Master's Thesis

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Tilings Of Additive Groups

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Introduction

In this thesis we look at tilings of finite, abelian additive groups. Tiling mathematical spaces is not so different from tiling for example your kitchen floor. You start with a fixed tile and place similar looking tiles all over the space until you have covered the whole floor. This tiling now consists of the starting tile and all shifts of this starting tile.

We can look at this from a mathematical point of view. We can apply this sense of tilings to finite spaces and it depends on the structure of the space or the tile how the tiling looks. We started by looking just at the vector spaces $\mathbb{F}_q^n$ for prime powers q. In this sense, the name tiling is a very natural choice. In these spaces, a tiling consists of a tile V and a set of coset representatives A.

Since the construction of tilings do not depend on the vector space structure of $\mathbb{F}_q^n$, the question arose whether or not we would be able to extend the definition. Via a detour through different articles, we discovered that some research on a similar structure has been done in group theory, using the "same" definition under a different name: factorization.

This makes tilings of finite vector spaces even more interesting. We can look at tilings from a metric point of view, like the tilings in $\mathbb{R}^n$, or as a factorization of a finite abelian group.

We say that a group G has the Rédei property if in every tiling (V, A) of G, then either V or A is contained in a strictly smaller subgroup of G. So if G does not have the Rédei property, then there exists a tiling (V, A) such that $\langle V \rangle = \langle A \rangle = G$. We say that such a tiling is full-rank. So, we would like to know what kind of groups admit full-rank tilings and thus, which vector spaces allow full-rank tilings?

It is interesting to see that a lot of work has been done independently. The work on full-rank tilings of $\mathbb{F}_2^n$ in [1] does not use results on the Rédei property. Also, work on the Rédei property does not mention any work done on tilings of vector spaces. Dinitz appears to be the first to combine the results from both fields in his article [3]. His work provides a good overview of the work that was done and that was not. Since this article was easier to understand, it became the base of this thesis.

Before looking at the full-rank property, we want to be able to create any tiling. Our approach was an algorithm executed in Magma. After finding a tiling, we would check if it could be full-rank. Of course, this is a naive idea and we had to somehow narrow down the search. We combined articles on the Rédei property and the full-rank property.

It is reasonable to wonder why we would even want full-rank tilings. This is because of a conjecture Minkowski expressed as an open problem in his book in 1907. It was reformulated by Hajós in 1942 in [5] on page 428

> Sind $\xi_1, \ldots, \xi_n$ homogene lineare Formen in den Veränderlichen $x_1, \ldots, x_n$, mit reellen Koeffizienten und der Determinante 1, besitzt weiter keine diezer Formen nur ganzzahlige Koeffizienten, so gibt er ganze Zahlen $x_1, \ldots, x_n$, die nicht alle verschwinden und für welche $|\xi_1| < 1, \ldots, |\xi_n| < 1$ ist.

In 1942, Hajós proved the following theorem that solved the Minkowski conjecture. He formulated five different, equivalent versions of the conjecture and proved the last.

> Let G be a finite abelian non-p-group with factorization $G = A_1 + \cdots + A_n$ where all A_i are cyclic subsets with prime order. Then at least one of the A_i is a subgroup of G.

This is the fourth equivalent form in [5], page 463. He moved from the homogeneous linear forms statement, to a more geometric statement, to three statements in group theory.

Then Hajós himself wondered for which groups G there exist only factorizations A, B such that at least one of the factors is periodic. We say that these groups have the Hajós property. This question raised by Hajós was solved by Sands, de Bruijn and Rédei and it was proven that if a group has the Hajós property, then it has the Rédei property.

Since we also looked at spaces with a vector space structure, it is natural to look at the span of a subset. The span of a subset might be different from an additive subgroup generated by this subset. We are interested how this would relate to the existing theory and what indicators this might have. In this way it could help understanding the previously published results. We were able to prove that every vector space $\mathbb{F}_q^n$ has a tiling where both the spans of both V and A generate the whole space, if q is a prime power but not a prime number. Even though this is a simple result, no one ever mentioned it, which is rather interesting.

In Chapter 1 we give the definitions of factorizations of finite abelian groups and tiling in more detail and explain that we look at the same definition under a different name. Then we look at the form of tilings and describe a recursive decomposition of a tiling. We also give an equivalence relation on tilings.

In Chapter 2 we look at more specific groups and describe which of those groups have the full-rank property and which vector spaces have the full-dimensional property.

In Chapter 3 we show how to use tilings of binary vector spaces to construct perfect binary codes.

In the last Chapter we list the open problems and possibilities for further research.

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First of all, I would like to thank my supervisor, Wieb Bosma, for his help and for steering me in the process. He never got tired of me bursting through his door with a small question or a program that did not work.

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Chapter 1

Definitions

In this chapter, we will give most of the important definitions. We start by defining tilings or factorizations of finite abelian groups and then look at tilings of finite vector spaces. The definition of a tiling does not depend on the multiplicative structure of a module or vector space, so we can generalize the definition and look at more specific properties.

Throughout this thesis, let $(G, +)$ be an additive, finite, abelian group with identity element 0. We say that a subset $A \subset G$ is called *normalized* if $0 \in A$. The subset is periodic if there exists $g \in G \setminus \{0\}$ such that $g + A = A$.

Recall that for subsets $A_1, \ldots, A_n \subset G$, the sum $A_1 + \cdots + A_n$ is given by the set

$$\{a_1 + \cdots + a_n \mid a_i \in A_i\}.$$

We say this sum is *direct* if every $a \in A_1 + \cdots + A_n$ has a unique representation $a = a_1 + \cdots + a_n$ for $a_i \in A_i$. If we have a direct sum $A_1 + \cdots + A_n$ that is equal to G, we say that the sum defines a *factorization* of G. We say that a factorization is normalized if each factor A_i is normalized.

Notation 1.1. Let A be a subset of the group G. By $\langle A \rangle$ we mean the subgroup generated by A in G.

Definition 1.2. Let G be a finite abelian group. We say that G has the *Rédei property* if for every normalized factorization $G = A + B$, one of the factors A and B is contained in a subgroup strictly smaller than G.

Definition 1.3. Let G be a finite abelian group. We say that G has the *Hajós property* if for every normalized factorization $G = A + B$, one of the factors A or B is periodic.

Lemma 1.4. *If a finite abelian group G has the Hajós property, then it has the Rédei property.*

This is proven using induction on the number of prime factors of $|G|$ by Szabó, Lemma 1 in [12].

Example 1.5. In this example we will see that the group $(\mathbb{Z}/4\mathbb{Z})^2$ has the Rédei property and the Hajós property.

We pick $A = \{(00), (10), (01), (11)\}$ and $B = \{(00), (20), (02), (22)\}$. It is clear that $A+B = \mathbb{Z}/4\mathbb{Z}$, so this gives a normalized factorization. Moreover, $\langle A \rangle = (\mathbb{Z}/4\mathbb{Z})^2$ but $\langle B \rangle \neq (\mathbb{Z}/4\mathbb{Z})^2$. Therefore, this group has the Rédei property. Furthermore, all nonzero elements of B are periodic points for B and we see that the group has the Hajós property as well.

Definition 1.6. A given subset $V \subset G$ is a *tile* of G if there exists a partition of G into disjoint additive cosets of V. We denote a set of coset representatives by A. The pair (V, A) is now a *tiling* of G.

We say that V is a *linear tile* if there exists a tiling (V, A) such that $\langle A \rangle = A$ or $\langle V \rangle = V$.

Without loss of generality, we may assume that both V and A contain the identity element. We will come back to this when we discuss equivalence of tilings. The set A is a representative system, it could be possible that (V, A) and (V, A') are both tilings of a group G while $A \neq A'$.

Furthermore, if V is a tile of G, with tiling (V, A), then A is also a tile of G with associated tiling (A, V).

This naturally raises the following question: given a subset V in a finite abelian group G, can we determine whether V can be a tile of G? This question is not easy to answer. In [2], Coppersmith and Miller try to answer this for subsets of $\mathbb{F}_2^n$. They provide two computational criteria to verify that certain subsets are not tiles of $\mathbb{F}_2^n$. They show how we can use bin packing and linear programming for the verification.

The definition of tilings is not very insightful. Therefore, in the next remark, we will give some evident properties and a more useful, equivalent way of saying that two subsets tile a space.

Remark 1.7. Let V and A be two subsets of G.

1. If (V, A) is a tiling of G, then every $x \in G$ has a unique representation $x = v + a$ for some $v \in V$, $a \in A$.
2. An alternative definition states that the pair (V, A) is a tiling of G if and only if

$$\{v_1 - v_2 \mid v_1, v_2 \in V\} \cap \{a_1 - a_2 \mid a_1, a_2 \in A\} = \{0\} \text{ and } V + A = G.$$

   In particular, we have that $|V| \cdot |A| = |G|$.

   Furthermore, if $G = C_2^n$ is an elementary 2-group of rank n, then (V, A) is a tiling of C_2^n if $V + A = C_2^n$ and $2V \cap 2A = \{0\}$.
3. If V and A both are subgroups of G, then (V, A) is a tiling of G if and only if $A = G/V$.

