0\pi(G)/K ,$$

where $\overline{\pi}_0\pi(G) := \pi_0\pi(G)/\pi_0 G(\mathbb{R})_+$. Passing to the limit one finds that the $G(\mathbb{A}_f)$-action on $Sh(G, X)_{\mathbb{C}}$ makes $\pi_0\bigl(Sh(G, X)_{\mathbb{C}}\bigr)$ a principal homogeneous space under $\overline{\pi}_0\pi(G) \cong G(\mathbb{A}_f)/G(\mathbb{Q})_+^-$.

1.6.7 Given a Shimura datum (G, X), we can define some other data as follows. Write $X^{\mathrm{ad}} \subset \mathrm{Hom}(\mathbb{S}, G^{\mathrm{ad}}_{\mathbb{R}})$ for the $G^{\mathrm{ad}}(\mathbb{R})$-conjugacy class containing the image of X under the map $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}}) \to \mathrm{Hom}(\mathbb{S}, G^{\mathrm{ad}}_{\mathbb{R}})$. The map $X \to X^{\mathrm{ad}}$ is not necessarily an isomorphism, but every connected component of X maps isomorphically to its image. The pair $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is a Shimura datum, called the adjoint Shimura datum. Similarly, the image X^{ab} of X in $\mathrm{Hom}(\mathbb{S}, G^{\mathrm{ab}}_{\mathbb{R}})$ is a $G^{\mathrm{ab}}(\mathbb{R})$-conjugacy class (necessarily a single point), and we have a Shimura datum $(G^{\mathrm{ab}}, X^{\mathrm{ab}})$.

Another construction that is sometimes useful is the following. Suppose the group G is of the form $G = \mathrm{Res}_{F/\mathbb{Q}}(H)$, where F is a totally real number field and H is an absolutely simple algebraic group over F. (Such is the case, for example, if G is a simple adjoint group.) Now take an extension $F \subset F'$ of totally real number fields, and set $G_2 = \mathrm{Res}_{F'/\mathbb{Q}}(H_{F'})$. There is a unique $G_2(\mathbb{R})$-conjugacy class $X_2 \subset \mathrm{Hom}(\mathbb{S}, G_{2,\mathbb{R}})$ such that the natural homomorphism $G \to G_2$ gives a closed immersion of Shimura data $(G, X) \hookrightarrow (G_2, X_2)$.

1.7 One might ask "how many" Shimura varieties there are. A possible approach is to begin by classifying the Shimura varieties of adjoint type. These are products of Shimura varieties $\mathit{Sh}(G, X)_{\mathbb{C}}$, where G is a $\mathbb{Q}$-simple adjoint group. The group $G_{\mathbb{R}}$ is an inner form of a compact group, of one of the types A, B, C, $\mathrm{D}^{\mathbb{R}}$, $\mathrm{D}^{\mathbb{H}}$, E_6 or E_7, and given $G_{\mathbb{R}}$, the possibilities for X are classified in terms of special nodes in the Dynkin diagram. We refer to [De3], sections 1.2 and 2.3 for more details.

Given $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$, we can list all possibilities for G^{der}. As we have seen, the pair (G^{der}, X^+) consisting of G^{der} and a connected component $X^+ \subseteq X$, determines the connected components of the Shimura variety. In particular, the "toric part" $(G^{\mathrm{ab}}, X^{\mathrm{ab}})$ does not contribute to the geometry of $\mathit{Sh}(G, X)_{\mathbb{C}}$, in the sense that it has no effect on $\mathit{Sh}^0_{\mathbb{C}}$, but only on $\pi_0\big(\mathit{Sh}(G, X)_{\mathbb{C}}\big)$. Finally, let us remark that "toric" Shimura data are in bijective correspondence to pairs (Y, μ) consisting of a free $\mathbb{Z}$-module Y of finite rank with a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (the cocharacter group of the torus), together with an element $\mu \in Y$.

1.8 The definition of a Shimura variety is set up in such a way that that if $\xi\colon G_{\mathbb{R}} \to \mathrm{GL}(V_{\mathbb{R}})$ is a representation, then we obtain a (direct sum of) polarizable $\mathbb{R}$-VHS $\mathcal{V}(\xi)_{\mathbb{R}}$ over X with underlying local system $X \times V_{\mathbb{R}}$. If $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$ for a $\mathbb{Q}$-vector space V, and if the weight $\xi \circ w\colon \mathbb{G}_{\mathrm{m},\mathbb{R}} \to \mathrm{GL}(V)_{\mathbb{R}}$ is defined over $\mathbb{Q}$, then $\mathcal{V}(\xi)_{\mathbb{R}}$ comes from a polarizable $\mathbb{Q}$-VHS $\mathcal{V}(\xi)$. Under some conditions on $G/\mathrm{Ker}(\xi)$, this VHS descends, for K sufficiently small, to a $\mathbb{Q}$-VHS on $\mathit{Sh}_K(G, X)$. (It suffices if the center of $G/\mathrm{Ker}(\xi)$ is the almost direct product of a $\mathbb{Q}$-split torus and a torus T for which $T(\mathbb{R})$ is compact.)

One expects (see [De5], [Mi4]) that these variations of Hodge structure are the Betti realizations of families of motives, and that Shimura varieties, at least those for which the weight is defined over $\mathbb{Q}$, have an interpretation (depending on the choice of a representation ξ) as moduli spaces for motives with certain additional structures. What is missing, at present, is a sufficiently good theory of motives. In certain cases, however, the dictionary between abelian varieties and certain Hodge structures (see 1.2 above) leads to a modular interpretation of $Sh(G,X)$. Let us briefly review some facts and terminology.

1.8.1 *Siegel modular varieties.* Let φ denote the standard symplectic form on $\mathbb{Q}^{2g}$, and set $G = \mathrm{CSp}_{2g}$. The homomorphisms $h\colon \mathbb{S} \to G_{\mathbb{R}}$ which determine a $\mathbb{Q}$-H.S. of type $(-1,0)+(0,-1)$ on $\mathbb{Q}^{2g}$ such that $\pm\varphi$ is a polarization, form a single $G(\mathbb{R})$-conjugacy class $\mathfrak{H}_g^{\pm}$. It can be identified with the Siegel double-$\frac{1}{2}$-space. The pair $(\mathrm{CSp}_{2g}, \mathfrak{H}_g^{\pm})$ is a Shimura datum with reflex field $\mathbb{Q}$. The associated Shimura variety is often referred to as the Siegel modular variety.

For $K \subset G(\mathbb{A}_f)$ a compact open subgroup, $Sh_K(\mathrm{CSp}_{2g}, \mathfrak{H}_g^{\pm})_{\mathbb{C}}$ is a moduli space for g-dimensional complex p.p.a.v. with a level K-structure (as defined, for instance, in [Ko2], §5). Here a couple of remarks should be added. The interpretation of $Sh_K(\mathrm{CSp}_{2g}, \mathfrak{H}_g^{\pm})_{\mathbb{C}}$ in terms of abelian varieties up to *isomorphism* depends on the choice of a lattice $\Lambda \subset \mathbb{Q}^{2g}$. This choice also determines the "type" of the polarization; if we want to work with principally polarized abelian varieties then we must choose Λ such that $\varphi_{|\Lambda}$ has discriminant 1 (e.g., $\Lambda = \mathbb{Z}^{2g}$). For further details see [De1], §4. In the sequel, we identify $Sh(\mathrm{CSp}_{2g}, \mathfrak{H}_g^{\pm})_{\mathbb{C}}$ and $\varprojlim_n \mathsf{A}_{g,1,n} \otimes \mathbb{C}$.

1.8.2 *Shimura varieties of PEL and of Hodge type.* By definition, a Shimura datum (G,X) (as well as the associated Shimura variety) is said to be of Hodge type, if there exists a closed immersion of Shimura data $j\colon (G,X) \hookrightarrow (\mathrm{CSp}_{2g}, \mathfrak{H}_g^{\pm})$ for some g. If this holds, the Shimura variety $Sh(G,X)_{\mathbb{C}} \hookrightarrow Sh(\mathrm{CSp}_{2g}, \mathfrak{H}_g^{\pm})_{\mathbb{C}}$ has an interpretation in terms of abelian varieties with certain "given Hodge classes". The precise formulation of such a modular interpretation is usually rather complicated.

This is already the case for Shimura varieties of PEL type (see [De1], 4.9–14, [Ko2]). Loosely speaking, these are the Shimura varieties parametrizing abelian varieties with a given algebra acting by endomorphisms.[2] Recall (1.2)

[2]For the reader who has not worked with Shimura varieties before, it may be instructive to read Shimura's paper [Sh]. Here certain Shimura varieties of PEL type are written

that endomorphisms of an abelian variety are particular examples of Hodge classes. On finite levels one thus looks at abelian varieties with a *P*olarization, certain given *E*ndomorphisms, and a *L*evel structure.

Shimura varieties of PEL type are more special in that they represent a moduli problem that can be formulated over an arbitrary basis. For more general Shimura varieties of Hodge type we can only do this if we assume the Hodge conjecture.

In 2.10, we will introduce two more classes of Shimura varieties: those of abelian and of pre-abelian type. Among these classes we have the following inclusions

$$\begin{pmatrix} \text{Sh. var. of} \\ \text{PEL type} \end{pmatrix} \subset \begin{pmatrix} \text{Sh. var. of} \\ \text{Hodge type} \end{pmatrix} \subset \begin{pmatrix} \text{Sh. var. of} \\ \text{abelian type} \end{pmatrix} \subset$$
$$\subset \begin{pmatrix} \text{Sh. var. of} \\ \text{pre-ab. type} \end{pmatrix} \subset \begin{pmatrix} \text{general} \\ \text{Sh. var.} \end{pmatrix}$$

All inclusions are strict; for the first one see 1.2. (A priory, the Shimura variety corresponding to a "Mumford example" could have a different realization for which it is of PEL type. By looking at the group involved over $\mathbb{R}$, one easily shows that this does not happen.)

1.9 *Compactifications; mixed Shimura varieties.* This is a whole subject in itself, and we cannot say much about it here. We will briefly indicate some important statements, referring to the literature for details.

The first compactification to mention is the Baily-Borel (or minimal) compactification, for which we write $Sh_K(G,X)^*_{\mathbb{C}}$. (References: [BB], see also [Br], §4 for a summary.) It was constructed by Baily and Borel in the setting of locally symmetric varieties. If $\Gamma\backslash X^+$ is a component of $Sh_K(G,X)_{\mathbb{C}}$, say with K neat so that $\Gamma\backslash X^+$ is non-singular, then its Baily-Borel compactification is given as a quotient $\Gamma\backslash X^*$. Here X^* is the Satake compactification of X^+; as a set it is the union of X^+ and its (proper) rational boundary components, which themselves are again hermitian symmetric domains. It is shown in [BB] that $\Gamma\backslash X^*$ has a natural structure of a normal projective variety. The stratification of X^* by its boundary components F induces a stratification of $\Gamma\backslash X^*$ by locally symmetric varieties $\Gamma_{\overline{F}}\backslash F$.

down "by hand". Both for understanding Shimura's paper and for understanding the abstract Deligne-formalism we are presenting here, it is a good exercise to translate the considerations of [Sh] to the "(G,X) language".

As the referee pointed out to us, it is worth noticing that the Baily-Borel compactification $Sh_K(G,X)^*_{\mathbb{C}}$ is not simply defined as the disjoint union of the $\Gamma\backslash X^*$. Instead, one starts with a Satake compactification of the whole of X at once, and one defines $Sh_K(G,X)^*_{\mathbb{C}}$ as a suitable quotient. Thus, for example, if $(G,X) = (\mathrm{GL}_2, \mathfrak{H}^{\pm})$, one does not adjoin the points of $\mathbb{P}^1(\mathbb{Q})$ to $\mathfrak{H}^+$ and $\mathfrak{H}^-$ separately but one works with $\mathbb{P}^1(\mathbb{Q}) \cup \mathfrak{H}^+ \cup \mathfrak{H}^-$. We refer to [Pi], Chap. 6 for further details.

The Baily-Borel compactification is canonical. In particular, it is easy to show (see [Br], p. 90) that it descends to a compactification of the canonical model $Sh_K(G,X)$ (to be discussed in the next section). In general, $Sh_K(G,X)^*$ is singular along the boundary.

Next we have the toroidal compactifications[3] studied in the monograph [Aea]. These are no longer canonical, as they depend on the choice of a certain cone decomposition. We will reflect this in our notation, writing $Sh_K(G,X;\mathcal{S})_{\mathbb{C}}$ for the toroidal compactification corresponding to a K-admissible partial cone decomposition $\mathcal{S}$ as in [Pi], Chap. 6. From the construction, we obtain a natural stratification of the boundary. For suitable choices of $\mathcal{S}$ (and K neat), one obtains a projective non-singular scheme $Sh_K(G,X;\mathcal{S})_{\mathbb{C}}$ such that the boundary is a normal crossings divisor—in this case one speaks of a smooth toroidal compactification.

Although both the Baily-Borel and the toroidal compactifications were initially studied in the setting of locally symmetric varieties, it was realized that they should be tied up with the theory of degenerating Hodge structures (e.g., see [Aea], p. iv). For certain Shimura varieties this was done by Brylinski in [Br], using 1-motives. Subsequently, Pink developed a general theory of mixed Shimura varieties and studied compactifications in this setting. Similar ideas, but in a less complete form, were presented by Milne in [Mi2]. It seems that several important ideas can actually be traced back to Deligne.

The main results of [Pi] include the following statements (some of which had been known before for pure Shimura varieties or some special mixed Shimura varieties). We refer to loc. cit. for definitions, more precise statements and of course for the proofs.

(i) Let $Sh_K(G,X)_{\mathbb{C}}$ be a pure Shimura variety. It has a canonical model $Sh_K(G,X)$ over the reflex field $E(G,X)$ (see §2 below). The Baily-Borel compactification descends to a compactification $Sh_K(G,X)^*$ of this canoni-

[3] Here we indulge in the customary abuse of terminology to call these compactifications, even though $Sh_K(G,X;\mathcal{S})$ is compact only if the cone decomposition $\mathcal{S}$ satisfies some conditions.

cal model. The boundary has a stratification by finite quotients of (canonical models of) certain pure Shimura varieties; each such stratum is a finite union of the natural boundary components in the Baily-Borel compactification. Which pure Shimura varieties occur in this way can be described directly in terms of the Shimura datum (G, X).

(ii) Next consider K-admissible cone decompositions $\mathcal{S}$ for (P, X). If $\mathcal{S}$ satisfies certain conditions (such $\mathcal{S}$ always exists if K is neat) then the following assertions hold. The toroidal compactification $Sh_K(P, X; \mathcal{S})_{\mathbb{C}}$ descends to a compactification $Sh_K(P, X; \mathcal{S})$ of the canonical model. It is a smooth projective scheme, and the boundary is a normal crossings divisor. The boundary has a stratification by finite quotients of (canonical models of) certain other mixed Shimura varieties; each such stratum is a finite union of the strata of $Sh_K(P, X; \mathcal{S})$ as a toroidal compactification. The natural morphism $\pi\colon Sh_K(P, X; \mathcal{S}) \to Sh_K(P, X)^*$ is compatible with the stratifications. If $Sh_K(P', X')$ is a mixed Shimura variety of which a finite quotient occurs as a boundary stratum $\mathcal{C} \subset Sh_K(P, X; \mathcal{S})$, then the restriction of π to $\mathcal{C}$ is induced by the canonical morphism of $Sh_K(P', X')$ to the associated pure Shimura variety.

Furthermore, Pink proves several results about the functoriality of the structures in (i) and (ii).

To conclude this section, let us remark that in some cases (modular curves: Deligne and Rapoport, [DR]; Hilbert modular surfaces: Rapoport, [R1]; Siegel modular varieties: Chai and Faltings, [FC]) we even have smooth compactifications of Shimura varieties over (an open part of) Spec($\mathbb{Z}$) or the ring of integers of a number field. As Chai and Faltings remark in the introduction to [FC], many of their ideas also apply to Shimura varieties of PEL type; they conclude: "…and as our ideas usually either carry over directly, or we are lead to hard new problems which require new methods, we leave these generalizations to the reader."

§2 Canonical models of Shimura varieties.

2.1 Before turning to more recent developments, we will discuss some aspects of the theory of canonical models of Shimura varieties (over number fields). Our motivation for doing so is twofold.

(i) For "most" Shimura varieties, the existence of a canonical model was shown by Deligne in his paper [De3]. As we will see, the same strategy of proof is useful in the context of integral canonical models.

(ii) The existence of canonical models in general, i.e., including the cases where the group G has factors of exceptional type, was claimed in [Mi1] (see also [La], [Bo2], [MS], [Mi2]) as a consequence of the Langlands conjecture on conjugation of Shimura varieties. It was pointed out to us by J. Wildeshaus that the argument given there is not complete. Below we will explain this in more detail, and we correct the proof.

2.2 Recall (1.6.6) that for a reductive group G over a global field K of characteristic 0 we have set $\pi(G) = G(\mathbb{A}_K)/G(K) \cdot \rho\widetilde{G}(\mathbb{A}_K)$. We have the following constructions, for which we refer to [De3], section 2.4.

(a) Given a finite field extension $K \subset L$, there is a norm homomorphism $\mathrm{Nm}_{L/K}\colon\ \pi(G_L) \longrightarrow \pi(G)$.

(b) If T is a torus over K and M is a $G(\overline{K})$-conjugacy class of homomorphisms $T_{\overline{K}} \to G_{\overline{K}}$ which is defined over K, then there is associated to M a homomorphism $q_M\colon\ \pi(T) \longrightarrow \pi(G)$.

If (G, X) is a Shimura datum with reflex field $E = E(G, X) \subset \mathbb{C}$, we use this to define a reciprocity homomorphism

$$r_{(G,X)}\colon\ \mathrm{Gal}(\overline{\mathbb{Q}}/E) \longrightarrow \overline{\pi}_0\pi(G) = G(\mathbb{A}_f)/G(\mathbb{Q})^-_+$$

as follows. Global class field theory provides us with an isomorphism

$$\mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}} \xrightarrow{\ \sim\ } \pi_0\pi(\mathbb{G}_{\mathrm{m},E})\,. \tag{2.2.1}$$

Applying (b) to the conjugacy class $M = \{h \circ \mu\colon\ \mathbb{G}_{\mathrm{m},\mathbb{C}} \to G_{\mathbb{C}} \mid h \in X\}$, which (by definition) is defined over E, we obtain a map $q_M\colon\ \pi(\mathbb{G}_{\mathrm{m},E}) \longrightarrow \pi(G_E)$. From (a) we get $\mathrm{Nm}_{E/\mathbb{Q}}\colon\ \pi(G_E) \longrightarrow \pi(G)$. Combining these maps we can now define the reciprocity map as

(2.2.2)

$$r_{(G,X)}\colon\ \mathrm{Gal}(\overline{\mathbb{Q}}/E) \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}} \xrightarrow{(2.2.1)} \pi_0\pi(\mathbb{G}_{\mathrm{m},E}) \xrightarrow{\pi_0(\mathrm{Nm}_{E/\mathbb{Q}}\circ q_M)} \pi_0\pi(G) \twoheadrightarrow \overline{\pi}_0\pi(G)\,.$$

For a Shimura datum $(T, \{h\})$ where T is a torus, the reciprocity map can be described more explicitly: if v is a place of E dividing p then

$$r_{(T,\{h\})}\colon\ \mathrm{Gal}(\overline{\mathbb{Q}}/E) \longrightarrow \overline{\pi}_0\pi(T) = T(\mathbb{A}_f)/T(\mathbb{Q})^-$$

sends a geometric Frobenius element $\Phi_v \in \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ at v to the class of the element $\mathrm{Nm}_{E/\mathbb{Q}}\big(h(\pi_v)\big) \in T(\mathbb{Q}_p) \hookrightarrow T(\mathbb{A}_f)$, where π_v is a uniformizer at v.

2.3 If $(T, \{h\})$ is a Shimura datum with T a torus, then for every compact open subgroup $K \subset T(\mathbb{A}_f)$ the Shimura variety $Sh_K(T, \{h\})_{\mathbb{C}}$ consists of finitely many points. To define a model of it over the reflex field $E = E(T, \{h\})$ it therefore suffices to specify an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$. Write $Sh_K(T, \{h\})$ for the model over E determined by the rule that $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ acts on $Sh_K(T, \{h\})_{\mathbb{C}}$ by sending $[h, tK]$ to $[h, r_{(T,\{h\})}(\sigma) \cdot tK]$. It is clear that the transition maps $Sh_{(K',K)}$ descend to E, and we define the canonical model of $Sh(T, \{h\})_{\mathbb{C}}$ to be

$$Sh(T, \{h\}) = \varprojlim_K Sh_K(T, \{h\}) \,.$$

2.4 Definition. Let (G, X) be a Shimura datum.

(i) A model of $Sh(G, X)_{\mathbb{C}}$ over a field $F \subset \mathbb{C}$ is a scheme S over F together with a continuous action of $G(\mathbb{A}_f)$ from the right and a $G(\mathbb{A}_f)$-equivariant isomorphism $S \otimes_F \mathbb{C} \xrightarrow{\sim} Sh(G, X)_{\mathbb{C}}$.

(ii) Let $F \subset \mathbb{C}$ be a field containing $E(G, X)$. A weakly canonical model of $Sh(G, X)$ over F is a model S over F such that for every closed immersion of Shimura data $i\colon (T, \{h\}) \hookrightarrow (G, X)$ with T a torus, the induced morphism $Sh(T, \{h\})_{\mathbb{C}} \hookrightarrow Sh(G, X)_{\mathbb{C}} \cong S_{\mathbb{C}}$ descends to a morphism $Sh(T, \{h\}) \otimes_E EF \hookrightarrow S \otimes_F EF$, where $E = E(T, \{h\})$, and where $Sh(T, \{h\})$ is the model defined in 2.3.

(iii) A canonical model of $Sh(G, X)$ is a weakly canonical model over the reflex field $E(G, X)$.

It should be noticed that if S is a model of $Sh(G, X)$ over the field $F \subset \mathbb{C}$, then we have an action of $G(\mathbb{A}_f) \times \mathrm{Gal}(\overline{F}/F)$ on $S_{\overline{F}}$ (i.e., two commuting actions of $G(\mathbb{A}_f)$ and $\mathrm{Gal}(\overline{F}/F)$.)

2.5 Let $f\colon (G_1, X_1) \to (G_2, X_2)$ be a morphism of Shimura data, and suppose there exist canonical models $Sh(G_1, X_1)$ and $Sh(G_2, X_2)$. Then, as shown in [De1], section 5, the morphism $Sh(f)$ descends uniquely to a morphism $Sh(G_1, X_1) \to Sh(G_2, X_2) \otimes_{E(G_2,X_2)} E(G_1, X_1)$, which we will also denote $Sh(f)$. In particular, it follows that a canonical model, if it exists, is unique up to isomorphism. (The isomorphism is also unique, since the isomorphism $Sh(G, X) \otimes_E \mathbb{C} \xrightarrow{\sim} Sh(G, X)_{\mathbb{C}}$ is part of the data.)

2.6 If $Sh(G, X)$ is a canonical model of a Shimura variety, then the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ acts on the set of connected components of $Sh(G, X)_{\mathbb{C}}$, which, as recalled in 1.6.6, is a principal homogeneous space under $\overline{\pi}_0\pi(G)$.

Deligne proves in [De3], section 2.6, that the homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/E) \to \overline{\pi}_0\pi(G)$ describing the Galois action on $\pi_0\big(Sh(G,X)_{\mathbb{C}}\big)$ is equal to the homomorphism $r_{(G,X)}$ defined above. (Strictly speaking, this is only true up to a sign: in [De3] the Galois action on $\pi_0(Sh_{\mathbb{C}})$ is described to be $r_{(G,X)}$; Milne pointed out in [Mi3], Remark 1.10, that the reciprocity law is given by $r_{(G,X)}$, not its inverse.)

2.7 An important technique for proving the existence of canonical models is the reduction to a problem about connected Shimura varieties. To explain this, let us assume that $Sh(G,X)$ is a canonical model of the Shimura variety associated to the datum (G,X), and let us inventory the available structures. As in all of this section, we are mainly repeating things from Deligne's paper [De3].

The group $G(\mathbb{A}_f)$ acts continuously on $Sh_{\mathbb{C}} = Sh(G,X)_{\mathbb{C}}$ from the right. If Z denotes the center of G then $Z(\mathbb{Q})^- \subset G(\mathbb{A}_f)$ acts trivially. Write $G^{\mathrm{ad}}(\mathbb{Q})_1 := G^{\mathrm{ad}}(\mathbb{Q}) \cap \mathrm{Im}\big(G(\mathbb{R}) \to G^{\mathrm{ad}}(\mathbb{R})\big)$. The action of G^{ad} on G by inner automorphisms induces (by functoriality) a left action of $G^{\mathrm{ad}}(\mathbb{Q})_1$ on $Sh(G,X)_{\mathbb{C}}$. For $g \in G(\mathbb{Q})$, the action of g through $G^{\mathrm{ad}}(\mathbb{Q})_1$ coincides with the one of g^{-1} considered as an element of $G(\mathbb{A}_f)$. In total we therefore obtain a continuous left action of the group

$$\Gamma := \big(G(\mathbb{A}_f)/Z(\mathbb{Q})^-\big) \underset{G(\mathbb{Q})/Z(\mathbb{Q})}{*} G^{\mathrm{ad}}(\mathbb{Q})_1 = \big(G(\mathbb{A}_f)/Z(\mathbb{Q})^-\big) \underset{G(\mathbb{Q})_+/Z(\mathbb{Q})}{*} G^{\mathrm{ad}}(\mathbb{Q})^+$$

(converting the operation of $G(\mathbb{A}_f)$ to a left action). The group Γ operates transitively on $\pi_0(Sh_{\mathbb{C}})$. For any connected component of $Sh_{\mathbb{C}}$, the stabilizer of this component is the subgroup

$$\big(G(\mathbb{Q})_+^-/Z(\mathbb{Q})^-\big) \underset{G(\mathbb{Q})_+/Z(\mathbb{Q})}{*} G^{\mathrm{ad}}(\mathbb{Q})^+ \cong G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge} \quad (\text{rel. } \tau(G^{\mathrm{der}}))\,,$$

where the completion $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ is taken relative to the topology $\tau(G^{\mathrm{der}})$. The profinite set $\pi_0(Sh_{\mathbb{C}})$ is a principal homogeneous space under the abelian group

$$G(\mathbb{A}_f)/G(\mathbb{Q})_+^- \cong \overline{\pi}_0\pi(G)\,.$$

(Cf. 1.6.5 and 1.6.6.)

From now on we fix a connected component $X^+ \subset X$, and we write $Sh^0_{\mathbb{C}} = Sh^0(G^{\mathrm{der}}, X^+)_{\mathbb{C}}$ for the corresponding connected Shimura variety, to be identified with a connected component of $Sh(G,X)_{\mathbb{C}}$. We have an action of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on $Sh_{\mathbb{C}}$. As mentioned in 2.6, it acts on $\pi_0(Sh_{\mathbb{C}})$

through the reciprocity homomorphism $r_{(G,X)}$. The subgroup $\mathcal{E}_E(G^{\mathrm{der}}, X^+) \subset \Gamma \times \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ which fixes the connected component $Sh^0_{\mathbb{C}}$ is an extension

$$0 \longrightarrow G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge} \longrightarrow \mathcal{E}_E(G^{\mathrm{der}}, X^+) \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/E) \longrightarrow 0\,.$$

With these notations, we have the following important remarks.

(i) The extension $\mathcal{E}_E(G^{\mathrm{der}}, X^+)$ depends only on the pair (G^{der}, X^+); in particular this justifies the notation. (See [De3], section 2.5.)

(ii) Galois descent (see also 2.15 below) tells us that it is equivalent to give a model of $Sh(G,X)_{\mathbb{C}}$ over E or to give a scheme S over $\overline{\mathbb{Q}}$ with a continuous action of $\Gamma \times \mathrm{Gal}(\overline{\mathbb{Q}}/E)$ and a Γ-equivariant isomorphism $S \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} Sh(G,X)_{\mathbb{C}}$.

(iii) Write $e \in \pi_0(Sh_{\mathbb{C}})$ for the class of the connected component $Sh^0_{\mathbb{C}}$. To give a $\overline{\mathbb{Q}}$-scheme S as in (ii), which, in particular, comes equipped with a Γ-equivariant isomorphism $\pi_0(S) \cong \pi_0(Sh_{\mathbb{C}})$, is equivalent to giving its connected component S^e corresponding to e together with a continuous action of $\mathcal{E}_E(G^{\mathrm{der}}, X^+)$. The idea here is that we can recover S from S^e by "induction" from $\mathcal{E}_E(G^{\mathrm{der}}, X^+)$ to $\Gamma \times \mathrm{Gal}(\overline{\mathbb{Q}}/E)$. (See [De3], section 2.7.)

2.8 Definition. (i) Let (G', X^+) be a pair defining a connected Shimura variety with reflex field E, let $F \subset \overline{\mathbb{Q}}$ be a finite extension of E, and write $\mathcal{E}_F(G', X^+)$ for the extension of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ by $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge}$ (completion for the topology $\tau(G')$) described in [De3], Def. 2.5.7. Then a weakly canonical model of the connected Shimura variety $Sh^0(G', X^+)_{\mathbb{C}}$ over F consists of a scheme S over $\overline{\mathbb{Q}}$ together with a continuous left action of the group $\mathcal{E}_F(G', X^+)$ and an isomorphism $i\colon\ S \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} Sh^0(G', X^+)_{\mathbb{C}}$ such that the following conditions are satisfied.

(a) The action of $\mathcal{E}_F(G', X^+)$ on S is semi-linear, i.e., compatible with the canonical action on $\mathrm{Spec}(\overline{\mathbb{Q}})$ through the quotient $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

(b) The isomorphism i is equivariant w.r.t. the action of $G^{\mathrm{ad}}(\mathbb{Q})^{+\wedge} \subset \mathcal{E}_F(G', X^+)$ (which by (i) acts linearly on S).

(c) Given a special point $h \in X^+$, factoring through a subtorus $h\colon \mathbb{S} \to H_{\mathbb{C}} \subset G^{\mathrm{ad}}_{\mathbb{C}}$ defined over $\mathbb{Q}$, let $E^{(h)}$ denote the field of definition of the cocharacter $h \circ \mu$. Delinge defines in loc. cit., 2.5.10, an extension $0 \to H(\mathbb{Q}) \to \mathcal{E}_F(h) \to \mathrm{Gal}(\overline{\mathbb{Q}}/EE^{(h)}) \to 0$, for which there is a natural homomorphism $\mathcal{E}_F(h) \to \mathcal{E}_F(G', X^+)$. Then we require that the point in $Sh^0(G', X^+)_{\mathbb{C}}$ defined by h is defined over $\overline{\mathbb{Q}}$ and is fixed by $\mathcal{E}_F(h)$.

(ii) A canonical model of the connected Shimura variety $Sh^0(G', X^+)_{\mathbb{C}}$ is a weakly canonical model over the reflex field E.

Although our formulation of condition (c) in (i) is a little awkward, it should be clear that this definition is just an attempt to formalize the above remarks. In fact, these remarks lead to the following result (= [De3], Prop. 2.7.13).

2.9 Proposition. *Let (G, X) be a Shimura datum, and choose a connected component X^+ of X. If $Sh(G, X)$ is a weakly canonical model of $Sh(G, X)_{\mathbb{C}}$ over $F \supseteq E(G, X)$, then the connected component $Sh^0(G, X)_{\overline{\mathbb{Q}}}$ determined by the choice of X^+ is a weakly canonical model of $Sh^0(G^{\mathrm{der}}, X^+)_{\mathbb{C}}$ over F. Conversely, if there exists a weakly canonical model of $Sh^0(G^{\mathrm{der}}, X^+)_{\mathbb{C}}$ over F, then it is obtained in this way from a weakly canonical model of $Sh(G, X)_{\mathbb{C}}$.*

2.10 The main result of [De3] is the existence of canonical models for a large class of Shimura varieties (see below). Since the strategy of proof also works for other statements about Shimura varieties, let us present it in an abstract form (following [Mi2], II.9). So, suppose we want to prove a statement $\mathcal{P}(G, X)$ about Shimura varieties.

(a) Prove $\mathcal{P}(\mathrm{CSp}_{2g}, \mathfrak{H}_g^{\pm})$ using the interpretation of $Sh(\mathrm{CSp}_{2g}, \mathfrak{H}_g^{\pm})_{\mathbb{C}}$ as a moduli space.

(b) For a closed immersion $i\colon (G_1, X_1) \hookrightarrow (G_2, X_2)$, prove the implication $\mathcal{P}(G_2, X_2) \Longrightarrow \mathcal{P}(G_1, X_1)$.

(c) Find a statement $\mathcal{P}^0(G', X^+)$ for pairs (G', X^+) defining a connected Shimura variety, such that, for any connected component $X^+ \subseteq X$, we have $\mathcal{P}(G, X) \Longleftrightarrow \mathcal{P}^0(G^{\mathrm{der}}, X^+)$.

