Adaptive composite grid methods for problems of plasticity

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Abstract: The incremental FE method, which is a widely used procedure for the solution of problems of elasto-plasticity, is considered in this paper. An adaptive variant of this method is described as a tool for efficient computation of sufficiently accurate approximate solution. The main investigation is then devoted to the use of composite FE grids within the incremental FE methods. The attention is devoted mainly to implementation and convergence analysis of the Newton–FAC/BEPS iterative procedures for the solution of nonlinear systems arising within the load steps.

Keywords: Elasto-plasticity; Incremental FE method; Composite grid FE method; FAC and BEPS methods; Newton-FAC/BEPS iterations

1 Introduction

The paper concerns the solution of problems of elasto–plasticity by the incremental FE method which seems to be the most widely used method for solving history dependent nonlinear problems in structural mechanics. The development of this method firstly concerned the use of various discretization procedures which differ mainly in the definition of the stress-strain operator within the load increments, see e.g. [2], [8], [17], [16]. Lot of the effort has been also devoted to constructing of efficient solution procedures for the nonlinear systems which appear in each load step. For the use of the Newton–type procedures, we can refer again to [2], [8], [16] and the references therein, for the use of other techniques as e.g. Newton–like procedures with successively constructed Jacobian, see e.g. [21].

Recent development of the incremental FE method includes also aposteriori error estimation and adaptivity. Restricting to the adaptivity in space, such development is described in the literature [13], [12], [19] and the references therein. Remarkable is the contemporary progress, see [18], [14], [1], [20] and [15].

The adaptivity developed in the above literature is based on the use of unstructured locally adapted FE grids. A contribution of this paper is to discuss the use of composite FE grids which may preserve some grid regularity which is advantageous for the computer implementation. The use of composite grids can bring also other benefits in the vectorization and parallelization of the computing.

This paper is organized as follows. In Section 2, we briefly describe some model elasto–plasticity problems. Section 3 describes the adaptive incremental procedure and its most important parts. Section 4 introduces the composite grids and describes their implementation within the incremental FE algorithm. It describes also two basic iterative procedures, FAC and BEPS, which are used for solving linear composite grid problems. These procedures are then used within the Newton–type methods for solving linear systems arising within the Newton iterations. The convergence of the

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arising Newton–FAC/BEPS iterations is investigated in Section 5 . In the Concluding remarks, we discuss the possibility of extension of the convergence results and the use of semi–structured composite grids.

2 Elasto–plasticity

In our paper, we shall consider *problems of elasto–plasticity* formulated as follows: for a given domain $\Omega \subset R^d$, $d = 2, 3$, and a history interval $I_t = [0, t^*]$ find the *displacements* $u : \Omega \times I \to R^d$, *stresses* $\sigma : \Omega \times I \to S \subset R^{d^2}$ and *hardening parameters* $\kappa : \Omega \times I \to H \subset R^m (m \geq 1)$ such that the following relations are valid in $\Omega \times I_t$ (dot denotes the derivation with respect to the history parameter):

$\varepsilon = \varepsilon^e + \varepsilon^p$	(strain decomposition)
$\varepsilon = \varepsilon(u)$	(compatibility, small strains)
$P(\sigma, \kappa) \leq 0$	(yield criterion)
$\sigma = D\varepsilon^e$	(generalized Hooke's law)
$\dot{\varepsilon}^p = \gamma p$	(flow rule, $p \in S$ is the flow direction, $\gamma \geq 0$ is the plastic multiplicator)
$p = \frac{\partial P}{\partial \sigma}(\sigma, \kappa)$	(associated plasticity)
$\dot{\kappa} = \gamma z$	($z \in H$, hardening rule)

As an *model example* we shall consider the *von Mises plasticity with isotropic strain hardening* given by

$$D\varepsilon = (3\lambda + 2\mu)\mathrm{vol}(\varepsilon) \; + \; 2\mu \mathrm{dev}(\varepsilon), \tag{1}$$

$$P(\sigma, \kappa) = \sqrt{\frac{3}{2}} \parallel \mathrm{dev}(\sigma) \parallel - \mathcal{H}(\kappa)\,, \tag{2}$$

$$\kappa = \sqrt{\frac{2}{3}} \int_0^t \parallel \dot{\varepsilon}^p \parallel \; dt, \tag{3}$$

$$p = \frac{\partial P}{\partial \sigma} = \sqrt{\frac{3}{2}} \frac{\mathrm{dev}(\sigma)}{\parallel \mathrm{dev}(\sigma) \parallel}\,, \quad q = \frac{\partial P}{\partial \kappa} \; = \; -\mathcal{H}^{'}\,, \quad z = 1\,. \tag{4}$$