So far, we have only looked at finite abelian groups in general. We want to make a start with looking into more specific groups. Let q be a prime power and $\mathbb{F}_q$ be the finite field with q elements. While the definition of a tiling of $\mathbb{F}_q^n$ might be the same as a tiling of $(\mathbb{Z}/q\mathbb{Z})^n$, the properties of the tiles are very different. First, we will give some more notation.

Notation 1.8. If G equals the additive group of $\mathbb{F}_q^n$ for a prime power q, we can also look at the vectorspace structure of $\mathbb{F}_q^n$. For a subset $V \subset G$, $\mathrm{Span}(V)$ is the linear subspace of G generated by V.

Of course, if $q = p$ for a prime number p, then $\mathrm{Span}(V) = \langle V \rangle$.

Definition 1.9. Let (V, A) be a tiling of G.

1. We say that V is *periodic tile* if V is a periodic subset of G.
2. We say that a tiling (V, A) is *full-rank* if

$$\langle V \rangle = \langle A \rangle = G.$$

3. If $G = \mathbb{F}_q^n$ for some prime power q, we say that a tiling (V, A) is *full-dimensional* if

$$\mathrm{Span}(V) = \mathrm{Span}(A) = \mathbb{F}_q^n.$$

In the next chapter, we will show that not all groups G admit a full-rank tiling or a full-dimensional tiling when defined. It is clear that a full-rank tiling is also full-dimensional.

In general, full-rank and full-dimensional are not the same thing. For example, if we look in $\mathbb{F}_4^2$, then $\operatorname{Span}(\{(10),(01)\}) = \mathbb{F}_4^2$ while $\langle\{(10),(01)\}\rangle = \{(00),(10),(01),(11)\}$.

If G is of the form $\mathbb{F}_p^n$ for a prime number p, then saying that a tiling is full-rank is equivalent to saying it is full-dimensional.

Example 1.10. In this example we will construct a tiling (V, A) of $\mathbb{F}_3^4$. We start with an arbitrary subset V of $\mathbb{F}_3^4$ of 3^2 elements containing the zero vector. We immediately see from Remark 1.7 that A should have 3^2 elements as well. Suppose V is given by

$$V = \{(0000),(1000),(0002),(2112),(0012),(2211),(2201),(0001),(1020)\}.$$

To find A, we can naively check all elements of $\mathbb{F}_3^4 \setminus V$ but that is not the most efficient way to construct tilings.

We do know that the zero vector will be contained in A, so we start with $A = \{0\}$. We set $U = V$ to keep track of which vectors we already have. We want find all coset representatives for V, so we loop through all elements s in the group and check if we already found all $s + v$ for v in V. If not, we add s to A and all $s + v$ to U. Now, if U and G have the same number of elements, then we are finished. This algorithm finds one tiling (V, A) if it exists and [0] if it does not.

Note that there might be a lot more tilings (V, A') of the group, but here we only want to find one.

Input: a subset V of the group G

1. `Set` $A := [0]$ `and` $U = V$
2. `for` s `in` G `do`
3.   `if` s `in` U `then continue`
4.   `Set` $Us = [\ ]$
5.   `for` v `in` V
6.   `if` $s + v$ `in` U `then continue in` s
7.   `else add` $s + v$ `to` Us
8.   `Add` s `to A and add` Us `to` U
9.   `if` $|U| = |G|$ `then return` A
10. `return [0]`

Output: a list A such that (V, A) gives one tiling of G or [0] if there does not exists a tiling (V, A).

This algorithm with input V as above and $G = \mathbb{F}_3^4$ finds

$$A = \{(0000),(2020),(1010),(0100),(0200),(2120),(2220),(1110),(1210)\}.$$

The tiling we find by the algorithm depends on the ordering of G. This is used in step 2 and step 5. We conclude that (V, A) is a tiling of $\mathbb{F}_3^4$.

It is easy to see that the tiling is not full-dimensional and not full-rank: $\operatorname{Span}(V) = \langle V\rangle = \mathbb{F}_3^4$, but $\operatorname{Span}(A) \neq \mathbb{F}_3^4$ and $\langle A\rangle = A$. Also, V is not periodic while A is. The periodic points of A are

$$(2020),(1010),(0100),(0200),(2120),(2220),(1110),(1210).$$

Classification of Tilings

Now we want to make some kind of classification of tilings of G. The first proposition in this section shows that we are mainly interested in tilings (V, A) such that V generates the whole group. We will see later in Theorem 1.13 that this proposition can be generalized: the classification of tilings of a group can be reduced to the tilings of all its subgroups.

We will look at the decomposition of tilings of modules into non-periodic full-rank tilings or trivial tilings and we will illustrate this with an example.

Proposition 1.11. *A subset $V \subset G$ tiles G if and only if it tiles $\langle V \rangle$.*

This proof is based on the proof of Proposition 2 in [3].

Proof. Suppose we have a tiling (V, A) of G. Set $A_0 = A \cap \langle V \rangle$, this A_0 is not empty because $0 \in A$. Then $A_0 \subset A$ and $V + A_0 \subset \langle V \rangle$. We claim that (V, A_0) is a tiling for $\langle V \rangle$. By definition, we have $(V - V) \cap (A - A) = \{0\}$ and it follows that $(V - V) \cap (A_0 - A_0) = \{0\}$. Since $\langle V \rangle \subset V + A$, for any $x \in \langle V \rangle$ we have $x = v + a$ for some $v \in V, a \in A$. But $\langle V \rangle$ is closed under addition in G, so $a = x - v \in \langle V \rangle$. This shows that $a \in A_0$. Then $\langle V \rangle \subset V + A_0$, so $V + A_0 = \langle V \rangle$. This proves our claim so we conclude that (V, A_0) is a tiling of $\langle V \rangle$.

For the other direction, suppose we have a tile V of $\langle V \rangle$ with tiling (V, A_0). Since $\langle V \rangle$ is a subgroup of G, it is a tile of G. Let $(\langle V \rangle , A_1)$ be this tiling of G. Then clearly $(V, A_0 + A_1)$ is a tiling of G: $V + A_0 + A_1 = \langle V \rangle + A_1 = G$ and $(V - V) \cap (A_0 + A_1 - (A_0 + A_1)) = \{0\}$ since $(V - V) \cap (A_i - A_i) = \{0\}$ for $i = 0, 1$. ☐

If we replace all $\langle V \rangle$ by $\mathrm{Span}(V)$ in the proof, we get the following proposition.

Proposition 1.12. *Let q be a prime power. A subset $V \subset \mathbb{F}_q^n$ tiles $\mathbb{F}_q^n$ if and only if it tiles Span(V).*

In the next theorem, we show that a tiling always has a certain fixed form.

Theorem 1.13. *Let V be a tile of G such that $\langle V \rangle \neq G$. Let $z = |G|/|V|$ and $m = |G|/| \langle V \rangle |$. The pair (V, A) is a tiling of G if and only if A has the following form:*

1. *For $i = 0, \dots, m - 1$, take $A_i \subset \langle V \rangle$ such that (V, A_i) is a tiling of $\langle V \rangle$;*
2. *Let $c_0 = 0, c_1, \dots, c_{m-1}$ be coset representatives of $G/ \langle V \rangle$.*

Then $A = A_0 \cup (c_1 + A_1) \cup \dots \cup (c_{m-1} + A_{m-1})$.

This proof is based on the proof of Theorem 3 in [3].

Proof. Suppose we have a tiling (V, A) of G. Let $c_0 = 0, c_1, \dots, c_{m-1}$ be representatives of $G/ \langle V \rangle$ and set $A_i = -c_i + (A \cap (c_i + \{V\}))$. If $0 \notin A_i$ for some i, we can change the representative c_i so that $A'_i = -a_i - c'_i + (A \cap (a_i + c'_i + \langle V \rangle)) = -a_i + A_i$. The last equality follows from the fact that $a_i \in \langle V \rangle$ which was proven in the previous proposition. Now, $a_i \in A_i$ so $0 \in A'_i$. But we would have gotten that if we picked c'_i instead of c_i. Therefore, we can assume that $0 \in A_i$ for all i.

For the other direction, assume that A is in the specific form. We see that $|A_i| = z/m$ so that $|A| = z$ and $|V| \cdot |A| = |G|$. To prove that (V, A) is a tiling of G, we only need to show that $(V - V) \cap (A - A) = \{0\}$.

For $a \in A - A$, we have three possible forms:

1. $(c_i + a_i) - (c_i + a_i)$;
2. $(c_i + a_{i1}) - (c_i + a_{i2})$ for $a_{i1} \neq a_{12}$;

3. $(c_i + a_i) - (c_j + a_j)$ for $i \neq j$

where $a_i, a_{i1}, a_{i2}, a_j \in A$. Clearly, all a of the first form equal 0.

For the set of elements of the second type, let us denote those by U. For $u \in U$, we see that $u = a_{i1} - a_{i2} \in A_i - A_i$. Since (V, A_i) is a tiling of $\langle V \rangle$ we have $(V - V) \cap (A_i - A_i) = \{0\}$. Then for all i, $(V - V) \cap U = \{0\}$.