(d) Given pairs (G'_i, X_i^+), $i = 1, \ldots, m$, prove that $\forall i\ \mathcal{P}^0(G'_i, X_i^+) \Longrightarrow \mathcal{P}^0(\prod_i G'_i, \prod_i X_i^+)$.

(e) For an isogeny $G' \to G''$, prove that $\mathcal{P}^0(G', X^+) \Longrightarrow \mathcal{P}^0(G'', X^+)$.

Roughly speaking, the class of Shimura varieties of abelian type is the largest class for which (a)–(e) suffice to prove statement $\mathcal{P}$. (As we will see below, this is not completely true: we may have to modify the strategy a bit, and even then it is not clear whether we obtain property $\mathcal{P}$ for all Shimura varieties of abelian type.) More precisely, a Shimura datum (G, X) is said to be of abelian type if there exists a Shimura datum (G_2, X_2) of Hodge type and an isogeny $G_2^{\mathrm{der}} \twoheadrightarrow G^{\mathrm{der}}$ which induces an isomorphism $(G_2^{\mathrm{ad}}, X_2^{\mathrm{ad}}) \xrightarrow{\sim} (G^{\mathrm{ad}}, X^{\mathrm{ad}})$. Deligne has analysed which simple Shimura data belong to this class. He showed that if (G, X) is of abelian type with G simple over $\mathbb{Q}$, then the following two conditions hold:

(i) The adjoint datum $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ is of type A, B or C, or of type $\mathrm{D}^{\mathbb{R}}$, or of type $D^{\mathbb{H}}$ (cf. [De3], section 1.2), and

(ii) For a datum $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ of type A, B, C or $D^{\mathbb{R}}$, let $G^{\sharp}$ denote the universal covering of G^{ad}; for $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ of type $D^{\mathbb{H}}_{\ell}$, let $G^{\sharp}$ be the double covering of G^{ad} which is an inner form of (a product of copies of) $\mathrm{SO}(2\ell)$, cf. ibid., 2.3.8, and notice that the case $D^{\mathbb{H}}_4$ is defined to exclude the case $D^{\mathbb{R}}_4$. Then G^{der} is a quotient of $G^{\sharp}$.

Conversely, if (G', X^+) is a pair defining a connected Shimura variety such that (i) and (ii) hold, then there exists a Shimura datum (G, X) of abelian type with $G^{\mathrm{der}} = G'$, $X^+ \subseteq X$.

Finally, we define (G, X) to be of pre-abelian type if condition (i) holds. We see that, as far as connected Shimura varieties is concerned, this class is only slightly larger than that of data of abelian type.

2.11 Let us check steps (a)–(e) above for the statement

$$\mathcal{P}(G, X): \quad \text{there exists a canonical model for } \; Sh(G, X)_{\mathbb{C}}\,.$$

(a) The scheme $\varprojlim_n \mathsf{A}_{g,1,n} \otimes \mathbb{Q}$ is a canonical model for $Sh(\mathrm{CSp}_{2g}, \mathfrak{H}_g^{\pm})_{\mathbb{C}}$. Given the definitions as set up above, this boils down to a theorem of Shimura and Taniyama—see [De1], section 4. (Needless to say, the theorem of Shimura and Taniyama historically came first. The definition of a canonical model was modelled after a number of examples, including the Siegel modular variety.)

(b) This is shown in ibid., section 5. We should note here that, using a modular interpretation, one can prove $\mathcal{P}(G, X)$ more directly for Shimura varieties of Hodge type. This was indicated in the introduction of [De3], and carried out in detail in [Br].

For steps (c)–(e), let us work with the statement

$$\mathcal{P}^0(G', X^+): \quad \text{there exists a canonical model for } \; Sh^0(G', X^+)_{\mathbb{C}}\,.$$

The (d) and (e) follow easily from the definitions (cf. [De3], 2.7.11) As for (c), we see that our strategy is not completely right: to prove $\mathcal{P}^0(G', X^+)$, we want to take a Shimura datum (G_2, X_2) of Hodge type (for which we know $\mathcal{P}(G_2, X_2)$ by (a) and (b)) with $(G_2^{\mathrm{der}}, X_2^+) \cong (G', X^+)$, and then we can apply Prop. 2.9. The problem here is that this only gives the existence of a weakly canonical model of $Sh^0(G', X^+)_{\mathbb{C}}$ over $E(G_2, X_2)$, which in general is a proper field extension of $E(G^{\mathrm{ad}}, X^+)$. (Notice that (G_2, X_2) is required to be of Hodge type—without this condition there would be no problem.) Thus we see that our "naive" strategy has to be corrected. This is done in two steps.

First one assumes that G is $\mathbb{Q}$-simple, and one considers the maximal covering $G^\sharp \to G^{\mathrm{ad}}$ (as in 2.10) which occurs as the semi-simple part in a Shimura datum of Hodge type. As explained, the Shimura data (G_2, X_2) of Hodge type with $G_2^{\mathrm{der}} \cong G^\sharp$ in general have $E(G^{\mathrm{ad}}, X^+) \subsetneq E(G_2, X_2)$. Deligne shows, however, that by "gluing in" a suitable toric part, the field extension $E(G_2, X_2)$ can be made in almost every "direction"; for a precise statement see [De3], Prop. 2.3.10. Finally one shows that this is enough to guarantee the existence of a canonical model of $Sh^0(G', X^+)_{\mathbb{C}}$; one proves (ibid., Cor. 2.7.19): if for every finite extension $F \subset \overline{\mathbb{Q}}$ of $E = E(G', X^+)$, there exists another finite extension $E \subseteq F' \subset \overline{\mathbb{Q}}$ which is linearly disjoint from F, and such that $Sh^0(G', X^+)_{\mathbb{C}}$ has a weakly canonical model over F', then it has a canonical model.

Putting everything together, one obtains the following result.

2.12 Theorem. (Deligne, [De3]) *Let (G, X) be a Shimura datum, and let $(G^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (G_1, X_1) \times \cdots \times (G_m, X_m)$ be the decomposition of its adjoint datum into simple factors. Suppose that, using the notations of 2.10, G^{der} is a quotient of $G_1^\sharp \times \cdots \times G_m^\sharp$. Then there exists a canonical model of $Sh(G, X)$.*

Notice that it is not clear whether this statement covers all data (G, X) of abelian type.

To extend this result to arbitrary Shimura data, additional arguments are needed. Since eventually we want to apply a Galois descent argument, it would be useful if we could first descend $Sh(G, X)_{\mathbb{C}}$ to a scheme over $\overline{\mathbb{Q}}$. Faltings has shown that this can be done using a rigidity argument.

2.13 Theorem. (Faltings, [Fa1]) *Let G be a semi-simple algebraic group over $\mathbb{Q}$, $K_\infty \subseteq G(\mathbb{R})$ a maximal compact subgroup, and $\Gamma \subset G(\mathbb{Q})$ a neat arithmetic subgroup. If $X = G(\mathbb{R})/K_\infty$ is a hermitian symmetric domain, then the locally symmetric variety $\Gamma\backslash X$ (with its unique structure of an algebraic variety) is canonically defined over $\overline{\mathbb{Q}}$. The special points on $\Gamma\backslash X$ are defined over $\overline{\mathbb{Q}}$. If $\Gamma_1, \Gamma_2 \subset G(\mathbb{Q})$ are neat arithmetic subgroups, $\gamma \in G(\mathbb{Q})$ an element with $\gamma\Gamma_1\gamma^{-1} \subseteq \Gamma_2$, then the natural morphism $\gamma\colon\ \Gamma_1\backslash X \to \Gamma_2\backslash X$ is also defined over $\overline{\mathbb{Q}}$.*

Next we have to recall Langlands's conjecture on the conjugation of Shimura varieties (now a theorem, due to work of Borovoi, Deligne, Milne, and Milne-Shih, among others). We will not go into details here; the interested reader can consult [Bo1], [Bo2], [Mi1], [MS].

2.14 Theorem. (Borovoi, Deligne, Milne, Shih, ...) *Given a Shimura datum* (G, X)*, a special point* $x \in X$*, and a* $\tau \in \mathrm{Aut}(\mathbb{C})$*, one can define a Shimura datum* $({}^{\tau,x}G, {}^{\tau,x}X)$*, a special point* ${}^{\tau}x \in {}^{\tau,x}X$*, and an isomorphism* $G(\mathbb{A}_f) \xrightarrow{\sim} {}^{\tau,x}G(\mathbb{A}_f)$*, denoted* $g \mapsto {}^{\tau,x}g$*, satisfying the following conditions (writing* $\mathcal{T}(g)$ *for the action of an element* $g \in G(\mathbb{A}_f)$ *on* $Sh(G, X)_{\mathbb{C}}$*)*

(i) There is a unique isomorphism $\varphi_{\tau,x}\colon {}^{\tau}Sh(G, X)_{\mathbb{C}} \xrightarrow{\sim} Sh({}^{\tau,x}G, {}^{\tau,x}X)_{\mathbb{C}}$ *with* $\varphi_{\tau,x}({}^{\tau}[x, 1]) = [{}^{\tau}x, 1]$ *and with* $\varphi_{\tau,x} \circ {}^{\tau}\mathcal{T}(g) = \mathcal{T}({}^{\tau,x}g) \circ \varphi_{\tau,x}$ *for all* $g \in G(\mathbb{A}_f)$*.*

(ii) If $x' \in X$ *is another special point then there is an isomorphism* $\varphi(\tau; x, x')\colon Sh({}^{\tau,x}G, {}^{\tau,x}X)_{\mathbb{C}} \xrightarrow{\sim} Sh({}^{\tau,x'}G, {}^{\tau,x'}X)_{\mathbb{C}}$ *such that* $\varphi(\tau; x, x') \circ \varphi_{\tau,x} = \varphi_{\tau,x'}$ *and such that* $\varphi(\tau; x, x') \circ \mathcal{T}({}^{\tau,x}g) = \mathcal{T}({}^{\tau,x'}g) \circ \varphi(\tau; x, x')$ *for all* $g \in G(\mathbb{A}_f)$*.*

As explained in [La], section 6 (see also [Mi2], section II.5), using the theorem one obtains a "pseudo" descent datum from $\mathbb{C}$ to $E = E(G, X)$ on $Sh(G, X)_{\mathbb{C}}$. By this we mean a collection of isomorphisms

$$\{f_\tau\colon {}^{\tau}Sh(G, X)_{\mathbb{C}} \xrightarrow{\sim} Sh(G, X)_{\mathbb{C}}\}_{\tau \in \mathrm{Aut}(\mathbb{C}/E)}$$

satisfying the cocycle condition $f_{\sigma\tau} = f_\sigma \circ {}^{\sigma}f_\tau$. At several places in the literature (e.g., [La], section 6, [Mi2], p. 340, [MS], §7) it is asserted that "by descent theory" this gives a model of $Sh(G, X)_{\mathbb{C}}$ over E. (Due to the properties of the f_τ, this model would then be a canonical model.) We think that this argument is not complete—let us explain why.

2.15 To descend a scheme $X_{\mathbb{C}}$ from $\mathbb{C}$ to a number field $E \subset \mathbb{C}$, it does in general not suffice to give a collection of isomorphisms $\{f_\tau\colon {}^{\tau}X_{\mathbb{C}} \xrightarrow{\sim} X_{\mathbb{C}}\}_{\tau \in \mathrm{Aut}(\mathbb{C}/E)}$ with $f_{\sigma\tau} = f_\sigma \circ {}^{\sigma}f_\tau$ (or, what is the same, a homomorphism of groups $\alpha\colon \mathrm{Aut}(\mathbb{C}/E) \to \mathrm{Aut}(X_{\mathbb{C}})$ sending τ to a τ-linear automorphism of $X_{\mathbb{C}}$). For instance, using the fact that $\mathbb{Q}$ is an injective object in the category of abelian groups, we easily see that there exist non-trivial group homomorphisms $c\colon \mathrm{Aut}(\mathbb{C}/E) \to \mathbb{Q}$. Taking $X_{\mathbb{C}} = \mathbb{A}^1_{\mathbb{C}}$, on which we let $\tau \in \mathrm{Aut}(\mathbb{C}/E)$ act as the τ-linear translation over $c(\tau)$, we get an example of a non-effective "pseudo" descent datum. The same remarks apply if we replace $\mathbb{C}$ by $\overline{\mathbb{Q}}$. (Thus, for example, [Mi4], Lemma 3.23 is not correct as it stands.)

In this context it seems useful to remark the following. Given a $\overline{\mathbb{Q}}$-scheme $X_{\overline{\mathbb{Q}}}$, one might expect that a descent datum on $X_{\overline{\mathbb{Q}}}$ relative to $\overline{\mathbb{Q}}/E$ can be expressed as a collection of isomorphisms $\{\varphi_\tau\colon {}^{\tau}X_{\overline{\mathbb{Q}}} \xrightarrow{\sim} X_{\overline{\mathbb{Q}}}\}_{\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/E)}$ for which, apart from the cocycle condition $\varphi_{\sigma\tau} = \varphi_\sigma \circ {}^{\sigma}\varphi_\tau$, a certain "continuity condition" holds. To see why a continuity condition should enter, one must realize that a scheme such as $\mathrm{Spec}(\overline{\mathbb{Q}} \otimes_E \overline{\mathbb{Q}})$ is not a disjoint union of copies of

$\mathrm{Spec}(\overline{\mathbb{Q}})$ indexed by $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ (which would not be a quasi-compact scheme), but rather a projective limit $\mathrm{Spec}(\overline{\mathbb{Q}}\otimes_E\overline{\mathbb{Q}}) = \varprojlim_F \mathrm{Spec}(\overline{\mathbb{Q}})^{\mathrm{Gal}(F/E)}$, where F runs through the finite Galois extensions of E in $\overline{\mathbb{Q}}$. (In other words: this is $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ as a pro-finite group scheme.) It seems though that it is not so easy to formulate the desired continuity condition directly. Even if one succeeds in doing this, however, it should be remarked that descent data relative to $\overline{\mathbb{Q}}/E$ are not necessarily effective (cf. [SGA1], Exp. VIII).

Since we are really only interested in *effective* descent data relative to $\overline{\mathbb{Q}}/E$, we take a slightly different approach. Let us call a (semi-linear) action $\alpha\colon \mathrm{Gal}(\overline{\mathbb{Q}}/E) \to \mathrm{Aut}(X')$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on a $\overline{\mathbb{Q}}$-scheme X' continuous if it is continuous as an action of a locally compact, totally disconnected group (see [De3], section 2.7). Since the Galois group is actually compact, the following statement is then a tautology.

2.15.1 *The functor $X \mapsto X' = X\otimes_E\overline{\mathbb{Q}}$ gives an equivalence of categories*

$$\left(\begin{array}{c}\text{quasi-projective}\\ \text{schemes } X \text{ over } E\end{array}\right) \xrightarrow{\text{eq.}} \left(\begin{array}{c}\text{quasi-projective schemes } X' \text{ over } \overline{\mathbb{Q}}\\ \text{with a continuous semi-linear action}\\ \text{of } \mathrm{Gal}(\overline{\mathbb{Q}}/E)\end{array}\right).$$

We thus see that, in order to prove the existence of canonical models in the general case, we need to show that Theorem 2.14 provides us with a *continuous* Galois action on $Sh(G,X)_{\overline{\mathbb{Q}}}$. For this we will use the following lemma.

2.16 Lemma. *Let (G,X) be a Shimura datum, $K \subset G(\mathbb{A}_f)$ a compact open subgroup, and let $S = \Gamma\backslash X^+$ be a connected component of $Sh_K(G,X)_{\mathbb{C}}$. Then we can choose finitely many special points $x_1,\ldots,x_n \in S^0$ such that S has no non-trivial automorphisms fixing the x_i.*

Proof. Let $j\colon S \hookrightarrow S^*$ denote the Baily-Borel compactification. Every automorphism of S extends to an automorphism of S^*. There exists an ample line bundle $\mathcal{L}$ on S^* such that $\alpha^*\mathcal{L} \cong \mathcal{L}$ for every $\alpha \in \mathrm{Aut}(S)$. In fact, if G has no simple factors of dimension 3 then we can take $\mathcal{L} := j_*\Omega^d_S$, where $d = \dim(S)$. In the general case one has to impose growth conditions at infinity: using the terminology of [BB] we can take for $\mathcal{L}$ the subsheaf of $j_*\Omega^d_S$ (now taken in the analytic sense) of automorphic forms which are integral at infinity. (So $\mathcal{L}$ is the bundle $\mathcal{O}(1)$ corresponding to the projective embedding of S^* as in loc. cit., §10. Mumford showed in [Mu3] that if $\overline{S}$ is a smooth toroidal compactification and $\pi\colon \overline{S} \to S^*$ is the canonical birational morphism, then $\pi^*\mathcal{L}$ is the sheaf $\Omega^d_{\overline{S}}(\log\partial\overline{S})$.)

Let P be the "doubled" Hilbert polynomial of $\mathcal{L}$, given by $P(x) = P_{\mathcal{L}}(2x)$. Recall from [FGA], Exposé 221, p. 20, that the scheme $\mathrm{Hom}(S^*, S^*)^P$ given by

$$\mathrm{Hom}(S^*, S^*)^P(T) = \{g\colon\ S^* \times_{\mathbb{C}} T \longrightarrow S^* \times_{\mathbb{C}} T \mid \chi\big((\mathcal{L} \otimes_{\mathcal{O}_T} g^*\mathcal{L})^{\otimes n}\big) = P(n)\}$$

is of finite type. With the obvious notations, it follows that $\mathrm{Aut}(S^*)^P$ is a scheme of finite type, being a locally closed subscheme of $\mathrm{Hom}(S^*, S^*)^P$. The lemma now follows from the following two trivial remarks:

(i) if $\alpha \in \mathrm{Aut}(S^*)$ fixes all special points of S then $\alpha = \mathrm{id}$,

(ii) if $x_1, \ldots, x_n$ are special points of S^0 then

$$\mathrm{Aut}(S^*; x_1, \ldots, x_n)^P := \{\alpha \in \mathrm{Aut}(S^*)^P \mid \alpha(x_i) = x_i \quad \text{for all } i = 1, \ldots, n\}$$

is a closed subgroup scheme of $\mathrm{Aut}(S^*)^P$. □

2.17 We now complete the argument showing that $\mathit{Sh}(G,X)_{\mathbb{C}}$ has a canonical model. Obviously, the first step is to use Theorem 2.13, so that we obtain a model $\mathit{Sh}(G,X)_{\overline{\mathbb{Q}}}$ over $\overline{\mathbb{Q}}$. We claim that the "pseudo" descent datum $\{f_\tau\colon\ {}^\tau\mathit{Sh}(G,X)_{\mathbb{C}} \xrightarrow{\ \sim\ } \mathit{Sh}(G,X)_{\mathbb{C}}\}_{\tau \in \mathrm{Aut}(\mathbb{C}/E)}$ considered in 2.14 induces a semi-linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on $\mathit{Sh}(G,X)_{\overline{\mathbb{Q}}}$, which is functorial. We can show this using the special points: if $\mathit{Sh}(T,\{h\})_{\mathbb{C}} \hookrightarrow \mathit{Sh}(G,X)_{\mathbb{C}}$ is a 0-dimensional sub-Shimura variety, then the canonical model $\mathit{Sh}(T,\{h\})$ over E' $= E(T,\{h\})$ gives rise to a collection of isomorphisms $\{\tilde{f}_\sigma\colon\ {}^\sigma\mathit{Sh}(T,\{h\})_{\mathbb{C}} \xrightarrow{\sim} \mathit{Sh}(T,\{h\})_{\mathbb{C}}\}_{\sigma \in \mathrm{Aut}(\mathbb{C}/E')}$, and for $\sigma \in \mathrm{Aut}(\mathbb{C}/E')$, the two maps f_σ and $\tilde{f}_\sigma$ are equal on ${}^\sigma\mathit{Sh}(T,\{h\})_{\mathbb{C}}$. Using the fact that the special points on $\mathit{Sh}(G,X)_{\mathbb{C}}$ are defined over $\overline{\mathbb{Q}}$ for the $\overline{\mathbb{Q}}$-structure $\mathit{Sh}(G,X)_{\overline{\mathbb{Q}}}$, one now checks that the f_σ induce a system

$$\{\varphi_\tau\colon\ {}^\tau\mathit{Sh}(G,X)_{\overline{\mathbb{Q}}} \xrightarrow{\ \sim\ } \mathit{Sh}(G,X)_{\overline{\mathbb{Q}}}\}_{\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/E)}$$

with $\varphi_{\sigma\tau} = \varphi_\sigma \circ {}^\sigma\varphi_\tau$. What we shall use is that the action on the special points agrees with the one obtained from the canonical models $\mathit{Sh}(T,\{h\})$.

Now for the continuity of the Galois action on $\mathit{Sh}(G,X)_{\overline{\mathbb{Q}}}$. First let us remark that it suffices to prove that the semi-linear Galois action on each of the $\mathit{Sh}_K(G,X)$ is continuous, since the transition morphisms then automatically descend. Here we may even restrict to "levels" Sh_K where K is neat. Furthermore, it suffices to show that there is an open subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ which acts continuously. In fact, if we assume this then $\mathit{Sh}_K(G,X)$ descends to a finite Galois extension F of E. On the model $\mathit{Sh}_K(G,X)_F$ thus obtained we still have a Galois descent datum relative to F/E, and since this is now a *finite* Galois extension, the descent datum is effective.

Since $Sh_K(G,X)_{\overline{\mathbb{Q}}}$ is a $\overline{\mathbb{Q}}$-scheme of finite type, there exists a finite extension E' of E and a model $S_{E'}$ of $Sh_K(G,X)$ over E'. This model gives rise to semi-linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E')$ on $Sh_K(G,X)_{\overline{\mathbb{Q}}}$, which we can describe as a collection of automorphisms

$$\{\psi_\tau\colon\, {}^\tau Sh_K(G,X)_{\overline{\mathbb{Q}}} \xrightarrow{\sim} Sh_K(G,X)_{\overline{\mathbb{Q}}}\}_{\tau\in\mathrm{Gal}(\overline{\mathbb{Q}}/E)}\,.$$

Observe that $\varphi_\tau \circ (\psi_\tau)^{-1}$ is a $\overline{\mathbb{Q}}$-*linear* automorphism of $Sh_K(G,X)_{\overline{\mathbb{Q}}}$, and that $\{\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/E) \mid \varphi_\tau = \psi_\tau\}$ is a subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$.

At this point we apply Lemma 2.16. It gives us special points $x_1,\ldots,x_n \in Sh_K(G,X)_{\overline{\mathbb{Q}}}$ such that there are no automorphisms of $Sh_K(G,X)_{\overline{\mathbb{Q}}}$ fixing all x_i. For each x_i, choose a closed immersion $j_i\colon (T_i,\{h_i\}) \hookrightarrow (G,X)$ and an element $g_i \in G(\mathbb{A}_f)$ such that x_i lies in $g_i \cdot Sh(T_i,\{h_i\})_{\overline{\mathbb{Q}}} \subseteq Sh(G,X)_{\overline{\mathbb{Q}}}$. Let $K_i := j_i^{-1}(K) \subset T_i(\mathbb{A}_f)$. There exists a finite extension E'' of E', containing the reflex fields $E(x_i)$, such that the x_i are all E''-rational on the chosen model $S_{E'}$ and such that furthermore all points of $Sh_{K_i}(T_i,\{h_i\})$ are rational over E'' (for every $i = 1,\ldots,n$). It now follows from what was said above that the two Galois actions on $Sh_K(G,X)_{\overline{\mathbb{Q}}}$, given by the φ_τ and the ψ_τ, respectively, are the same when restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/E'')$. This finishes the proof of the following theorem.

2.18 Theorem. *Let (G,X) be a Shimura datum. Then there exists a canonical model $Sh(G,X)$ of the associated Shimura variety.*

2.19 Remark. In [Pi], the notion of a canonical model is generalized to the mixed case, and the existence of such canonical models is proven for arbitrary mixed Shimura varieties. Pink's proof essentially reduces the problem to statement 2.18; once we have 2.18, the mixed case does not require any further corrections.

2.20 Remark. There is also a theory of a canonical models for automorphic vector bundles on Shimura varieties. The interested reader is referred to [Ha] and [Mi2].

§3 Integral canonical models

3.1 Let (G,X) be a Shimura datum with reflex field $E = E(G,X)$, and let v be a prime of E dividing $p > 0$. We want to study models of the Shimura variety $Sh(G,X)$ over the local ring $\mathcal{O}_{E,(v)}$ of E at v. In our personal view, the theory of such models is still in its infancy. How to set up the definitions, what properties to expect, etc., are dictated by the examples where the Shimura variety represents a moduli problem that can be formulated in mixed characteristics (notably Shimura varieties of PEL type). Even in the case where G is unramified over $\mathbb{Q}_p$, this leaves open some subtle questions.

Some of the rules of the game become clear already from looking at Siegel modular varieties. We have seen that the canonical model in this case can be identified with the projective limit $\varprojlim_n \mathsf{A}_{g,1,n} \otimes \mathbb{Q}$. Fixing a prime number p, we see that, for constructing a model over $\mathbb{Z}_{(p)}$, we run into problems at the levels $\mathsf{A}_{g,1,n}$ with $p \mid n$. By contrast, if we only consider levels with $p \nmid n$, then we have a natural candidate model, viz. $\varprojlim_{p\nmid n} \mathsf{A}_{g,1,n} \otimes \mathbb{Z}_{(p)}$, which has all good properties we can expect.

Returning to the general case, this suggests the following set-up. Let (G,X), E and v be as above. We fix a compact open subgroup $K_p \subset G(\mathbb{Q}_p)$, and we consider

$$Sh_{K_p}(G,X) = \varprojlim_{K^p} Sh_{K_p\times K^p}(G,X)\,,$$

where K^p runs through the compact open subgroups of $G(\mathbb{A}_f^p)$. It is this scheme $Sh_{K_p}(G,X)$, the quotient of $Sh(G,X)$ for the action of K_p, of which we shall study models. Notice that we can expect to find a smooth model (to be made precise in a moment) only for special choices of K_p.

3.2 Definition. Let (G,X) be a Shimura datum, $E = E(G,X)$, v a finite prime of E dividing p, and let K_p be a compact open subgroup of $G(\mathbb{Q}_p)$. Let $\mathcal{O}$ be a discrete valuation ring which is faithfully flat over $\mathcal{O}_{(v)}$. Write F for the quotient field of $\mathcal{O}$.

(i) An integral model of $Sh_{K_p}(G,X)$ over $\mathcal{O}$ is a faithfully flat $\mathcal{O}$-scheme $\mathcal{M}$ with a continuous action of $G(\mathbb{A}_f^p)$ and a $G(\mathbb{A}_f^p)$-equivariant isomorphism $\mathcal{M}\otimes F \cong Sh_{K_p}(G,X)\otimes_E F$.

(ii) An integral model $\mathcal{M}$ of $Sh_{K_p}(G,X)$ over $\mathcal{O}$ is said to be smooth (respectively normal) if there exists a compact open subgroup $C \subset G(\mathbb{A}_f^p)$, such that for every pair of compact open subgroups $K_1^p \subseteq K_2^p \subset G(\mathbb{A}_f^p)$ contained in C, the canonical map $\mathcal{M}/K_1^p \to \mathcal{M}/K_2^p$ is an étale morphism between smooth (resp. normal) schemes of finite type over $\mathcal{O}$.

It should be clear that an integral model $\mathcal{M}$, if it exists, is by no means unique. For example, given one such model, we could delete a $G(\mathbb{A}_f^p)$-orbit properly contained in the special fibre, or we could blow up in such an orbit, to obtain a different integral model. To arrive at the notion of an integral canonical model, we will impose the condition that $\mathcal{M}$ satisfy an "extension property", similar to the Néron mapping property in the theory of Néron models (cf. [BLR], section 1.2). This idea was first presented by Milne in [Mi3]. As we shall see, one of the main difficulties in this approach is to find a good class of "test schemes" for which the extension property should hold. Given a base ring $\mathcal{O}$, we will work with a class of $\mathcal{O}$-schemes that we call "admissible test schemes over $\mathcal{O}$", abbreviated "a.t.s.". We postpone the precise definition of the class that we will work with until 3.5.

3.3 Definition. Let (G, X), E, v, K_p, $\mathcal{O}$ and F be as in 3.2.

(i) An integral model $\mathcal{M}$ of $Sh_{K_p}(G, X)$ over $\mathcal{O}$ is said to have the extension property if for every admissible test scheme S over $\mathcal{O}$, every morphism $S_F \to \mathcal{M}_F$ over F extends uniquely to an $\mathcal{O}$-morphism $S \to \mathcal{M}$.

(ii) An integral canonical model of $Sh_{K_p}(G, X)$ at the prime v is a separated smooth integral model over $\mathcal{O}_{(v)}$ which has the extension property. A local integral canonical model is a separated smooth integral model over $\mathcal{O}_v := \mathcal{O}_{(v)}^{\wedge}$ having the extension property.

3.4 Comments. A definition in this form was first given by Milne in [Mi3]. As admissible test schemes over $\mathcal{O}$ he used all regular $\mathcal{O}$-schemes S for which S_F is dense in S. Later it was seen that this is not the right class to work with (cf. [Mi4], footnote on p. 513); the reason for this is the following. One wants to set up the theory in such a way that $\varprojlim_{p \nmid n} \mathsf{A}_{g,1,n} \otimes \mathbb{Z}_{(p)}$ is an integral canonical model for the Siegel modular variety. Using Milne's definition, this boils down to [FC], Cor. V.6.8, which, however, is false as it stands. Recall that this concerns the following question: suppose given a regular scheme S with maximal points of characteristic 0, a closed subscheme $Z \hookrightarrow S$ of codimension at least 2, and an abelian scheme over the complement $U = S \setminus Z$. Does this abelian scheme extend to an abelian scheme over S? In loc. cit. it is claimed that the answer is "yes"—this is not correct in general. A counterexample, due to Raynaud-Ogus-Gabber, is discussed in [dJO], section 6. Let us try to explain the gist of the example, referring to loc. cit. for details.

As base scheme we take $S = \mathrm{Spec}(R)$, where $R = W(\overline{\mathbb{F}}_p)[\![x, y]\!]/\big((xy)^{p-1} - p\big)$. There exists a primitive pth root of unity ζ_p in R. Let $s \in S$ be the closed point, and set $U_1 = D(x)$, $U_2 = D(y)$, $U = S \setminus \{s\} = U_1 \cup U_2$, $U_{12} = D(xy) = U_1 \cap U_2$. We obtain a finite locally free group scheme G_U of rank p^2 over U by gluing

$$G_1 = \big(\mu_p \times \mathbb{Z}/p\mathbb{Z}\big)_{U_1} \quad \text{and} \quad G_2 = \big(\mu_p \times \mathbb{Z}/p\mathbb{Z}\big)_{U_2}$$

via the isomorphism

$$\varphi\colon\ G_{1|U_{12}} \xrightarrow{\sim} G_{2|U_{12}} \quad \text{given by the matrix} \quad \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix},$$

where $\beta\colon\ \mu_p \xrightarrow{\sim} \mathbb{Z}/p\mathbb{Z}$ (over U_{12}) maps $\zeta_p \in \Gamma(U_{12}, \mu_p)$ to $\bar{1} \in \Gamma(U_{12}, \mathbb{Z}/p\mathbb{Z})$. One easily sees from the construction that we have an exact sequence

$$0 \longrightarrow (\mathbb{Z}/p\mathbb{Z})_U \xrightarrow{\gamma_U} G_U \longrightarrow \mu_{p,U} \longrightarrow 0\,,$$

and that this extension is not trivial.

The group scheme G_U extends uniquely to a finite locally free group scheme G over S. Also, the homomorphism γ_U extends uniquely to a homomorphism $\gamma\colon\ (\mathbb{Z}/p\mathbb{Z})_S \to G$, which, however, is not a closed immersion. (The whole point!) To get the desired example, one only has to embed G into an abelian scheme X over S (using the theorem [BM3], Thm. 3.1.1 by Raynaud), and take $Y_U := X_U/(\mathbb{Z}/p\mathbb{Z})_U$, where $(\mathbb{Z}/p\mathbb{Z})_U$ is viewed as a subgroup scheme of X_U via γ_U and the chosen embedding $G \hookrightarrow X$.

To understand what is going on, the following remarks may be of help. One can show that the fibre G_s is isomorphic to $\alpha_p \times \alpha_p$. There is a blowing up $\pi\colon\ \widetilde{S} \to S$ with center in s such that $(\mathbb{Z}/p\mathbb{Z})_U \hookrightarrow G_U$ extends to a closed flat subgroup scheme $N \hookrightarrow G_{\widetilde{S}}$. Over $\widetilde{S}$, the abelian scheme Y_U extends to the abelian scheme $Y_{\widetilde{S}} := X_{\widetilde{S}}/N$. When restricted to the exceptional fibre E, we have $Y_{\widetilde{S}|E} \cong (X_s \times E)/N_E$, where $N_E \hookrightarrow (\alpha_p \times \alpha_p)_E$ is a non-constant subgroup scheme isomorphic to α_p. Therefore, we cannot blow down $Y_{\widetilde{S}}$ to an abelian scheme over S.