Note that $\mathrm{vol}(\sigma)$ and $\mathrm{dev}(\sigma)$ denote respectively the volumetric and deviatoric part of σ, λ, μ are Lamè moduli and $\mathcal{H}$ is assumed to be a twice differentiable function, $\mathcal{H}(\kappa) \geq \mathcal{H}_0 > 0$, $\mathcal{H}'(\kappa) \geq \mathcal{H}_1 > 0$, $\mid \mathcal{H}''(\kappa) \mid \leq \mathcal{H}_2$ for all $\kappa \geq 0$. In the sequel, we shall also use the estimate

$$\frac{p^T D p}{p^T D p - q^T z} \; = \; \frac{3\mu}{3\mu + \mathcal{H}'} \; \leq \; \frac{3\mu}{3\mu + \mathcal{H}_0} \; = \; \nu_0 \; < \; 1. \tag{5}$$

The problem of elasto–plasticity can be then formulated as an initial - boundary value problem with respect to the history parameter $t \in I_t$, cf. [8].

3 Adaptive incremental FE method

To construct an approximation to the solution of the initial - boundary value problem of elasto-plasticity

$$\chi : \Omega \times I_t \; \to \; R^d \times S \times H, \quad \chi \; = \; \chi(x, t) \; = \; (u, \sigma, \kappa),$$

we shall use the *incremental finite element method.*

For the standard incremental FE method, see [8], [21], [2] or [4], we have to choose apriori a division of the history interval

$$I_t \; = \; \langle 0, t^* \rangle \; = \; \bigcup_{k=1}^{k^*} \langle t_{k-1}, t_k \rangle,$$

and replace the space V of admissible displacements by a finite element space V_h with the corresponding spaces of stresses S_h and hardening parameters H_h.

$\chi_h^0 \;=\; 0,\;$ i.e. $u_h^0 = 0, \quad \sigma_h^0 = 0, \quad \kappa_h^0 = 0 \quad$ (initial step)

for $k = 1, \cdots, k^*$ (load steps) **do**

(S1) compute $\triangle\chi \;=\; (\triangle u_h,\, \triangle\sigma_h,\, \triangle\kappa_h) \,\in\, V_h \times S_h \times H_h$:

$$\int_\Omega \triangle\sigma_h^T\, \varepsilon(v_h)\, dx = (\triangle F^k, v_h) \qquad \text{for all } v_h \epsilon V_h\,,$$

$$\triangle\sigma_h = T_\sigma(\sigma_h^{k-1}, \kappa_h^{k-1}, \triangle u_h)$$

(S2) compute

$$\triangle\kappa_h = T_\kappa(\sigma_h^{k-1}, \kappa_h^{k-1}, \triangle u_h)$$

(S3) put $u_h^k \,=\, u_h^{k-1} + \triangle u_h, \quad \sigma_h^k \,=\, \sigma_h^{k-1} + \triangle\sigma_h, \quad \kappa_h^k \,=\, \kappa_h^{k-1} + \triangle\kappa_h\,.$

end

Figure 1: Incremental FE algorithm.

Set $V_h = V_{h_0},\, S_h = S_{h_0},\, H_h = H_{h_0}$

$\chi_h^0 \;=\; 0,\;$ i.e. $u_h^0 = 0, \quad \sigma_h^0 = 0, \quad \kappa_h^0 = 0 \quad$ (initial step)

for $k = 1, \cdots, k^*$ (load steps) **do**

(S1) set $V_h = V_{h_k} = V_{h_{k-1}},\, S_h = S_{h_k} = S_{h_{k-1}},\, H_h = H_{h_k} = H_{h_{k-1}}$