Now we look at the third possibility. Let W denote the set of the elements of the last type. We know that $c_i - c_j \notin \langle V \rangle$ for all $i \neq j$ and $A_i \subset \langle V \rangle$ by construction. Then $W \cap \langle V \rangle = \emptyset$. It immediately follows that $(V - V) \cap W = \emptyset$.

Now we combine the results:

$$(V - V) \cap (A - A) = (V - V) \cap (\{0\} + U + W) = \{0\}$$

which shows us that (V, A) is a tile of G. □

This theorem implies that we can construct all tilings of G, given all tiles V such that $\langle V \rangle = G$. But we can decompose these tilings even further if we switch the roles of A and V in the theorem. If we have a tiling (V, A) of G such that $\langle V \rangle = G$, we can look at the tiling (A, V). If $\langle A \rangle \neq G$ we can give $d_1, \ldots, d_{m-1}$ such that

$$V = V_0 \cup (d_1 + V_1) \cup \cdots \cup (d_{m-1} + V_{m-1}). \tag{1.1}$$

Now, (A, V_i) is a tiling of $\langle A \rangle$ and the $d_1, \ldots, d_{m-1}$ are representatives of $\langle V \rangle / \langle A \rangle$. Using Equation 1.1, we can decompose each of the tilings (V, A_i) of $\langle V \rangle$ into smaller tilings unless $\langle V \rangle = \langle A_i \rangle$. We can continue this process until we find full-rank tilings of subgroups of G or tilings (V', A') where $\langle V \rangle = V$ and $A = 0$. This leads to the next proposition.

Proposition 1.14. *Any tiling of G can recursively be decomposed into smaller tilings that are either full-rank or trivial.*

In this example, we will look at a rather trivial tiling and see how we can decompose it.

Example 1.15. Suppose that (V, A) is a tiling of $\mathbb{F}_2^5$ given by

$$V = \{(00000), (10000), (01000), (00100)\}$$

and

$$A = \{(00000), (00010), (00001), (00011), (11100), (11110), (11101), (11111)\}.$$

The coset representatives of V are given by $c_0 = (00000), c_1 = (00010), c_2 = (00001), c_3 = (00011)$. Now we set $B_i = A \cap \{c_i + w \mid w \in \langle V \rangle\}$ and construct $A_i = \{-c_i + b \mid b \in B_i\}$. Since we are working in a group of characteristic 2, we have $-c_i = c_i$.

We need to check now whether $(00000) \in A_i$ for $i = 0, 1, 2, 3$. In this case,

$$A_0 = A_1 = A_2 = A_3 = \{(00000), (11100)\}.$$

This shows us that $(00000) \in A_i$, so we are done.

We have to check that this gives a decomposition:

$$\begin{aligned} A_0 \cup (c_1 + A_1) \cup (c_2 + A_2) \cup (c_3 + A_3) =& \{(00000), (11100)\} \cup \{(11110), (00010)\} \\ & \cup \{(11101), (00001)\} \cup \{(11111), (00011)\} \\ =& A. \end{aligned}$$

So this gives us a decomposition of A.

We can continue and look at tilings (A_i, V) and split V in the same manner. Since all A_i are the same, we only need to look at A_0. The coset representatives are given by $d_0 = (00000), d_1 = (10000), d_2 = (01000), d_3 = (11000)$. We calculate that $V_i = \{(00000)\}$, so we are done.

Therefore, we see that the initial tiling (V, A) can be built up from 16 tilings of the form

$$(\{(00000), (11100)\}, \{(00000)\}).$$

The recursive decomposition of Proposition 1.14 breaks down tilings into trivial tilings or full-rank tilings. But we can also decompose full-rank tilings. These can be broken down into non-periodic full-rank tilings. We will give some notation and propositions first and then show how this decomposition works.

Notation 1.16. For any subset $A \subset G$, we say $A_0 = \{x \in G \mid x + A = A\}$. This is the set of all periodic points of A.

Clearly, A is non-periodic if and only if $A_0 = \{0\}$ and $A_0 \subset A$ if $0 \in A$.

Proposition 1.17. *If $0 \in A$ then $A_0 \subset A$ is a subgroup of G and A_0 is a tile of A.*

Proof. For $a, b \in A_0$ we have $(a + b) + A = a + (b + A) = a + A = A$ and $-a + A = -a + a + A = A$ so $a + b, -a \in A_0$. This shows that A_0 is a subgroup.

For $a \in A$ clearly $a + A_0 \in A$ so we can write A as a union of cosets of A_0. The cosets are disjoint because A_0 is a subgroup of G. □

From now on, we take $A' \subset A$ to be the set of representatives of A/A_0. By the previous Proposition, (A_0, A') is a tiling of A.

Theorem 1.18. *Let (V, A) be a tiling of G and let A_0 be the set of all periodic points of A. Then $(V/A_0, A/A_0)$ is a tiling of G/A_0. Furthermore, if (V, A) is a full-rank tiling, then so is $(V/A_0, A/A_0)$. The set A/A_0 is non-periodic and V/A_0 is periodic if V is periodic.*

The proof is given by Dinitz in [3]. The proof is not difficult, it is rather straightforward with quite some technical details.

It would be very insightful to include an example of how this theorem works. But the groups that allow full-rank tilings are quite big as we will see in the next chapter. Due to this size any example illustrating the use of this Theorem will not be insightful.

Equivalence

Also, we want to know something about equivalence of tilings. When are two tiles, or two tilings, equivalent? Does equivalence preserve the full-rank condition?

Definition 1.19. Let A, A' be two subsets of a group $G = \prod_{i=1}^{n} G_i$. We say that the two subsets are *equivalent* if we can obtain A' out of A after applying transformations of the following types:

- a permutation of the coordinates;
- a permutation of the elements of G_i in all coordinates i.

This definition of equivalence of subsets is clearly an equivalence relation.

If A, A' are two equivalent subsets of a group G, then there exists a permutation π in S_n and n permutations ϕ_i in S_{G_i} for each coordinate i. Then the set

$$\{\pi(\phi_1(a_1), \dots, \phi_n(a_n)) \mid (a_1, \dots, a_n) \in A\}$$

equals A'.

This definition is normally used only in vector spaces or coding theory.

Remark 1.20. In $\mathbb{F}_2^n$, we have a simpler definition. We can replace the condition of the permutation of elements of $\mathbb{F}_2$ on each coordinate by adding a fixed element of $\mathbb{F}_2^n$ to all elements in the subset.

So, if A, A' are two equivalent subsets of $\mathbb{F}_2^n$, then there exists a permutation π in S_n and a vector v in $\mathbb{F}_2^n$ such that the set

$$\{\pi(a) + v \mid a \in A\}$$

equals A'.

Now we can deduce from this definition, the definition of equivalence of tilings of finite abelian groups G.

Definition 1.21.
- We say that two tiles V, V' are equivalent if they are equivalent as subsets of G.
- We say that two tilings $(V, A), (V, A')$ are equivalent if A, A' are equivalent as subsets of G.
- We say that two tilings $(V, A), (V', A')$ are equivalent if V, V' and A, A' both are equivalent with the same coordinate transformation.

Since a tilings (V, A) is not an ordered pair, we say that the two tilings (V, A) and (A, V) are also equivalent.

Example 1.22. Consider the set $V = \{(0,0,0), (0,1,0), (0,0,1), (0,1,1)\}$ in $\mathbb{F}_2^3$. This set is equivalent to $V' = \{(1,0,0), (1,1,0), (1,0,1), (1,1,1)\}$.

We see that $\langle V \rangle \neq \mathbb{F}_2^3$ while $\langle V' \rangle = \mathbb{F}_2^3$. The set V' does not contain the zero vector. It still is a tile: if we consider the set $A = \{(0,0,0), (1,0,0)\}$ then (V', A) tiles $\mathbb{F}_2^3$.

Example 1.23. In this example we will write down all non-trivial tilings of $\mathbb{F}_2^3$ up to equivalence.

We have the following ten non-trivial tilings. The sets $\{(000), (100)\}$, $\{(000), (110)\}$ and $\{(000), (111)\}$ are pairwise not equivalent. Therefore, the ten tilings are all pairwise not equivalent.

$$\begin{pmatrix} 000 & 000 \\ 100 & 001 \\ & 010 \\ & 111 \end{pmatrix} \quad \begin{pmatrix} 000 & 000 \\ 100 & 001 \\ & 110 \\ & 111 \end{pmatrix} \quad \begin{pmatrix} 000 & 000 \\ 100 & 001 \\ & 010 \\ & 011 \end{pmatrix} \quad \begin{pmatrix} 000 & 000 \\ 100 & 011 \\ & 101 \\ & 110 \end{pmatrix}$$

$$\begin{pmatrix} 000 & 000 \\ 110 & 001 \\ & 100 \\ & 101 \end{pmatrix} \quad \begin{pmatrix} 000 & 000 \\ 110 & 010 \\ & 011 \\ & 111 \end{pmatrix} \quad \begin{pmatrix} 000 & 000 \\ 110 & 011 \\ & 100 \\ & 111 \end{pmatrix}$$

$$\begin{pmatrix} 000 & 000 \\ 111 & 001 \\ & 010 \\ & 100 \end{pmatrix} \quad \begin{pmatrix} 000 & 000 \\ 111 & 001 \\ & 100 \\ & 101 \end{pmatrix} \quad \begin{pmatrix} 000 & 000 \\ 111 & 011 \\ & 101 \\ & 110 \end{pmatrix}$$

Chapter 2

Full-Rank And Full-Dimensional Tilings

In this chapter, we look at more specific groups. From now on, by $\mathcal{F}_m$ we denote a finite group on m elements. So if m is a prime number p, then $\mathcal{F}_m$ equals the additive group of $\mathbb{F}_p$. If m is a composite number, then $\mathcal{F}_m$ equals the additive group of $\mathbb{Z}/m\mathbb{Z}$. If m is a prime power q, then we look at both the additive groups of $\mathbb{F}_q$ and of $\mathbb{Z}/q\mathbb{Z}$.