In order to guarantee that $\varprojlim_{p\nmid n} \mathsf{A}_{g,1,n} \otimes \mathbb{Z}_{(p)}$ is an i.c.m., we want our a.t.s. to satisfy the following condition. (Here $\mathcal{O}$ is a d.v.r. with field of fractions F and S is an $\mathcal{O}$-scheme.)

(3.4.1) for every closed subscheme $Z \hookrightarrow S$, disjoint from S_F and of codimension at least 2 in S, every abelian scheme over the complement $U = S \setminus Z$ extends to an abelian scheme over S.

On the other hand, we want that an integral canonical model, if it exists, is unique up to isomorphism. Thus we want it to be an a.t.s. over $\mathcal{O}_{(v)}$ itself.

The notion that we will work with in this paper is the following.

3.5 Definition. Let $\mathcal{O}$ be a discrete valuation ring. We call an $\mathcal{O}$-scheme S an admissible test scheme (a.t.s.) over $\mathcal{O}$ if every point of S has an open neigbourhood of the form $\mathrm{Spec}(A)$, such that there exist $\mathcal{O} \subseteq \mathcal{O}' \subseteq A_0 \subseteq A$, where

—$\mathcal{O} \subseteq \mathcal{O}'$ is a faithfully flat and unramified extension of d.v.r. with $\mathcal{O}'/(\pi)$ separable over $\mathcal{O}/(\pi)$,

—A_0 is a smooth $\mathcal{O}'$-algebra, and where

—$\mathrm{Spec}(A) \to \mathrm{Spec}(A_0)$ is a pro-étale covering.

We write $\mathrm{ATS}_{\mathcal{O}}$ for the class of a.t.s. over $\mathcal{O}$.

We want to stress that this should be seen as a working definition, see also the remarks in 3.9 below. Clearly, a smooth model of a Shimura variety over $\mathcal{O}$ belongs to $\mathrm{ATS}_{\mathcal{O}}$. In particular, we have unicity of integral canonical models:

3.5.1 Proposition. *Let (G, X) be a Shimura datum, v a prime of its reflex field E dividing the rational prime p, and let K_p be a compact open subgroup of $G(\mathbb{Q}_p)$. If there exists an integral canonical model of $Sh_{K_p}(G, X)$ over $\mathcal{O}_{(v)}$, then it is unique up to isomorphism.*

Furthermore, we have the following properties.

(3.5.2) If $S \in \mathrm{ATS}_{\mathcal{O}}$ then S is a regular scheme, formally smooth over $\mathcal{O}$. (To prove that the local rings of S are noetherian, we can follow the arguments of [Mi3], Prop. 2.4.)

(3.5.3) If $\mathcal{O} \subseteq \mathcal{O}'$ is an unramified faithfully flat extension of d.v.r., then $S \in \mathrm{ATS}_{\mathcal{O}'} \Rightarrow S \in \mathrm{ATS}_{\mathcal{O}}$, and $S \in \mathrm{ATS}_{\mathcal{O}} \Rightarrow (S \otimes_{\mathcal{O}} \mathcal{O}') \in \mathrm{ATS}_{\mathcal{O}'}$.

Next we investigate whether (3.4.1) holds. For this we use the following two lemmas.

3.6 Lemma. (Faltings) *Let $\mathcal{O}$ be a d.v.r. of mixed characteristics $(0, p)$ with $p > 2$. Suppose that the ramification index e satisfies $e < p - 1$. Then every regular formally smooth $\mathcal{O}$-scheme S satisfies condition (3.4.1).*

Proof (sketch). As mentioned above, some statements in [FC], section V.6, are not correct. The mistake can be found on p. 182: the map $p^{-\dim(G)} \cdot \mathrm{trace}_{G[p^{n+1}]/G[p^n]}$ is not a splitting of the map $\mathcal{O}_{G[p^n]} \subset \mathcal{O}_{G[p^{n+1}]}$, as claimed. Most arguments in the rest of the section are correct however, and with some modifications we can use them to prove the lemma. Let us provisionally write $\mathrm{RFS}_{\mathcal{O}}$ for the class of regular, formally smooth $\mathcal{O}$-schemes. For $S \in \mathrm{RFS}_{\mathcal{O}}$, we have the following version of [FC], Thm. V.6.4'.

3.6.1 *Let S be a regular, formally smooth $\mathcal{O}$-scheme ($\mathcal{O}$ as above, with $e < p-1$), and let $U \hookrightarrow S$ be the complement of a closed subscheme $Z \hookrightarrow S$ of codimension at least 2. Then every p-divisible group $\mathcal{G}_U$ over U extends uniquely to a p-divisible group $\mathcal{G}$ over S.*

The only step in the proof of [FC], Thm. V.6.4' that we have to correct is the one showing the existence of an extension $\mathcal{G}$ in case $\dim(S) = 2$ (loc. cit., top of p. 183). So we may assume $S = \mathrm{Spec}(R) \hookleftarrow U = S \setminus \{s\}$, where R is a 2-dimensional regular local ring, and where s is the closed point of S. The $\mathcal{G}_{U,n} := \mathcal{G}_u[p^n]$ extend uniquely to an inductive system of finite flat group schemes $\{\mathcal{G}_n; i_n \colon \mathcal{G}_n \to \mathcal{G}_{n+1}\}$. (See [FC], Lemma V.6.2.) We have to prove that the sequences

$$0 \longrightarrow \mathcal{G}_n \xrightarrow{i_n} \mathcal{G}_{n+1} \xrightarrow{p^n} \mathcal{G}_1 \longrightarrow 0 \tag{3.6.2}$$

are exact. That i_n is a closed immersion needs to be checked only on the closed fibre. The formal smoothness of R over $\mathcal{O}$ guarantees that there exists an unramified faithfully flat extension of d.v.r. $\mathcal{O} \subset \mathcal{O}'$ such that S has a section over $\mathcal{O}'$ with s contained in the image. Pulling back to $\mathcal{O}'$, it then follows from [Ra1], Cor. 3.3.6, that i_n is a closed immersion. Finally, this implies that $\mathcal{G}_{n+1}/\mathcal{G}_n$ is a finite flat extension of $\mathcal{G}_{U,1}$, and because of the unicity of such an extension it follows that (3.6.2) is exact.

It remains to be checked that, using 3.6.1 to replace [FC] Thm. V.6.4', all steps in the proof of ibid., Thm. 6.7 go through for $S \in \mathrm{RFS}_{\mathcal{O}}$. One has to note that in carrying out the various reduction steps, we stay within the class $\mathrm{RFS}_{\mathcal{O}}$. At some points one furthermore needs arguments similar to the above ones, i.e., taking sections over an extension $\mathcal{O}'$ and using [Ra1], Cor. 3.3.6. We leave it to the reader to verify the details. □

3.7 Lemma. *Let (G, X) be a Shimura datum, and let v be a prime of $E(G, X)$ dividing p. Assume that $G_{\mathbb{Q}_p}$ is unramified (see 3.11 below). Then v is an unramified prime (in the extension $E(G, X) \supset \mathbb{Q}$).*

Proof. See [Mi4], Cor. 4.7. □

3.8 Corollary. *Notations as in 3.2. If $p > 2$ then every $S \in \mathrm{ATS}_{\mathcal{O}_{(v)}}$ satisfies (3.4.1). In particular, if $p > 2$ then $\varprojlim_{p\nmid n} \mathsf{A}_{g,1,n} \otimes \mathbb{Z}_{(p)}$ is an integral canonical model of $Sh_{K_p}(\mathrm{CSp}_{2g,\mathbb{Q}}, \mathfrak{H}_g^{\pm})$, where $K_p = \mathrm{CSp}_{2g}(\mathbb{Z}_p)$.*

Proof. We can follow Milne's proof of [Mi3], Thm. 2.10, except that we have to modify the last part of the proof in the obvious way. Notice that the group CSp_{2g} is unramified everywhere, so that Lemmas 3.6 and 3.7 apply. □

3.9 Remarks. (i) We do not know whether the corollary is also true for $p = 2$. (Note that in the example in 3.4, the base scheme S is not an a.t.s. over W or $W[\zeta_p]$.) This is one of the reasons why we do not pretend that Def. 3.5 is in its final form.

(ii) Our definitions differ from those used in [Va2]. Vasiu's definition of an integral canonical model is of the above form, but the class $\mathrm{ATS}_{\mathcal{O}}$ he works with is the class of all regular schemes S over $\mathrm{Spec}(\mathcal{O})$, for which the generic fibre S_F is Zariski dense and such that condition (3.4.1) holds. As we have seen above, this contains the class we are working with if $p > 2$ and $e(\mathcal{O}) < p - 1$.

It seems to us that Vasiu's definition is more difficult to work with. For example, it is not clear to us whether his notion of an a.t.s. is a local one, and whether it satisfies $S \in \mathrm{ATS}_{\mathcal{O}} \Rightarrow (S \otimes_{\mathcal{O}} \mathcal{O}') \in \mathrm{ATS}_{\mathcal{O}'}$. (This *is* important for some of the constructions.) On the other hand, if we want that the extension property is preserved under extension of scalars from $\mathcal{O}_{(v)}$ to $\mathcal{O}_v$ or to $W\big(\overline{\kappa(v)}\big)$, then this forces us to work with a class $\mathrm{ATS}_{\mathcal{O}}$ which is not "too small". Here we should draw a comparison with the theory of Néron models: we note that the proof of [BLR], Thm. 7.2.1 (ii) makes essential use of Weil's theorem, ibid. Thm. 4.4.1, for which we see no analogue in the context of Shimura varieties. This may help to explain why we set up the Def. 3.5 the way we did.

3.10 Proposition. *Let (G, X), $E = E(G, X)$, v and K_p be as in 3.2.*

(i) There exists an integral canonical model of $Sh_{K_p}(G, X)$ at v if and only if there exists a local integral canonical model.

(ii) Suppose that $p > 2$ and that the prime v is (absolutely) unramified. Write B for the fraction field of $W(\overline{\mathbb{F}}_p)$, and choose an embedding $\mathcal{O}_v \hookrightarrow W(\overline{\mathbb{F}}_p)$, where $\mathcal{O}_v = \mathcal{O}_{(v)}^{\wedge}$ is the completed local ring of $\mathcal{O}_E$ at v. Suppose there exists a smooth integral model $\overline{\mathcal{M}}$ for $Sh_{K_p}(G, X) \otimes B$ over $W(\overline{\mathbb{F}}_p)$ having the extension property. Then there exists an integral canonical model of $Sh_{K_p}(G, X)$ over $\mathcal{O}_{(v)}$.

Proof. (i) In the "only if" direction this readily follows from (3.5.3). For the converse, suppose that $\mathcal{M}^\sharp$ is a local integral canonical model of $Sh_{K_p}(G,X)$ over $\mathcal{O}_v$. We have $\mathcal{M}^\sharp = \varprojlim \mathcal{M}^\sharp_{K^p}$, where K^p runs through the compact open subgroups of $G(\mathbb{A}_f^p)$. Write $S' = \mathrm{Spec}(\mathcal{O}_v) \to S = \mathrm{Spec}(\mathcal{O}_{(v)})$ and $\eta' = \mathrm{Spec}(E_v) \to \eta = \mathrm{Spec}(E)$. Also write $S'' = S' \times_S S'$, $\eta'' = \eta' \times_\eta \eta'$, and write p_i $(i = 1, 2)$ for the ith projection $S'' \to S'$ (resp. $\eta'' \to \eta'$). On the generic fibre $\mathcal{M}^\sharp \otimes E_v$ we have an effective descent datum relative to $\eta' \to \eta$. If we consider $p_1^*(\mathcal{M}^\sharp \otimes E_v) \to \eta''$ as a η'-scheme via $p_2\colon\ eta'' \to \eta'$, then this descent datum is equivalent to giving a morphism $p_1^*(\mathcal{M}^\sharp \otimes E_v) \to \mathcal{M}^\sharp \otimes E_v$ over η'. (Here we ignore the cocycle condition for a moment.) Since $p_1^*\mathcal{M}^\sharp$, considered as a S'-scheme via $p_2\colon\ S'' \to S'$, is an a.t.s. over S', and since $\mathcal{M}^\sharp$ was assumed to have the extension property, the descent datum on $\mathcal{M}^\sharp \otimes E_v$ extends to one on $\mathcal{M}^\sharp$ relative to $S' \to S$. (It is clear that the extended descent datum again satisfies the cocycle condition, $\mathcal{M}^\sharp$ being separated.)

By the arguments of [BLR], pp. 161–162, the extended descent datum is effective. (We can work with each of the $\mathcal{M}^\sharp_{K^p}$ separately, and since $\mathcal{M}^\sharp$ is a smooth model, we may furthermore restrict our attention to those $\mathcal{M}^\sharp_{K^p}$ which are smooth over $\mathcal{O}_v$.) Thus we obtain a smooth model $\mathcal{M}$ over $\mathcal{O}_{(v)}$. It remains to be shown that this model again has the extension property. This follows easily from property (3.5.3) and the fact that descent data for morphisms are effective ([BLR], Prop. D.4(b) in section 6.2).

(ii) The descent from $\overline{\mathcal{M}}$ to a local i.c.m. $\mathcal{M}^\sharp$ is done following the same argument. By (i) this suffices. □

3.11 From now on, we will concentrate on the case where $K_p \subset G(\mathbb{Q}_p)$ is a hyperspecial subgroup. This means that there exists a reductive group scheme $\mathcal{G}_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$ (uniquely determined by K_p) with generic fibre $G_{\mathbb{Q}_p}$ such that $K_p = \mathcal{G}(\mathbb{Z}_p)$. Hyperspecial subgroups of $G(\mathbb{Q}_p)$ exist if and only if $G_{\mathbb{Q}_p}$ is unramified, i.e., quasi-split over $\mathbb{Q}_p$ and split over an unramified extension. For more on hyperspecial subgroups we refer to [Ti], [Va2].

One can show ([Va2], Lemma 3.13) that the group $\mathcal{G}_{\mathbb{Z}_p}$ is obtained by pull-back from a group scheme $\mathcal{G}$ over $\mathbb{Z}_{(p)}$. This suggests that we define an integral Shimura datum to be a pair $(\mathcal{G}, X)$, where $\mathcal{G}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$, and where, writing $G = \mathcal{G}_\mathbb{Q}$, the pair (G,X) is a Shimura datum in the sense of 1.4[4]. To $(\mathcal{G}, X)$ we associate the Shimura variety

[4] We hasten to add that one has to be careful about morphisms: if we have two pairs $(\mathcal{G}_1, X_1)$ and $(\mathcal{G}_2, X_2)$ plus a morphism $f\colon (G_1,X_1) \to (G_2,X_2)$ such that $f(K_{p,1}) \subseteq K_{p,2}$ then it is *not* true in general that f extends to a morphism $\tilde{f}\colon \mathcal{G}_1 \to \mathcal{G}_2$; cf. [BT], 1.7 and

$Sh(\mathcal{G}, X) := Sh_{K_p}(G, X)$, where of course $K_p := \mathcal{G}(\mathbb{Z}_p)$.

Suppose $G_{\mathbb{Q}_p}$ is unramified. Whether there exists an integral canonical model of $Sh_{K_p}(G, X)$ does not depend on the choice of the hyperspecial subgroup $K_p \subset G(\mathbb{Q}_p)$. This is a consequence of the fact that the hyperspecial subgroups of $G(\mathbb{Q}_p)$ are conjugate under $G^{\mathrm{ad}}(\mathbb{Q}_p)$, see [Va2], 3.2.7.

3.12 Examples. (i) Let $(T, \{h\})$ be a Shimura datum with T a torus. The group $T_{\mathbb{Q}_p}$ is unramified precisely if the character group $X^*(T)$ is unramified at p as a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module. If this is the case then $T_{\mathbb{Q}_p}$ extends uniquely to a torus $\mathcal{T}$ over $\mathbb{Z}_p$, and $K_p := \mathcal{T}(\mathbb{Z}_p)$ is the unique hyperspecial subgroup of $T(\mathbb{Q}_p)$. Let $K^p \subset T(\mathbb{A}_f^p)$ be a compact open subgroup. It follows from the description given in 2.2 and 2.3 that $Sh_{K_p \times K^p}(T, \{h\}) = \mathrm{Spec}(L_1 \times \cdots \times L_r)$ for certain number fields $L_i \supset E$ which are unramified above p. Now set $\mathcal{M}_{K_p \times K^p} = \mathrm{Spec}(\mathcal{O}_1 \times \cdots \times \mathcal{O}_r)$, where $\mathcal{O}_i$ is the normalization of $\mathcal{O}_{(v)}$ in L_i. Then $\varprojlim_{K^p} \mathcal{M}_{K_p \times K^p}$ is an integral canonical model of $Sh_{K_p}(T, \{h\})$ over $\mathcal{O}_{(v)}$.

(ii) If (G, X) defines a Shimura variety of PEL type, then we can use the modular interpretation of $Sh(G, X)$ to study integral canonical models. As mentioned before, the precise formulation of a moduli problem requires a lot of data, and we refer to [Ko2] for details. We remark that the Shimura varieties that we are interested in, in general only form an open subscheme of the moduli space studied in loc. cit., section 5. The arguments given there (see also [LR], §6) show that, for primes p satisfying suitable conditions which imply the existence of a hyperspecial subgroup $K_p \subset G(\mathbb{Q}_p)$, the Shimura variety $Sh_{K_p}(G, X)$ has an i.c.m. over $\mathcal{O}_{(v)}$ for all primes v of $E(G, X)$ above p.

3.13 Remark. If there exists an i.c.m. $\mathcal{M}$ for $Sh(\mathcal{G}, X)$, then one expects that each "finite level" $\mathcal{M}_{K^p}$ is a quasi-projective $\mathcal{O}_{(v)}$-scheme. This is certainly the case for the examples in 3.8 and 3.12. Moreover, one easily checks that the quasi-projectivity is preserved under all constructions presented in this section.

3.14 Our next goal is to show that if $G_{\mathbb{Q}_p}$ is unramified, then we can adapt [De3], 2.1.5–8 (which we summarized in 1.6.5) to the present context. The connected component of $Sh_{K_p}(G, X)_{\overline{\mathbb{Q}}}$ containing the image of $X^+ \times \{e\}$ is the projective limit $\varprojlim \Gamma \backslash X^+$, where $\Gamma = \mathrm{Im}\big([G^{\mathrm{der}}(\mathbb{Q})^+ \cap (K_p \times K^p)] \to G^{\mathrm{ad}}(\mathbb{Q})^+\big)$ for some compact open subgroup $K^p \subset G^{\mathrm{der}}(\mathbb{A}_f^p)$. (Here we use [De3], 2.0.13.)

[Va2], 3.1.2.

Formalizing this, we are led to consider pairs $(\mathcal{G}', X^+)$ consisting of a semi-simple group $\mathcal{G}'$ over $\mathbb{Z}_{(p)}$ and a $\mathcal{G}^{\mathrm{ad}}(\mathbb{R})^+$-conjugacy class of homomorphisms $h\colon \mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ (writing $G' = \mathcal{G}'_{\mathbb{Q}}$, $G^{\mathrm{ad}} = \mathcal{G}^{\mathrm{ad}}_{\mathbb{Q}} := (\mathcal{G}')^{\mathrm{ad}}_{\mathbb{Q}}$) such that conditions (i), (ii) and (iii) in 1.4 are satisfied. For such a pair we define the topology $\tau(\mathcal{G}')$ on $\mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})$ as the linear topology having as a fundamental system of neighbourhoods of 1 the images of the $\{p,\infty\}$-congruence subgroups $\mathcal{G}'(\mathbb{Z}_{(p)}) \cap K^p$, where K^p is a compact open subgroup of $\mathcal{G}'(\mathbb{A}_f^p)$. We then write $Sh^0(\mathcal{G}', X^+)_{\mathbb{C}} := \varprojlim \Gamma\backslash X^+$, where Γ runs through the $\{p,\infty\}$-arithmetic subgroups of $\mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})$ which are open in $\tau(\mathcal{G}')$.

On $Sh^0(\mathcal{G}', X^+)_{\mathbb{C}}$ we have a continuous action of $\mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+\wedge}$ (completion rel. $\tau(\mathcal{G}')$), and by 2.13, these data are all canonically defined over $\overline{\mathbb{Q}}$. (Even over a much smaller field, as we shall see next.) For an integral Shimura datum $(\mathcal{G}, X)$ and a connected component $X^+ \subseteq X$, the corresponding connected component of $Sh(\mathcal{G}, X)_{\overline{\mathbb{Q}}}$ is a scheme with continuous $\mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+\wedge}$-action, isomorphic to $Sh^0(\mathcal{G}^{\mathrm{der}}, X^+)_{\overline{\mathbb{Q}}}$. Note that $Sh^0(\mathcal{G}', X^+)$ is an integral scheme (use [EGA], IV, Cor. 8.7.3).

3.15 Lemma. *Let (G, X) be a Shimura datum, $E = E(G, X)$, v a prime of E dividing p. Assume that $G_{\mathbb{Q}_p}$ is unramified, and let $K_p \subset G(\mathbb{Q}_p)$ be a hyperspecial subgroup. Then the connected components of $Sh_{K_p}(G, X)$ are defined over an abelian extension $\widetilde{E}$ of E which is unramified above p.*

Proof. First we prove this under the additional assumption that G^{der} is simply connected. The $G(\mathbb{C})$-conjugacy class of homomorphisms $\mu_x\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to G_{\mathbb{C}}$ (for $x \in X$) gives rise to a well-defined cocharacter $\mu^{\mathrm{ab}}\colon \mathbb{G}_{\mathrm{m},\mathbb{C}} \to G^{\mathrm{ab}}_{\mathbb{C}}$, which has field of definition $E(G^{\mathrm{ab}}, X^{\mathrm{ab}}) \subseteq E$. Writing $T_E = \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{\mathrm{m},E}$, we get a homomorphism

$$\rho = \operatorname*{Nm}_{E/\mathbb{Q}} \circ \mu^{\mathrm{ab}}_E \colon T_E \to G^{\mathrm{ab}},$$

inducing a map $\rho(\mathbb{A}/\mathbb{Q})\colon \mathbb{A}_E^*/E^* \to G^{\mathrm{ab}}(\mathbb{A})/G^{\mathrm{ab}}(\mathbb{Q}) = \pi(G^{\mathrm{ab}})$. The assumption that G^{der} is simply connected implies (see [De1], 2.7) that $\overline{\pi}_0\pi(G)$ is a quotient of $\pi(G^{\mathrm{ab}})$. Moreover, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)^{\mathrm{ab}}$ on $\pi_0\big(Sh(G, X)\big)$ factors through $\pi_0\rho(\mathbb{A}/\mathbb{Q})\colon \pi_0\pi(T_E) \to \pi_0\pi(G^{\mathrm{ab}})$. By class field theory it therefore suffices to show that the image under $\rho(\mathbb{Q}_p)$ of $C_p := \prod_{v|p} \mathcal{O}_v^* \subset T_E(\mathbb{Q}_p)$ in $G^{\mathrm{ab}}(\mathbb{Q}_p)$ is contained in $K_p^{\mathrm{ab}} := \mathrm{Im}\big(K_p \subset G(\mathbb{Q}_p) \to G^{\mathrm{ab}}(\mathbb{Q}_p)\big)$.

The fact that $G_{\mathbb{Q}_p}$ is unramified implies ([Mi4], Cor. 4.7) that T_E is unramified over $\mathbb{Q}_p$, so it extends to a torus $\mathcal{T}_E$ over $\mathbb{Z}_p$. Clearly, $C_p = \mathcal{T}_E(\mathbb{Z}_p)$. Write $\mathcal{G}$ for the extension of $G_{\mathbb{Q}_p}$ to a reductive group scheme over $\mathbb{Z}_p$ with

$K_p = \mathcal{G}(\mathbb{Z}_p)$. The map ρ extends to a homomorphism $\mathcal{T}_E \to \mathcal{G}^{\mathrm{ab}}$ over $\mathbb{Z}_p$, hence we are done if we show that $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p) = K_p^{\mathrm{ab}}$, i.e., $\mathcal{G}(\mathbb{Z}_p)$ maps surjectively to $\mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$. Again using that G^{der} is simply connected we have $\mathrm{H}^1(\mathbb{Q}_p, G^{\mathrm{der}}) = \{1\}$, hence $G(\mathbb{Q}_p) \twoheadrightarrow G^{\mathrm{ab}}(\mathbb{Q}_p)$. For $s \in \mathcal{G}^{\mathrm{ab}}(\mathbb{Z}_p)$ we thus can lift the corresponding $s_\eta \in G^{\mathrm{ab}}(\mathbb{Q}_p)$ to $\widetilde{s_\eta} \in G(\mathbb{Q}_p)$. Taking the Zariski closure of the image of $\widetilde{s_\eta}$ inside $\mathcal{G}$ then gives the desired $\mathbb{Z}_p$-valued point $\tilde{s}$ of $\mathcal{G}$ mapping to s.

The general case is reduced to the previous one. An easy generalization of [MS], Application 3.4 shows that there exists a morphism of Shimura data $f\colon (G_1, X_1) \to (G, X)$ such that $f^{\mathrm{der}}\colon G_1^{\mathrm{der}} \to G^{\mathrm{der}}$ is the universal covering of G^{der}, such that $E(G_1, X_1) = E(G, X)$, and such that there is a hyperspecial subgroup $\widetilde{K_p} \subset G_1(\mathbb{Q}_p)$ with $f(\widetilde{K_p}) \subseteq K_p$. This suffices to prove the lemma, since the components of $Sh_{\widetilde{K_p}}(G_1, X_1)$ map surjectively to components of $Sh_{K_p}(G, X)$ and since all components have the same field of definition (being permuted transitively under the $G(\mathbb{A}_f)$-action). □

3.16 Consider a pair $(\mathcal{G}', X^+)$ as in 3.14. Write $G' = \mathcal{G}'_{\mathbb{Q}}$, and write $\widetilde{E}$ for the maximal subfield of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})^{\mathrm{ab}}$ which is unramified above p. The lemma implies that the connected Shimura variety $Sh^0(\mathcal{G}', X^+)$ has a well-defined "canonical" model over $\widetilde{E}$. Indeed, we can choose an integral Shimura datum $(\mathcal{G}, X)$ with $\mathcal{G}' = \mathcal{G}^{\mathrm{der}}$, $X^+ \subseteq X$ and $E(\mathcal{G}, X) = E(\mathcal{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$, and take $Sh^0(\mathcal{G}^{\mathrm{der}}, X^+)_{\widetilde{E}}$ (which makes sense, grace to the lemma) as the desired model. That this does not depend on the chosen pair $(\mathcal{G}, X)$ follows from the facts in 2.7.

3.17 Definition. Write $Sh^0(\mathcal{G}', X^+)_{\widetilde{E}}$ for the model over $\widetilde{E}$ just defined, and let w be a prime of $\widetilde{E}$ above p. We adapt Def. 3.2 to connected Shimura varieties, replacing E by $\widetilde{E}$ and $G(\mathbb{A}_f)$ by $\mathcal{G}(\mathbb{Z}_{(p)})^{+\wedge}$. Then an integral canonical model (resp. local i.c.m.) for $Sh^0(\mathcal{G}', X^+)_{\widetilde{E}}$ at w is a separated smooth integral model over $\mathcal{O}_{(w)}$ (resp. $\mathcal{O}_w$) which has the extension property.

Of course, the point of this definition is that a Shimura variety can be recovered from the (or rather: some) corresponding connected Shimura variety by an "induction" procedure. This will enable us to follow the same strategy as in 2.10. We consider the properties

$$\mathcal{P}(\mathcal{G}, X; v): \quad \text{there exists an i.c.m. for } Sh(\mathcal{G}, X) \text{ over } \mathcal{O}_{(v)}$$

(for $(\mathcal{G}, X)$ an integral Shiumura datum, v a prime of $E = E(G, X)$ above p), and

$$\mathcal{P}^0(\mathcal{G}', X^+; w): \quad \text{there exists a local i.c.m. for } Sh^0(\mathcal{G}', X^+)_{\widetilde{E}} \text{ over } \mathcal{O}_{(w)}$$

(for $(\mathcal{G}', X^+)$, $\widetilde{E}$ and w as above). Using the induction technique of [De3], Lemma 2.7.3 and Prop. 3.10, we can prove the following statement. We leave the details of the proof to the reader.

3.18 Proposition. *Notations as above, with $\mathcal{G}' = \mathcal{G}^{\mathrm{der}}$, $X^+ \subseteq X$. Suppose that v and w restrict to the same prime of $E \cap \widetilde{E}$. Then $\mathcal{P}(\mathcal{G}, X; v) \iff \mathcal{P}^0(\mathcal{G}^{\mathrm{der}}, X^+; w)$.*

From now on we restrict our attention to the case $p > 2$. Recall that it is implicit in our notations that we are working at a prime where the group is unramified, since $\mathcal{G}$ and $\mathcal{G}'$ are supposed to be *reductive* group schemes over $\mathbb{Z}_{(p)}$. Write $\mathcal{P}(\mathcal{G}, X)$ for "$\mathcal{P}(\mathcal{G}, X; v)$ holds for all primes v of E above p", and similarly for $\mathcal{P}^0(\mathcal{G}', X^+)$. We have shown that statements (a) and (c) in 2.10 hold. Furthermore, statement (d) is almost trivially true. By contrast, it is not at all obvious how to prove (b). The only thing we get more or less for free is a good *normal* model.

3.19 Proposition. *Let $i\colon (G_1, X_1) \hookrightarrow (G_2, X_2)$ be a closed immersion of Shimura data such that there exist hyperspecial subgroups $K_{j,p} \subset G_j(\mathbb{Q}_p)$ with $i(K_{1,p}) \subseteq K_{2,p}$. Suppose there exists an i.c.m. $\mathcal{M}$ for $Sh_{K_{2,p}}(G_2, X_2)$ over $\mathcal{O}_{E_2,(v)}$. If w is a prime of $E_1 = E(G_1, X_1)$ above v then there exists a normal integral model $\mathcal{N}$ of $Sh_{K_{1,p}}(G_1, X_1)$ over $\mathcal{O}_{E_1,(w)}$ which has the extension property (see Def. 3.3).*

Proof. Let $\mathcal{G}_j$ $(j = 1, 2)$ denote the extension of G_j to a reductive group scheme over $\mathbb{Z}_{(p)}$ with $\mathcal{G}_j(\mathbb{Z}_p) = K_{j,p}$. Write $\mathcal{K}$ for the set of pairs (K_1^p, K_2^p) of compact open subgroups $K_j^p \subset G_j(\mathbb{A}_f^p)$ such that $i(K_1^p) \subset K_2^p$, partially ordered by $(K_1^p, K_2^p) \preceq (L_1^p, L_2^p)$ iff $K_1^p \supseteq L_1^p$ and $K_2^p \supseteq L_2^p$. Given $(K_1^p, K_2^p) \in \mathcal{K}$, we have a morphism

$$i(K_1^p, K_2^p)\colon\ Sh_{K_1^p}(\mathcal{G}_1, X_1) \longrightarrow Sh_{K_2^p}(\mathcal{G}_2, X_2) \hookrightarrow \mathcal{M}_{K_2^p} \otimes \mathcal{O}_{E_1,(w)}\,.$$

Write $N(K_1^p, K_2^p)$ for the (scheme-theoretical) image of $i(K_1^p, K_2^p)$, and let $\mathcal{N}(K_1^p, K_2^p)$ be its normalization. For fixed K_1^p we set

$$N_{K_1^p} = \varprojlim_{K_2^p} N(K_1^p, K_2^p)\,, \quad \mathcal{N}_{K_1^p} = \varprojlim_{K_2^p} \mathcal{N}(K_1^p, K_2^p)\,, \quad \mathcal{M}_{K_1^p} = \varprojlim_{K_2^p} \mathcal{M}_{K_2^p}\,,$$

where the limits run over all K_2^p such that $(K_1^p, K_2^p) \in \mathcal{K}$. Also we set

$$N := \varprojlim_{K_1^p} N_{K_1^p}\,, \quad \mathcal{N} := \varprojlim_{K_1^p} \mathcal{N}_{K_1^p}\,.$$

First we show that, for $K_1^p \supseteq L_1^p$ sufficiently small, the canonical morphism $\mathcal{N}_{L_1^p} \to \mathcal{N}_{K_1^p}$ is étale. For this, we take compact open subgroups $C_j^p \subset G_j(\mathbb{A}_f^p)$ with $i(C_1^p) \subseteq C_2^p$, and such that for all $K_j^p \supseteq L_j^p$ contained in C_j^p $(j = 1, 2)$, the transition morphisms $Sh_{L_1^p}(\mathcal{G}_1, X_1) \to Sh_{K_1^p}(\mathcal{G}_1, X_1)$ and $\mathcal{M}_{L_2^p} \to \mathcal{M}_{K_2^p}$ are étale morphisms of smooth schemes over E_1 and $\mathcal{O}_{E_2,(v)}$ respectively. One checks that for all such $K_1^p \supseteq L_1^p$, the morphism $t\colon\ \mathcal{M}_{L_1^p} \to \mathcal{M}_{K_1^p}$ is again étale, of degree $[K_1^p : L_1^p]$. It follows that $N_{L_1^p} \to N_{K_1^p}$ is a pull-back of t, hence étale. Now $N_{K_1^p}$ has finitely many irreducible components, being a scheme-theoretical image of $Sh_{K_1^p}(\mathcal{G}_1, X_1)$, and the normalization of $N_{K_1^p}$ is just $\mathcal{N}_{K_1^p}$. Using this remark, it follows that $\mathcal{N}_{L_1^p} \to \mathcal{N}_{K_1^p}$ is étale, so that $\mathcal{N}$ is a normal model of $Sh(\mathcal{G}_1, X_1)$ over $\mathcal{O}_{E_1,(w)}$.