(S2) compute $\triangle\chi \;=\; (\triangle u_h,\, \triangle\sigma_h,\, \triangle\kappa_h) \,\in\, V_h \times S_h \times H_h$:

$$\int_\Omega \triangle\sigma_h^T\, \varepsilon(v_h)\, dx = (\triangle F^k, v_h) \qquad \text{for all } v_h \epsilon V_h\,,$$

$$\triangle\sigma_h = T_\sigma(\sigma_h^{k-1}, \kappa_h^{k-1}, \triangle u_h)$$

(S3)
- (a) evaluate the error indicators
- (b) **if** error indicators show sufficient accuracy **then** go to (S4)
- (c) refine the mesh, construct $V_{h_k}(new), S_{h_k}(new), H_{h_k}(new)$
- (d) interpolate the data $u_h^{k-1}, \sigma_h^{k-1}, \kappa_h^{k-1}$ from $V_h \times S_h \times H_h$ to $V_{h_k}(new) \times S_{h_k}(new) \times H_{h_k}(new)$
- (e) go to (S2) with $V_h = V_{h_k}(new), S_h = S_{h_k}(new), H_h = H_{h_k}(new)$

(S4) compute

$$\triangle\kappa_h = T_\kappa(\sigma_h^{k-1}, \kappa_h^{k-1}, \triangle u_h)$$

(S5) put $u_h^k \,=\, u_h^{k-1} + \triangle u_h, \quad \sigma_h^k \,=\, \sigma_h^{k-1} + \triangle\sigma_h, \quad \kappa_h^k \,=\, \kappa_h^{k-1} + \triangle\kappa_h\,.$

end

Figure 2: Adaptive incremental FE algorithm.

For $\theta \in \langle 0, 1\rangle$, we assume that $\chi(t_k + \theta(t_{k+1} - t_k))$ is approximated by $\chi(t_k) + \theta(\chi(t_{k+1}) - \chi(t_k))$ with $\chi(t_k)$ approximated by χ_h^k where

$$\chi_h^k \;=\; (u_h^k, \sigma_h^k, \kappa_h^k) \,\in V_h \times S_h \times H_h$$

are computed by the incremental FE algorithm, see Figure 1, where ΔF^k represents the k-th load increment, T_σ and T_κ are operators given by the adopted discretization scheme, cf. [2] where two alternatives given by the elasto-plastic incremental relations and the return mapping are described, see also e.g. [17], [16] and [8].

The accuracy of the approximate solution, which is computed by the incremental FE algorithm of Figure 1, depends on the adopted discretization of the history interval I_t and the problem domain Ω. Two questions arise here, how to estimate the accuracy and how to efficiently compute sufficiently accurate approximate solution.

A progress in this direction can be obtained by using the adaptive incremental FE method of Figure 2, cf. [12], [19], [18]. This method updates the FE discretization during the incremental loading and produces a sequence of values χ_h^k where $\chi_h^k \in V_{h_k} \times S_{h_k} \times H_{h_k}$. The adaptivity within this algorithm concerns only the space discretization and consists in three procedures within the step (S3): evaluating of some error indicators, constructing of new FE spaces and transferring the data computed in the previous load step to a newly constructed FE spaces.

As error indicators, it is possible to use similar error indicators as for elasticity problems. It means error indicators based on the Zienkiewicz-Zhu technique [13], on the residual estimators [1] or on the duality [14], [15] which are applied to the energy norm defined by the corresponding elasticity operator. Also the hardening terms can be included into the energy norm. We can also define indicators based on the dissipation energy functional or the plastic work, see [13].

The construction of FE grids can be done similarly as in elasticity although it is also possible to give a special care to the boundary between the elastic and plastic regions.

The transfer of data from the previous load steps also requires a special care, the interpolation itself is not sufficiently accurate, see [12], [19], [18].

4 Incremental FE with composite grids

In this Section, we shall discuss the use of composite FE grids within the incremental FE algorithms. We shall start with a brief description of the concept of *composite FE grid* in a domain Ω.

1. Let $\mathcal{T}_H$ be a FE division of Ω and U_H be the corresponding FE space, $U_H = \{v \in C(\bar{\Omega}) : v\,|_E \in \mathcal{P}\ \forall E \in \mathcal{T}_H\}$. Here $\mathcal{P}$ is a set of polynomials. We shall assume lagrangian FE spaces, i.e. the functions from U_H can be represented by their values in the set $N(\mathcal{T}_H)$ of nodes corresponding to $\mathcal{T}_H$.