In the first section, we look only at the case of composite m and in the second to the case of prime numbers. Then we want to look at the prime powers.

But first, we will give three theorems that apply in all cases since they apply to finite, abelian, additive groups.

Theorem 2.1. *Let G be a finite, cyclic, abelian group and let (V, A) be a tiling of G. If the order of both V and A is either a prime power or a product of two primes, then one factor must be periodic.*

This is proven by Sands in [10].

Corollary 2.2. *If G is a finite, cyclic abelian group with tiling (V, A) and the order of both V and A are prime powers or the product of two primes, then (V, A) is not a full-rank tiling.*

This Corollary follows from Lemma 1.4.

In the next theorem, we will see that once we find a full-rank tiling, there exist full-rank tilings in all modules of higher dimension over the same group.

Theorem 2.3. *If a nontrivial finite abelian group G admits a full-rank tiling, then there exists a full-rank tiling of $G \times H$ where H is any finite abelian group.*

This is Theorem 9 as stated by Dinitz in [3].

Corollary 2.4. *Let $m \geq 2$ and $n \geq 1$. If $\mathcal{F}_m^n$ admits a full-rank tiling, then there exists a full-rank tiling of $\mathcal{F}_m^{n+1}$.*

To understand how this works, we give a special case of the proof Dinitz gives in [3] for the proof of the Theorem.

Proof. Let (V, A) be a full-rank tiling of $\mathcal{F}_m^n$. We will construct a full-rank tiling of $\mathcal{F}_m^{n+1}$.

Suppose there exists $a \in A \setminus \{0\}$ such that $\langle A \setminus \{a\} \rangle = \mathcal{F}_m^n$. Set $V' = \{(v, x) \mid v \in V, x \in \mathcal{F}_m\}$ and $A' = \{(a', 0) \mid a \in (A \setminus \{a\})\} \cup \{(a, 1)\}$. Now, (V', A') is a full-rank tiling of $\mathcal{F}_m^{n+1}$.

If such a does not exist, then $A \setminus \{0\}$ and $V \setminus \{0\}$ are minimal generating sets for $\mathcal{F}_m^n$. Dinitz then proves that $\mathcal{F}_m^n$ can not have more then 36 elements if it has a full-rank tiling. He checks all possibilities and concludes that $\mathcal{F}_m^n$ can not have a full-rank tiling and not have such a.

$\square$

Theorem 2.5. *Let $m \geq 3$ and $n \geq 3$. Then $\mathcal{F}_m^{2n}$ admits a full-rank tiling.*

The proof of this theorem is based on Theorem 2 of [13].

Proof. In this proof, we will construct a full-rank tiling.
Let $x_{11}, x_{12}, \ldots, x_{n1}, x_{n2}$ basis elements of $\mathcal{F}_m^{2n}$. We set $K_i = \langle x_{i1} \rangle$, $H_i = \langle x_{i2} \rangle$ and construct $A_i = (K_i \setminus \{(m-1)x_{i1}\}) \cup \{(m-1)x_{i1} + x_{i2}\}$. Now, we take $H = H_1 + \cdots + H_n$ and $A = A_1 + \cdots + A_n$.

Then (H, A) gives a tiling of $\mathcal{F}_m^{2n}$. This can be seen from

$$\begin{aligned} H + A &= H_1 + A_1 + \cdots + H_n + A_n \\ &= \langle x_{11}, x_{12} \rangle + \cdots + \langle x_{n1}, x_{n2} \rangle \\ &= \mathcal{F}_m^{2n}. \end{aligned}$$

It is clear that $\langle H \rangle \neq \mathcal{F}_m^{2n}$, so we will construct a set V out of H such that (V, A) is a full-rank tiling.

We remove the sets

$$(m-1)x_{12} + H_2, \quad (m-1)x_{22} + H_3, \quad \ldots \quad (m-1)x_{n2} + H_1$$

from H and then we add the sets

$$(m-1)x_{12} + H_2 + x_{21}, \quad (m-1)x_{22} + H_3 + x_{31}, \quad \ldots \quad (m-1)x_{n2} + H_1 + x_{11}.$$

This set we will call V. We claim that (V, A) is a tiling of $\mathcal{F}_m^{2n}$.

We will first prove that

$$(m-1)x_{12} + H_2 + A = (m-1)x_{12} + x_{21} + H_2 + A, \quad (m-1)x_{22} + H_3 + A = (m-1)x_{22} + x_{31} + H_3 + A, \quad \ldots$$

$$(m-1)x_{n2} + H_1 + A = (m-1)x_{n2} + x_{11} + H_1 + A.$$

We will verify the first equality, the other equations can be proved similarly.

$$\begin{aligned} (m-1)x_{12} + x_{21} + H_2 + A &= (m-1)x_{12} + x_{21} + H_2 + A_1 + \cdots + A_n \\ &= (m-1)x_{12} + x_{21} + (H_2 + A_2) + A_1 + A_3 + \ldots A_n \\ &= (m-1)x_{12} + x_{21} + \langle x_{21}, x_{22} \rangle + A_1 + A_3 + \cdots + A_n \\ &= (m-1)x_{12} + \langle x_{21}, x_{22} \rangle + A_1 + A_3 + \cdots + A_n \\ &= (m-1)x_{12} + (H_2 + A_2) + A_1 + A_3 + \ldots A_n \\ &= (m-1)x_{12} + H_2 + A. \end{aligned}$$

Furthermore, we need the sets

$$(m-1)x_{12} + H_2 + A, \quad (m-1)x_{22} + H_3 + A, \quad \ldots \quad (m-1)x_{n2} + H_1 + A$$

to be disjoint. Suppose that

$$((m-1)x_{12} + H_2 + A) \cap ((m-1)x_{22} + H_3 + A) \neq \emptyset,$$

which is equivalent to stating that the sets

$$((m-1)x_{12} + \langle x_{21}, x_{22}\rangle + A_1 + A_3 + \cdots + A_n)$$

and

$$((m-1)x_{22} + \langle x_{31}, x_{32}\rangle + A_1 + A_2 + A_4 + \cdots + A_n)$$

are not disjoint. So pick x in the intersection. Since $x \in (m-1)x_{12} + \langle x_{21}, x_{22}\rangle + A_1 + A_3 + \cdots + A_n$, its x_{11}, x_{12} component is of the form $(m-1)x_{11}$ where $0 \le a \le m-2$ or of the form $ax_{11} + (m-1)x_{12}$. We also have $x \in (m-1)x_{22} + \langle x_{31}, x_{32}\rangle + A_1 + A_2 + A_4 + \cdots + A_n$ and this tells us that the x_{11}, x_{12} component is of the form $(m-1)x_{11} + x_{12}$ or bx_{11} for $0 \le b \le m-2$. Such x clearly cannot exist.

Now we have shown that (V, A) gives a tiling of $\mathcal{F}_m^{2n}$. We only need to show that the tiling is full-rank.

Since $x_{i1}, (m-1)x_{i1} + x_{i2} \in A_i$ and $A_i \subset A$ for all $1 \le i \le n$, we immediately see that $\langle A\rangle = \mathcal{F}_m^{2n}$. By construction of H, $x_{i2} \in V$ for all $1 \le i \le n$ and we added the elements

$$(m-1)x_{12} + x_{21}, \quad (m-1)x_{22} + x_{31}, \quad \ldots \quad (m-1)x_{n2} + x_{11}$$

to V. Therefore, $\langle V\rangle = \mathcal{F}_m^{2n}$.

We can now conclude that (V, A) is a full-rank tiling of $\mathcal{F}_m^{2n}$. □

Combining the previous Theorem and Corollary 2.4 we get the following result.

Corollary 2.6. *For $m \ge 3$ and $n \ge 6$, $\mathcal{F}_m^n$ has a full-rank tiling.*

Full-rank tilings of $(\mathbb{Z}/m\mathbb{Z})^n$

In this section, let m be a composite number.

Theorem 2.7. *Let a, b, c be composite numbers. Then $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z} \times \mathbb{Z}/c\mathbb{Z}$ has a full-rank tiling.*

This is Theorem 10 in [3]. The next Corollary follows immediately and shows that we can always find a full-rank tiling of $(\mathbb{Z}/m\mathbb{Z})^3$.

Corollary 2.8. *The group $\mathcal{F}_m^3$ has a full-rank tiling.*

This proof is based on Theorem 10 in [3] and the construction given in section 2 of [11].

Proof. In this proof, we will construct a full-rank tiling of $\mathcal{F}_m^3$.