That $\mathcal{N}$ has the extension property is seen as follows. We consider an $S \in \mathrm{ATS}_{\mathcal{O}}$ (with $\mathcal{O} = \mathcal{O}_{E_1,(w)}$) and a morphism $\alpha_{E_1}\colon\ S_{E_1} \to \mathcal{N}_{E_1}$ on the generic fibre. The fact that $\mathcal{O}_{E_1,(w)}$ is an unramified extension of $\mathcal{O}_{E_2,(v)}$ implies, using (3.5.3), that $\mathcal{M} \otimes \mathcal{O}_{E_1,(w)}$ has the extension property over $\mathcal{O}_{E_1,(w)}$, hence α_E extends to a morphism

$$\alpha\colon\ S \longrightarrow N \hookrightarrow \mathcal{M} \otimes \mathcal{O}_{E_1,(w)}\,.$$

Now fix $(K_1^p, K_2^p) \in \mathcal{K}$, and set

$$\widetilde{S} = \widetilde{S}(K_1^p, K_2^p) := S \underset{N(K_1^p, K_2^p)}{\times} \mathcal{N}(K_1^p, K_2^p) \xrightarrow{\ \rho\ } S\,.$$

Then $\widetilde{S}$ is integral over S, since ρ is a pull-back of the normalization map $\mathcal{N}(K_1^p, K_2^p) \to N(K_1^p, K_2^p)$. On the generic fibre, ρ is an isomorphism. Since S is a normal scheme (being an a.t.s.), it follows that ρ is an isomorphism, hence α lifts to $\tilde{\alpha}\colon\ S \to \mathcal{N}$. □

3.20 Remark. Suppose $\mathcal{M}$ is an integral model of a Shimura variety over a d.v.r. $\mathcal{O}$. We will say that $\mathcal{M}$ has the extended extension property (e.e.p.), if it satisfies the condition that for every $S = \mathrm{Spec}(\mathcal{O}_1)$ with $\mathcal{O} \subset \mathcal{O}_1$ a faithfully flat extension of d.v.r., setting $F = \mathrm{Frac}(\mathcal{O}_1)$, every morphism $\alpha_F\colon \mathrm{Spec}(F) \to \mathcal{M}_F$ over $\mathcal{O}$ extends to an $\mathcal{O}$-morphism $\alpha\colon\ S \to \mathcal{M}$.

It follows from the Néron-Ogg-Shafarevich criterion that the model of the Siegel modular variety as in 3.8 enjoys the e.e.p. Also it is clear that in the situation of 3.19, we have the implication "$\mathcal{M}$ has the e.e.p. $\Rightarrow \mathcal{N}$ has the e.e.p.", if $\mathcal{N}$ is the model constructed in the proof.

3.21 The last step in our strategy is statement (e). So, we consider a pair (G', X^+) defining a connected Shimura variety and an isogeny $\pi\colon G' \twoheadrightarrow G''$. We assume that G' (hence also G'') is unramified over $\mathbb{Q}_p$, so that π extends to an isogeny $\pi\colon \mathcal{G}' \to \mathcal{G}''$ of semi-simple groups over $\mathbb{Z}_{(p)}$.

Let us also assume that there exists an i.c.m. $\mathcal{M}$ of $Sh^0(\mathcal{G}', X^+)_{\widetilde{E}}$ over $\mathcal{O}_{(w)}$, where w is a prime of the field $\widetilde{E}$ (as in 3.16) above p. We want to show that there exists an i.c.m. $\mathcal{N}$ of $Sh^0(\mathcal{G}'', X^+)_{\widetilde{E}}$ over $\mathcal{O}_{(w)}$.

3.21.1 Set

$$\Delta := \operatorname{Ker}\big[\mathcal{G}^{\rm ad}(\mathbb{Z}_{(p)})^{+\wedge} \quad \text{rel. } \tau(\mathcal{G}') \longrightarrow \mathcal{G}^{\rm ad}(\mathbb{Z}_{(p)})^{+\wedge} \quad \text{rel. } \tau(\mathcal{G}'')\big]\,.$$

This is a finite group which acts freely on $Sh^0(\mathcal{G}', X^+)_{\widetilde{E}}$. The canonical morphism $Sh(\pi)\colon Sh^0(\mathcal{G}', X^+)_{\widetilde{E}} \to Sh^0(\mathcal{G}'', X^+)_{\widetilde{E}}$ is a quotient morphism for this action. (Cf. [De3], 2.7.11 (b).) Since $\mathcal{M}$ has the extension property, the action of Δ on $\mathcal{M}_{\widetilde{E}}$ extends uniquely to an action on $\mathcal{M}$. The natural candidate for an i.c.m. of $Sh^0(\mathcal{G}'', X^+)_{\widetilde{E}}$ is the quotient $\mathcal{N} := \mathcal{M}/\Delta$.

3.21.2 Problem. *Consider a faithfully flat extension of d.v.r. $\mathbb{Z}_{(p)} \subseteq \mathcal{O}$. Let Δ be a finite (abstract) group acting on a faithfully flat $\mathcal{O}$-scheme $\mathcal{M}$ which is locally noetherian and formally smooth over $\mathcal{O}$. Assume the action of Δ on the generic fibre of $\mathcal{M}$ is free. Under what further conditions does it follow that the action of Δ on all of $\mathcal{M}$ is free?*

3.21.3 It follows from a result of Edixhoven ([Ed1], Prop. 3.4) that, under the previous assumptions, the action of Δ on all of $\mathcal{M}$ is free if p does not divide the order of Δ. On the other hand, if p does divide $|\Delta|$, then extra assumptions are needed.

Example 1: take $\mathcal{O} = \mathbb{Z}_p[\zeta_p]$, $\mathcal{M} = \operatorname{Spec}(\mathcal{O}[\![x]\!])$ with the automorphism of order p given by $x \mapsto \zeta_p \cdot x - (\zeta_p - 1)$. In this case, the action of $\mathbb{Z}/p\mathbb{Z}$ on the generic fibre is free (note that $x - 1$ is a unit in $\mathcal{O}[\![x]\!]$), but the action on the special fibre is trivial.

In order to avoid examples of this kind, we can add the assumption that $p > 2$ and $e(\mathcal{O}/\mathbb{Z}_{(p)}) < p - 1$. (In the situation where we want to use it, this holds anyway.) That this is not a sufficient condition is shown by the following example that was communicated to us by Edixhoven.

Example 2: write Λ for the $\mathbb{Z}_p$-module $\mathbb{Z}_p \oplus \mathbb{Z}_p[\zeta_p]$, and consider the automorphism of order p given by $(x, y) \mapsto (x, \zeta_p \cdot y)$. This induces a $\mathbb{Z}_p$-linear automorphism of order p on $\mathbb{P}(\Lambda) = \mathbb{P}^{p-1}_{\mathbb{Z}_p}$. On the generic fibre there are (geometrically) p fixed points. On the special fibre there is an $\mathbb{F}_p$-rational line of fixed points. By removing the closure of the fixed points in the generic fibre we obtain a $\mathbb{Z}_p$-scheme $\mathcal{M}$ with a $\mathbb{Z}/p\mathbb{Z}$-action as in 3.21.2, such that the action is *not* free on the special fibre.

3.21.4 Proposition. *In the situation of 3.21, suppose that (i) the action of Δ on $\mathcal{M}$ is free, and (ii) $\mathcal{M}$ has the extended extension property (see 3.20). Then $\mathcal{N} := \mathcal{M}/\Delta$ is an i.c.m. of $Sh^0(\mathcal{G}'', X^+)_{\widetilde{E}}$ over $\mathcal{O}_{(w)}$.*

Proof. Condition (i) implies that $\mathcal{N}$ is a smooth model, so it remains to be shown that it has the extension property. Consider an a.t.s. S over $\mathcal{O}_{(w)}$ plus a morphism $\alpha_{\widetilde{E}}\colon S_{\widetilde{E}} \to \mathcal{N}_{\widetilde{E}}$. Let

$$T_{\widetilde{E}} := (S_{\widetilde{E}} \underset{\mathcal{N}_{\widetilde{E}}}{\times} \mathcal{M}_{\widetilde{E}}) \xrightarrow{\beta_{\widetilde{E}}} \mathcal{M}_{\widetilde{E}}\,,$$

and write T for the integral closure of S in the fraction ring of $T_{\widetilde{E}}$. We have a canonical morphism $\rho\colon T \to S$. If $U \subseteq S$ is an open subscheme such that $\alpha_{\widetilde{E}|U_{\widetilde{E}}}$ extends to $\alpha_U\colon U \to \mathcal{N}$, then $\rho^{-1}(U) \cong U \times_{\mathcal{N}} \mathcal{M}$, so that $\rho^{-1}(U) \to U$ is étale.

We now first consider the special case where $S = \mathrm{Spec}(A)$ for some d.v.r. A which is faithfully flat over $\mathcal{O}_{(w)}$. It then follows from the e.e.p. of $\mathcal{M}$ that $\beta_{\widetilde{E}}$ extends to a morphism $\beta\colon T \to \mathcal{M}$ which is equivariant for the action of Δ. On quotients this gives the desired extension α of $\alpha_{\widetilde{E}}$.

Back to the general case, it follows from the special case, the remarks preceding it and the Zariski-Nagata purity theorem of [SGA1] Exp. X, 3.1, that $\rho\colon T \to S$ is étale, so that $T \in \mathrm{ATS}_{\mathcal{O}_{(w)}}$. This again gives an extension β of $\beta_{\widetilde{E}}$ and, on quotients, an extension α as desired. ☐

3.21.5 For a reductive group G over $\mathbb{Q}$, define δ_G as the degree of the covering $\widetilde{G} \to G^{\mathrm{ad}}$. (In other words: δ_G is the "connectedness index" of the root system of $G_{\overline{\mathbb{Q}}}$.) By definition, δ_G depends only on G^{ad}. We claim that, in the situation of 3.21 and 3.21.1, the order of Δ is invertible in $\mathbb{Z}[1/\delta]$, where $\delta = \delta_{G'} = \delta_{G''}$. To prove this, we need some facts and notations. We write $\rho_1\colon \widetilde{G} \to G'$ and $\rho_2\colon \widetilde{G} \to G''$ for the canonical maps from the universal covering. Writing $\Gamma_1 := \rho_1\widetilde{G}(\mathbb{A}_f^p) \cap \mathcal{G}'(\mathbb{Z}_{(p)})$, $\Gamma_2 := \rho_2\widetilde{G}(\mathbb{A}_f^p) \cap \mathcal{G}''(\mathbb{Z}_{(p)})$, we have

$$\mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+\wedge} \quad \text{rel. } \tau(\mathcal{G}') \ = \rho_1\widetilde{G}(\mathbb{A}_f^p) \underset{\Gamma_1}{*} \mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+\,,$$

and similarly for $\mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^{+\wedge}$ rel. $\tau(\mathcal{G}'')$. (Cf. [De3], (2.1.6.2).)

Write $\mathcal{K} := \mathrm{Ker}(\rho_2 \colon \widetilde{\mathcal{G}} \to \mathcal{G}'')$. We claim there is an exact sequence

$$\mathcal{K}(\mathbb{A}_f^p) \xrightarrow{\ t\ } \Delta \xrightarrow{\ u\ } \Gamma_2/\rho_2\widetilde{\mathcal{G}}(\mathbb{Z}_{(p)})\,. \tag{3.21.6}$$

Here the map t sends an element $g \in \widetilde{\mathcal{G}}$ with $\rho_2(g) = e_{\mathcal{G}''}$ to the element $\rho_1(g) *_{\Gamma_1} e_{\mathcal{G}^{\mathrm{ad}}}$, which obviously lies in Δ. The map u sends an element $x *_{\Gamma_1} y \in \Delta \subset \rho_1\widetilde{G}(\mathbb{A}_f^p) *_{\Gamma_1} \mathcal{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$ to $\pi(x) \bmod \rho_2\widetilde{\mathcal{G}}(\mathbb{Z}_{(p)})$; notice that $x *_{\Gamma_1} y \in \Delta$ means that $(\pi(x), y) = \big(\gamma^{-1}, \mathrm{ad}(\gamma)\big)$ for some $\gamma \in \Gamma_2$. If $x *_{\Gamma_1} y \in \mathrm{Ker}(u)$ then we can take $\gamma = \rho_2(g)$ for some $g \in \widetilde{\mathcal{G}}(\mathbb{Z}_{(p)})$, in which case $x *_{\Gamma_1} y = \big(x \cdot \rho_1(g)\big) *_{\Gamma_1} e_{\mathcal{G}^{\mathrm{ad}}} \in \mathrm{Im}(t)$. This proves the exactness of (3.21.6).

It follows from the definitions that every element of $\mathcal{K}(\mathbb{A}_f^p)$ has a finite order dividing δ. On the other hand, $\Gamma_2/\rho_2\widetilde{\mathcal{G}}(\mathbb{Z}_{(p)})$ is a subgroup of $\mathrm{H}^1_{\mathrm{fppf}}(\mathbb{Z}_{(p)}, \mathcal{K})$, in which again all elements are killed by δ. This proves our claim that $|\Delta| \in \mathbb{Z}[1/\delta]^*$.

For simple groups G, the number δ_G is given by $\delta(\mathrm{A}_\ell) = \ell + 1$, $\delta(\mathrm{B}_\ell) = 2$, $\delta(\mathrm{C}_\ell) = 2$, $\delta(\mathrm{D}_\ell) = 4$, $\delta(\mathrm{E}_6) = 3$, $\delta(\mathrm{E}_7) = 2$. (The other three simple types have $\delta = 1$ but do not occur as part of a Shimura datum.) In particular, we see that $|\Delta|$ is invertible in $\mathbb{Z}[1/6]$ if G does not contain factors of type A_ℓ.

After the technical problems encountered in our discussion of steps (b) and (e), the good news is that we can prove the converse of (e).

3.22 Proposition. *Consider the situation as in the first paragraph of 3.21, and assume that $Sh^0(\mathcal{G}'', X^+)_{\widetilde{E}}$ has an i.c.m. $\mathcal{N}$ over $\mathcal{O}_{(w)}$. Then the normalization $\mathcal{M}$ of $\mathcal{N}$ in the fraction field of $Sh^0(\mathcal{G}', X^+)_{\widetilde{E}}$ is an i.c.m. of $Sh^0(\mathcal{G}', X^+)_{\widetilde{E}}$ over $\mathcal{O}_{(w)}$.*

Proof. First we remark that the action of the group Δ on $\mathcal{M}_{\widetilde{E}}$ extends to an action on $\mathcal{M}$ and that $\mathcal{M}/\Delta \xrightarrow{\sim} \mathcal{N}$. (We have a map $\mathcal{M}/\Delta \to \mathcal{N}$ which is an isomorphism on generic fibres; now use that $\mathcal{N}$ is normal.) We claim that the action of Δ on $\mathcal{M}$ is free. On the generic fibre we know this. The important point now is that the purity theorem applies, so that possible fixed points must occur in codimension 1. So, suppose Δ has fixed points. Without loss of generality we may assume that Δ is cyclic of order p (cf. 3.21.3). Restricting to a suitable open part $\mathrm{Spec}(A) \subset \mathcal{M}$, we then obtain a non-trivial automorphism of order p of the $\mathcal{O}_{(w)}$-module A which (using purity and the fact that the action is free on the generic fibre) is the identity modulo p. But now we have the following fact from algebra, probably well-known and in any case not difficult to prove: if R is a principal ideal domain, $p > 2$ a prime number with $(p) \neq R$, M a flat R-module, and α an R-module automorphism of M with $\alpha^p = \mathrm{id}_M$ and $(\alpha \bmod p\colon M/pM \to M/pM) = \mathrm{id}_{M/pM}$, then $\alpha = \mathrm{id}_M$. Applying this fact we obtain a contradiction, and it follows that $\mathcal{M}$ is a smooth model.

For the extension property, consider an $S \in \mathrm{ATS}_{\mathcal{O}_{(w)}}$ and a morphism $\alpha_{\widetilde{E}}\colon S_{\widetilde{E}} \to \mathcal{M}_{\widetilde{E}}$. The projection $\beta_{\widetilde{E}}\colon S_{\widetilde{E}} \to \mathcal{N}_{\widetilde{E}}$ of $\alpha_{\widetilde{E}}$ to $\mathcal{N}$ extends to a morphism $\beta\colon S \to \mathcal{N}$. Set $T := S \times_{\mathcal{N}} \mathcal{M}$, then $T \to S$ is a finite étale Galois covering with group Δ. The section $T_{\widetilde{E}} \leftarrow S_{\widetilde{E}}$ on the generic fibres (corresponding to $\alpha_{\widetilde{E}}$) therefore extends to a section on all of S (recall that S and T are flat over $\mathcal{O}_{(w)}$ and normal), which means that $\alpha_{\widetilde{E}}$ extends to a morphism α. □

Combining all the results in this section, we arrive at the following conclusion.

3.23 Corollary. *Fix a prime number $p > 2$. Let (H, Y) be a Shimura datum of pre-abelian type with $p \nmid \delta_H$, and let v be a prime of $E(H, Y)$ above p. Suppose that for each simple factor $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ of the adjoint datum $(H^{\mathrm{ad}}, Y^{\mathrm{ad}})$, there exist:*

(i) a Shimura datum (G, X) covering $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$,

(ii) a closed immersion $i\colon (G, X) \hookrightarrow (\mathrm{CSp}_{2g}, \mathfrak{H}_g^{\pm})$,

(iii) a prime w of $E(G, X)$ such that v and w restrict to the same prime of $E(G^{\mathrm{ad}}, X^{\mathrm{ad}})$,

(iv) a hyperspecial subgroup $K_p \subset G(\mathbb{Q}_p)$ with $i(K_p) \subseteq \mathrm{CSp}_{2g}(\mathbb{Z}_p)$,

such that the normal model $\mathcal{N}$ of $Sh_{K_p}(G, X)$ constructed in 3.19 is a formally smooth $\mathcal{O}_{(w)}$-scheme. Then for every hyperspecial subgroup $L_p \subset H(\mathbb{Q}_p)$ there exists an integral canonical model of $Sh_{L_p}(H, Y)$ over $\mathcal{O}_{(v)}$.

3.24 Remark. In this section, we have tried to follow the strategy of [De3] very closely, adapting results to the p-adic context whenever possible. We wish to point out that our presentation of the above material is very different from the treatment in Vasiu's paper [Va2]. In particular, our definitions are different (see 3.9), and models of connected Shimura varieties (which play a central role in our discussion) do not appear in [Va2]. Vasiu claims 3.23 (using his definitions) *without* the condition that $p \nmid \delta_H$. We were not able to understand his proof of this (in which one step is postponed to a future publication). It seems to us that at several points the arguments are incomplete, and that Vasiu's proof furthermore contains some arguments which are not correct as they stand.

3.25 Remark. We should mention Morita's paper [Mor]. (See also Carayol's paper [Ca].) Of particular interest, in connection with the material discussed in this section, are the following two aspects.

(i) Morita proves that certain Shimura varieties (of dimension 1) have good reduction by relating them to other Shimura varieties which are of Hodge type. (The example is classical—see §6, "Modèles étranges" in Deligne's Bourbaki paper [De3].) In Morita's method of proof we recognize several results that have reappeared in this section in an abstract and somewhat more general form.

(ii) The Shimura varieties in question ($M_0' = M'_{\mathbb{Z}_p^*\times K_\mathfrak{p}}(G', X')$ in the notations of [Ca]) are shown to have good reduction at certain primes $\mathfrak{p}$ of the reflex field. This includes cases where the group $G'_{\mathbb{Q}_p}$ in question is ramified. Thus we see that good reduction is possible also if the group K_p (the "level at p") is not hyperspecial.

§4 Deformation theory of p-divisible groups with Tate classes

In the next section, we will try to approach the smoothness problem appearing in 3.23 using deformation theory. The necessary technical results are due to Faltings and are the subject of the present section. Here we work out some details of a series of remarks in Faltings's paper [Fa3].

4.1 To begin with, let us recall a result from crystalline Dieudonné theory. For an exposition of this theory, we refer to the work of Berthelot-Messing and Berthelot-Breen-Messing ([BM1], [BBM], [BM3]); some further results can be found in [dJ].

Let k be a perfect field of characteristic $p > 2$, let $W = W(k)$ be its ring of infinite Witt vectors, and write σ for the Frobenius automorphism of W. We will be working with rings of the form $A = W[\![t_1, \ldots, t_n]\!]$. For such a ring, set $A_0 = k[\![t_1, \ldots, t_n]\!]$, $\mathfrak{m} = \mathfrak{m}_A = (p, t_1, \ldots, t_n)$, $J = J_A = (t_1, \ldots, t_n)$, let $e_A\colon A \to W$ be the zero section, and define a Frobenius lifting ϕ_A by $\phi_A = \sigma$ on W, $\phi_A(t_i) = t_i^p$.

With these notations we have the following fact: the category of p-divisible groups over $\mathrm{Spf}(A)$ is equivalent to that of p-divisible groups over $\mathrm{Spec}(A)$ (see [dJ], Lemma 2.4.4), and these categories are equivalent to the category of 4-tuples $(M, \mathrm{Fil}^1, \nabla, F)$, where

—M is a free A-module of finite rank,

—$\mathrm{Fil}^1 \subset M$ is a direct summand,

—$\nabla\colon M \to M \otimes \hat{\Omega}^1_{A/W}$ is an integrable, topologically quasi-nilpotent connection,

—$F\colon M \to M$ is a ϕ_A-linear horizontal endomorphism,

such that, writing $\widetilde{M} = M + p^{-1}\mathrm{Fil}^1$,

$$\text{(4.1.1)} \qquad F \quad \text{induces an isomorphism} \quad F\colon \phi_A^*\widetilde{M} \xrightarrow{\sim} M\,, \quad \text{and}$$

$$\text{(4.1.2)} \quad \mathrm{Fil}^1 \otimes_A A_0 = \mathrm{Ker}(F \otimes \mathrm{Frob}_{A_0}\colon M \otimes_A A_0 \longrightarrow M \otimes_A A_0).$$

(Here, as often in the sequel, we write ϕ_A^*- for $-\otimes_{A,\phi_A} A$.) Notice that (4.1.1) implies that there is a ϕ_A^{-1}-linear endomorphism $V\colon M \to M$ such that

$$\text{(4.1.3)} \qquad F \circ V = p \cdot \mathrm{id}_M = V \circ F\,.$$

This equivalence is an immediate corollary to [Fa2], Thm. 7.1. One also obtains it by combining the following results:

—the description of a Dieudonné crystal on $\mathrm{Spf}(A_0)$ in terms of a 4-tuple (M, ∇, F, V), see [BBM], [BM3], [dJ],

—the Grothendieck-Messing deformation theory of p-divisible groups, see [Me],

—the results of de Jong, saying that over formal $\mathbb{F}_p$-schemes satisfying certain smoothness conditions, the crystalline Dieudonné functor for p-divisible groups is an equivalence of categories, see [dJ].

If $(M, \mathrm{Fil}^1, \nabla, F)$ corresponds to a p-divisible group $\mathcal{H}$ over A then

$$\mathrm{rk}_A(M) = \mathrm{height}(\mathcal{H}) \quad , \quad \mathrm{rk}_A(\mathrm{Fil}^1) = \dim(\mathcal{H})\,.$$

4.2 The 4-tuples $(M, \mathrm{Fil}^1, \nabla, F)$ form a category $\mathsf{MF}^\nabla_{[0,1]}(A)$ similar to the category $\mathcal{MF}^\nabla_{[0,1]}(A)$ as in [Fa2], except that we are working here with p-adically complete, torsion-free modules, rather than with p-torsion modules. More generally, let us write $\mathsf{MF}^\nabla_{[a,b]}(A)$ for the category of 4-tuples $(M, \mathrm{Fil}^\bullet, \nabla, F)$, where M and ∇ are as in 4.1, where F is a ϕ_A-linear endomorphism $M \otimes A[1/p] \to M \otimes A[1/p]$, and where $\mathrm{Fil}^\bullet$ is a descending filtration of M such that

$$\mathrm{Fil}^{i+1} \text{ is a direct summand of } \mathrm{Fil}^i\,, \quad \mathrm{Fil}^a M = M\,, \quad \mathrm{Fil}^{b+1} M = 0\,,$$

and such that, writing $\widetilde{M} = \sum_{i=a}^b p^{-i}\mathrm{Fil}^i M$,

$$F \text{ induces an isomorphism } F\colon \phi_A^*\widetilde{M} \xrightarrow{\sim} M\,.$$

The arguments of [Fa2], Thm. 2.3, show that for $p > 2$ and $0 \leq b - a \leq p-1$, the category $\mathsf{MF}^{\nabla}_{[a,b]}(A)$ is independent, up to canonical isomorphism, of the chosen Frobenius lifting ϕ_A. Every morphism in $\mathsf{MF}^{\nabla}_{[a,b]}(A)$ (the definition of which, we hope, is clear) is strictly compatible with the filtrations (cf. [Wi], Prop. 1.4.1(i), in which the subscript "lf" should be replaced by "tf").

For $a' \leq a$ and $b' \geq b$, we have a natural inclusion $\mathsf{MF}^{\nabla}_{[a,b]}(A) \subseteq \mathsf{MF}^{\nabla}_{[a',b']}(A)$. We will write $\mathsf{MF}^{\nabla}_{[\,,\,]}(A)$ for the union of these categories, i.e., $M \in \mathsf{MF}^{\nabla}_{[\,,\,]}(A)$ means that $M \in \mathsf{MF}^{\nabla}_{[a,b]}(A)$ for *some* a and b.

The Tate object $A(-n) \in \mathsf{MF}^{\nabla}_{[n,n]}(A)$ is given by the A-module A with $\nabla = \mathrm{d}$, $\mathrm{Fil}^n = A\{n\} \supset \mathrm{Fil}^{n+1} = (0)$ and $F(a) = p^n \cdot \phi_A(a)$.

4.3 Before we turn to the deformation theory of p-divisible groups, we need to discuss some properties of 4-tuples $(M, \mathrm{Fil}^1, \nabla, F)$ as in 4.1.

4.3.1 The connection ∇ induces a connection $\widetilde{\nabla}$ on $\phi_A^*\widetilde{M}$ (not on $\widetilde{M}$ itself): if $m \in M$ and $\nabla(m) = \sum m_\alpha \otimes \omega_\alpha$, then $\widetilde{\nabla}(m \otimes 1) = \sum (m_\alpha \otimes 1) \otimes \mathrm{d}\phi_A(\omega_\alpha)$. One checks that this gives a well-defined integrable connection $\widetilde{\nabla}$. The horizontality of F can be expressed by saying that $\widetilde{\nabla}$ is the pull-back of ∇ via $F\colon\ \phi_A^*\widetilde{M} \xrightarrow{\sim} M$.

4.3.2 Given (M, Fil^1, F) satisfying (4.1.1) and (4.1.2), there is at most one connection ∇ for which F is horizontal. Indeed, the difference of two such connections ∇ and ∇' is a linear form $\delta \in \mathrm{End}(M) \otimes \hat{\Omega}^1_{A/W}$ satisfying $\mathrm{Ad}(F)(\delta) = \delta$. Here $\mathrm{Ad}(F)(\delta) = (F \otimes \mathrm{id}) \circ \tilde{\delta} \circ F^{-1}$, where $\tilde{\delta} = \widetilde{\nabla} - \widetilde{\nabla}'$. One checks that if $\delta \in J^t\mathrm{End}(M) \otimes \hat{\Omega}^1_{A/W}$, then $\mathrm{Ad}(F)(\delta) \in p \cdot J^{t+1}\mathrm{End}(M) \otimes \hat{\Omega}^1_{A/W}$, so that $\mathrm{Ad}(F)(\delta) = \delta$ implies $\delta = 0$. Similar arguments show that any connection δ for which F is horizontal, is integrable and topologically quasi-nilpotent.

4.3.3 Suppose $A = W[\![t_1, \ldots t_n]\!]$ and $B = W[\![u_1, \ldots u_m]\!]$ are two rings of the kind considered above. Let $f\colon\ A \to B$ be a W-homomorphism. If $\mathcal{H}$ is a p-divisible group over A corresponding to the 4-tuple $\mathbb{D} = (M, \mathrm{Fil}^1_M, \nabla_M, F_M)$, then the pull-back $f^*\mathcal{H}$ corresponds to a 4-tuple $f^*\mathbb{D} = (N, \mathrm{Fil}^1_N, \nabla_N, F_N)$ described as follows:

(i) $N = f^*M := M \otimes_{A,f} B$, $\quad \mathrm{Fil}^1_N = f^*\mathrm{Fil}^1_M$, $\quad \nabla_N = f^*\nabla_M$.

(ii) To describe F_N, we have to take into account that f may not be compatible with the two chosen Frobenius liftings ϕ_A and ϕ_B. First we use the connection ∇_M to construct an isomorphism

$$c = c(\phi_B \circ f, f \circ \phi_A) \colon \phi_B^* f^* M \xrightarrow{\sim} f^* \phi_A^* M \,,$$

which, using multi-index notations

$$\nabla(\partial)^{\underline{i}} = \nabla(\partial t_1)^{i_1} \cdot \dots \cdot \nabla(\partial t_n)^{i_n} \quad , \quad z^{\underline{i}} = z_1^{i_1} \cdot \dots \cdot z_n^{i_n} \quad , \quad \text{etc.}\,,$$

is given (for $m \in M$) by

$$c(m \otimes 1) = \sum_{\underline{i}} \nabla(\partial)^{\underline{i}}(m) \otimes p^{|\underline{i}|} \cdot \frac{z^{\underline{i}}}{\underline{i}!} \,,$$

where $z_i = \big(\phi_B \circ f(t_i) - f \circ \phi_A(t_i)\big)/p$. Then one defines F_N as the composition

$$F_N \colon \quad \phi_B^* N = \phi_B^* f^* M \xrightarrow[\sim]{c} f^* \phi_A^* M \xrightarrow{f^* F_M} f^* M = N \,.$$

4.4 Theorem. (Faltings) *Let $A = W[\![t_1, \dots t_n]\!]$ and consider a p-divisible group $\mathcal{H}$ over A with filtered Dieudonné crystal $\mathbb{D}(\mathcal{H}) = (\mathcal{M}, \mathrm{Fil}^1_{\mathcal{M}}, \nabla_{\mathcal{M}}, F_{\mathcal{M}})$. Write $H = e_A^* \mathcal{H}$, which has Dieudonné module $\mathbb{D}(H) = \big(M, \mathrm{Fil}^1_M, F_M\big) = e_A^*(\mathcal{M}, \mathrm{Fil}^1_{\mathcal{M}}, F_{\mathcal{M}})$. Assume that $\mathcal{H}$ is a versal deformation of H in the sense that the Kodaira-Spencer map*

$$\kappa \colon W\partial t_1 + \dots + W\partial t_n \longrightarrow \mathrm{Hom}_W\big(\mathrm{Fil}^1_M, M/\mathrm{Fil}^1_M\big)$$

is surjective.

Next consider a ring $B = W[\![u_1, \dots u_m]\!]$ and a 3-tuple $\mathbb{E}' = (\mathcal{N}, \mathrm{Fil}^1_{\mathcal{N}}, F_{\mathcal{N}})$ satisfying (4.1.1) and (4.1.2), and such that $\mathbb{E}' \otimes_{B, e_B} W \cong \mathbb{D}(H)$. Then there exists a W-homomorphism $f \colon A \to B$ such that $\mathbb{E}'$ is isomorphic to the pull-back of $(\mathcal{M}, \mathrm{Fil}^1_{\mathcal{M}}, F_{\mathcal{M}})$. In particular, $\mathbb{E}'$ can be completed to a filtered Dieudonné crystal $\mathbb{E}$ by setting $\nabla_{\mathcal{N}} = f^* \nabla_{\mathcal{M}}$, and therefore corresponds to a deformation of H.