2. Let Ω_R be a subdomain of Ω where we need to refine the grid. Let us assume $\Omega_R = \bigcup\{E \in \mathcal{T}_R \subset \mathcal{T}_H\}$.

3. Let $\mathcal{T}_h$ be a FE division of Ω_R constructed by a refinement of the elements from $\mathcal{T}_R$. Let U_h be the corresponding FE space, $U_h = \{v \in C(\bar{\Omega}_R) : v\,|_E \in \mathcal{P}\ \forall\, E \in \mathcal{T}_h\}$.

4. The composite grid division $\mathcal{T}_{H,h}$ of the whole domain Ω can be now defined as $\mathcal{T}_{H,h} = (\mathcal{T}_H \setminus \mathcal{T}_R) \cup \mathcal{T}_h$ and the corresponding composite grid FE space as $U_{H,h} = \{v \in C(\bar{\Omega}) : v\,|_E \in \mathcal{P}\ \forall E \in \mathcal{T}_{H,h}\}$.

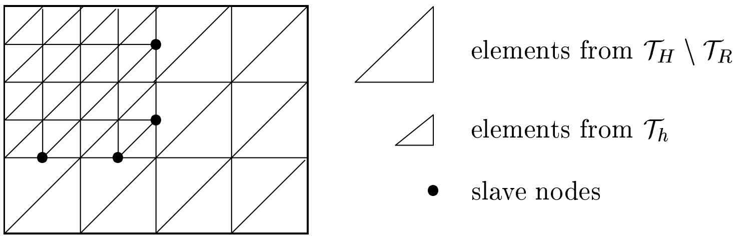

Figure 3: Composite grid FE division of the domain.

The set of nodes corresponding to the composite grid $\mathcal{T}_{H,h}$ will be $N(\mathcal{T}_{H,h}) = N(\mathcal{T}_H) \cup N(\mathcal{T}_h)$. On the contrary to the standard FE method, $N(\mathcal{T}_{H,h})$ will contain a nonempty subset $N_S(\mathcal{T}_{H,h})$ of slave or hanged nodes. For $v \in U_{H,h}$, the value of v in a slave node is determined by the value of v in the neighbour nodes due to the continuity requirement. This continuity requirement defines an interpolation

$$I_S : N(\mathcal{T}_{H,h}) \setminus N_S(\mathcal{T}_{H,h}) \;\rightarrow\; N(\mathcal{T}_{H,h})\,.$$

which will be important for the implementation of the *composite grid FE method.*

For the numerical solution of problems of elasto–plasticity in a domain Ω which is fixed along a part Γ_0 of its boundary and loaded on the remaining part Γ_1 of the boundary, we introduce the space $V_{H,h} \;=\; \left\{ v \in [U_{H,h}]^d : \quad v \;=\; 0 \quad \text{on} \quad \Gamma_0 \right\}$. After definition of the standard FE basis in $V_{H,h}$, we get an isomorphism $I_{H,h} : R_{H,h} \;\equiv\; R^{n_1} \rightarrow V_{H,h}$.

This isomorphism allows us to write the algebraic form of the incremental FE method introduced in the previous Section. Numerically most work consuming part of the incremental FE method will be the step (S1) which represents solution of the nonlinear system

$$\mathcal{F}_k(\mathbf{\Delta u}) \;=\; \mathbf{\Delta f}^k \qquad \mathbf{\Delta u}, \mathbf{\Delta f^k} \in R_{H,h} \tag{6}$$

where the operator $\mathcal{F}_k$ is given by the identity

$$\langle \mathcal{F}_k(\mathbf{u}), \mathbf{v} \rangle \;=\; \int\limits_{\Omega} \langle T_\sigma \left(\sigma_h^{k-1}, \kappa_h^{k-1}, I_{H,h}\mathbf{u} \right), \varepsilon\left(I_{H,h}\mathbf{v} \right) \rangle \, dx \tag{7}$$

which must hold for all $\mathbf{u}, \mathbf{v} \in R_{H,h}$.