Let 1 be a generator of $\mathcal{F}_m$, let u be the smallest prime number dividing m and set $v = m/u$. For $x \in G$ and $n \ge 1$, let $[x]_n$ be the set $\{0, x, \ldots, (n-1)x\}$.

Set $V = [e_1]_u + [e_2]_u + [e_3]_u$ and $B = [ue_1]_v + [ue_2]_c + [ue_3]_v$, where $e_1 = (100), e_2 = (010), e_3 = (001)$. Clearly, (V, B) is a tiling of $\mathcal{F}_m^3$ and $\langle B\rangle \neq \mathcal{F}_m^3$, while $\langle V\rangle = \mathcal{F}_m^3$

Now, we will construct A out of B such that (V, A) gives a full-rank tiling. Let π be any cyclic permutation of $\{1, 2, 3\}$.

We construct

$$X = \bigcup_{i=1}^{3} \left\{(a_1, a_2, a_3) \mid a_i \in \{0, 1, \ldots, (v-1)\}, a_{\pi(i)} = u, a_{\pi^{-1}(i)} = 0\right\}$$

and

$$Y = \bigcup_{i=1}^{3} \left\{(a_1, a_2, a_3) \mid a_i \in \{1, (u+1), \ldots, (v-1)(u+1)\}, a_{\pi(i)} = u, a_{\pi^{-1}(i)} = 0\right\}.$$

Now, we set $A = (B \cup Y) \setminus X$. We claim that (V, A) is a full-rank tiling.
First, we will show that $V + [ue_i]_v + ue_{\pi(i)} = V + [ue_i]_v + ue_{\pi(i)} + e_i$.

$$\begin{aligned}
V + [ue_i]_v + ue_{\pi(i)} &= (\sum_{j=1}^{3} [e_j]_u) + [ue_i]_v + ue_{\pi(i)} \\
&= (\sum_{j=1, j\neq i}^{3} [e_j]_u) + [e_i]_u + [ue_i]_v + ue_{\pi(i)} \\
&= (\sum_{j=1, j\neq i}^{3} [e_j]_u) + [e_i]_{uv} + ue_{\pi(i)} \\
&= (\sum_{j=1, j\neq i}^{3} [e_j]_u) + [e_i]_{uv} + ue_{\pi(i)} + e_i \\
&= (\sum_{j=1, j\neq i}^{3} [e_j]_u) + [e_i]_u + [ue_i]_v + ue_{\pi(i)} + e_i \\
&= (\sum_{j=1}^{3} [e_j]_u) + [ue_i]_v + ue_{\pi(i)} + e_i \\
&= V + [ue_i]_v + ue_{\pi(i)} + e_i.
\end{aligned}$$

Here, we repeatedly used that $[e_i]_m = [e_i]_m + e_i$. We now proved that we can replace $[ue_i]_v + ue_{\pi(i)} + e_i$ by $[ue_i]_v + ue_{\pi(i)}$. We only have to show that

$$(V + [ue_i]_v + ue_{\pi(i)}) \cap (V + [ue_j]_v + ue_{\pi(j)}) = \emptyset$$

for $1 \leq i, j \leq 3$ and $i \neq j$.

Suppose that the two sets are not disjoint. Then there exist $a = x_1e_1 + x_2e_2 + x_3e_3 \in A$, $a' = x'_1e_1 + x'_2e_2 + x'_3e_3 \in A'$ and $0 \leq y, y' < v$ such that

$$a + yue_i + ue_{\pi(i)} = a' + y'ue_j + ue_{\pi(j)}.$$

If $i \neq j$ then clearly $\pi(i) \neq \pi(j)$. Then either $\pi(i) \neq j$ or $\pi(j) \neq i$. We can not have $\pi(i) \neq j$ and $\pi(j) \neq i$, because π is a cyclic permutation of $\{1, 2, 3\}$. We also can not have $\pi(i) = j$ and $\pi(j) = i$ as then $\pi = (i, \pi(i))$ and this is not possible.

So, suppose that $\pi(j) = k \neq i$. Then $\pi(i) = j$ and $\pi(k) = i$. Then the equation becomes

$$(x_i + yu)e_i + (x_j + u)e_j + x_ke_k = x'_ie_i + (y'u + x'_j)e_j + (x'_k + u)e_k.$$

It follows that $x_k = x'_k + u$, but we had $x_k < x'_k + u$ by construction.

If $\pi(j) = k, \pi(k) = i$ and $\pi(i) = j$, in the same way we get $x_j + u = x'_j$ but $x_{\pi(i)} + u > x'_{\pi(i)}$. Therefore, such $x_1, x'_1, x_2, x'_2, x_3, x'_3, y, y'$ can not exist and the intersection is empty.

This shows us that (V, A) is a tiling of $\mathcal{F}_m^3$. We will look at the full-rank property now. By construction, $e_1, e_2, e_3 \in V$ so $\langle V \rangle = \mathcal{F}_m^3$.

If $v = 2$ then $u = 2$ and $m = 4$. We compute V and A:

$$V = \{(111), (010), (011), (100), (101), (000), (001), (110)\}$$

and

$$A = \{(222), (012), (201), (000), (320), (203), (032), (120)\}.$$

Then $\langle V \rangle = \langle A \rangle = \mathcal{F}_m^3$, so this construction gives a full-rank tiling if $m = 4$.

Now we look at $v \geq 3$. For $1 \leq i \leq 3$ there exists a j such that $\pi(i) = j$. If $v = 3$, then we know that $0, ue_j + e_i, 2ue_j \in A$ and it follows that $2ue_j + (ue_j + e_i) = e_i \in \langle A \rangle$. If $v > 3$, then $3ue_j \in A$ and $3ue_j - 2ue_j = ue_j \in \langle A \rangle$ and we see that $ue_j + e_i - ue_j = e_i \in \langle A \rangle$. Since this is true for al $1 \leq i \leq 3$, we can conclude that $\langle A \rangle = \mathcal{F}_m^3$ and therefore, this construction gives a full-rank tiling for all composite m. $\square$

Conjecture 2.9. *Let m be a composite number. Then $(\mathbb{Z}/m\mathbb{Z})^2$ does not have a full-rank tiling.*

We tried finding a full-rank tiling for $m = 4, 6, 9, 10$ and did not find any. For $m = 4$, we started with a sets $V_x = \{(00), (10), (01), x\}$ for $x \in (\mathbb{Z}/4\mathbb{Z})^2$ such that $|V| = 4$ and computed all possible tilings (V_x, A). We checked and we never found any A such that $\langle A \rangle = (\mathbb{Z}/4\mathbb{Z})^2$.

For $m = 6$, we looked at $V = \{(00), (10), (01)\}$, a tile of 3 elements, $V_x = \{(00), (10), (01), x\}$, a family of tiles of 4 elements, and $V_{xyz} = \{(00), (10), (01), x, y, z\}$ a family of tiles of 6 elements. Again we computed all possible different tilings and never found a full-rank tiling.

For $m = 9$, we looked at $V = \{(00), (10), (01)\}$ and all possible tiles of 9 elements containing $(00), (10), (01)$. Computing all possibilities, we never find a full-rank tiling. In the same way, for $n = 10$ we took all possible sets of size 5 and size 10 containing $(00), (10), (01)$.

Full-rank tilings of $\mathbb{F}_p^n$

Theorem 2.10. *For $p = 2$, $\mathbb{F}_2^n$ admits a full-rank tiling if and only if $n \geq 10$*

Cohen, Litsyn, Vardy and Zemor proved in [1] that full-rank tilings do not exist for $n \leq 7$ and that they do exist for $n \geq 112$. Later, Etzion and Vardy gave a construction for full-rank tilings for $n \geq 14$ in [4]. LeVan and Phelps used this construction to get a full-rank tiling for $n = 10$.

Trachtenberg and Vardy used a computer to prove that there does not exist a full-rank tiling for $n = 8$ in [17]. Östergård and Vardy have extended this computer proof to show that $n = 9$ does not admit a full-rank tiling either in [8]. They used to different methods to show this. The first method uses group characters to show that the two sets in the tiling must have certain properties. This calculation was rather slow, it took them 10 days. The second method used the classification of all $[14, 5, 3]$ binary codes and the Dancing Links Algorithm by Donald Knuth in [6]. This calculation was much faster: it took only 18 minutes, assuming that all $[14, 5, 3]$ binary codes were already available.