Proof. For every W-homomorphism $f_1 \colon A \to B$ there is an isomorphism of filtered B-modules $g_1 \colon f_1^*(\mathcal{M}, \mathrm{Fil}^1_{\mathcal{M}}) \xrightarrow{\sim} (\mathcal{N}, \mathrm{Fil}^1_{\mathcal{N}})$, which is unique up to an element of $\mathrm{Aut}(\mathcal{N}, \mathrm{Fil}^1_{\mathcal{N}})$. The map g_1 induces an isomorphism $\tilde{g}_1 \colon f_1^* \widetilde{\mathcal{M}} \xrightarrow{\sim} \widetilde{\mathcal{N}}$. By induction on $n \geq 1$ we may assume that the two Frobenii

$$F_{\mathcal{N}} \colon \phi_B^* \widetilde{\mathcal{N}} \xrightarrow{\sim} \mathcal{N}$$

and (with $c_1 = c(\phi_B \circ f_1, f_1 \circ \phi_A)$ as in 4.3.3)

$$F'_{\mathcal{N}} \colon \phi_B^* \widetilde{\mathcal{N}} \xrightarrow[\sim]{\phi_B^* \tilde{g}_1^{-1}} \phi_B^* f_1^* \widetilde{\mathcal{M}} \xrightarrow[\sim]{c_1} f_1^* \phi_A^* \widetilde{\mathcal{M}} \xrightarrow[\sim]{f_1^* F_{\mathcal{M}}} f_1^* \mathcal{M} \xrightarrow[\sim]{g_1} \mathcal{N}$$

are congruent modulo J_B^n. (For $n = 1$ this is so by our assumptions.) Because B is J_B-adically complete, it suffices to show that we can modify f_1 and g_1 such that the new $F'_{\mathcal{N}}$ is congruent to $F_{\mathcal{N}}$ modulo J_B^{n+1}.

Consider an $f_2\colon A \to B$ which is congruent to f_1 modulo J_B^n. Notice that $f_1 \circ \phi_A \equiv f_2 \circ \phi_A$ and $\phi_B \circ f_1 \equiv \phi_B \circ f_2$ modulo J_B^{n+1}, so we have canonical isomorphisms

$$\phi_B^* f_1^* \widetilde{\mathcal{M}} \otimes B/J_B^{n+1} \cong \phi_B^* f_2^* \widetilde{\mathcal{M}} \otimes B/J_B^{n+1}$$

and

$$f_1^* \phi_A^* \widetilde{\mathcal{M}} \otimes B/J_B^{n+1} \cong f_2^* \phi_A^* \widetilde{\mathcal{M}} \otimes B/J_B^{n+1}\,.$$

Next we choose an isomorphism $h\colon f_2^*(\mathcal{M}, \mathrm{Fil}^1_{\mathcal{M}}) \xrightarrow{\sim} f_1^*(\mathcal{M}, \mathrm{Fil}^1_{\mathcal{M}})$ which reduces to the canonical isomorphism modulo J_B^n, and we set $g_2 = g_1 \circ h$.

The first important remark is that, given the above identifications, the two maps

$$c_i\colon\ \phi_B^* f_i^* \widetilde{\mathcal{M}} \xrightarrow{\ \sim\ } f_i^* \phi_A^* \widetilde{\mathcal{M}} \qquad (i = 1, 2)$$

are equal modulo J_B^{n+1}. One can check this using the description of the maps c_i given in 4.3.3 and using that $f_i(t_j) \in J_B$.

Write ν for the automorphism of $\mathcal{N}$ such that $F_{\mathcal{N}} = \nu \circ F'_{\mathcal{N}}$. The induction hypothesis gives us that $\nu \equiv \mathrm{id}_{\mathcal{N}} \bmod J^n \cdot \mathrm{End}(\mathcal{N})$. It follows from the previous remarks that we are done if we can choose f_2 and h such that the diagram

(4.4.1)

$$\begin{array}{ccccc}
f_1^* \phi_A^* \widetilde{\mathcal{M}} \otimes B/J^{n+1} & \xrightarrow[\sim]{f_1^* F_{\mathcal{M}}} & f_1^* \mathcal{M} \otimes B/J^{n+1} & \xrightarrow[\sim]{\nu \circ g_1} & \mathcal{N} \otimes B/J^{n+1} \\
\wr\Vert & & & & \Vert \\
f_2^* \phi_A^* \widetilde{\mathcal{M}} \otimes B/J^{n+1} & \xrightarrow[\sim]{f_2^* F_{\mathcal{M}}} & f_2^* \mathcal{M} \otimes B/J^{n+1} & \xrightarrow[\sim]{g_2} & \mathcal{N} \otimes B/J^{n+1}
\end{array}$$

commutes. Note that the diagram is commutative modulo J_B^n and that, given f_2, we can still change h (and consequently g_2) by an element of $\mathrm{Aut}(f_2^* \mathcal{M}, f_2^* \mathrm{Fil}^1_{\mathcal{M}})$.

The composition $g_1^{-1} \circ \nu^{-1} \circ g_2$ induces a W-linear map

$$\xi\colon\ f_1^* \mathrm{Fil}^1_{\mathcal{M}} \otimes B/J \overset{\text{can}}{\cong} f_2^* \mathrm{Fil}^1_{\mathcal{M}} \otimes B/J \longrightarrow f_1^*\big(\mathcal{M}/\mathrm{Fil}^1_{\mathcal{M}}\big) \otimes J^n/J^{n+1}\,,$$

which is independent of the choice of h. Similarly, $f_1^* F_{\mathcal{M}} \circ (f_2^* F_{\mathcal{M}})^{-1}$ induces a W-linear map

$$\eta\colon\ f_1^* \mathrm{Fil}^1_{\mathcal{M}} \otimes B/J \longrightarrow f_1^*\big(\mathcal{M}/\mathrm{Fil}^1_{\mathcal{M}}\big) \otimes J^n/J^{n+1}\,.$$

The assumption that $\mathcal{H}$ is a versal deformation of H now implies that we can choose f_2 such that $\eta = \xi$. This means precisely that we can modify g_2 by something in $\mathrm{Aut}(f_2^*\mathcal{M}, f_2^*\mathrm{Fil}^1_{\mathcal{M}})$ such that the diagram (4.4.1) commutes. This proves the induction step. $\square$

4.5 Let H be a p-divisible group over W, with special fibre H_0. Write $n = \dim(H_0) \cdot \dim(H_0^D)$, and let $A = W[\![t_1, \ldots, t_n]\!]$. The formal deformation functor of H_0 is pro-represented by A (see [Il]), where we may choose the coordinates such that H corresponds to the zero section e_A. Write $\mathcal{H}$ for the universal p-divisible group over A, and let $\mathbb{D}(\mathcal{H}) = (\mathcal{M}, \mathrm{Fil}^1_{\mathcal{M}}, \nabla_{\mathcal{M}}, F_{\mathcal{M}})$ be its filtered Dieudonné crystal. We will use the previous result to give a more explicit description of $\mathbb{D}(\mathcal{H})$.

Let $(M, \mathrm{Fil}^1_M, F_M) = e_A^*\mathbb{D}(\mathcal{H})$ be the filtered Dieudonné module of H. Choose a complement M' for $\mathrm{Fil}^1_M \subseteq M$. Inside the reductive group $\mathrm{GL}(M)$ over W, consider the parabolic subgroup of elements g with $gM' = M'$, and let U be its unipotent radical. Notice that U is (non-canonically) isomorphic to $\mathbb{G}^n_{\mathrm{a},W}$. Let $\widehat{U} = \mathrm{Spf}(B)$ be the formal completion of U along the identity, and choose coordinates $B \cong W[\![u_1, \ldots, u_n]\!]$ such that e_B gives the identity section. Over B we define a filtered Dieudonné crystal $\mathbb{E} = (\mathcal{N}, \mathrm{Fil}^1_{\mathcal{N}}, \nabla_{\mathcal{N}}, F_{\mathcal{N}})$ as follows. We set

$$\mathcal{N} = M \otimes_W B\,, \quad \mathrm{Fil}^1_{\mathcal{N}} = \mathrm{Fil}^1_M \otimes_W B\,, \quad F_{\mathcal{N}} = g \cdot (F_M \otimes \phi_B)\,,$$

where $g\colon \mathcal{N} \xrightarrow{\sim} \mathcal{N}$ is the "universal" automorphism, i.e., the automorphism given by the canonical B-valued point of $U \subset \mathrm{GL}(M)$. At this point we apply the theorem. This gives us a connection $\nabla_{\mathcal{N}}$ and a W-homomorphism $f\colon A \to B$ such that $\mathbb{E} \cong f^*\mathbb{D}(\mathcal{H})$.

We claim that the map f is an isomorphism. Since A and B are formally smooth W-algebras of the same dimension, it suffices for this to show that $\mathbb{E}$ is a versal deformation of $(M, \mathrm{Fil}^1_M, F_M)$. Now we have an isomorphism

(4.5.1)
$$\widetilde{\mathcal{N}} \otimes_{B,\phi_B} B/\phi(J_B) = \big((\widetilde{\mathcal{N}} \otimes_{B,e_B} W) \otimes_{W,\sigma} W\big) \otimes_W B/\phi(J_B) \stackrel{\mathrm{can}}{\cong} (\widetilde{M \otimes_{W,\sigma} W}) \otimes_W B/\phi(J_B)\,.$$

On the left hand term we have the connection $\widetilde{\nabla}$; on the right we take $1 \otimes \mathrm{d}$. It follows easily from the definition of $\widetilde{\nabla}$ that (4.5.1) is horizontal modulo J_B^{p-1}. Composing with the isomorphisms $F_{\mathcal{N}}$ and F_M then gives a horizontal isomorphism

$$\bar{g}\colon \mathcal{N} \otimes_B B/J^{p-1} \xleftarrow{\sim} M \otimes_W B/J^{p-1}\,,$$

which, as is clear from the constructions, is just the reduction modulo J^{p-1} of the automorphism g. Since $p > 2$, it then follows from the choice of U that $\mathbb{E}$ is a versal deformation.

4.6 Our next goal is to redo some of the above constructions for p-divisible groups with given Tate classes. We keep the notations of 4.5. For $r_1, r_2 \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}$, set

$$M(r_1, r_2; s) := M^{\otimes r_1} \otimes (M^*)^{\otimes r_2} \otimes W(s)\,,$$

with its induced structure of an object of $\mathsf{MF}_{[,]}(W)$. We will refer to any direct sum of such objects as a tensor space $T = T(M)$ obtained from M.

We assume given a polarization $\psi\colon\ M \otimes_W M \to W(-1)$, i.e., a morphism in $\mathsf{MF}_{[0,2]}(W)$ which on modules is given by a perfect symplectic form. We let $\mathrm{CSp}(M, \psi)$ act on the Tate twist $W(-1)$ through the multiplier character. Then we consider a closed reductive subgroup $\mathcal{G} \subseteq \mathrm{CSp}(M, \psi)$ such that

(4.6.1) there exists a tensor space T and an element $t \in T$ such that $L = W \cdot t$ is a subobject of T in $\mathsf{MF}_{[,]}(W)$ isomorphic to $W(0)$, and such that $\mathcal{G} \subseteq \mathrm{CSp}(M, \psi)$ is the stabilizer of the line L.

4.7 Remark. In [Fa3], Faltings gives an argument which shows that, for (4.6.1) to hold, it suffices if the Lie algebra $\mathfrak{g} \subset \mathrm{End}(M)$ is a subobject in $\mathsf{MF}_{[-1,-1]}(W)$. Since, by assumption, $\mathcal{G}$ is a smooth group, an easy argument then shows that (4.6.1) is equivalent to the condition that $\mathfrak{g}$ is stable under the Frobenius on $\mathrm{End}(M)$.

4.8 We can now construct a "universal" deformation of H such that the Tate class t remains a Tate class. The procedure is essentially the same as in 4.5. First, however, we have to find the right unipotent subgroup $U_{\mathcal{G}} \subset \mathcal{G}$. For this we use the canonical decomposition $M = M^0 \oplus \mathrm{Fil}^1$ defined by Wintenberger in [Wi]. The corresponding cocharacter

$$\mu\colon\ \mathbb{G}_{\mathrm{m},W} \longrightarrow \mathrm{GL}(M)\,; \quad \mu(z) = \begin{cases} \mathrm{id} & \text{on} \quad M^0 \\ z^{-1}\cdot \mathrm{id} & \text{on} \quad \mathrm{Fil}^1 M \end{cases}$$

factors through $\mathcal{G}$. In 4.5 we now take $M' = M^0$, and we set $U_{\mathcal{G}} = U \cap \mathcal{G}$. Then $U_{\mathcal{G}}$ is a smooth unipotent subgroup of $\mathcal{G}$, whose Lie algebra is a complement of $\mathrm{Fil}^0\mathfrak{g} \subseteq \mathfrak{g}$. (Here we use that $\mathcal{G}$ is reductive.)

Taking formal completions of $U_{\mathcal{G}} \hookrightarrow U$ along the origin corresponds, on rings, to a surjection

$$B = W[\![u_1, \ldots, u_n]\!] \twoheadrightarrow C = W[\![v_1, \ldots, v_q]\!]\,,$$

where $q = \dim_W(\mathfrak{g}/\mathrm{Fil}^0\mathfrak{g})$. We set

$$\mathcal{P} = M \otimes_W C\,, \quad \mathrm{Fil}^1_{\mathcal{P}} = \mathrm{Fil}^1_M \otimes_W C\,, \quad F_{\mathcal{P}} = h \cdot (F_M \otimes \phi_C)\,,$$

where $h\colon\ \mathcal{P} \xrightarrow{\sim} \mathcal{P}$ is the universal element of $U_{\mathcal{G}}$. As in 4.5, applying Theorem 4.4 gives a connection $\nabla_{\mathcal{P}}$ and a homomorphism $f_{\mathcal{G}}\colon\ A \to C$ such that

$$\mathbb{E}_{\mathcal{G}} := (\mathcal{P}, \mathrm{Fil}^1_{\mathcal{P}}, \nabla_{\mathcal{P}}, F_{\mathcal{P}})$$

is the Dieudonné crystal of a deformation $\mathcal{H}_{\mathcal{G}} = f_{\mathcal{G}}^*\mathcal{H}$ of H over $\mathrm{Spf}(C) = \widehat{U_{\mathcal{G}}}$.

From the fact that $F_{\mathcal{P}}$ is horizontal w.r.t. $\nabla_{\mathcal{P}}$, one can derive that $\nabla_{\mathcal{P}}$ is of the form $\nabla_{\mathcal{P}} = \mathrm{d} + \beta$ with $\beta \in \mathfrak{g}_C \otimes \hat{\Omega}^1_{C/W} \subseteq \mathrm{End}(M) \otimes \hat{\Omega}^1_{C/W}$. It follows that if we extend the space T to an object $\mathcal{T} \in \mathsf{MF}^{\nabla}_{[a,b]}(C)$ by applying to $\mathcal{P} = M \otimes C$ the same linear algebra construction as was used to obtain T from M, then the line $L \subset T$ extends to a subobject $\mathcal{L} \subset \mathcal{T}$ in $\mathsf{MF}^{\nabla}_{[a,b]}(C)$.

To finish, let us prove that, conversely, every deformation of H over a ring $D = W[\![x_1, \ldots, x_r]\!]$ such that the tensor t deforms as a Tate class (i.e., the line $L \subset T$ extends to an inclusion $\mathcal{L} \subset \mathcal{T}$ in $\mathsf{MF}^{\nabla}_{[a,b]}(D)$), can be obtained by pullback from $\mathcal{H}_{\mathcal{G}}$. The map $\mathrm{End}(M) \to T/L$ obtained by sending $\alpha \in \mathrm{End}(M)$ to the evaluation at t of the induced $T(\alpha) \in \mathrm{End}(T)$ is a morphism in $\mathsf{MF}_{[\,,\,]}(W)$, hence strictly compatible with the filtrations. It follows that

(4.8.1) $\qquad$ if $T(\alpha)$ maps L into Fil^0T, then $\alpha \in \mathrm{Fil}^0\mathrm{End}(M) + \mathfrak{g}$.

To prove the claim we can now follow the same reasoning as in 4.4, making use of (4.8.1). Alternatively, it follows from what we did in 4.5 that our deformation of H over D is obtained by pulling back the universal deformation $\mathcal{H}_B$ over B via a homomorphism $\pi\colon\ B \to D$. It then suffices to show that $\pi_n\colon\ B/J_B^n \to D/J_D^n$ factors via C/J_C^n for every n. For this we can argue by induction, and because of the way we have chosen $U_{\mathcal{G}}$ and U, the induction step easily follows from (4.8.1). This proves:

4.9 Proposition. *Notations and assumptions as above. We have a formally smooth deformation space* $\widehat{U_{\mathcal{G}}} = \mathrm{Spf}(C) \hookrightarrow \widehat{U}$ *of relative dimension equal to* $\dim_W(\mathfrak{g}/\mathrm{Fil}^0\mathfrak{g})$ *which parametrizes the deformations of* H *such that the horizontal continuation of* t *remains a Tate class.*

§5 Vasiu's strategy for proving the existence of integral canonical models

After our excursion to deformation theory, we return to the problem of the existence of integral canonical models. Our aim in this section is to explain the principal ideas in Vasiu's paper [Va2] (which is a revised version of part of [Va1]). We add that, to our understanding, some technical points are not treated correctly in loc. cit, to the effect that the main conclusions remain conjectural.

5.1 Consider a closed immersion of Shimura data $i\colon (G,X) \hookrightarrow (\mathrm{CSp}_{2g}, \mathfrak{H}_g^{\pm})$. Let v be a prime of $E = E(G,X)$ above $p > 2$, and assume that there is a hyperspecial subgroup $K_p \subset G(\mathbb{Q}_p)$ with $i(K_p) \subset C_p := \mathrm{CSp}_{2g}(\mathbb{Z}_p)$. (In particular, $G_{\mathbb{Q}_p}$ is unramified.) Write $\mathcal{A} := \varprojlim_{p\nmid n} \mathsf{A}_{g,1,n} \otimes \mathbb{Z}_{(p)}$ which, as we have seen, is an i.c.m. of $Sh_{C_p}(\mathrm{CSp}_{2g}, \mathfrak{H}_g^{\pm})$ over $\mathbb{Z}_{(p)}$, and let $\mathcal{N} \twoheadrightarrow N \subset \mathcal{A} \otimes \mathcal{O}_{(v)}$ be the normal integral model constructed in the proof of 3.19 (so $\mathcal{N}$ is the normalization of N). Choose embeddings $\overline{\mathbb{Q}} \subset \overline{E_v} = \overline{\mathbb{Q}}_p \subset \mathbb{C}$, and write $\overline{\mathcal{N}} \to \overline{N} \subset \overline{\mathcal{A}}$ for the base-change of $\mathcal{N}$, N and $\mathcal{A}$ to $\overline{W} := W\big(\overline{\kappa(v)}\big)$. Let $\tilde{x}_0 \in \overline{\mathcal{N}}$ be a closed point mapping to $x_0 \in \overline{N} \subset \overline{\mathcal{A}}$.

We would like to show that $\overline{\mathcal{N}}$ is formally smooth at $\tilde{x}_0$. If this holds (for all $\tilde{x}_0$) then $\mathcal{N}$ is an i.c.m. of $Sh_{K_p}(G,X)$ over $\mathcal{O}_{(v)}$. To achieve this, we would like to use Prop. 4.9. This is a reasonable idea: over our Shimura variety we have certain Hodge classes, which, by a results of Blasius and Wintenberger, give crystalline Tate classes (in the sense used in §4). The corresponding formal deformation space of p-divisible groups with these Tate classes is formally smooth and has a dimension equal to that of $\overline{\mathcal{N}}$. Arguing along these lines one could hope to prove that $\overline{\mathcal{N}}$ is formally smooth at $\tilde{x}_0$.

We see at least two obstacles in this argument: (i) in §4 we started from a p-divisible group over a ring of Witt-vectors, and (ii) we need a reductive group $\mathcal{G} \subset \mathrm{GL}(M)$ (the generic fibre of which should essentially be our group G). To handle these problems, we will first try to prove the formal smoothness of $\mathcal{N}$ under an additional hypothesis (5.6.1). In rough outline, the argument runs as follows. We start with a lifting of x_0 to a V-valued point of $\overline{N}$, where V is a purely ramified extension of $\overline{W}$. If (X,λ) is the corresponding p.p.a.v. over V then the associated filtered Frobenius crystal can be described as a module M over some filtered ring R_e. We have a closely related ring $\widetilde{R}_e$ which is a projective limit of nilpotent PD-thickenings of V/pV and on which we have sections $i_0\colon \mathrm{Spec}(\overline{W}) \hookrightarrow \mathrm{Spec}(\widetilde{R}_e)$ and $i_\pi\colon \mathrm{Spec}(V) \hookrightarrow \mathrm{Spec}(\widetilde{R}_e)$.

We will construct a deformation $(\widetilde{X}, \widetilde{\lambda})$ over $\widetilde{R}_e$ which corresponds to an $\widetilde{R}_e$-valued point of $\overline{N}$ and such that $i_\pi^*(\widetilde{X}, \widetilde{\lambda}) = (X, \lambda)$. Then $i_0^*(\widetilde{X}, \widetilde{\lambda})$ will give a lifting of x_0 to a $\overline{W}$-valued point of $\overline{N}$, which takes care of problem (i).

To construct $(\widetilde{X}, \widetilde{\lambda})$, we will use the Grothendieck-Messing deformation theory; the essential problem is to find the right Hodge filtration on the module $\widetilde{M} := M \otimes_{R_e} \widetilde{R}_e$. (We remark that the filtration on M cannot be used directly for this purpose: M is filtered free over the filtered ring R_e, whereas the desired Hodge filtration should be a direct summand.) One of the key steps in the argument is to show that the Zariski closure of a certain reductive group $G_{1,R_e[1/p]} \hookrightarrow \mathrm{GL}(M[1/p])$ inside $\mathrm{GL}(M)$ is a reductive group scheme—this will also take care of problem (ii). To achieve this, we have to keep track of Hodge classes on X in various cohomological realizations. At a crucial point we use a result of Faltings which permits to compare étale and crystalline classes with integral coefficients.

Once we have shown that the closure of $G_{1,R_e[1/p]}$ is reductive, an argument about reductive group schemes leads to the definition of the desired Hodge filtration on $\widetilde{M}$. After checking that it has the right properties, this brings us in a situation where the deformation theory of § 4 can be applied. The formal smoothness of $\overline{\mathcal{N}}$ at $\tilde{x}_0$ is then a relatively simple consequence of Prop. 4.9.

Sections 5.2 and 5.5 contain the necessary definitions and a brief description of the crystalline theory with values in R_e-modules. In 5.6 the argument that we just sketched is carried out, resulting in Thm. 5.8.3. What then remains to be shown is that there exist "enough" Shimura data for which (5.6.1) is satisfied. Vasiu's strategy to solve this problem is discussed briefly from 5.9 on.

5.2 Let $\mathcal{O}$ be a d.v.r. with uniformizer π and field of fractions F. Let W be a finite dimensional F-vector space with a non-degenerate symplectic form ψ. Write $F(-n)$ for the vector space F on which $\mathrm{CSp}(W, \psi)$ acts through the nth power of the multiplier character, and consider tensor spaces $W(r_1, r_2; s) := W^{\otimes r_1} \otimes (W^*)^{\otimes r_2} \otimes F(s)$. The fact that $\psi \in W(0, 2; -1)$ is non-degenerate implies that there exists a class $\psi^* \in W(2, 0; 1)$ such that $\langle \psi, \psi^* \rangle = 1 \in F = W(0, 0; 0)$.

Consider a faithfully flat $\mathcal{O}$-algebra R and a free R-module M with a given identification $M \otimes_R R[1/\pi] = W \otimes_F R[1/\pi]$. An element $t \in W(r_1, r_2; s)$ is said to be M-integral if, with the obvious notations, $t \otimes 1$ lies in the subspace $M(r_1, r_2; s)$ of $W(r_1, r_2; s) \otimes_F R[1/\pi]$. For example, ψ and ψ^* are both M-integral precisely if ψ induces a perfect form $\psi_M \colon M \times M \to R$.

If $t \in W(r_1, r_2; s)$ then we shall say that t is of type $(r_1, r_2; s)$ and has degree $r_1 + r_2$. In the sequel we shall often use a notation $T(W)$ for direct sums of spaces $W(r_1, r_2; s)$, and we call such a space a "tensor space obtained from W".

5.3 Definition. Let $G \subset \mathrm{CSp}(W, \psi)$ be a reductive subgroup, and consider a collection $\{t_\alpha\}_{\alpha \in \mathcal{J}}$ of G-invariants in spaces $T_\alpha(W)$. We say that $\{t_\alpha\}_{\alpha \in \mathcal{J}}$ is a well-positioned family of tensors for the group G over the d.v.r. $\mathcal{O}$ if, for every R and M as above, we have

$$\begin{matrix} \psi, \psi^* \text{ and } \{t_\alpha\} \\ \text{are } M\text{-integral} \end{matrix} \Longrightarrow \begin{matrix} \text{the Zariski closure of } G_{R[1/\pi]} \text{ inside } \mathrm{CSp}(M, \psi_M) \\ \text{is a reductive group scheme over } R \end{matrix} .$$

If in addition there exists an $\mathcal{O}$-lattice $M \subset W$ such that ψ, ψ^* and all t_α are M-integral, then we say that $\{t_\alpha\}_{\alpha \in \mathcal{J}}$ is a very well-positioned family of tensors.

5.4 Remarks. (i) One should *not* think of a well-positioned family of tensors as some special family of tensors which cut out the group G (i.e., such that G is the subgroup of $\mathrm{CSp}(W, \psi)$ leaving invariant all t_α), since G may be strictly contained in the group cut out by the t_α. We only use the well-positioned families of tensors to guarantee that certain models of G are again reductive groups.

(ii) For general reductive $G \subset \mathrm{CSp}(W, \psi)$, the main difficulty with this notion is to prove the existence of (very) well-positioned families of tensors. We will come back to this point in 5.9 below.

5.5 Consider a purely ramified extension of d.v.r. $\overline{W} = W(\overline{\mathbb{F}}_p) \subset V$. Write $\mathrm{Frac}(\overline{W}) = K_0 \subset K = \mathrm{Frac}(V)$, fix $K \subset \overline{K} \hookrightarrow \mathbb{C}$, and let $e = e(V/\overline{W}) = [K : K_0]$. Suppose we have a p.p.a.v. (X, λ) over V. We write

$$\begin{aligned} &\mathrm{H}^1_{\mathrm{B},\mathbb{Z}} := \mathrm{H}^1_{\mathrm{B}}(X(\mathbb{C}), \mathbb{Z})\,, \quad \mathrm{H}^1_{\mathrm{B}} := \mathrm{H}^1_{\mathrm{B},\mathbb{Z}} \otimes \mathbb{Q}\,, \\ &\mathrm{H}^1_{\mathrm{dR},K} := \mathrm{H}^1_{\mathrm{dR}}(X_K/K)\,, \quad \mathrm{H}^1_{\mathrm{dR},\mathbb{C}} := \mathrm{H}^1_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C}) = \mathrm{H}^1_{\mathrm{dR},K} \otimes_K \mathbb{C} \quad \text{and} \\ &\mathrm{H}^1_{\text{ét},\mathbb{Z}_p} := \mathrm{H}^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)\,, \quad \mathrm{H}^1_{\text{ét}} := \mathrm{H}^1_{\text{ét},\mathbb{Z}_p} \otimes \mathbb{Q}_p\,. \end{aligned}$$

Let $T_{\mathrm{B}} = T(\mathrm{H}^1_{\mathrm{B}})$ be a tensor space as in 5.2, obtained from $\mathrm{H}^1_{\mathrm{B}}$. We adopt the notational convention that T_{dR}, $T_{\text{ét}}$ etc. stand for "the same" tensor space built from the corresponding first cohomology group $\mathrm{H}^1_{\mathrm{dR}}$, $\mathrm{H}^1_{\text{ét}}$ etc. In each case, $T_?$ naturally comes equipped with additional structures (Hodge structure/filtration/Galois action/$\cdots$), where we interpret $F(n)$ as a Tate twist.

(Cf. [De4], Sect. 1 and Sect. 5.5.8 below.) In each theory, the polarization λ gives rise to a symplectic form on $\mathrm{H}^1_?$, which, if there is no risk of confusion, we denote by ψ without further indices.

5.5.1 Choose a uniformizer π of V, and write $g = T^e + a_{e-1}T^{e-1} + \cdots + a_0$ for its minimum polynomial over K_0, which is an Eisenstein polynomial. The PD-hull (compatible with the standard PD-structure on (p)) of $\overline{W}[\![T]\!] \twoheadrightarrow \overline{W}[\![T]\!]/(g) = V$ is the ring S_e obtained from $\overline{W}[\![T]\!]$ by adjoining all $T^{en}/n!$. Let $I := (p, g) = (p, T^e) \subset S_e$. We define R_e as the p-adic completion of S_e and $\widetilde{R}_e$ as the completion of S_e w.r.t. the filtration by the ideals $I^{[n]}$. (Thus $\widetilde{R}_e$ is the nilpotent PD-hull of $\overline{W}[\![T]\!] \twoheadrightarrow V/pV$.) Notice that these rings only depend on the ramification index e, which justifies the notation. We can identify $\widetilde{R}_e$ (resp. R_e) with the subring of $K_0[\![T]\!]$ consisting of all formal power series $\sum a_n \cdot T^n$ such that all $\lfloor n/e \rfloor! \cdot a_n$ are integral (resp. the coefficients $\lfloor n/e \rfloor! \cdot a_n$ are integral and p-adically convergent to zero for $n \to \infty$). On R_e we have

—a filtration by the ideals $\mathrm{Fil}^n(R_e) := (g)^{[n]}$,

—a σ-linear Frobenius endomorphism $\phi = \phi_{R_e}$ given by $T \mapsto T^p$,

—a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K_0)$, commuting with ϕ and respecting the filtration.

5.5.2 Next we briefly recall the definition of the ring A_{crys} as in [Fo]. (In [Fa3] and [Va2] the notation $B^+(V)$ is used.) Write $\mathcal{O}_C$ for the p-adic completion of the integral closure of $\mathbb{Z}_p$ in $\overline{K} = \overline{\mathbb{Q}}_p$, and let C ($= \mathbb{C}_p$) be its fraction field. Let

$$R_{\mathcal{O}_C} := \varprojlim(\mathcal{O}_C/p\mathcal{O}_C \leftarrow \mathcal{O}_C/p\mathcal{O}_C \leftarrow \cdots \leftarrow \mathcal{O}_C/p\mathcal{O}_C \leftarrow \cdots)\,,$$

where the transition maps are given by $x \mapsto x^p$. It is a perfect ring of characteristic p. Choose a sequence of elements $\pi^{(n)} \in \mathcal{O}_C$ with $\pi^{(1)} = \pi$ (the chosen uniformizer of V) and $(\pi^{(m+1)})^p = \pi^{(m)}$, and set $\underline{\pi} = \big(\pi^{(1)} \bmod p, \pi^{(2)} \bmod p, \ldots\big) \in R_{\mathcal{O}_C}$. There is a surjective homomorphism $\theta\colon W(R_{\mathcal{O}_C}) \twoheadrightarrow \mathcal{O}_C$ whose kernel is the principal ideal generated by $\xi := g([\underline{\pi}])$, where $[\underline{\pi}]$ is the Teichmüller representative of $\underline{\pi}$ and where g is the polynomial as in 5.5.1 (see [Fa3], sect. 4).

Define A_{crys} as the p-adic completion of the PD-hull of $W(R_{\mathcal{O}_C}) \twoheadrightarrow \mathcal{O}_C$, compatible with the canonical PD-structure on (p). Then A_{crys} is a $\overline{W}$-algebra which comes equipped with

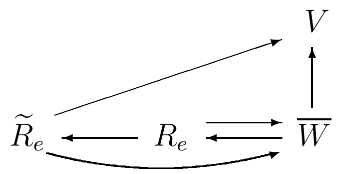
—a filtration by the ideals $\mathrm{Fil}^n(A_{\mathrm{crys}}) := (\xi)^{[n]}$,

—a σ-linear Frobenius endomorphism $\phi = \phi_{A_{\mathrm{crys}}}$,

—a continuous action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K_0)$ commuting with ϕ and respecting the filtration.

There is a $\mathbb{Z}_p$-linear homomorphism $\mathbb{Z}_p(1) \hookrightarrow \mathrm{Fil}^1(A_{\mathrm{crys}})$. We let β (called t in [Fo]) denote the image of a generator of $\mathbb{Z}_p(1)$; we have $\phi(\beta) = p \cdot \beta$.

5.5.3 We have ring homomorphisms



where the sections $R_e \to \overline{W}$ and $\widetilde{R}_e \to \overline{W}$ are given by $T \mapsto 0$, and where $\widetilde{R}_e \to V$ is given by $T \mapsto \pi$. Also we have a homomorphism

$$\iota\colon\ R_e \hookrightarrow A_{\mathrm{crys}}$$

given by $T \mapsto [\underline{\pi}]$ (hence $g \mapsto \xi$). This map is strictly compatible with the filtrations and induces ([Fa3], sect. 4) an isomorphism $\mathrm{gr}^{\bullet}(R_e) \otimes_{\overline{W}} \mathcal{O}_C \xrightarrow{\sim} \mathrm{gr}^{\bullet}(A_{\mathrm{crys}})$. Also, ι is compatible with the Frobenii, but in general *not* with the Galois-actions.

5.5.4 Let H be a p-divisible group over V. Its Dieudonné crystal can be described ([BBM], Thm. 1.2.7) as a free R_e-module $M = M(H)$ of finite rank with an integrable, topologically nilpotent connection ∇ on M as a $\overline{W}[\![T]\!]$-module. On M we have

—a filtration by R_e-submodules $\mathrm{Fil}^{\bullet}_M \subset M$,

—a ϕ_{R_e}-linear horizontal endomorphism F,

such that

—there exists a basis $m_1^{(0)}, \ldots, m_{r_0}^{(0)}, m_1^{(1)}, \ldots, m_{r_1}^{(1)}$ such that $\mathrm{Fil}^j M = \sum_{a+b=j} \mathrm{Fil}^a R_e \cdot m_i^{(b)}$,

—the connection satisfies Griffiths transversality,

—F is divisible by p^j on $\mathrm{Fil}^j M$, and we can choose the basis $\{m_i^{(j)}\}$ as above in such a way that the elements $F(m_i^{(j)})/p^j$ form a new R_e-basis of M.

(Modules M with these additional structures are the objects of a category $\mathsf{MF}^{\nabla}_{[0,1]}(V)$, analogous to the categories considered in §4; see [Fa3], §3.)

The Dieudonné module of the special fibre $H_0 := H \otimes_V \overline{\mathbb{F}}_p$ is a free $\overline{W}$-module M_0 with a σ-linear Frobenius endomorphism F_0. We have a canonical isomorphism of Frobenius crystals

$$M_0 \cong M \otimes_{R_e} R_e/T\cdot R_e\,. \tag{5.5.5}$$

(Recall that $R_e/T\cdot R_e \xrightarrow{\sim} \overline{W}$.) On the other hand, the reduction $(H \bmod p)$ on $\mathrm{Spec}(V/pV)$ is isogenous, via some power of Frobenius, to $H_0 \otimes_{\overline{\mathbb{F}}_p} V/pV$, so that

$$M \otimes_{R_e} R_e[1/p] \cong M_0 \otimes_{\overline{W}} R_e[1/p] \tag{5.5.6}$$

as Frobenius crystals.

Although the homomorphism ι from 5.5.3 is in general not compatible with the Galois actions, there is a canonical action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ on $M \otimes_{R_e,\iota} A_{\mathrm{crys}}$. (Here one uses that M is a crystal over R_e, see [Fa3], §4.) We also define

$$\mathrm{H}^1_{\text{ét},\mathbb{Z}_p}(H) := T_p(H)^*\,,$$

which is a free $\mathbb{Z}_p$-module with $\mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$-action.

5.5.7 Theorem. (Faltings, [Fa3]) *There is a functorial injection*

$$\rho\colon\ M(H)\otimes_{R_e} A_{\mathrm{crys}} \hookrightarrow \mathrm{H}^1_{\text{ét},\mathbb{Z}_p}(H)\otimes_{\mathbb{Z}_p} A_{\mathrm{crys}}\,,$$

which after extension of scalars to $B_{\mathrm{crys}} := A_{\mathrm{crys}}[1/\beta]$, and using the isomorphism $M\otimes_{R_e} R_e[1/p] \cong M_0 \otimes_{\overline{W}} R_e[1/p]$ gives back Faltings's comparison isomorphism $\tilde{\rho}\colon M_0\otimes_{\overline{W}} B_{\mathrm{crys}} \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(H)\otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}$ of [Fa2]. The map ρ is compatible with the Frobenii, filtrations and Galois actions on both sides. Its cokernel is annihilated by $\beta\in A_{\mathrm{crys}}$.

5.5.8 We shall try to be precise about Tate twists and polarization forms. We have

—$\mathbb{Z}_{\mathrm{B}}(1) := 2\pi i\cdot\mathbb{Z}\subset\mathbb{C}$, with H.S. purely of type $(-1,-1)$,

—$K_{\mathrm{dR}}(1) := K$ with filtration $\mathrm{Fil}^{-1} = K \supset \mathrm{Fil}^0 = (0)$ (similarly for other fields than K),

—$\mathbb{Z}_p(1) := \varprojlim \mu_{p^n}(\overline{K})$ as $\mathrm{Gal}(\overline{K}/K)$-module (similarly for other fields).

Fixing $i\in\mathbb{C}$ with $i^2=-1$ we have generators $2\pi i$ for $\mathbb{Z}_{\mathrm{B}}(1)$ and 1 for $K_{\mathrm{dR}}(1)$. Also, the choice of i determines a generator $\big(\exp(2\pi i/p^n)\big)_{n\in\mathbb{N}}$ for $\mathbb{Z}_p(1)$ over $\mathbb{C}$. Via the chosen embedding $\overline{K}\hookrightarrow\mathbb{C}$ this gives a generator ζ for $\mathbb{Z}_p(1)$ over $\overline{K}$. We have comparison isomorphisms (over $\mathbb{C}$)

$$\mathbb{Z}_{\mathrm{B}}(1)\otimes_{\mathbb{Z}}\mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}_p(1) \quad\text{and}\quad \mathbb{Z}_{\mathrm{B}}(1)\otimes_{\mathbb{Z}}\mathbb{C} \xrightarrow{\sim} \mathbb{C}_{\mathrm{dR}}(1)\,,$$

(see [De4], Sect. 1) mapping generators to generators.

If (X,λ) is a p.p.a.v. over $\mathbb{C}$ then λ gives rise to a perfect symplectic form $\psi_{\mathrm{B}}\colon \mathrm{H}^1_{\mathrm{B}} \times \mathrm{H}^1_{\mathrm{B}} \longrightarrow \mathbb{Z}_{\mathrm{B}}(-1)$. For de Rham and étale cohomology we have an analogous statement, and the various forms $\psi_?$ are compatible via the comparison isomorphisms. Using the chosen generators for Tate objects, we can view the forms $\psi_?$ as "ordinary" bilinear forms with values in the coefficient ring corresponding to $? \in \{\mathrm{B}, \mathrm{dR}, \text{ét}\}$. This is consistent with our usage of the notation $F(n)$ in 5.2.

For crystalline cohomology (or Dieudonné modules) with values in R_e, the Tate twist is given by

—$R_e(-1) := R_e$ with filtration $\mathrm{Fil}^j\big(R_e(-1)\big) = \mathrm{Fil}^{j-1}R_e$ and Frobenius $F = p\cdot\phi_{R_e}$.

(This should be thought of as an object of a category $\mathsf{MF}(V)$, see [Fa3]. We have $R_e(-1) \cong M(\widehat{\mathbb{G}}_{\mathrm{m}})$.) The generator $\zeta \in \mathbb{Z}_p(1)$ over $\overline{K}$ determines a generator ζ^* of $\mathbb{Z}_p(-1)$ and an element $\beta \in A_{\mathrm{crys}}$ as in 5.5.2 (see [Fo], 1.5.4). On Tate twists, the comparison map of Thm. 5.5.7 is (after a suitable normalization) the map $\delta\colon\ R_e(1)\otimes_{R_e} A_{\mathrm{crys}} \xrightarrow{\sim} \mathbb{Z}_p(1)\otimes_{\mathbb{Z}_p} A_{\mathrm{crys}}$ with $1\otimes 1 \mapsto \zeta\otimes\beta$. Here we see a factor β entering. In other words: if (X,λ) is a p.p.a.v. over V with associated p-divisible group $H = X[p^\infty]$, then λ gives polarization forms

$$\psi_{\text{ét}}\colon\ \mathrm{H}^1_{\text{ét},\mathbb{Z}_p} \times \mathrm{H}^1_{\text{ét},\mathbb{Z}_p} \longrightarrow \mathbb{Z}_p(-1) \quad\text{and}\quad \psi_{\mathrm{crys}}\colon\ M(H)\times M(H) \longrightarrow R_e(-1)\,,$$

so that under the map ρ from 5.5.7 we have $\delta\circ(\psi_{\mathrm{crys}}\otimes 1) = (\psi_{\text{ét}}\otimes\beta)\circ(\rho\times\rho)$.

5.6 *Vasiu's strategy—first part.* We return to the situation considered in 5.1. We shall first try to prove the formal smoothness of $\overline{\mathcal{N}}$ under the following assumption. Here we recall that we write CSp_{2g} for the Chevalley group scheme $\mathrm{CSp}(\mathbb{Z}^{2g},\psi)$, where ψ is the standard symplectic form on $\mathbb{Z}^{2g}$.

(5.6.1) There is a collection of tensors $\{t_\alpha \in (\mathbb{Z}^{2g}_{(p)})(r_\alpha, r_\alpha; 0)\}_{\alpha\in\mathcal{J}_1}$ of degrees $2r_\alpha \leq 2(p-2)$ such that this collection is very well-positioned over the d.v.r. $\mathbb{Z}_{(p)}$ for the group G (considered as a subgroup of $\mathrm{CSp}_{2g,\mathbb{Q}}$ via the given closed embedding i).

Also, we shall consider a larger collection $\{t_\alpha\}_{\alpha\in\mathcal{J}}$ (with $\mathcal{J}_1 \subset \mathcal{J}$) of tensors which, together with the tensor ψ, cut out the group G. (The t_α with $\alpha \in \mathcal{J}\setminus\mathcal{J}_1$ again of types $(r_\alpha, r_\alpha, 0)$ but not neccesarily $\mathbb{Z}^{2g}_{(p)}$-integral, nor of degree $\leq 2(p-2)$.)

5.6.2 Consider triplets $(X_{\mathbb{C}}, \lambda_{\mathbb{C}}, \theta^p)$ consisting of a g-dimensional p.p.a.v. over $\mathbb{C}$ with a compatible system of Jacobi level n structures for all n with $p \nmid n$ (which we represent by the single symbol θ^p). The modular interpretation

$$Sh_{C_p}(\mathrm{CSp}_{2g,\mathbb{Q}}, \mathfrak{H}_g^{\pm})(\mathbb{C}) \xrightarrow{\sim} \{(X_{\mathbb{C}}, \lambda_{\mathbb{C}}, \theta^p)\}/\cong$$

is given as follows. If $(h, \gamma) \in \mathfrak{H}_g^{\pm} \times \mathrm{CSp}_{2g}(\mathbb{A}_f)$, then we can view γ as an isomorphism $\gamma\colon \mathbb{Q}^{2g} \otimes_{\mathbb{Q}} \mathbb{A}_f \xrightarrow{\sim} \hat{\mathbb{Z}}^{2g} \otimes_{\hat{\mathbb{Z}}} \mathbb{A}_f$. For $X_{\mathbb{C}}$ we take the abelian variety determined by the lattice $\Lambda := \mathbb{Q}^{2g} \cap \gamma^{-1}(\hat{\mathbb{Z}}^{2g})$ and the Hodge structure h. There is a unique $q \in \mathbb{Q}^*$ such that $q \cdot \psi$ is the Riemann form of a principal polarization $\lambda_{\mathbb{C}}$ on $X_{\mathbb{C}}$, and the system of level structures is given by the isomorphism $\gamma\colon \Lambda \otimes (\prod_{\ell \neq p} \mathbb{Z}_\ell) \xrightarrow{\sim} \prod_{\ell \neq p} \mathbb{Z}_\ell^{2g}$. One checks that this gives a well-defined bijection as claimed.

By construction of the model $\mathcal{N}$, there exists a purely ramified extension $\overline{W} \subset V$ as in 5.5 such that the closed point $\tilde{x}_0 \in \overline{\mathcal{N}}$ lifts to a V-valued point $x\colon \mathrm{Spec}(V) \to \overline{\mathcal{N}}$. Considered as a point of $\overline{\mathcal{A}}$ it corresponds to a p.p.a.v. with a system of level structures (X, λ, θ^p) over V. We shall use the notations and assumptions of 5.5; in particular we obtain a triplet $(X_{\mathbb{C}}, \lambda_{\mathbb{C}}, \theta^p)$ over $\mathbb{C}$ via base-change over the chosen embedding $V \subset K \hookrightarrow \mathbb{C}$. The corresponding point of $Sh_{C_p}(\mathrm{CSp}_{2g,\mathbb{Q}}, \mathfrak{H}_g^{\pm})(\mathbb{C})$ can be represented by a pair $(h, e) \in \mathfrak{H}_g^{\pm} \times \mathrm{CSp}_{2g}(\mathbb{A}_f^p)$. In particular, we get an identification $\mathrm{H}^1_{\mathrm{B}}(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{Z}_{(p)} \cong \mathbb{Z}_{(p)}^{2g}$. The fact that x factors through $\overline{N}$ now implies that the tensors t_α as in (5.6.1) correspond to Hodge classes $t_{\alpha,\mathrm{B}}$ on $X_{\mathbb{C}}$ which for $\alpha \in \mathcal{J}_1 \subset \mathcal{J}$ are integral w.r.t. the $\mathbb{Z}_{(p)}$-lattice $\mathrm{H}^1_{\mathrm{B},\mathbb{Z}} \otimes \mathbb{Z}_{(p)} \subset \mathrm{H}^1_{\mathrm{B}}$. Notice that ψ gives an isomorphism $(\mathrm{H}^1_{\mathrm{B},\mathbb{Z}})^* \cong \mathrm{H}^1_{\mathrm{B},\mathbb{Z}}(1)$, so that it is no restriction to assume that all t_α live in spaces $(\mathbb{Z}^{2g})(r_\alpha, r_\alpha; 0)$.

By [De4], Prop. 2.9, the de Rham realizations $t_{\alpha,\mathrm{dR}}$ ($\alpha \in \mathcal{J}$) are defined over a finite extension of K. Possibly after replacing V by a finite extension, we may therefore assume that the $t_{\alpha,\mathrm{dR}}$ are defined over K (i.e., they are elements of tensor spaces of the form $\mathrm{H}^1_{\mathrm{dR},K}(r_\alpha, r_\alpha; 0)$). In particular, we obtain a subgroup $G_{1,K} \subset \mathrm{CSp}(\mathrm{H}^1_{\mathrm{dR},K}, \psi)$ such that $G_{1,K} \otimes_K \mathbb{C} = G \otimes_{\mathbb{Q}} \mathbb{C}$.

Write $t_{\alpha,\text{ét}}$ for the (p-adic) étale realization of $t_{\alpha,\mathrm{B}}$, which is an element of some tensor space $T_{\alpha,\text{ét}}$. For $\alpha \in \mathcal{J}_1$, the class $t_{\alpha,\text{ét}}$ is $\mathrm{H}^1_{\text{ét},\mathbb{Z}_p}$-integral. Since the $t_{\alpha,\mathrm{B}}$ are Hodge classes on $X_{\mathbb{C}}$, the $t_{\alpha,\mathrm{dR}}$ and $t_{\alpha,\text{ét}}$ correspond to each other via the p-adic comparison isomorphism. More precisely, we have the following result, which was obtained independently by Blasius (see [Bl]) and Wintenberger. A simplified proof was given by Ogus in [Og].

5.6.3 Theorem. (Blasius, Wintenberger) *Let $\mathcal{O}$ be a complete d.v.r. of characteristic $(0, p)$ with perfect residue field. Set $F = \mathrm{Frac}(\mathcal{O})$, and let*

X_F be an abelian variety over F with good reduction over $\mathcal{O}$. Let $t_{\mathrm{dR}} \in \mathrm{H}^1_{\mathrm{dR}}(r_1, r_2; s)$ and $t_{\text{ét}} \in \mathrm{H}^1_{\text{ét}}(r_1, r_2; s)$ be the de Rham component and the p-adic étale component respectively of an absolute Hodge class on X. Under the comparison isomorphism

$$\gamma\colon\ \mathrm{H}^1_{\mathrm{dR}}(X_F/F) \otimes_F B_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$

we have $\gamma(t_{\mathrm{dR}} \otimes 1) = t_{\text{ét}} \otimes \beta^{-s}$.

We remark that in [Bl] and [Og], this results is only proven under the additional assumption that X_F is obtained via base-change from an abelian variety over a number field. It can be shown that this condition, which appears in the proof of a version of Deligne's "Principle B", is superfluous. In [Va2], Vasiu shows this by using a trick of Lieberman. One can also remark that the "Principle B" is needed only in the situation where we have a family of abelian varieties $\mathfrak{X} \to S$ over a variety S over $\overline{\mathbb{Q}}$, such that X_F occurs as the fibre over an F-valued point of S. (The variety S constructed in [De4], Sect. 6 is a component of a Shimura variety.) In this situation, the arguments given in [Bl], Sect. 3 and [Og], Prop. 4.3 suffice.

We also remark that the factor β^{-s} appears because we choose $K_{\mathrm{dR}}(1) \otimes B_{\mathrm{dR}} \xrightarrow{\sim} \mathbb{Q}_p(1) \otimes B_{\mathrm{dR}}$ to be the map $1 \otimes 1 \mapsto \zeta \otimes \beta^{-1}$. (In [Bl] a different normalization is used.)

5.6.4 Set $H := X[p^\infty]$, where X is as in 5.6.2. Notice that $\mathrm{H}^1_{\text{ét},\mathbb{Z}_p}(H) \cong \mathrm{H}^1_{\text{ét},\mathbb{Z}_p}(X_{\overline{K}})$. There are well-defined F_0-invariants $\tau_{\alpha,\mathrm{crys}} \in M_0[1/p](r_\alpha, r_\alpha; 0)$ such that $\tilde{\rho}(\tau_{\alpha,\mathrm{crys}} \otimes 1) = t_{\alpha,\text{ét}} \otimes 1$. Using (5.5.6), we then obtain horizontal F-invariant classes $t_{\alpha,\mathrm{crys}} \in M[1/p](r_\alpha, r_\alpha; 0)$. Also we have polarization forms ψ_{crys} and $\psi_{\text{ét}}$ as already mentioned in 5.5.8.

We claim that, writing $T_{\alpha,\mathrm{crys}}$ for the tensor spaces obtained from $M = M(H)$, the $t_{\alpha,\mathrm{crys}}$ lie in $\mathrm{Fil}^0(T_{\alpha,\mathrm{crys}}[1/p])$. To see this, we use that $t_{\alpha,\mathrm{crys}}$ is a lifting of $t_{\alpha,\mathrm{dR}}$ in the following sense. By (5.5.5) and the isomorphism $M_0 \otimes_{\overline{W}} K \cong \mathrm{H}^1_{\mathrm{dR},K}$ from [BO], we have $M \otimes_{R_e} K \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR},K}$. Combining this with the isomorphism $R_e(-1) \otimes_{R_e} K \xrightarrow{\sim} K_{\mathrm{dR}}(-1)$ by $1 \otimes 1 \mapsto 1$, we obtain maps $M(H)(r_1, r_2; s) \otimes_{R_e} K \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR},K}(r_1, r_2; s)$. The functoriality of the map ρ in Theorem 5.5.7 implies that $t_{\alpha,\mathrm{crys}} \otimes 1 \mapsto t_{\alpha,\mathrm{dR}}$ and $\psi_{\mathrm{crys}} \otimes 1 \mapsto \psi_{\mathrm{dR}}$. That $t_{\alpha,\mathrm{crys}} \in \mathrm{Fil}^0 T_{\alpha,\mathrm{crys}}$ now follows from the fact that $\mathrm{Fil}^1 M(H)$ is the inverse image of $\mathrm{Fil}^1 \mathrm{H}^1_{\mathrm{dR},K}$ under the map $M(H) \to M(H) \otimes_{R_e} K \cong \mathrm{H}^1_{\mathrm{dR},K}$. In this way we see that ψ_{crys} and the $t_{\alpha,\mathrm{crys}}$ are crystalline Tate classes, in the sense that they are horizontal, in Fil^0 and invariant under Frobenius.

We will be able to exploit assumption (5.6.1) by using the following supplement to Thm. 5.5.7.

5.6.5 Theorem. (Faltings, [Fa3]) *Suppose that $r \leq (p-2)$, and consider Tate classes*

$$t_{\mathrm{crys}} \in M(H)(r,r;0) \otimes_{R_e} R_e[1/p] \quad \textit{and} \quad t_{\text{ét}} \in \mathrm{H}^1_{\text{ét},\mathbb{Z}_p}(H)(r,r;0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\,,$$

with $\rho(t_{\mathrm{crys}} \otimes 1) = (t_{\text{ét}} \otimes 1)$. Then t_{crys} is $M(H)$-integral if and only if $t_{\text{ét}}$ is $\mathrm{H}^1_{\text{ét},\mathbb{Z}_p}(H)$-integral.

It follows from this result that the $t_{\alpha,\mathrm{crys}}$ with $\alpha \in \mathcal{J}_1$ are $M(H)$-integral classes. (Notice that we assumed these classes to have degree $\leq 2(p-2)$, as required in Faltings's theorem.)

5.6.6 We are now ready to prove one of the key steps in the argument. Since at this point we were not able to follow [Va2], we present our own explanation of what is going on.

The tensors $\tau_{\alpha,\mathrm{crys}}$ cut out a subgroup G_1 of $\mathrm{CSp}(M_0[1/p], \psi_{M_0})$. By Thm. 5.6.3 we have $G_1 \otimes_{K_0} K = G_{1,K}$ (where the latter is the group cut out by the $t_{\alpha,\mathrm{dR}}$ that was introduced in 5.6.2 above), so that G_1 is reductive. What we would like to show now is that the Zariski closure $\mathcal{G}_{1,R_e}$ of $G_{1,R_e[1/p]}$ inside $\mathrm{CSp}(M(H), \psi_{\mathrm{crys}})$ is a reductive group scheme over R_e. Here we use the identification (5.5.6) to identify $G_{1,R_e[1/p]}$ as a subgroup of $\mathrm{CSp}(M(H), \psi_{\mathrm{crys}})$.

Obviously, we will try to achieve our goal by using (5.6.1). If we look at Def. 5.3 then we see that we already know (grace to Thm. 5.6.5) that ψ_{crys}, ψ^*_{crys} and the $t_{\alpha,\mathrm{crys}}$ with $\alpha \in \mathcal{J}_1$ are $M(H)$-integral, and therefore it only remains to show that there exists an isomorphism $\mathbb{Q}^{2g} \otimes R_e[1/p] \xrightarrow{\sim} M(H)[1/p]$ such that ψ and the t_α ($\alpha \in \mathcal{J}$) are sent to ψ_{crys} and the $t_{\alpha,\mathrm{crys}}$ respectively. The first step is that we have an isomorphism $\mathbb{Q}^{2g} \otimes \mathbb{Q}_p \xrightarrow{\sim} \mathrm{H}^1_{\text{ét}}$ sending ψ and the t_α to their étale realizations $\psi_{\text{ét}}$ and $t_{\alpha,\text{ét}}$.

Notice that we now only have to consider rings with p inverted. Since the $t_{\alpha,\mathrm{crys}}$ were obtained from the $\tau_{\alpha,\mathrm{crys}}$ using (5.5.6), it suffices (and will actually be easier) to show that there exists an isomorphism $\nu\colon \mathrm{H}^1_{\text{ét}} \otimes_{\mathbb{Q}_p} K_0 \xrightarrow{\sim} M_0[1/p]$ such that $\psi_{\text{ét}} \mapsto \psi_{M_0}$ and $t_{\alpha,\text{ét}} \mapsto \tau_{\alpha,\mathrm{crys}}$. (This isomorphism is of course not required to have any "meaning".)

By what was explained before, we can compare $\mathrm{H}^1_{\text{ét}}$ and $M_0[1/p]$ after extension of scalars to the ring B_{crys}, in such a way that the tensors $t_{\alpha,\text{ét}}$ and the $\tau_{\alpha,\mathrm{crys}}$ correspond. We have to be a little more careful about the polarization forms, since ψ_{crys} and $\psi_{\text{ét}}$ correspond to each other only up to a factor β (see 5.5.8). The ring to work over therefore is $B_{\mathrm{crys}}[\sqrt{\beta}]$, since the factor $\sqrt{\beta}$ allows us to modify the isomorphism $M_0[1/p] \otimes B_{\mathrm{crys}} \cong \mathrm{H}^1_{\mathrm{\acute{e}t}} \otimes B_{\mathrm{crys}}$ in such a way that the forms ψ_{crys} and $\psi_{\mathrm{\acute{e}t}}$ do correspond. This does not affect the tensors t_α, since these are of type $(r_\alpha, r_\alpha; 0)$. In any case, we see that there exists a field extension $K_0 \subset \Omega$ such that the desired comparison isomorphism exists after extension of scalars to Ω. Writing $\mathfrak{V} = (\mathrm{H}^1_{\mathrm{\acute{e}t}}; \psi, \{t_{\alpha,\mathrm{\acute{e}t}}\}_{\alpha\in\mathcal{J}})$ and $\mathfrak{V}' = (M_0[1/p]; \psi_{M_0}, \{\tau_{\alpha,\mathrm{crys}}\}_{\alpha\in\mathcal{J}})$ the torsor $\mathcal{I}som(\mathfrak{V}, \mathfrak{V}')$ is therefore non-empty. Since the automorphism group of the system $\mathfrak{V}$ is precisely $G_{\mathbb{Q}_p}$, the obstruction for finding ν then is a class in $\mathrm{H}^1\big(\mathrm{Gal}(\overline{\mathbb{Q}}_p/K_0), G(\overline{\mathbb{Q}}_p)\big)$. Now the fact that K_0 is a field of dimension ≤ 1, together with [Se2], Thm. 1 in Chap. III, §2.2, proves that the obstruction vanishes, whence the existence of an isomorphism ν as desired. By applying (5.6.1) this gives the following statement.

5.6.7 Proposition. *The Zariski closure $\mathcal{G}_{1,R_e}$ of $G_{1,R_e[1/p]}$ inside the scheme $\mathrm{CSp}(M(H), \psi_{\mathrm{crys}})$ is a reductive group scheme over R_e.*

It will now rapidly become clear why 5.6.7 is important. For this, we set

$$\widetilde{M} := M \otimes_{R_e} \widetilde{R}_e\,, \quad \text{and } M_V := M \otimes_{R_e} V = \mathrm{H}^1_{\mathrm{dR}}(X/V)\,,$$

using the maps $R_e \subset \widetilde{R}_e \to V$ from 5.5.3. Also we set

$$\mathcal{G}_{1,\widetilde{R}_e} := \mathcal{G}_1 \times_{R_e} \widetilde{R}_e\,, \quad \mathcal{G}_{1,V} := \mathcal{G}_1 \times_{R_e} V\,,$$

which are reductive groups over $\widetilde{R}_e$ and V respectively. We write $\mathrm{Fil}^1(M_V) = \mathrm{Fil}^1(M) \otimes_{R_e} V$ for the Hodge filtration on M_V.

5.7 Lemma. *(i) There exists a complement M'_V for $\mathrm{Fil}^1(M_V) \subset M_V$ such that the cocharacter $\mu\colon \mathbb{G}_{\mathrm{m},V} \to \mathrm{GL}(M_V)$ given by*

$$\mu(z) = \begin{cases} \mathrm{id} & \text{on } M'_V \\ z^{-1}\cdot \mathrm{id} & \text{on } \mathrm{Fil}^1(M_V) \end{cases}$$

factors through $\mathcal{G}_{1,V}$.

(ii) The cocharacter $\mu\colon \mathbb{G}_{\mathrm{m},V} \to \mathcal{G}_{1,V}$ lifts to a cocharacter $\widetilde{\mu}\colon \mathbb{G}_{\mathrm{m},\widetilde{R}_e} \to \mathcal{G}_{1,\widetilde{R}_e}$.

We will admit this lemma, referring to [Va2], sect. 5.3 for a proof. We remark that the reductiveness of $\mathcal{G}_{1,V}$ and $\mathcal{G}_{1,\widetilde{R}_e}$ is used in an essential way.

The cocharacter $\widetilde{\mu}$ yields a direct sum decomposition $\widetilde{M} = \widetilde{M}' \oplus \widetilde{M}''$ with $\widetilde{M}'' \otimes_{\widetilde{R}_e} V = \mathrm{Fil}^1(M_V)$. Since $\widetilde{R}_e$ is a projective limit of nilpotent PD-thickenings of V/pV, we can apply the Grothendieck-Messing deformation theory of abelian varieties (see [Me], in particular Chap. V). This gives us a formal p.p.a.v. $(\mathfrak{X}, \boldsymbol{\lambda})$ over $\mathrm{Spf}(\widetilde{R}_e)$ (with the I-PD-adic topology on $\widetilde{R}_e$), the de Rham cohomology of which is given by $\mathrm{H}^1_{\mathrm{dR}}(\mathfrak{X}/\widetilde{R}_e) = \widetilde{M}$ with Hodge filtration $\widetilde{M''}$ and Gauß-Manin connection induced from the connection on $M(H)$ as a crystal. The fact that we have a polarization on $\mathfrak{X}$ implies that $(\mathfrak{X}, \boldsymbol{\lambda})$ algebraizes to a p.p.a.v. $(\widetilde{X}, \widetilde{\lambda})$ over $\mathrm{Spec}(\widetilde{R}_e)$. We have $(\widetilde{X}, \widetilde{\lambda}) \otimes_{\widetilde{R}_e} V = (X, \lambda)$. Since we only consider level n structures with $p \nmid n$, the system of level structures θ^p extends to a system $\widetilde{\theta^p}$ on $(\widetilde{X}, \widetilde{\lambda})$. We claim that the morphism $\mathrm{Spec}(\widetilde{R}_e) \to \overline{\mathcal{A}}$ corresponding to $(\widetilde{X}, \widetilde{\lambda}, \widetilde{\theta^p})$ factors through $\overline{N} \subset \overline{\mathcal{A}}$. To prove this, we will use the following lemma.

5.8 Lemma. *Notations as above. Let $R := \mathbb{C}[\![z]\!]$ with its (z)-adic topology, and let $y\colon \mathrm{Spf}(R) \to \mathcal{A} \otimes \mathbb{C}$ be a morphism corresponding to a (formal) p.p.a.v. (Y, μ, η^p) over $\mathrm{Spf}(R)$. Let $i_0\colon \mathrm{Spec}(\mathbb{C}) \to \mathrm{Spf}(R)$ be the unique $\mathbb{C}$-valued point (given by $z \mapsto 0$), and assume that $y_0 := y \circ i_0$ is a point of $Sh(G,X)_{\mathbb{C}} \hookrightarrow \mathcal{A} \otimes \mathbb{C}$. As in 5.6.2, we obtain de Rham classes $t_{\alpha,\mathrm{dR},0} \in T_{\alpha,\mathrm{dR},0}$ for $\alpha \in \mathcal{J}$, where the subscript "$_0$" refers to the fact that these are classes on the special fibre Y_0. Assume that the formal horizontal continuations of the classes $t_{\alpha,\mathrm{dR},0}$ over $\mathrm{Spf}(R)$ remain inside $\mathrm{Fil}^0 T_{\alpha,\mathrm{dR}}$. Then y factors through $Sh(G,X)$.*

Proof (sketch). There exists a p.p.a.v. $(\widetilde{Y}, \widetilde{\mu})$ over an algebraic curve S such that the formal completion at some non-singular point $s_0 \in S$ gives back (Y, μ). (In this sketch of the argument we will forget about the level structures.) Over some open disc $U \hookrightarrow S^{\mathrm{an}}$ around s_0, we can choose a symplectic basis of $\mathrm{H}^1_{\mathrm{dR}}(\widetilde{Y}_U/U)$. By virtue of the Hodge filtration, this gives rise to a map $q\colon U \to \mathfrak{H}_g^\vee$, where $\mathfrak{H}_g^\vee$ (the compact dual of $\mathfrak{H}_g$) is the domain parametrizing g-dimensional subspaces $\mathrm{Fil}^1 \subset \mathbb{C}^{2g}$ which are totally isotropic for the standard symplectic form ψ. The point $q(s_0)$ lies on a subvariety $\check{X} \subset \mathfrak{H}_g^\vee$ (where $\check{X}$ is the compact dual of the hermitian symmetric domains $X^+ \subseteq X$ as in the given Shimura datum) parametrizing those flags for which the horizontal continuations $\tilde{t}_{\alpha,\mathrm{dR}}$ of the $t_{\alpha,\mathrm{dR},0}$ remain in the filtration step Fil^0. By consideration of the Taylor series development of the map q at s_0 one shows that q maps U into $\check{X}$, and this implies the assertion. ☐

5.8.1 Proposition. *The morphism $\tilde{x}\colon \mathrm{Spec}(\widetilde{R}_e) \to \overline{\mathcal{A}}$ factors through $\overline{N} \subset \overline{\mathcal{A}}$.*

Proof (sketch). It suffices to show that the generic point of $\mathrm{Spec}(\widetilde{R}_e)$ maps to $Sh(G,X)$. Consider the homomorphism $j\colon \widetilde{R}_e \hookrightarrow \mathbb{C}[\![z]\!]$ with $T \mapsto z + \sigma(\pi)$

(using the chosen embedding $K_0 \subseteq K \overset{\sigma}{\hookrightarrow} \mathbb{C}$). Note that if we set $(Y, \mu, \eta^p) := j^*(\widetilde{X}, \widetilde{\lambda}, \widetilde{\theta}^p)$, then $(Y_0, \mu_0, \eta_0^p) = \sigma^*(X, \lambda, \theta)$. The de Rham classes $j^* t_{\alpha,\mathrm{dR}}$ on Y are formally horizontal, since $\partial/\partial z \cdot (j^* t_{\alpha,\mathrm{dR}}) = j^*(\partial/\partial T \cdot t_{\alpha,\mathrm{dR}}) = 0$. The claim now follows by applying the lemma. $\square$

5.8.2 The rest of the argument is easy. Pulling back $(\widetilde{X}, \widetilde{\lambda}, \widetilde{\theta}^p)$ via the morphism $\mathrm{Spec}(\overline{W}) \hookrightarrow \mathrm{Spec}(\widetilde{R}_e)$ we obtain a lifting of the closed point x_0 that we started off with, to a $\overline{W}$-valued point of $\overline{N}$. If H_1 is the corresponding p-divisible group over $\overline{W}$ then, by specialization, we have a collection $\{t_{\alpha,\mathrm{crys},1}\}_{\alpha \in \mathcal{J}}$ of crystalline Tate classes on H_1. Writing $M_1 := M(H_1) = \widetilde{M} \otimes_{\widetilde{R}_e} \overline{W}$ and $\mathcal{G} := \mathcal{G}_{1,\widetilde{R}_e} \times_{\widetilde{R}_e} \overline{W} \hookrightarrow \mathrm{CSp}(M_1, \psi_1)$, the group $\mathcal{G}$ is reductive and is precisely the group fixing all tensors $t_{\alpha,\mathrm{crys},1}$. This brings us in a situation where we can apply the results of §4. Writing $\overline{\mathcal{N}}^\wedge$, $\overline{N}^\wedge$ and $\overline{\mathcal{A}}^\wedge$ for the formal completions at $\tilde{x}_0$ and x_0, and using the notations of 4.5–4.9, the same reasoning as in 5.8.1 above shows that the composition $\mathrm{Spf}(C) = \widehat{U_{\mathcal{G}}} \longrightarrow \widehat{U} \cong \overline{\mathcal{A}}^\wedge$ factors through $\overline{N}^\wedge$. Now C and $\widehat{\mathcal{O}}_{\overline{N},x_0}$ are local $\overline{W}$-algebras of the same dimension, hence $\widehat{U_{\mathcal{G}}} \hookrightarrow \overline{N}^\wedge$ is dominant onto a component of $\overline{N}^\wedge$ and lifts to a morphism $\widehat{U_{\mathcal{G}}} \hookrightarrow \overline{\mathcal{N}}^\wedge$. Then $\widehat{\mathcal{O}}_{\overline{\mathcal{N}},\tilde{x}_0} \twoheadrightarrow C$ is a surjective homomorphism between local domains of the same dimension, hence an isomorphism. This concludes the proof of the following result.