There exists a full-rank tiling of $\mathbb{F}_2^{10}$, given by

$$V = \begin{array}{c}
(0000000000)\\
(1000000000)\\
(0100000000)\\
(0010000000)\\
(0001000000)\\
(0000100000)\\
(0000010000)\\
(0000001000)\\
(0000000100)\\
(0000000010)\\
(0000000001)\\
(0111110001)\\
(1100010111)\\
(1011001011)\\
(0101101111)\\
(1010111101)
\end{array} \quad \text{and} \quad A = \begin{array}{cccc}
(0000000000) & (1100000100) & (1000000011) & (1010000110)\\
(0001100110) & (1100010010) & (1001010000) & (1010001001)\\
(0000011001) & (1101100010) & (1000101010) & (1011100000)\\
(0001111111) & (1100101001) & (1000010101) & (1011001100)\\
(0000101101) & (1100100111) & (1001100101) & (1010110010)\\
(0001101000) & (1101110100) & (1000111100) & (1010011111)\\
(0000010110) & (1101011001) & (1001110011) & (1011111001)\\
(0001011010) & (1101011110) & (1001001111) & (1011111110)\\
(0101001100) & (0110100000) & (0011000101) & (1111000001)\\
(0101100001) & (0111010000) & (0010001010) & (1110011000)\\
(0101010101) & (0110000011) & (0010011100) & (1111001010)\\
(0100110011) & (0110001101) & (0010110001) & (1110101100)\\
(0100111000) & (0111000110) & (0011110100) & (1110110101)\\
(0101001011) & (0110110110) & (0011010011) & (1111010111)\\
(0100101110) & (0111111010) & (0010100111) & (1110111011)\\
(0100011111) & (0111111101) & (0011101011) & (1111101111).
\end{array}$$

This tiling had been discovered by LeVan and Phelps, but was printed in [8] because LeVan and Phelps never published it themselves.

In the next theorem we look at $p = 3$, so $\mathbb{F}_3^n$. Here, the dimension of $\mathbb{F}_3^n$ to admit a full-rank tiling is lower than for $p = 2$.

Theorem 2.11. *The vector space $\mathbb{F}_3^n$ has a full-rank tiling if and only if $n \geq 6$.*

Szabó and Ward proved in [16] that $\mathbb{F}_3^n$ does not admit a full-rank tiling for $n \leq 4$.

A different approach was used for $n = 5$. Östergård and Szabó used the Dancing Links Algorithm by Donald Knuth described in [6] to show that we can not find any full-rank tilings of $\mathbb{F}_3^5$. In [7], Östergård and Szabó explained that only 9015 possible tiles of $\mathbb{F}_3^5$ needed to be checked and none of them were full-rank. We tried to apply the same idea and we narrowed down the search to only 197 possible tiles by removing equivalent tiles. Still, the algorithm we implemented in magma was too inefficient to run completely during a regular working day.

For $n = 6$, we use the construction in the proof of Theorem 2.5. This gives us a full-rank tiling of $\mathbb{F}_3^6$:

$$V = \begin{matrix} (000000) & (010001) & (120002) \\ (010000) & (010101) & (000212) \\ (000100) & (020101) & (021200) \\ (010100) & (000101) & (100002) \\ (010200) & (000102) & (000211) \\ (010201) & (010102) & (021000) \\ (020201) & (020102) & (021100) \\ (020001) & (020202) & (000210) \\ (000001) & (010202) & (110002) \end{matrix} \quad \text{and} \quad A = \begin{matrix} (000000) & (101000) & (210010) \\ (212110) & (101010) & (100021) \\ (102121) & (211021) & (002100) \\ (100000) & (102100) & (002110) \\ (100010) & (102110) & (000021) \\ (001021) & (001000) & (211000) \\ (210021) & (001010) & (211010) \\ (002121) & (212121) & (101021) \\ (000010) & (210000) & (212100). \end{matrix}$$

Now, using induction and Corollary 2.4, we see that $\mathbb{F}_3^n$ admits a full-rank tiling for $n \geq 6$.

In the next example, we look at $\mathcal{F}_4^2$. We can show that $\mathbb{F}_4^2$ has a full-rank tiling, while we can not find one in $(\mathbb{Z}/4\mathbb{Z})^2$, starting with a tile that looks the same.

Example 2.12. Let $\mathbb{F}_4^2 = \{0, 1, a, a^2\}$ and look at $\mathbb{F}_4^2$. Set $V = \{(00), (10), (01), (11)\}$ and $A = \{(00), (a0), (0a), (aa)\}$. Then clearly $V + A = \mathbb{F}_4^2$ and $(V - V) \cap (A - A) = \{(00)\}$. So (V, A) is a tiling. It is full-rank: $\langle V \rangle = \langle A \rangle = \mathbb{F}_4^2$.

We can try the same for $(\mathbb{Z}/4\mathbb{Z})^2$. Set $V' = \{(00), (10), (01), (11)\}$ and $A' = \{(00), (20), (02), (22)\}$. Then clearly $V' + A' = (\mathbb{Z}/4\mathbb{Z})^2$ and $(V' - V') \cap (A' - A') = \{(00)\}$. So (V', A') is a tiling. It is not full-rank: $\langle V' \rangle = (\mathbb{Z}/4\mathbb{Z})^2$, while $\langle A' \rangle \neq (\mathbb{Z}/4\mathbb{Z})^2$.

In the next two theorems we look at prime numbers greater or equal to 5. There are no exact lower bounds on the dimension of $\mathbb{F}_p^n$ to allow full-rank tilings. Dinitz first proved that $\mathbb{F}_p^{p+1}$ has a full-rank tiling and used this and to show that $\mathbb{F}_p^4$ has a full-rank tiling. We tried to generalize these proofs so that it would also apply to prime powers or all composite numbers greater than 3, but that was not possible because of the zero-divisors.

Theorem 2.13. *For $p \geq 5$ a prime number, $\mathbb{F}_p^{p+1}$ has a full-rank tiling.*

This is Proposition 11 given by Dinitz in [3]. He gave a proof using code theory. We can give a much simpler proof using previous results.

Proof. Since $p \geq 5$ is a prime number, it is odd and $n = \frac{p+1}{2} \geq 3$. Then we can apply Theorem 2.5 to see that $\mathbb{F}_p^{2n}$ has a full-rank tiling. $\square$

Theorem 2.14. *For $p \geq 5$ a prime number, $\mathbb{F}_p^4$ has a full-rank tiling.*

This is Proposition 12 given by Dinitz in [3]. He applied Theorem 1.18 to the tiling he found in the proof of the previous Theorem.

Is this the smallest bound? Rédei conjectured in Problem 5 in the section Open Problems in [9] that $\mathcal{F}_p^3$ does not have a full-rank tiling for any p. This is still an open problem but it has been verified for $p \leq 11$ using Latin squares in [15].

Conjecture 2.15 (Rédei's Conjecture). *For a prime number p, $\mathbb{F}_p^3$ does not have a full-rank tiling.*

Theorem 2.16. *Let p be a prime number, $p \leq 11$. Then $\mathbb{F}_p^3$ does not admit a full-rank tiling.*

Szabó and Ward show this by contradiction. They start with assuming that a full-rank tiling (V, A) of $\mathcal{F}_p^3$ exists for a prime number p, where $|V| = p$ and $|A| = p^2$. Then $\langle V \rangle = \mathcal{F}_p^3$ and there are elements $x, y, z \in V$ generating $\mathcal{F}_p^3$. This means that can write all elements in the set in the form of $ix + jy + kz$ for $i, j, k \in \{0, \ldots, p-1\}$.

This is an immediate proof that for $p = 2, 3$ we cannot have a full-rank tiling. If $p = 2$, then $V = \{0, x\}$ and if $p = 3$ then $V = \{0, x, y\}$. So the tile V contains only one, if $p = 2$, or two, if $p = 3$, nontrivial elements while at least three non-trivial elements are required to generate $\mathcal{F}_p^3$. So $\langle V \rangle \neq \mathcal{F}_p^3$ and the tiling is clearly not full-rank.

So now we look at $5 \leq p \leq 11$. We can code the elements $ix + jy + kz$ by the triple (i, j, k) and consider the p^2 elements of A as a $p \times p$ table T, where $T_{i,j} = k$. Using some possible replacements that are written down in the article, we can consider T to be a Latin square.

At this point, some restrictions are given for the properties of the Latin square. If π is a permutation of $\{0, 1, \ldots, p-1\}$, they say that "the ith row of the Latin square contains (is) permutation π" [15], page 1201. A permutation σ of a finite abelian group $(G, +)$ is said to be a complete mapping of G if $g \mapsto g + \sigma(g)$ for $g \in G$, is again a permutation of G. Szabó and Ward continue to show that the rows and columns of the Latin square are complete mappings of $(\mathbb{F}_p, +)$.

They give one last definition on permutations. The kth transversal of the Latin square T is given by the permutation on $\{0, 1, \ldots, p-1\}$ which maps i to j precisely when $T_{i,j} = k$. These transversals are complete mappings.

The authors then make a claim that if there exists a full-rank tiling of $\mathcal{F}_p^3$, then there is one such that the Latin square belonging to the tiling contains "nonlinear complete mappings in the first column, first row and the 0th transversal", page 1202 in [15].

For $p = 5$, it is shown that all complete mappings are linear, in all possible Latin squares. So this contradicts the claim and the theorem is proven.

For $p = 7, 11$, it requires more work to show that $\mathcal{F}_p^3$ does not have a full-rank tiling. They list all complete mappings fixing 0 and compute that that there do not exist Latin squares of size p that meet the requirements of the claim.

For $p = 13$, the idea of using Latin squares in this way does not work. There are too many Latin Squares possible and it is not possible to proceed in the same way. Szabó and Ward end with saying that further reductions must be made to prove Rédei's conjecture. However, in 2011, Szabó described a computer search to show that $\mathbb{F}_{13}^3$ does not admit full-rank tilings in [14]. He used graph theory combined with the Dancing Links Algorithm to show that there are no full-rank tilings possible.

Full-rank tilings of $\mathbb{F}_q^n$

In this section we want to give an overview for which prime powers q and which n we have a full-rank tiling of $\mathbb{F}_q^n$.

We know that we have an isomorphism of additive groups

$$(\mathbb{F}_{p^k}^n, +) \cong (\mathbb{F}_p^{kn}, +).$$

Using Corollary 2.4 and other results from the previous section, we can make an overview for which q and which n we have a full-rank tiling.

- If $q = 2^k$, then $\mathbb{F}_q^n$ has a full-rank tiling if and only if $kn \geq 10$ by Theorem 2.10;
- If $q = 3^k$, then $\mathbb{F}_q^n$ has a full-rank tiling if and only if $kn \geq 6$ by Theorem 2.11;
- If $q = p^k$ for a prime number $p \geq 5$, then $\mathbb{F}_q^n$ has a full-rank tiling if $kn \geq 4$ by Theorem 2.14.

Example 2.17. In this example we give a full-rank-tiling of $\mathbb{F}_9^3$. Let $\mathbb{F}_9 = \left\{0, 1, a, a^2, a^3, 2, a^5, a^6, a^7\right\}$ and $b = a^2$ and $a^2 = a + 1$.

Then $\phi\colon \mathbb{F}_3^6 \to \mathbb{F}_9^3$ given by $\phi(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 + x_2 b, x_3 + x_4 b, x_5 + x_6 b)$ is an isomorphism of additive abelian groups.

If we look at the full-rank tiling of $\mathbb{F}_3^6$ as given in the proof of Theorem 2.11, we can construct a tiling of $\mathbb{F}_9^3$. We get the sets

$$\phi(V) = \begin{matrix} (0 & 0 & 0) & (a^2 & 0 & a^2) & (a^5 & 0 & a^6) \\ (a^2 & 0 & 0) & (a^2 & a^2 & a^2) & (0 & a^6 & a^5) \\ (0 & a^2 & 0) & (a^6 & a^2 & a^2) & (a^6 & a^5 & 0) \\ (a^2 & a^2 & 0) & (0 & a^2 & a^2) & (1 & 0 & a^6) \\ (a^2 & a^6 & 0) & (0 & a^2 & a^6) & (0 & a^6 & a^7) \\ (a^2 & a^6 & a^2) & (a^2 & a^2 & a^6) & (a^6 & 1 & 0) \\ (a^6 & a^6 & a^2) & (a^6 & a^2 & a^6) & (a^6 & a^7 & 0) \\ (a^6 & 0 & a^2) & (a^6 & a^6 & a^6) & (0 & a^6 & 1) \\ (0 & 0 & a^2) & (a^2 & a^6 & a^6) & (a^7 & 0 & a^6) \end{matrix} \quad \text{and} \quad \phi(A) = \begin{matrix} (0 & 0 & 0) & (1 & 1 & 0) & (a & 0 & 1) \\ (a & a & 1) & (1 & 1 & 1) & (1 & 0 & a) \\ (1 & a & a) & (a & 1 & a) & (0 & a & 0) \\ (1 & 0 & 0) & (1 & a & 0) & (0 & a & 1) \\ (1 & 0 & 1) & (1 & a & 1) & (0 & 0 & a) \\ (0 & 1 & a) & (0 & 1 & 0) & (a & 1 & 0) \\ (a & 0 & a) & (0 & 1 & 1) & (a & 1 & 1) \\ (0 & a & a) & (a & a & a) & (1 & 1 & a) \\ (0 & 0 & 1) & (a & 0 & 0) & (a & a & 0). \end{matrix}$$

Then $\phi(V) + \phi(A) = \mathbb{F}_9^3$ and $(\phi(V) - \phi(V)) \cap (\phi(A) - \phi(A)) = \{(000)\}$. We can verify that $\langle \phi(V) \rangle = \langle \phi(A) \rangle = \mathbb{F}_9^3$ and we conclude that we get a full-rank tiling of $\mathbb{F}_9^3$.

Using the results from the last three sections and especially Theorem 2.3, for most groups we can answer the question whether an arbitrary finite abelian group G has the full-rank property.

Full-dimensional tilings

In this section we look at tilings of $\mathbb{F}_q^n$ for prime powers q. When are they full-dimensional?

Throughout this section, let q be a prime power p^k, where $k \geq 2$.

Theorem 2.18. *If $\mathbb{F}_q^n$ has a full-dimensional tiling, then so has $\mathbb{F}_q^{n+1}$.*

Proof. Let (V, A) be a full-dimensional tiling of $\mathbb{F}_q^n$. We will construct a full-dimensional tiling (V', A') of $\mathbb{F}_q^{n+1}$ out of (V, A).

Suppose there exists $\mathbf{a} \in A$ such that $\mathrm{Span}(A \setminus \{\mathbf{a}\}) = \mathbb{F}_q^{n+1}$. Then set

$$V' = \{(\mathbf{v}, x) \mid \mathbf{v} \in V, x \in \mathbb{F}_q\}$$

and

$$A' = \left\{(\mathbf{a}', 0) \mid \mathbf{a}' \in (A \setminus \{a\})\right\} \cup \{(\mathbf{a}, 1)\}.$$

Clearly, (V', A') is a tiling of $\mathbb{F}_q^{n+1}$. It is also easy to see that it is full-dimensional: $\mathrm{Span}(V') = \mathrm{Span}(A') = \mathbb{F}_q^{n+1}$.

Now, suppose that there does not exists an $\mathbf{a} \in A$ such that $\mathrm{Span}(A \setminus \{\mathbf{a}\}) = \mathbb{F}_q^{n+1}$. Then $V \setminus \{0\}$ and $A \setminus \{0\}$ are both minimal generating sets for $\mathbb{F}_q^n$. This means that $|V| = |A| = n+1$. Since (V, A) tiles $\mathbb{F}_q^n$, we know that

$$q^n = |\mathbb{F}_q^n| = |V| \cdot |A| = (n+1)^2.$$

Since the right-hand side is an even power, the left-hand side must be an even power as well. We assumed that $q = p^k$ and then k must be even, $k = 2k'$. We can rewrite the equation in

$$p^{2k'n} = (n+1)^2.$$

This does not have a solution for prime numbers p and $k', n > 1$. Therefore, we always can find an $\mathbf{a} \in A$ such that $\mathrm{Span}(A \setminus \{\mathbf{a}\}) = \mathbb{F}_q^{n+1}$ and construct a full-dimensional tiling in $\mathbb{F}_q^{n+1}$. □

Theorem 2.19. *If q is a prime power but not a prime number, the vector space $\mathbb{F}_q^n$ has a full-dimensional tiling for all $n \geq 1$.*

Proof. Using the previous Theorem, we only need to prove that $\mathbb{F}_q$ has a full-dimensional tiling.

Take $V = \{0, 1, \ldots, p-1\}$ a subgroup of order p in $\mathbb{F}_q$. Then we immediately see that $\mathrm{Span}(V) = \mathbb{F}_q$. For A, we need p^{k-1} elements. We take A to be the set of all coset representatives of V, then $\mathrm{Span}(A) = \mathbb{F}_q$ as well. Clearly, (V, A) tiles $\mathbb{F}_q$ and it is full-dimensional.

□

For $q = 4$, $\mathbb{F}_q = \{0, 1, a, a^2\}$ is the splitting field of $x^2 + x + 1$ over $\mathbb{F}_2[x]$. The sets $V = \{0, 1\}$ and $A = \{0, a\}$ give a full-dimensional tiling of $\mathbb{F}_4$.

A full-dimensional tiling of $\mathbb{F}_4^2$ is given by

$$V = \begin{matrix} (0 & 0) \\ (0 & 1) \\ (1 & 0) \\ (1 & 1) \end{matrix} \qquad \text{and} \qquad A = \begin{matrix} (0 & 0) \\ (0 & a) \\ (a & 0) \\ (a & a). \end{matrix}$$

For $q = 8$, $\mathbb{F}_q = \{0, 1, a, a^2, a^3, a^4, a^5, a^6\}$, that is the splitting field of $x^3 + x + 1$ over $\mathbb{F}_2[x]$. A full-dimensional tiling of $\mathbb{F}_8^2$ is given by

$$V = \begin{matrix} (0 & 0) \\ (1 & 0) \\ (0 & 1) \\ (a & 0) \\ (0 & a) \\ (1 & a^2) \\ (a^2 & 1) \\ (a^4 & a^4) \end{matrix} \qquad \text{and} \qquad A = \begin{matrix} (0 & 0) \\ (0 & a^4) \\ (a^6 & 0) \\ (a^6 & a^4) \\ (a^4 & a^6) \\ (a^4 & a^3) \\ (a^3 & a^6) \\ (a^3 & a^3). \end{matrix}$$

For $q = 9$, $\mathbb{F}_q = \{0, 1, a, a^2, a^3, 2, a^5, a^6, a^7\}$ is the splitting field of $x^2 + 2x + 2$ over $\mathbb{F}_3[x]$. A full-dimensional tiling of $\mathbb{F}_9^2$ is given by

$$V = \begin{array}{cc} (0 & 0) \\ (1 & 0) \\ (0 & 1) \\ (1 & 1) \\ (a & 0) \\ (0 & a) \\ (a & a) \\ (a & 1) \\ (1 & a) \end{array} \qquad \text{and} \qquad A = \begin{array}{cc} (0 & 0) \\ (2 & a^6) \\ (1 & a^2) \\ (a^2 & 0) \\ (a^5 & a^2) \\ (a^3 & a^6) \\ (a^6 & 0) \\ (a & a^6) \\ (a^7 & a^2) \end{array}.$$

The tiling we found for $\mathbb{F}_4^2$, can be sent to $\mathbb{F}_{16}$. The additive groups are isomorphic via $\phi\colon \mathbb{F}_4^2 \to \mathbb{F}_{16}$, $\phi(x, y) = x + y\alpha$. Then $\phi(V) = \{0, 1, \alpha, \alpha^4\}$ and $\phi(A) = \{0, \alpha^6, \alpha^5, \alpha^9\}$. This gives a full-dimensional tiling of $\mathbb{F}_{16}$.