5.8.3 Theorem. *In the situation of 5.1, assume that (5.6.1) holds. Then the model $\mathcal{N}$ is an integral canonical model of $Sh_{K_p}(G, X)$ over $\mathcal{O}_{(v)}$.*

5.9 *Vasiu's strategy—second part.* We continue our discussion of the paper [Va2]. What remains to be done to complete Vasiu's program is to show that, in the situation of Corollary 3.23, there exists a covering (G, X) with $i\colon (G, X) \hookrightarrow (\mathrm{CSp}_{2g}, \mathfrak{H}_g^\pm)$ for which the assumption (5.6.1) holds. This is a highly non-trivial problem, and it is not clear to us if one can expect to solve this with the definition of a well-positioned family of tensors as in 5.3. The presentation of this material as it is presently available is too sketchy to convince us of the correctness of all arguments[5]; we will indicate by a marginal symbol $\boxed{?}$ statements of which we have not seen a complete proof.

In the rest of this section we shall only indicate the main line of Vasiu's arguments, without much further explanation.

[5]As remarked before, we strongly encourage the reader to read Vasiu's original papers, some versions of which appeared after we completed this manuscript.

5.9.1 Let W be a finite dimensional vector space over a field F of characteristic zero, and consider a semi-simple subgroup $G \subset \mathrm{GL}(W)$. On Lie algebras we have $\mathfrak{gl}(W) = \mathfrak{g} \oplus \mathfrak{g}^\perp$, where $\mathfrak{g}^\perp$ is the orthogonal of $\mathfrak{g} := \mathrm{Lie}(G)$ w.r.t. the form $(A_1, A_2) \mapsto \mathrm{Tr}(A_1A_2)$ on $\mathfrak{gl}(W)$. Write $\pi_\mathfrak{g}$ for the projector onto $\mathfrak{g}$; we view $\pi_\mathfrak{g}$ as an element of $W(2,2;0)$. Next we consider the Killing form $\beta_\mathfrak{g}\colon \mathfrak{g} \times \mathfrak{g} \to F$. Since G is semi-simple, the form $\beta_\mathfrak{g}$ is non-degenerate, so that there exists a form $\beta_\mathfrak{g}^*\colon \mathfrak{g}^* \times \mathfrak{g}^* \to F$ with $\langle \beta_\mathfrak{g}, \beta_\mathfrak{g}^* \rangle = 1$. Using the direct sum decomposition $\mathfrak{gl}(W) = \mathfrak{g} \oplus \mathfrak{g}^\perp$ and the induced isomorphism $\mathfrak{gl}(W)^* = \mathfrak{g}^* \oplus (\mathfrak{g}^\perp)^*$, we can view $\beta_\mathfrak{g}$ and $\beta_\mathfrak{g}^*$ as elements of $W(2,2;0)$. Clearly the tensors $\pi_\mathfrak{g}$, $\beta_\mathfrak{g}$ and $\beta_\mathfrak{g}^*$ are G-invariant. Even better: if G is the derived subgroup of a reductive group $H \subset \mathrm{GL}(W)$ then $\pi_\mathfrak{g}$, $\beta_\mathfrak{g}$ and $\beta_\mathfrak{g}^*$ are also H-invariant.

Finally we define an integer $s(\mathfrak{g}, W)$. For this we fix an algebraic closure $\overline{F}$ of F and we choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}_{\overline{F}}$. For a root $\alpha \in R(\mathfrak{g}_{\overline{F}}, \mathfrak{t})$, let $\mathfrak{s}_\alpha \subset \mathfrak{g}_{\overline{F}}$ denote the Lie subalgebra (isomorphic to $\mathfrak{sl}_2$) generated by $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$. We write

$$d_\alpha := \max\{\dim(Y) \mid Y \subset W_{\overline{F}} \text{ is an irreducible } \mathfrak{s}_\alpha\text{-submodule}\}\,,$$

and we define $s(\mathfrak{g}, W) := \max\{d_\alpha \mid \alpha \in R(\mathfrak{g}, \mathfrak{t})\}$. If Ξ is the set of weights occurring in the $\mathfrak{g}$-module W then $d_\alpha = 1 + \max\{\alpha^\vee(\xi) \mid \xi \in \Xi\}$. So, if $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_r$ is a basis of $R(\mathfrak{g}, \mathfrak{t})$ and if W is irreducible with highest weight $\varpi = n_1 \cdot \varpi_1 + \cdots + n_r \cdot \varpi_r$, where ϖ_i is the fundamental dominant weight corresponding to α_i, then $d_\alpha = d_{\alpha_1} = 1 + n_1$.

? **5.9.2 Claim.** *Let W be a finite dimensional $\mathbb{Q}$-vector space with a non-degenerate symplectic form ψ. If $G \subset \mathrm{CSp}(W, \psi)$ is a semi-simple subgroup and if $p \geq s(\mathfrak{g}, W)$ then $\{\pi_\mathfrak{g}, \beta_\mathfrak{g}, \beta_\mathfrak{g}^*\}$ is a well-positioned family of tensors for the group G over the d.v.r. $\mathbb{Z}_{(p)}$.*

If one tries to prove a statement like this then a priori one would have to consider an arbitrary faithfully flat $\mathbb{Z}_{(p)}$-algebra R and a free R-module M with $M \otimes_R R[1/p] \cong W \otimes_\mathbb{Q} R[1/p]$. Since we are dealing with a finite collection of tensors, however, one easily reduces to the case that R is of finite type over $\mathbb{Z}_{(p)}$. Also we may replace R by a faithfully flat covering, since taking a Zariski closure of something quasi-compact commutes with flat base-change. This allows one to reduce to the case that R is a complete local noetherian ring.

It should be noted that in general the Zariski closure of $G_{R[/p]}$ inside $\mathrm{GL}(M)$ is *not* a subgroup scheme, even if R is a regular local ring. We refer to the work [BT] of Bruhat and Tits, especially loc. cit., 3.2.15, for further theory and a very instructive example.

?

5.9.3 Corollary. *Assume 5.9.2 to hold. Consider a closed immersion of Shimura data* $i\colon (G,X) \hookrightarrow (\mathrm{CSp}_{2g,\mathbb{Q}}, \mathfrak{H}_g^{\pm})$. *Let* p *be a prime number, with* $p \geq 5$. *Assume that the Zariski closure of* G *inside* $\mathrm{CSp}_{2g,\mathbb{Z}_{(p)}}$ *is reductive and that the tensors* $\pi_{\mathfrak{g}^{\mathrm{der}}}$, $\beta_{\mathfrak{g}^{\mathrm{der}}}$ *and* $\beta^*_{\mathfrak{g}^{\mathrm{der}}}$ *are* $\mathbb{Z}_{(p)}^{2g}$*-integral. Then condition (5.6.1) is satisfied. In particular: for every prime* v *of* $E = E(G,X)$ *above* p *and every hyperspecial subgroup* $K_p \subset G(\mathbb{Q}_p)$, *there exists an i.c.m. of* $Sh_{K_p}(G,X)$ *over* $\mathcal{O}_{E,(v)}$.

Up to one technical detail, we can derive this corollary from the previous claim by the following argument. By the results of [De3], Sect. 1.3, all highest weights in the representation $i\colon G \hookrightarrow \mathrm{GL}(\mathbb{Q}^{2g})$ are miniscule in the sense of [Bou], Chap. VIII, §7, n° 3. It follows from this that $s(\mathfrak{g}^{\mathrm{der}}, \mathbb{Q}^{2g}) = 2$. Since $p \geq 5$, the set of tensors $\{\pi_{\mathfrak{g}^{\mathrm{der}}}, \beta_{\mathfrak{g}^{\mathrm{der}}}, \beta^*_{\mathfrak{g}^{\mathrm{der}}}\}$ is a well-positioned set of G-invariant tensors (of degree 4) for the group G^{der} over $\mathbb{Z}_{(p)}$.

Next we consider the Zariski closure $\mathcal{G}$ of G inside $\mathrm{CSp}_{2g,\mathbb{Z}_{(p)}}$. By assumption, it is reductive. Let $\mathcal{Z} := Z(\mathcal{G})^0$ be the connected center of $\mathcal{G}$, which is a torus over $\mathbb{Z}_{(p)}$ with generic fibre $Z := Z(G)^0$. Also write $\mathcal{C} \subset \mathrm{End}(\mathbb{Z}_{(p)}^{2g})$ for the subalgebra of endomorphisms which commute with the action of $\mathcal{G}$. We claim that the elements of $\mathcal{C}$ form a well-positioned collection of G-invariant tensors (of degree 2) for the group Z over $\mathbb{Z}_{(p)}$. We will not prove this; the essential idea is to reduce to the situation where Z is a split torus. For details we refer to [Va2].

Now we take $\mathcal{T} := \{\pi_{\mathfrak{g}^{\mathrm{der}}}, \beta_{\mathfrak{g}^{\mathrm{der}}}, \beta^*_{\mathfrak{g}^{\mathrm{der}}}\} \cup \mathcal{C}$ as our collection of G-invariant tensors. Notice that the condition $2r_\alpha \leq 2(p-2)$ in (5.6.1) is satisfied, since we are only using tensors of degrees 2 and 4 and since $p \geq 5$. To conclude the proof of the corollary, one considers a faithfully flat $\mathbb{Z}_{(p)}$-algebra R and a free R-module M with an identification $M \otimes_R R[1/p] \cong \mathbb{Q}^{2g} \otimes_{\mathbb{Q}} R[1/p]$ such that ψ and ψ^*, as well as all tensors in our collection $\mathcal{T}$ are M-integral. Then we know that ψ induces a perfect form ψ_M on M, that the Zariski closure $\mathcal{G}_1$ of $G^{\mathrm{der}} \otimes_{\mathbb{Q}} R[1/p]$ inside $\mathrm{CSp}(M, \psi_M)$ is semi-simple, and that the Zariski closure $\mathcal{Z}_1$ of $Z \otimes_{\mathbb{Q}} R[1/p]$ inside $\mathrm{CSp}(M, \psi_M)$ is a torus. We are therefore left with the following question. (In [Va2] it is used implicitly that the answer is affirmative.)

5.9.4 Problem. *Let* R *be a faithfully flat* $\mathbb{Z}_{(p)}$*-algebra and let* M *be a free* R*-module of finite rank. If* $G_{R[1/p]} \subseteq \mathrm{GL}(M[1/p])$ *is a reductive subgroup* *scheme such that the Zariski closures $\mathcal{G}_1$ and $\mathcal{Z}_1$ of respectively its derived subgroup and its connected center are reductive subgroup schemes of* $\mathrm{GL}(M)$, *does it follow that the Zariski closure of* $G_{R[1/p]}$ *inside* $\mathrm{GL}(M)$ *is flat over* R *and therefore again a reductive subgroup scheme?*

Perhaps the answer to this question is known to experts in this field, in which case we would be interested to hear about it. If we assume that the answer is affirmative then Cor. 5.9.3 follows by the arguments given above.

? **5.9.5 Claim.** *Let* $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ *be an adjoint Shimura datum of abelian type, and let* $p \geq 5$ *be a prime number such that* $G^{\mathrm{ad}}_{\mathbb{Q}_p}$ *is unramified. Then there exists a Shimura datum* (G, X) *covering* $(G^{\mathrm{ad}}, X^{\mathrm{ad}})$ *and a closed immersion* $i\colon (G, X) \hookrightarrow (\mathrm{CSp}_{2g,\mathbb{Q}}, \mathfrak{H}_g^{\pm})$ *such that condition (5.6.1) holds for* G.

In [Va2] this statement is claimed as a consequence of a whole chain of constructions, reducing the problem to Cor. 5.9.3.

? **5.9.6 Corollary. (Assuming 5.9.2—5.9.5)** *Let* (G, X) *be a Shimura datum of pre-abelian type. Let* $p \geq 5$ *be a prime number such that (notations of 3.21.5)* $p \nmid \delta_G$ *and such that* $G_{\mathbb{Q}_p}$ *is unramified. Let* $K_p \subset G(\mathbb{Q}_p)$ *be a hyperspecial subgroup and let* v *be a prime of* $E(G, X)$ *above* p. *Then there exists an integral canonical model* $\mathcal{M}$ *of* $Sh_{K_p}(G, X)$ *over* $\mathcal{O}_{E,(v)}$. *As a scheme,* $\mathcal{M}$ *is the projective limit of smooth quasi-projective* $\mathcal{O}_{E,(v)}$*-schemes with étale coverings as transition maps.*

§6 Characterizing subvarieties of Hodge type; conjectures of Coleman and Oort

6.1 We now turn to a couple of problems of a somewhat different flavour. Consider a Shimura variety $Sh_K(G, X)$. We have seen in §1 that, depending on the choice of a representation of G, we can view it, loosely speaking, as a "moduli space" for Hodge structures with some given Hodge classes. In this interpretation, the "Shimura subvarieties" would be components of the loci where the Hodge structures have certain additional classes. The type of question that we are interested in here is: "can we give a direct description of these Shimura subvarieties?", and "given an arbitrary subvariety of $Sh_K(G, X)$, can we say something about "how often" it intersects a Shimura subvariety?". More specific questions will be formulated below. First, however, let us make the notion of a Shimura subvariety more precise.

6.2 Definition. Let (G, X) be a Shimura datum. An irreducible algebraic subvariety $S \subseteq Sh_K(G, X)_{\mathbb{C}}$ is called a subvariety of Hodge type if there exist an algebraic subgroup $H \subseteq G$ (defined over $\mathbb{Q}$), an element $\eta \in G(\mathbb{A}_f)$ and a connected component Y_H^+ of the locus

$$Y_H := \{h \in X \mid h\colon\ \mathbb{S} \to G_{\mathbb{R}} \text{ factors through } H_{\mathbb{R}}\}$$

such that $S(\mathbb{C})$ is the image of $Y_H^+ \times \eta K$ in $Sh_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$.

If $E(G, X) \subseteq F \subseteq \mathbb{C}$, then an algebraic subvariety $S \subseteq Sh_K(G, X)_F$ is called a subvariety of Hodge type if all components of $S_{\mathbb{C}}$ are of Hodge type. (If S is irreducible then it suffices to check this for *one* component of $S_{\mathbb{C}}$.)

For example: a point x of $Sh_K(G, X)$, considered as a 0-dimensional subvariety, is of Hodge type if and only if x is a special point. If $Sh_K(G, X) \hookrightarrow \mathsf{A}_{g,1,n}$ is a Shimura subvariety of Hodge type then these conditions on x are equivalent to saying that x corresponds to an abelian variety of CM-type (in which case we say that x is a CM-point.)

If $f\colon\ (G_1, X_1) \hookrightarrow (G_2, X_2)$ is a closed immersion of Shimura varieties and if we have compact open subgroups $K_i \subset G_i(\mathbb{A}_f)$ $(i = 1, 2)$ with $f(K_1) \subseteq K_2$, then the connected components of the image of $Sh(f)\colon\ Sh_{K_1}(G_1, X_1) \to Sh_{K_2}(G_2, X_2)$ are called subvarieties of Shimura type. The subvarieties of Hodge type are precisely the irreducible components of Hecke translates of subvarieties of Shimura type. For further details see [Mo1], Chap. I or [Mo2], section 1.

Now for some of the concrete problems that we are interested in.

6.3 Conjecture. (Coleman, cf. [Co]) *For $g \geq 4$, there are only finitely many smooth projective genus g curves C over $\mathbb{C}$ (taken up to isomorphism) for which $\mathrm{Jac}(C)$ is of CM-type.*

6.4 Conjecture. (Oort, cf. [Oo3]) *Let $Z \hookrightarrow \mathsf{A}_{g,1,n} \otimes \mathbb{C}$ be an irreducible algebraic subvariety such that the CM-points on Z are dense for the Zariski topology. Then Z is a subvariety of Hodge type.*

6.5 Let us first make some remarks on the status of these conjectures. Coleman's conjecture, as we phrased it here, is *false* for $g = 4$ and $g = 6$: there exist families of curves $\mathcal{C} \to S$ of genus 4 and 6, such that the image of S in $\mathsf{A}_{g,1}$ corresponding to the family of Jacobians $\mathrm{Jac}(\mathcal{C}/S) \to S$ is (an open part of) a subvariety of Hodge type of dimension > 0. The known examples of this type are given by explicit polynomial equations. For example, let S be the affine line with coordinate λ, and let $\mathcal{C}_N$ be the smooth curve over S with affine equation $y^N = x(x-1)(x-\lambda)$. If $3 \nmid N$ then $\mathcal{C}$ is a family of curves of genus $N-1$ with an automorphism ζ_N of order N given by $(x,y) \mapsto (x, e^{2\pi i/N} \cdot y)$. For $N = 5$ (resp. $N = 7$) we obtain a family of Jacobians $J_N \to S$ with complex multiplication by $\mathbb{Q}[\zeta_5]$ (resp. $\mathbb{Q}[\zeta_7]$), and one computes that the complex embedding given by $\zeta_N \mapsto e^{k \cdot 2\pi i/N}$ has multiplicity 2,1,1,0 for $k = 1,2,3,4$ (resp. multiplicity $2,2,1,1,0,0$ for $k = 1,\ldots,6$) on the tangent space. Now the Shimura variety of PEL type parametrizing abelian 4-folds (resp. 6-folds) with complex multiplication by an order of $\mathbb{Q}[\zeta_5]$ (resp. $\mathbb{Q}[\zeta_7]$) and the given multiplicities on the tangent space is 1-dimensional, so the image of S in $\mathsf{A}_{4,1}$ (resp. $\mathsf{A}_{6,1}$) is an open part of such a subvariety of PEL type. It follows that there are infinitely many values of λ such that $\mathrm{Jac}(\mathcal{C}_\lambda)$ is of CM-type. For further details, and another example of this kind, we refer to [dJN].

For genera $g = 5$ and $g \geq 7$, Coleman's conjecture remains, to our knowledge, completely open. It is plausible that the known counter examples are exceptional, and that examples of such kind only exist for certain "low" genera. Let us point out here that in the above example, we do not find a subvariety of Hodge type if $3 \nmid N$ and $N \geq 8$; this follows from [dJN], Prop. 5.7 and the results of Noot in [No2] (see 6.15 below).

6.6 Oort's conjecture was studied by the author in [Mo1]. The results here are based on a characterization of subvarieties of Hodge type in terms of certain "linearity properties". We will discuss this in more detail below. One of the results in loc. cit., is a proof of Oort's conjecture under an additional assumption. This is a general result, which provides further evidence for the conjecture. Unfortunately, the extra assumption is difficult to verify in practice.

In another direction, one can try to prove the conjecture in concrete cases. The first non-trivial case is to consider subvarieties of a product of two modular curves. After some reduction steps first proved by Chai, André and Edixhoven (see [Ed2]) both found a proof for the conjecture in this case under an additional hypothesis. Both their methods and the hypotheses involved were rather different. Recently, André found an unconditional proof, so that we now have the following result (see [An2]).

6.6.1 Theorem. (André) *Let S_1 and S_2 be modular curves over $\mathbb{C}$, and let $C \subset S_1 \times S_2$ be an irreducible algebraic curve containing infinitely many points (x_1, x_2) such that both $x_1 \in S_1$ and $x_2 \in S_2$ are CM-points (in other* *words, C contains a Zariski dense set of CM-points). Then C is a subvariety of Hodge type, i.e., either $C = S_1 \times \{x_2\}$, where x_2 is a CM-point of S_2, or $C = \{x_1\} \times S_2$, where x_1 is a CM-point of S_1, or C is a component of a Hecke correspondence.*

6.7 One of the motivations for Oort to formulate his conjecture is its analogy with the Manin-Mumford conjecture, now a theorem of Raynaud (see [Ra2]). We recall the statement:

6.7.1 Theorem. (Raynaud) *Let X be a complex abelian variety, and let $Z \hookrightarrow X$ be an algebraic subvariety which contains a Zariski dense collection of torsion points. Then Z is the translate of an abelian subvariety over a torsion point.*

The analogy is obtained by using the following dictionary:

Oort's conjecture	"Manin-Mumford" = Raynaud's thm.
Shimura variety	abelian variety
CM-point (or special point)	torsion point
subvariety of Hodge type	translate of an abelian subvariety over a torsion point

To push the analogy even further, let us mention that one can formulate a conjecture which contains both Oort's conjecture and "Manin-Mumford" as special cases. The idea here is to look at mixed Shimura varieties. Since we have not discussed these in detail, let us mention the following fact: if $S \hookrightarrow \mathsf{A}_{g,1,n}$ is a subvariety of Hodge type, and if $X \to S$ is the universal abelian scheme over it, then X can be described as a (component of a) mixed Shimura variety. (See [Pi] and [Mi2] for further examples and details.) The special points on X are the torsion points on fibres X_s of CM-type. However, the axioms of mixed Shimura varieties are too restrictive for our purposes, since, for example, an abelian variety X which is not of CM-type, cannot be described as a mixed Shimura variety. By loosening the axioms somewhat, we are led to what might be called "mixed Kuga varieties" and to the following conjecture, proposed by Y. André in [An1]. (André adds the remark that this is only a tentative statement, which may have to be adjusted.)

6.7.2 Conjecture. *Let G be an algebraic group over $\mathbb{Q}$, let K_∞ be a maximal compact subgroup of $G(\mathbb{R})$, and let Γ be an arithmetic subgroup of $G(\mathbb{Q})$. Suppose that K_∞ is defined over $\mathbb{Q}$, that $G(\mathbb{R})/K_\infty$ has a $G(\mathbb{R})$-invariant* *complex structure and that the complex analytic space* $\Gamma\backslash G(\mathbb{R})/K_\infty$ *is algebraizable. Let us call an irreducible algebraic subvariety* $S \hookrightarrow \Gamma\backslash G(\mathbb{R})/K_\infty$ *a special subvariety if there exists an algebraic subgroup* $H \subseteq G$ *defined over* $\mathbb{Q}$ *and an element* $g_0 \in G(\mathbb{Q})$ *such that* $S = \{[g_0 \cdot h] \in \Gamma\backslash G(\mathbb{R})/K_\infty \mid h \in H(\mathbb{R})\}$. *Then* S *is a special subvariety if and only if it contains a Zariski dense collection of special points.*

6.8 Assume Oort's conjecture to be true. Then Coleman's conjecture becomes the question of whether there are positive-dimensional subvarieties of Hodge type $S \hookrightarrow \mathsf{A}_{g,1} \otimes \mathbb{C}$ of which an open part is contained in the open Torelli locus $\mathcal{T}^0_{\mathbb{C}}$ (:= the image of the Torelli morphism $\mathcal{M}_g \otimes \mathbb{C} \to \mathsf{A}_{g,1} \otimes \mathbb{C}$). This seems a difficult question, also if we replace the open Torelli locus by its closure. Hain's paper [H] contains interesting new results about this.[6]

To state Hain's results, let us first consider an algebraic group G over $\mathbb{Q}$ which gives rise to a hermitian symmetric domain X (i.e., an algebraic group of hermitian type), and consider a locally symmetric (or arithmetic) variety $S = \Gamma\backslash X$, where Γ is an arithmetic subgroup of $G(\mathbb{Q})$. If G is $\mathbb{Q}$-simple then we call S a simple arithmetic variety. We say that S is *bad* if it contains a locally symmetric divisor (examples: $G = \mathrm{SO}(n,2)$ or $G = \mathrm{SU}(n,1)$, as well as the case $\dim(S) = 1$); otherwise call S *good*. This is a really a property of G, i.e., it does not depend on Γ and the resulting S. In the next statement we only consider the simple case; this is not a serious restriction since every arithmetic variety has a finite cover which is a product of simple ones.

6.8.1 Theorem. (Hain) *Let* S *be a simple arithmetic variety which is good in the above sense.*

(i) Suppose $p\colon \mathcal{C} \to S$ *is a family of stable curves over* S *such that the Picard group* $\mathrm{Pic}^0(\mathcal{C}_s)$ *of every fibre is an abelian variety (i.e., every fibre is a "good" curve: its dual graph is a tree), such that the generic fibre* $\mathcal{C}_\eta$ *is smooth, and such that the period map* $S \to \mathsf{A}_{g,1}$ *is a finite map of locally symmetric varieties. Then* S *is a quotient of the open complex* n*-ball for some* n*. (So* $G^{\mathrm{ad}}_{\mathbb{R}} = \mathrm{PSU}(n,1) \times$ *(compact factors).)*

(ii) Suppose $q\colon Y \to S$ *is a family of abelian varieties, such that every fibre* Y_s *is the Jacobian of a good curve, and such that the period map* $S \to \mathsf{A}_{g,1}$ *is a finite map of locally symmetric varieties. Write* S^{red} *(resp.* S^{hyp}*) for the locus of points such that* Y_s *is the Jacobian of a reducible (resp. hyperelliptic) curve, and let* S^* *be the complement of* S^{red}*, which we assume to be non-*

[6] We thank R. Hain for sending us a preliminary version of this paper.

empty. Then either S is the quotient of the complex n-ball for some n, or $g \geq 3$, each component of S^{red} has codimension ≥ 2 and $S^* \cap S^{\mathrm{hyp}}$ is a non-empty smooth divisor in S^*.

(We point out that a family $Y \to S$ as in (ii) is not necessarily of the form $\mathrm{Jac}(\mathcal{C}/S) \to S$ for a family $\mathcal{C} \to S$ as in (i), due to the fact that the Torelli morphism is ramified along the hyperelliptic locus. If, in (i), all fibres are smooth then the condition that S is good can be ommitted.)

6.9 The next issue that we want to discuss is the characterization of subvarieties of Hodge type by their property of being "formally linear". Here we owe the reader some explanation. Let us first do the theory over $\mathbb{C}$, which works for arbitrary Shimura varieties.

Consider a Shimura variety $Sh_K = Sh_K(G,X)_{\mathbb{C}}$ over $\mathbb{C}$, and let $S \hookrightarrow Sh_K$ be a subvariety of Hodge type. Then S is a totally geodesic subvariety: if $u\colon X^+ \to Sh_K^0$ is the uniformization of the component $Sh_K^0 \subseteq Sh_K$ containing S, and if $\widetilde{S} \subseteq X^+$ is a component of $u^{-1}(S)$, then $\widetilde{S}$ is a totally geodesic submanifold of the hermitian symmetric domain X^+. This property does not characterize subvarieties of Hodge type; for a trivial example: any point $x \in S$ forms a totally geodesic algebraic subvariety, but $\{x\} \subseteq S$ is a subvariety of Hodge type if and only if x is a special point. Essentially, however, we are dealing with the well-known distinction between "Kuga subvarieties" and subvarieties of Hodge type. In a somewhat less general setting, this distinction was clarified by Mumford in [Mu1]. The same idea works in general, and we have the following characterization (see [Mo1], Thm. II.3.1, or [Mo2]).

6.9.1 Theorem. *Let $S \hookrightarrow Sh_K(G,X)_{\mathbb{C}}$ be an irreducible algebraic subvariety. Then S is a subvariety of Hodge type if and only if (i) S is totally geodesic, and (ii) S contains at least one special point.*

Let us mention that one can also give a description of totally geodesic subvarieties in general (i.e., not necessarily containing a special point). It turns out that they are intimately connected with non-rigidity phenomena. For example, let $Sh_K(G,X)_{\mathbb{C}} \hookrightarrow Sh_{K'}(G',X')_{\mathbb{C}}$ be a closed immersion of Shimura varieties, and suppose that the adjoint group G^{ad} decomposes (over $\mathbb{Q}$) as a product, say $G^{\mathrm{ad}} = G_1 \times G_2$. Correspondingly, there is a decomposition $X^{\mathrm{ad}} = X_1 \times X_2$ of X as a product of (finite unions of) hermitian symmetric domains. Fix a component $X_1^+ \subseteq X_1$, a point $x_2 \in X_2$, and a class $\eta K \in G(\mathbb{A}_f)/K$, and let $S_{\eta K}(X_1^+, x_2)$ denote the image of $X_1^+ \times \{x_2\}$ in $Sh_K(G,X)$ under the map $X \ni x \mapsto [x \times \eta K]$. One can show that $S_{\eta K}(X_1^+, x_2)$

is a totally geodesic algebraic subvariety of $Sh_{K'}(G', X')_{\mathbb{C}}$, and that, conversely, all totally geodesic algebraic subvarieties of $Sh_{K'}(G', X')_{\mathbb{C}}$ are of this form.

After passing to a suitable level (i.e., replacing K by a suitable subgroup of finite index) we can arrange that the component Sh^0_K of $Sh_K(G, X)_{\mathbb{C}}$ containing $S_{\eta K}(X_1^+, x_2)$ is a product variety $Sh^0_K = S_1 \times S_2$, with $S_{\eta K}(X_1^+, x_2) = S_1 \times \{s_2\}$ for some point $s_2 \in S_2$. Now assume that G_2 is not trivial, so that $\dim(S_2) > 0$. We see that $S_1 \times \{s_2\}$ is non-rigid: global deformations are obtained by moving the point $s_2 \in S_2$. If (G, X) is of Hodge type, say with $Sh_{K'}(G', X')_{\mathbb{C}} = \mathsf{A}_{g,1,n} \otimes \mathbb{C}$ in the above, then we obtain a non-rigid abelian scheme over S_1. (Notice, however, that the non-rigidity may be of a trivial nature, in the sense that all non-rigid factors of the abelian scheme in question are isotrivial.) The gist of the results in [Mo1], §II.4 (see also [Mo2]) is that all non-rigid abelian schemes, and all their deformations, can be described via the above procedure. We refer to loc. cit. for further details.

6.9.2 We can jazz-up the above characterization of subvarieties of Hodge type. This will lead to a formulation very analogous to the results in mixed characteristics, to be discussed next.

The first important remark is that total geodesicness needs to be tested only at one point. More precisely: if $Z \hookrightarrow Sh_K(G, X)_{\mathbb{C}}$ is an irreducible algebraic subvariety, and if $x \in Z$ is a non-singular point of $Sh_K(G, X)_{\mathbb{C}}$, then Z is totally geodesic (globally) if and only if it is totally geodesic at the point x. This is true because $Sh_K(G, X)_{\mathbb{C}}$ has constant curvature.

Next we define a Serre-Tate group structure on the formal completion $\mathfrak{Sh}_x$ of $Sh_K(G, X)_{\mathbb{C}}$ at an arbitrary point x. Here we assume that K is neat, so that $Sh_K(G, X)$ is non-singular. The procedure is the following.