For $q = 25$, $x^2 + 4x + 2$

$$V = \begin{array}{cc} (0 & 0) \\ (1 & 0) \\ (0 & 1) \\ (1 & 1) \\ (a & 0) \\ (0 & a) \\ (a & a) \\ (1 & a) \\ (a & 1) \\ (a^{13} & 0) \\ (0 & a^{13}) \\ (a^{13} & a^{13}) \\ (1 & a^{13}) \end{array} \; \begin{array}{cc} (a^{13} & 1) \\ (a & a^{13}) \\ (a^{13} & a) \\ (a^7 & 0) \\ (0 & a^7) \\ (a^7 & a^7) \\ (1 & a^7) \\ (a^7 & 1) \\ (a^7 & a^{13}) \\ (a^{13} & a^7) \\ (a^7 & a) \\ (a & a^7) \end{array} \qquad \text{and} \qquad A = \begin{array}{cc} (0 & 0) \\ (a^2 & 0) \\ (a^8 & 0) \\ (a^{14} & 0) \\ (a^{20} & 0) \\ (0 & a^2) \\ (a^2 & a^2) \\ (a^8 & a^2) \\ (a^{14} & a^2) \\ (a^{20} & a^2) \\ (0 & a^8) \\ (a^2 & a^8) \\ (a^8 & a^8) \end{array} \; \begin{array}{cc} (a^{14} & a^8) \\ (a^{20} & a^8) \\ (0 & a^{14}) \\ (a^2 & a^{14}) \\ (a^8 & a^{14}) \\ (a^{14} & a^{14}) \\ (a^{20} & a^{14}) \\ (0 & a^{20}) \\ (a^2 & a^{20}) \\ (a^8 & a^{20}) \\ (a^{14} & a^{20}) \\ (a^{20} & a^{20}) \end{array}.$$

Chapter 3

Tilings Of Binary Spaces And Perfect Binary Codes

In this section, we will consider tilings of binary spaces. We look at the correspondence between tilings and perfect binary codes. We will show that for each tiling (V, A) of $\mathbb{F}_2^n$ we can construct a perfect binary code of length $|V|-1$ in a unique way. Furthermore, we will show that given a perfect binary code and a given linear subcode, we can construct a tiling (V, A) such that $\langle V \rangle = \mathbb{F}_2^n$.

Theorem 3.1. *Let (V, A) be a tiling of $\mathbb{F}_2^n$ and let $\nu = |V| - 1$. Let $H(V)$ be the $n \times \nu$ matrix consisting of elements of $V \setminus \{0\}$, in some fixed order, as its columns. Let $C = \{c \in \mathbb{F}_2^\nu \mid H(V)c^T \in A\}$. Then C is a perfect binary code of length ν.*

This is Proposition 7.1 in [1].

The code constructed in this way is not unique. In the next example we will look at two different tilings that give the same code.

Example 3.2. Let (V_1, A_1) be the tiling of $\mathbb{F}_2^3$ given by

$$V_1 = \{(000), (001), (010), (111)\}, \ A_1 = \{(000), (100)\}$$

and let (V_2, A_2) be the tiling of $\mathbb{F}_2^3$ given by

$$V_2 = \{(000), (101), (011), (111)\}, \ A_2 = \{(000), (001)\}.$$

The matrix $H(V_1)$ belonging to V_1 is given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

and $H(V_2)$ is given by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Now we see that

$$\{c \in \mathbb{F}_2^\nu \mid H(V_1)c^T \in A_1\} = \{c \in \mathbb{F}_2^\nu \mid H(V_2)c^T \in A_2\} = \{(000), (111)\}.$$

So the perfect codes belonging to (V_1, A_1) and (V_2, A_2) are the same while the tilings (V_1, A_2) and (V_2, A_2) are not equivalent.

In the next example we show that this theorem might not work on all other finite vector spaces.

Example 3.3. Let (V, A) be the tiling of $\mathbb{F}_3^3$ given by

$$V = \{(000), (100), (020)\}, A = \{(000), (220), (211), (102), (212), (021), (110), (022), (101)\}.$$

We compute

$$H(V) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

and see that

$$C = \left\{c \in \mathbb{F}_3^2 \mid Hc^T \in A\right\} = \{(00), (11), (22)\}.$$

It is clear that this is a code but that it is not perfect.

The following theorem is somewhat the converse of Theorem 3.1: given a perfect code and a linear subcode, we can construct a tiling.

Theorem 3.4. *Let C be a perfect binary code of length ν and let Γ be a linear subcode of C such that $\Gamma + C = C$. Set $\gamma = \dim(\Gamma)$ and let $H(\Gamma)$ be the $(\nu - \gamma) \times \nu$ parity-check matrix of Γ. Take $V = \{0\} \cup \{\text{columns of } H(\Gamma)\}$ and define $A = \left\{H(\Gamma)c^T \mid c \in C\right\}$. Then (V, A) is a tiling of $\mathbb{F}_2^{\nu-\gamma}$ and $\langle V \rangle = \mathbb{F}_2^{\nu-\gamma}$.*

This is Proposition 7.6 in [1].

Chapter 4

Open Problems

- Conjecture 2.15 is still open. It would be nice to give an all-encompassing proof that $\mathbb{F}_p^3$ does not allow full-rank tilings for prime numbers p.
- We would like to prove Conjecture 2.9, or even its generalization that $\mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/b\mathbb{Z}$ does not have a full-rank tiling for all $a, b \geq 2$. This is proven for prime numbers less than 17, but not for all numbers.

Bibliography

[1] Gerard Cohen, Simon Litsyn, Alexander Vardy, and Gilles Zémor. Tilings Of Binary Spaces. SIAM J. Discrete Mathematics, 9:393–412, 1996.

[2] Don Coppersmith and Victor Miller. Binary Non-Tiles. arXiv:0911.1388v2, 2011.

[3] Michael Dinitz. Full-Rank Tilings Of Finite Abelian Groups. SIAM J. Discrete Mathematics, 20(1):160–170, 2006.

[4] Tuvi Etzion and Alexander Vardy. On Perfect Codes And Tilings: Problems And Solutions. SIAM J. Discrete Mathematics, 11(2):205–223, 1998.

[5] Georg Hajós. Über Einfache Und Mehrfache Bedeckung Des n-dimensionalen Raumes Mit einem Würfelgitter. Mathematische Zeitschrift, 47:427–467, 1942.

[6] Donald Knuth. Dancing Links. Millenial Perspectives In Computer Science, pages 187–214, 2000.

[7] Patric Östergård and Sándor Szabó. Elementary p-groups With The Rédei Property. International Journal Of Algebra And Computation, 17(1):171–178, April 2006.

[8] Patric Östergård and Alexander Vardy. Resolving The Existence Of Full-Rank Tilings Of Binary Hamming Spaces. SIAM J. Discrete Mathematics, 18(2):382–387, 2004.

[9] László Rédei. Lacunary Polynomials Over Finite Fields. Akadémia Kiadó, Budapest, Hungary, January 1973.

[10] Arthur Sands. Factoring Finite Abelian Groups. Journal Of Algebra, 275(2):540–549, 2004.

[11] Sándor Szabó. A Type Of Factorization Of Finite Abelian Groups. Discrete Mathematics, 54:121–124, 1985.

[12] Sándor Szabó. Groups With The Rédei Property. Le Matematiche, 52(2):357–364, 1997.

[13] Sándor Szabó. Constructions Related To The Rédei Property Of Groups. J. London Mathematical Society, 73(2):701–706, 2006.

[14] Sándor Szabó. Verifying A Conjecture Of L. Rédei For $p = 13$. Mathematics of Computation, 80(274):1155–1162, 2011.

[15] Sándor Szabó and Coburn Ward. Factoring Elementary Groups Of Prime Cube Order Into Subsets. Mathematics Of Computation, 67:1199–1206, 1998.

[16] Sandór Szabó and Coburn Ward. Factoring Groups Having Periodic Maximal Subgroups. Boletín de la Sociedad Matemática Mexicana, 5(3):327–333, 1999.

[17] Ari Trachtenberg and Alexander Vardy. Full-Rank Tilings Of $\mathbb{F}_2^8$ Do Not Exist. SIAM J. Discrete Mathematics, 16:390–392, 2003.