The point x lies in the image Sh^0 of a uniformization map $u\colon X^+ \to Sh_K(G, X)$, which, by our assumption on K, is a topological covering. Choose $\tilde{x} \in X^+$ with $u(\tilde{x}) = x$. We have a Borel embedding

$$X^+ \hookrightarrow \check{X} = G^{\mathrm{ad}}(\mathbb{C})/P_{\tilde{x}}(\mathbb{C})\,,$$

where $P_{\tilde{x}} \subset G^{\mathrm{ad}}_{\mathbb{C}}$ is the parabolic subgroup stabilizing the point $\tilde{x}$. Using the Hodge decomposition of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathrm{Ad} \circ h_{\tilde{x}}$, we obtain a parabolic subgroup $P^-_{\tilde{x}} \subset G^{\mathrm{ad}}_{\mathbb{C}}$ opposite to $P_{\tilde{x}}$. Write $U^-_{\tilde{x}}$ for the unipotent radical of $P^-_{\tilde{x}}$, which is isomorphic to $\widehat{\mathbb{G}}_{\mathrm{a}}^d$ for $d = \dim(X)$. The natural map $U^-_{\tilde{x}}(\mathbb{C}) \to \check{X}$ gives an isomorphism of $U^-_{\tilde{x}}(\mathbb{C})$ onto its image $\mathcal{U} \subset \check{X}$ which is the complement of a divisor $D \subset \check{X}$. On formal completions we obtain an isomorphism

$$\mathfrak{U}_{\tilde{x}} := U^-_{\tilde{x}\ /\{1\}} \xrightarrow{\sim} \mathcal{U}_{/\{\tilde{x}\}} = \check{X}_{/\{\tilde{x}\}} \xrightarrow[\sim]{u} Sh^0_{/\{x\}} =: \mathfrak{Sh}_x\,,$$

and in this way $\mathfrak{Sh}_x$ inherits the structure of a formal vector group. This we call the Serre-Tate group structure on $\mathfrak{Sh}_x$. One checks that it is independent of the choice of $\tilde{x}$ above x.

If Z is a subvariety as above, then by taking the formal completion at x, we obtain a formal subscheme $\mathfrak{Z}_x \hookrightarrow \mathfrak{Sh}_x$, and we call Z formally linear at x if $\mathfrak{Z}_x$ is a formal vector subgroup of $\mathfrak{Sh}_x$. Using this terminology we have the following result. (See [Mo2], §5.)

6.9.3 Theorem. *Let $Z \hookrightarrow Sh_K(G,X)_{\mathbb{C}}$ be an irreducible algebraic subvariety. If Z is totally geodesic then it is formally linear at all its points. Conversely, if Z is formally linear at some point $x \in Z$, then it is totally geodesic. In particular, Z is a subvariety of Hodge type if and only if (i) Z is formally linear at some point $x \in Z$, and (ii) Z contains at least one special point.*

6.10 In mixed characteristics, our notion of formal linearity is based on Serre-Tate deformation theory of ordinary abelian varieties. Almost everything we need is treated in Katz' paper [Ka]; additional references are [DI] and [Me]. Without proofs, we record some statements that are most relevant for our discussion.

Let k be a perfect field of characteristic $p > 0$, and let X_0 be an ordinary abelian variety over k. Set $W = W(k)$, and write $\mathcal{C}_W$ for the category of artinian local W-algebras R with $W/(p) = k \xrightarrow{\sim} R/\mathfrak{m}_R$. The formal deformation functor $\mathcal{D}efo_{X_0}\colon \mathcal{C}_W \to \mathsf{Sets}$ is given by

$$\mathcal{D}efo_{X_0}(R) = \{(X,\varphi) \mid X \text{ an abelian scheme over } R;\ \varphi\colon\ X \otimes k \xrightarrow{\sim} X_0\}/\cong\ .$$

By the general Serre-Tate theorem, this functor is isomorphic to the formal deformation functor of the p-divisible group $X_0[p^\infty]$. Since X_0 was assumed to be ordinary, the latter is a direct sum $X_0[p^\infty] = G_\mu \oplus G_{\text{ét}}$ of a toroidal and an étale part. For $R \in \mathcal{C}_W$, these two summands both have a unique lifting, say $\widetilde{G}_\mu$ and $\widetilde{G}_{\text{ét}}$ respectively, to a p-divisible group over R. We therefore have

$$\mathcal{D}efo_{X_0}(R) = \{\alpha \in \mathrm{Ext}_R(\widetilde{G}_{\text{ét}}, \widetilde{G}_\mu) \mid \alpha_{|\mathrm{Spec}(k)} \text{ is trivial}\}\,,$$

and in particular we see that $\mathcal{D}efo_{X_0}$ has a natural structure of a group functor.

Fix an algebraic closure $\bar{k}$ of k, write $\overline{W} := W(\bar{k})$, and write T_pX_0 for the "physical" Tate module of X_0. The formal deformation functor of $X_0 \otimes \bar{k}$ can be given "canonical coordinates": if $(X, \varphi) \in \mathcal{D}\mathit{efo}_{X_0\otimes\bar{k}}(R)$ for some $R \in \mathcal{C}_{W(\bar{k})}$, then one associates to X a $\mathbb{Z}_p$-bilinear form

$$q(X/R; -, -)\colon\ T_pX_0 \times T_pX_0^t \longrightarrow \widehat{\mathbb{G}}_{\mathrm{m}}(R) = 1 + \mathfrak{m}_R$$

and it can be shown that this yields an isomorphism of functors

$$\mathcal{D}\mathit{efo}_{X_0\otimes\bar{k}} \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbb{Z}_p}(T_pX_0 \otimes T_pX_0^t, \widehat{\mathbb{G}}_{\mathrm{m}})\,.$$

If we identify the double dual X^{tt} and X, then we have a symmetry formula $q(X/R; \alpha, \alpha_t) = q(X^t/R; \alpha_t, \alpha)$. Furthermore, if $f_0\colon X_{0,\bar{k}} \to Y_{0,\bar{k}}$ is a homomorphism of ordinary abelian varieties over $\bar{k}$, then f_0 lifts to a homomorphism $f\colon X \to Y$ over $R \in \mathcal{C}_{\overline{W}}$ if and only if $q\big(X/R; \alpha, f^t(\beta)\big) = q\big(Y/R; f(\alpha), \beta\big)$ for every $\alpha \in T_pX_0$, $\beta \in T_pY_0$.

Let $\lambda_0\colon X_0 \to X_0^t$ be a principal polarization. Using the induced isomorphism $T_pX_0 \xrightarrow{\sim} T_pX_0^t$ we have $\mathcal{D}\mathit{efo}_{X_{0,\bar{k}}} \cong \mathrm{Hom}(T_pX_0^{\otimes 2}, \widehat{\mathbb{G}}_{\mathrm{m}})$, and by the previous remarks the formal deformation functor $\mathcal{D}\mathit{efo}_{(X_{0,\bar{k}},\lambda_0)}$ of the pair $(X_{0,\bar{k}}, \lambda_0)$ is isomorphic to the closed subfunctor $\mathrm{Hom}(\mathrm{Sym}^2(T_pX_0), \widehat{\mathbb{G}}_{\mathrm{m}})$.

6.11 Let κ be a perfect field of characteristic p with $p \nmid n$, and let $x \in (\mathsf{A}_{g,1,n} \otimes \kappa)^{\mathrm{ord}}$ be a closed ordinary moduli point with residue field k. Write $(X_0, \lambda_0, \theta_0)$ for the corresponding p.p.a.v. plus level structure over $\mathrm{Spec}(k)$. The formal completion $\mathfrak{A}_x := \big(\mathsf{A}_{g,1,n} \otimes W(\kappa)\big)_{/\{x\}}$ is a formal torus over $\mathrm{Spf}\big(W(k)\big)$; since we consider level n structures with $p \nmid n$, it represents the formal deformation functor $\mathcal{D}\mathit{efo}_{(X_0,\lambda_0)}$. By the above, $\mathfrak{A}_x$ has the structure of a formal torus over $W(k)$, called the Serre-Tate group structure.

Choose a basis $\{\alpha_1, \ldots, \alpha_g\}$ for T_pX_0, and set $q_{ij} = q\big(-; \alpha_i, \lambda_0(\alpha_j)\big)$. We have $\mathfrak{A}_x\widehat{\otimes}\overline{W} \cong \mathrm{Spf}(A)$, where $A = \overline{W}[\![q_{ij} - 1]\!]/(q_{ij} - q_{ji})$ with its $\mathfrak{m}$-adic topology, $\mathfrak{m} = (p, q_{ij} - 1)$. If $\mathfrak{X} \to \mathfrak{A} := \mathfrak{A}_x\widehat{\otimes}\overline{W}$ is the universal formal deformation, then there is an explicit description of the Hodge F-crystal $H = \mathrm{H}^1_{\mathrm{dR}}(\mathfrak{X}/\mathfrak{A})$: to the chosen basis $\{\alpha_1, \ldots, \alpha_g\}$ one associates an A-basis $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ of H such that

(i) the Hodge filtration is given by $\mathrm{Fil}^0 = H \supset \mathrm{Fil}^1 = A\cdot b_1 + \cdots + A\cdot b_g \supset \mathrm{Fil}^2 = (0)$,

(ii) the Gauß-Manin connection is given by $\nabla(a_i) = 0$, $\nabla(b_j) = \sum_i a_i \otimes \mathrm{dlog}(q_{ij})$,

(iii) the Frobenius Φ_H is the φ_A-linear map determined by $\Phi_H(a_i) = a_i$, $\Phi_H(b_j) = p\cdot b_j$.

We will need to work with $\mathfrak{A}_x$ is a slightly more general setting. For this, consider a number field F, a finite prime v of F above p, and write $\mathcal{A}_g = \mathsf{A}_{g,1,n} \otimes \mathcal{O}_{(v)}$. Let $x \in \big(\mathcal{A}_g \otimes \kappa(v)\big)^{\mathrm{ord}}$ be a closed ordinary moduli point with residue field k. Set $\mathcal{O}_v = \mathcal{O}_{(v)}^{\wedge}$, $\Lambda := W(k) \otimes_{W(\kappa(v))} \mathcal{O}_v$, $\overline{\Lambda} := \overline{W} \otimes_{W(\kappa(v))} \mathcal{O}_v$. The formal completion $\mathfrak{A}_x := (\mathcal{A}_g)_{/\{x\}}$ now is a formal torus over Λ. It is simply the pull-back via $\mathrm{Spf}(\Lambda) \to \mathrm{Spf}(W)$ of the formal torus considered above.

6.12 The lifting of X_0 corresponding to the identity element $1 \in \mathfrak{A}_x\big(W(k)\big)$ is called the canonical lifting, and will be denoted X_0^{can}. The liftings over $W(k)[\zeta_{p^n}]$ corresponding to the torsion points of $\mathfrak{A}_x$ are called the quasi-canonical liftings.

Suppose that k is a finite field, so that X_0 is an abelian variety of CM-type. The canonical lifting X_0^{can} is the unique lifting of X_0 such that all endomorphisms of X_0 lift to X_0^{can}. The quasi-canonical liftings of X_0 are precisely the liftings of X_0 which are of CM-type; they are mutually all isogenous. For proofs see [dJN], section 3, [Me], Appendix, [Mo1], §III.1.

6.13 Definition. Suppose, with the above notations, that $Z \hookrightarrow \mathsf{A}_{g,1,n} \otimes F$ is an algebraic subvariety. Let $\mathcal{Z} \hookrightarrow \mathcal{A}_g$ denote its Zariski closure inside $\mathcal{A}_g$. Suppose that the closed ordinary moduli point x is a point of $\big(\mathcal{Z} \otimes \kappa(v)\big)^{\mathrm{ord}} \hookrightarrow \big(\mathcal{A}_g \otimes \kappa(v)\big)^{\mathrm{ord}}$. Then we say that $\mathcal{Z}$ is formally linear (resp. formally quasi-linear) at x if its formal completion $\mathfrak{Z}_x := \mathcal{Z}_{/\{x\}} \hookrightarrow \mathfrak{A}_x$ is a formal subtorus (resp. if all its (formal) irreducible components are the translate of a formal subtorus over a torsion point).

6.14 Example. Suppose Z is a component of a subvariety of PEL type, parametrizing p.p.a.v. with an action of a given order R in a semi-simple $\mathbb{Q}$-algebra. In particular, we have $\iota_0\colon R \hookrightarrow \mathrm{End}(X_0)$. Consider the formal subscheme of $\mathfrak{A}_x$ parametrizing liftings X of X_0 such that ι_0 lifts to $\iota\colon R \hookrightarrow \mathrm{End}(X)$. It follows from the facts in 6.10 that this is a union of translates of formal subtori of $\mathfrak{A}_x$ over torsion points. (The reader is encouraged to verify this.) It follows that $\mathcal{Z}$ is formally quasi-linear at x. Moreover, if Z is absolutely irreducible and the order R is maximal at p then $\mathcal{Z}$ is formally linear at x.

The relation between formal linearity and subvarieties of Hodge type is expressed by the following two results, which were obtained by Noot in [No1] (see also [No2]) and the author in [Mo1] (see also [Mo3]), respectively.

6.15 Theorem. (Noot) *Let F be a number field, and let $S \hookrightarrow \mathsf{A}_{g,1,n} \otimes F$ be a subvariety of Hodge type. Let v be a prime of F above p, and write $\mathcal{S}$ for the Zariski closure of S inside $\mathsf{A}_{g,1,n} \otimes \mathcal{O}_{(v)}$. Let x be a closed point in the ordinary locus $\big(\mathcal{S} \otimes \kappa(v)\big)^{\mathrm{ord}}$. Then $\mathcal{S}$ is formally quasi-linear at x. For v outside a finite set of primes of $\mathcal{O}_F$, the formal completion $\mathfrak{S}_x$ of $\mathcal{S}$ at x is a union of formal subtori of $\mathfrak{A}_x$.*

6.16 Theorem. *Let $Z \hookrightarrow \mathsf{A}_{g,1,n} \otimes F$ be an irreducible algebraic subvariety over a number field F. Suppose there is a prime v of $\mathcal{O}_F$ such that the model $\mathcal{Z}$ of Z (as above) has formally quasi-linear components at some closed ordinary point $x \in \big(\mathcal{Z} \otimes \kappa(v)\big)^{\mathrm{ord}}$. Then Z is of Hodge type.*

We refer to [Mo1] and [Mo2] for some applications of 6.16 to Oort's conjecture. Given Z as in 6.4 (which then is defined over a number field), one tries to prove that Z is formally linear at some ordinary point in characteristic p. In general, we do not know how to do this; the main difficulty is that we have little control over the CM-points on Z. With certain additional assumptions, which we will not specify here, one can, however, prove such a statement. See in particular [Mo2], §5.

Notice that 6.16 is a "local" version of Oort's conjecture: an algebraizable irreducible formal subscheme of $\mathfrak{A}_x$ comes from a subvariety of Hodge type if and only if it contains a dense collection of CM-points (= torsion points). (The adjective "algebraizable" is essential.) We think of this local version and of Raynaud's "Manin-Mumford" theorem as "abelian" cases. Morally, the global case of Oort's conjecture is more difficult because it involves non-abelian group structures.

6.17 To finish, let us take one more look at Coleman's conjecture. A naive attempt to disprove it runs as follows: consider the ordinary locus of $\mathcal{M}_{g,\overline{\mathbb{F}}_p}$, and try to lift the corresponding curves to characteristic zero such that the Jacobian remains of CM-type. This does not work so easily, due to the well-known fact that the canonical lifting of a Jacobian in general no longer is a Jacobian. In [DO], Dwork and Ogus give an "abstract" proof of this. ("Abstract" as opposed to the explicit examples demonstrating this fact given by Oort and Sekiguchi in [OS].) They call an ordinary (smooth projective) curve C over a perfect field k of char. p a pre-W_n-canonical curve if, setting $X_0 = \mathrm{Jac}(C)$, the canonical lifting $X_0^{\mathrm{can}} \bmod p^{n+1}$ over $W_n(k)$ is a Jacobian. They then show that the locus Σ_{W_1} of pre-W_1-canonical curves (pre-W_2-canonical in their notations) forms a constructible part of $\mathcal{M}^{\mathrm{ord}}_{g,\overline{\mathbb{F}}_p}$ which is nowhere dense if $g \geq 4$.

For our "naive attempt" this still leaves hope, though. As Dwork and Ogus write, "It would be interesting to study the "deeper" subschemes Σ_{W_n} for higher n ...". Coleman's conjecture suggests that Σ_{W_∞} should be a finite set of points. Oort's conjecture together with 6.16 lead to another suggestion. Namely, if we write $\tau\colon \mathcal{M}_g \to \mathsf{A}_{g,1}$ for the Torelli morphism, and if $x \in \Sigma_{W_\infty}$ then "locally around $\tau(x)$", the locus $\tau(\Sigma_{W_\infty})$ should be the largest subvariety which is contained in the Torelli locus $\tau(\mathcal{M}_{g,\overline{\mathbb{F}}_p})$ and which is formally linear (purely in characteristic p). It seems that one can prove this by "iterating" the method of [DO]. Unfortunately, our control of the higher-order deformation theory is as yet insufficient to use this to show that Σ_{W_∞} is 0-dimensional.

References

[An1] Y. André, *Distribution des points CM sur les sous-variétés des variétés de modules de variétés abéliennes*, manuscript, April 1997.

[An2] ——, *Finitude des couples d'invariants modulaires singuliers sur une courbe algébrique plane non modulaire*, manuscript, April 1997.

[Aea] A. Ash et al., *Smooth compactification of locally symmetric varieties*, Lie groups: history, frontiers and applications, Vol. IV, Math. Sci Press, Brookline, 1975.

[BB] W.L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math., 84 (1966), pp. 442–528.

[BBM] P. Berthelot, L. Breen and W. Messing, *Théorie de Dieudonné cristalline II*, Lecture Notes in Mathematics 930, Springer-Verlag, Berlin, 1982.

[BO] P. Berthelot and A. Ogus, *F-isocrystals and de Rham cohomology, I*, Inventiones Math., 72 (1983), pp. 159–199.

[BM1] P. Berthelot and W. Messing, *Théorie de Dieudonné cristalline I*, in: Journées de géométrie algébrique de Rennes, Part I, Astérisque 63 (1979), pp. 17-37.

[BM3] ——, *Théorie de Dieudonné cristalline III*, in: The Grothendieck Festschrift, Vol. I, P. Cartier et al., eds., Progress in Math., Vol. 86, Birkhäuser, Boston, 1990, pp. 173–247.

[Bl] D. Blasius, *A p-adic property of Hodge classes on abelian varieties*, in: Motives, Part 2, U. Jannsen, S. Kleiman, and J-P. Serre, eds., Proc. of

Symp. in Pure Math., Vol. 55, American Mathematical Society, 1994, pp. 293–308.

[Bo1] M.V. Borovoi, *The Langlands conjecture on the conjugation of Shimura varieties*, Functional Anal. Appl., 16 (1982), pp. 292–294.

[Bo2] ——, *Conjugation of Shimura varieties*, in: Proc. I.C.M. Berkeley, 1986, Part I, pp. 783–790.

[BLR] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 21, Springer-Verlag, Berlin, 1990.

[Bou] N. Bourbaki, *Groupes et algèbres de Lie, Chap. 7 et 8 (Nouveau tirage)*, Éléments de mathématique, Masson, Paris, 1990.

[BT] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local, II: Schémas en groupes; Existence d'une donnée radicielle valuée*, Publ. Math. de l'I.H.E.S., N° 60 (1984), pp. 5–84.

[Br] J.-L. Brylinski, *"1-motifs" et formes automorphes (Théorie arithmétique des domaines de Siegel)*, in: Journées automorphes, Publ. Math. de l'université Paris VII, Vol. 15, Paris, 1983, pp. 43–106.

[Ca] H. Carayol, *Sur la mauvaise réduction des courbes de Shimura*, Compositio Math., 59 (1986), pp. 151–230.

[Ch] W. Chi, *ℓ-adic and λ-adic representations associated to abelian varieties defined over number fields*, American J. of Math., 114 (1992), pp. 315–353.

[Co] R. Coleman, *Torsion points on curves*, in: Galois representations and arithmetic algebraic geometry, Y. Ihara, ed., Adv. Studies in Pure Math., Vol. 12, North-Holland, Amsterdam, 1987, pp. 235–247.

[dJ] A.J. de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Publ. Math. de l'I.H.E.S., N° 82 (1996), pp. 5–96.

[dJN] A.J. de Jong and R. Noot, *Jacobians with complex multiplication*, in: Arithmetic Algebraic Geometry, G. van der Geer, F. Oort and J. Steenbrink, eds., Progress in Math., Vol. 89, Birkhäuser, Boston, 1991, pp. 177–192.

[dJO] A.J. de Jong and F. Oort, *On extending families of curves*, J. Alg. Geom., 6 (1997), pp. 545–562.

[De1] P. Deligne, *Travaux de Shimura*, in: Séminaire Bourbaki, Exposé 389, Février 1971, Lecture Notes in Mathematics 244, Springer-Verlag, Berlin, 1971, pp. 123–165.

[De2] ——, *Théorie de Hodge II*, Publ. Math. de l'I.H.E.S., N° 40 (1972), pp. 5–57.

[De3] ——, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, in: Automorphic Forms, Representations, and L-functions, Part 2, A. Borel and W. Casselman, eds., Proc. of Symp. in Pure Math., Vol. XXXIII, American Mathematical Society, 1979, pp. 247–290.

[De4] ——, *Hodge cycles on abelian varieties (Notes by J.S. Milne)*, in: Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics 900, Springer-Verlag, Berlin, 1982, pp. 9–100.

[De5] ——, *A quoi servent les motifs?*, in: Motives, Part 1, U. Jannsen, S. Kleiman, and J-P. Serre, eds., Proc. of Symp. in Pure Math., Vol. 55, American Mathematical Society, 1994, pp. 143–161.

[DI] P. Deligne and L. Illusie, *Cristaux ordinaires et coordonnées canoniques (with an appendix by N. Katz)*, Exposé V, in: Surfaces algébriques, J. Giraud, L. Illusie, and M. Raynaud, eds., Lecture Notes in Mathematics 868, Springer-Verlag, Berlin, 1981, pp. 80–137.

[DR] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, in: Modular functions of one variable II, Proc. Intern. summer school, Univ. of Antwerp, RUCA, P. Deligne and W. Kuyk, eds., Lecture Notes in Mathematics 349, Springer-Verlag, Berlin, 1973, pp. 143–316.

[DO] B. Dwork and A. Ogus, *Canonical liftings of Jacobians*, Compositio Math., 58 (1986), pp. 111-131.

[Ed1] B. Edixhoven, *Néron models and tame ramification*, Compositio Math., 81 (1992), pp. 291–306.

[Ed2] ——, *Special points on the product of two modular curves*, Univ. Rennes 1, Prépublication 96-26, Octobre 1996.

[Fa1] G. Faltings, *Arithmetic varieties and rigidity*, in: Séminaire de théorie des nombres de Paris, 1982-83, Progress in Math., Vol. 51, Birkhäuser, Boston, 1984, pp. 63–77.

[Fa2] ——, *Crystalline cohomology and p-adic Galois representations*, in: Algebraic analysis, geometry and number theory, Proc. JAMI inaugural conference, J.-I. Igusa, ed., Johns-Hopkins Univ. Press, Baltimore, 1989, pp. 25–80.

[Fa3] ——, *Integral crystalline cohomology over very ramified base rings*, preprint, Princeton University (1993?).

[FC] G. Faltings and C-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 22, Springer-Verlag, Berlin, 1990.

[Fo] J.-M. Fontaine, *Le corps des périodes p-adiques*, Exposé II, in: Périodes p-adiques, Séminaire de Bures, 1988, Astérisque 223 (1994), pp. 59–101.

[FGA] A. Grothendieck, *Fondements de la géométrie algébrique*, Extraits du séminaire Bourbaki 1957–1962. Paris: Secrétariat math., 11, R. Pierre Curie, 5.

[EGA] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique*, Publ. Math. de l'I.H.E.S., N^{os} 4, 8, 11, 17, 20, 24 and 32 (1960–67).

[SGA1] A. Grothendieck, *SGA 1; Revêtements étales et groupe fondamental*, Lecture Notes in Mathematics 224, Springer-Verlag, Berlin, 1971.

[H] R. Hain, *Locally symmetric families of curves and Jacobians*, manuscript, June 1997.

[Ha] M. Harris, *Arithmetic vector bundles and automorphic forms on Shimura varieties, I*, Inventiones Math., 82 (1985), pp. 151–189.

[Haz] F. Hazama, *Algebraic cycles on certain abelian varieties and powers of special surfaces*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math., 31 (1984), pp. 487–520.

[Il] L. Illusie, *Déformations de groupes de Barsotti-Tate (d'après A. Grothendieck)*, in: Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell, L. Szpiro, ed., Astérisque 127 (1985), pp. 151–198.

[Ka] N.M. Katz, *Serre-Tate local moduli*, Exposé V$^{\text{bis}}$, in: Surfaces algébriques, J. Giraud, L. Illusie, and M. Raynaud, eds., Lecture Notes in Mathematics 868, Springer-Verlag, Berlin, 1981, pp. 138–202.

[Ko1] R. Kottwitz, *Isocrystals with additional structure*, Compositio Math., 56 (1985), pp. 201–220.

[Ko2] ——, *Points on some Shimura varieties over finite fields*, J. of the A.M.S., 5 (1992), pp. 373–444.

[Ko3] ——, *Isocrystals with additional structure. II*, preprint.

[Ku] V. Kumar Murty, *Exceptional Hodge classes on certain abelian varieties*, Math. Ann., 268 (1984), pp. 197–206.

[La] R.P. Langlands, *Automorphic representations, Shimura varieties, and motives. Ein Märchen.*, in: Automorphic Forms, Representations, and L-functions, Part 2, A. Borel and W. Casselman, eds., Proc. of Symp. in Pure Math., Vol. XXXIII, American Mathematical Society, 1979, pp. 205–246.

[LR] R.P. Langlands and M. Rapoport, *Shimuravarietäten und Gerben*, J. reine angew. Math., 378 (1987), pp. 113–220.

[Me] W. Messing, *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes*, Lecture Notes in Mathematics 264, Springer-Verlag, Berlin, 1972.

[Mi1] J.S. Milne, *The action of an automorphism of $\mathbb{C}$ on a Shimura variety and its special points*, in: Arithmetic and geometry, Vol. 1, M. Artin and J. Tate, eds., Progress in Math., Vol. 35, Birkhäuser, Boston, 1983, pp. 239–265.

[Mi2] ——, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, in: Automorphic forms, Shimura varieties, and *L*-functions, L. Clozel and J. S. Milne, eds., Persp. in Math., Vol. 10(I), Academic Press, Inc., 1990, pp. 283–414.

[Mi3] ——, *The points on a Shimura variety modulo a prime of good reduction*, in: The zeta functions of Picard modular surfaces, R. P. Langlands and D. Ramakrishnan, eds., Les Publications CRM, Montréal, 1992, pp. 151–253.

[Mi4] ——, *Shimura varieties and motives*, in: Motives, Part 2, U. Jannsen, S. Kleiman, and J-P. Serre, eds., Proc. of Symp. in Pure Math., Vol. 55, American Mathematical Society, 1994, pp. 447–523.

[MS] J.S. Milne and K.-y. Shih, *Conjugates of Shimura varieties*, in: Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics 900, Springer-Verlag, Berlin, 1982, pp.280–356.

[Mo1] B.J.J. Moonen, *Special points and linearity properties of Shimura varieties*, Ph.D. thesis, University of Utrecht, 1995.

[Mo2] ——, *Linearity properties of Shimura varieties, I*, to appear in J. Alg. Geom.

[Mo3] ——, *Linearity properties of Shimura varieties, II*, to appear in Compositio Math.

[MZ1] B.J.J. Moonen and Yu.G. Zarhin, *Hodge classes and Tate classes on simple abelian fourfolds*, Duke Math. Journal, 77 (1995), pp. 553–581.

[MZ2] ——, *Weil classes on abelian varieties*, preprint `alg-geom/9612017`, to appear in J. reine angew. Math.

[Mor] Y. Morita, *Reduction mod $\mathfrak{P}$ of Shimura curves*, Hokkaido Math. J., 10 (1981), pp. 209–238.

[Mu1] D. Mumford, *A note of Shimura's paper "Discontinuous groups and abelian varieties"*, Math. Ann., 181 (1969), pp. 345–351.

[Mu2] ——, *Abelian varieties*, Tata inst. of fund. res. studies in math., Vol. 5, Oxford University Press, Oxford, 1970.

[Mu3] ——, *Hirzebruch's proportionality in the non-compact case*, Inventiones Math., 42 (1977), pp. 239–272.

[No1] R. Noot, *Hodge classes, Tate classes, and local moduli of abelian varieties*, Ph.D. thesis, University of Utrecht, 1992.

[No2] ——, *Models of Shimura varieties in mixed characteristic*, J. Alg. Geom., 5 (1996), pp. 187–207.

[Og] A. Ogus, *A p-adic analogue of the Chowla-Selberg formula*, in: p-adic analysis, Proceedings, Trento, 1989, F. Baldassari, S. Bosch and B. Dwork, eds., Lecture Notes in Mathematics 1454, Springer-Verlag, Berlin, 1990, pp. 319–341.

[Oo1] F. Oort, *Moduli of abelian varieties and Newton polygons*, C. R. Acad. Sci. Paris, Ser. 1, Math., 312 (1991), pp. 385–389.

[Oo2] ——, *Moduli of abelian varieties in positive characteristic*, in: Barsotti symposium in algebraic geometry, V. Christante and W. Messing, eds., Persp. in Math., Vol. 15, Academic Press, Inc., 1994, pp. 253–276.

[Oo3] ——, *Canonical liftings and dense sets of CM-points*, in: Arithmetic geometry, Proc. Cortona symposium 1994, F. Catanese, ed., Symposia Math., Vol. XXXVII, Cambridge University Press, 1997, pp. 228–234.

[OS] F. Oort and T. Sekiguchi, *The canonical lifting of an ordinary Jacobian variety need not be a Jacobian variety*, J. Math. Soc. Japan, 38 (1986), pp. 427–437.

[Pi] R. Pink, *Arithmetical compactification of mixed Shimura varieties*, Ph.D. thesis, Rheinische Friedrich-Wilhelms-Universität Bonn, 1989.

[R1] M. Rapoport, *Compactifications de l'espace de modules de Hilbert-Blumenthal*, Compositio Math., 36 (1978), pp. 255–335.

[R2] M. Rapoport, *On the bad reduction of Shimura varieties*, in: Automorphic forms, Shimura varieties, and L-functions, L. Clozel and J. S. Milne, eds., Persp. in Math., Vol. 10(II), Academic Press, Inc., 1990, pp. 253–321.

[RR] M. Rapoport and M. Richartz, *On the classification and specialization of F-crystals with additional structure*, Compositio Math., 103 (1996), pp. 153–181.

[RZ] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of mathematical studies 141, Princeton University Press, Princeton, 1996.

[Ra1] M. Raynaud, *Schémas en groupes de type $(p, \ldots, p)$*, Bull. Soc. math. France, 102 (1974), pp. 241–280.

[Ra2] ——, *Sous-variétés d'une variété abélienne et points de torsion*, in: Arithmetic and geometry, Vol. 1, M. Artin and J. Tate, eds., Progress in Math., Vol. 35, Birkhäuser, Boston, 1983, pp. 327–352.

[Ri] K. Ribet, *Hodge classes on certain types of abelian varieties*, Am. J. of Math., 105 (1983), pp. 523–538.

[Se2] J-P. Serre, *Cohomologie Galoisienne, Cinquième édition, révisée et complétée*, Lecture Notes in Mathematics 5, Springer-Verlag, Berlin, 1973 & 1994.

[Se2] ——, Letter to John Tate, 2 January, 1985.

[Sh] G. Shimura, *On analytic families of polarized abelian varieties and automorphic functions*, Annals of Math., 78 (1963), pp. 149–192.

[Ta] S. Tankeev, *On algebraic cycles on surfaces and abelian varieties*, Math. USSR Izvestia, 18 (1982), pp. 349–380.

[Ti] J. Tits *Reductive groups over local fields*, in: Automorphic Forms, Representations, and L-functions, Part 1, A. Borel and W. Casselman, eds., Proc. of Symp. in Pure Math., Vol. XXXIII, American Mathematical Society, 1979, pp. 29–69.

[Va1] A. Vasiu, *Integral canonical models for Shimura varieties of Hodge type*, Ph.D. thesis, Princeton University, November 1994.

[Va2] ——, *Integral canonical models for Shimura varieties of preabelian type*, manuscript, E.T.H. Zürich, 31st of October, 1995.

[We] A. Weil, *Abelian varieties and the Hodge ring*, Collected papers, Vol. III, [1977c], pp. 421–429.

[Wi] J-P. Wintenberger, *Un scindage de la filtration de Hodge pour certains variétés algébriques sur les corps locaux*, Annals of Math., 119 (1984), pp. 511–548.

Address: Ben Moonen, Westfälische Wilhelms-Universität Münster, Mathematisches Institut, Einsteinstraße 62, 48149 Münster, Germany.

E-mail: moonen@math.uni-muenster.de