The system (6) can be solved by the Newton–type methods of the following form

$\mathbf{\Delta u}^0$ given

for $i \;=\; 0, 1, ...$ **until** convergence **do**

$$\mathbf{r}^{\,i} \;=\; \mathbf{\Delta f}^k - \mathcal{F}_k(\mathbf{\Delta u}^{\,i})$$

$$\text{solve} \qquad A_{k,i}\mathbf{w}^{\,i} \;=\; \mathbf{r}^{\,i}$$

$$\mathbf{\Delta u}^{i+1} \;=\; \mathbf{\Delta u}^{\,i} + \omega_{\,i}\mathbf{w}^{\,i}$$

end

Figure 4: Newton-type algorithm.

Note that:

1. As the initial guess $\mathbf{\Delta} u^0$, we can take zero or a part of the initial elastic displacement or the value $\mathbf{\Delta u}$ from the previous load increment with some scaling if Δf^{k-1} and Δf^k are not equal.

2. As the matrix $A_{k,i}$, we can take the stiffness matrix A_e of the corresponding elasticity problem or so called elasto–plastic tangent matrix (see e.g. [8], [21], [2], [4]) or the Jacobian of the mapping $\mathcal{F}_k$ with the exception of the states on the boundary between elastic and plastic regions where this Jacobian does not exist. In the last case, this Jacobian can be computed analytically or numerically (see [2], [3], [4], [19]).

3. $\omega_{\,i} \in (0, 1\rangle$ is a damping factor. According to the analysis in [2], [8], [3], [4] such damping is mostly not necessary with an exception of the Newton method.

Now, let us consider the implementation of the two main steps from the above Newton–type algorithm in the case of composite grids.

The first step is computation of the residual. This computation can be performed in an element–by–element way due to the expression of $\mathcal{F}_k$ from (7). To be more precise, let us for example restrict to the linear finite elements with constant strain and stress fields. Then

$$\mathcal{F}_k(\mathbf{\Delta u}) \;=\; I_S^T \left(\sum_{E \in \mathcal{T}_{H,h}} B_E^T \sigma_E \;\mid E \mid \right) \tag{8}$$

where I_S is the previously defined interpolation to all composite grid nodes, $B_E \mathbf{v} = \varepsilon_E$, ε_E is the value of $\varepsilon\,(I_{H,h}\,\mathbf{v})$ in E for $\mathbf{v} \in R_{H,h}$ and $E \in \mathcal{T}_{H,h}$, σ_E is the value of the stress in E which is computed by means of T_σ and $\mid E \mid$ is the volume or area of E.

The second main step of the Newton–type algorithm is the solution of the linear system,

$$A_{k,i}\, \mathbf{w}^i \;=\; \mathbf{r}^{\,i}. \tag{9}$$

Here, the matrix $A_{k,i}$ is given by a similar expression as $\mathcal{F}_k$,

$$\langle A_{ki}\mathbf{u}, \mathbf{v}\rangle \;=\; a_{ki}\,(I_{H,h}\mathbf{u},\, I_{H,h}\mathbf{v}) \qquad \forall \mathbf{u}, \mathbf{v} \in R_{H,h} \tag{10}$$

where

$$a_{ki}(u,v) \;=\; \int_\Omega \langle D_{ki}\, \varepsilon(u), \varepsilon(v)\rangle \, d\Omega\,. \tag{11}$$

Here, $D_{k,i}$ is the elasticity (Hooke's) matrix or the elasto–plastic tangent matrix or the Jacobian of T_σ in $\left(\sigma_h^{k-1}, \kappa_h^{k-1}, \Delta u_h^i\right)$. It means that the inner residuals (residuals of the linear systems)

$$\mathbf{r}^{\,i,j} \;=\; \mathbf{r}^{\,i} - A_{ki}\, \mathbf{w}^{i,j} \tag{12}$$

can be again computed in the element-by–element way and that the system (9) can be solved by the Richardson or conjugate gradient iterative methods without necessity of assembling A_{ki} .

To construct efficient preconditioners for the above iterative methods, we shall consider the grids $\mathcal{T}_H$ and $\mathcal{T}_h$ associated with $\mathcal{T}_{H,h}$ and the corresponding FE spaces

$$V_H \;=\; \left\{ v \in U_H^d : \quad v \;=\; 0 \quad \text{on} \quad \Gamma_0 \right\}, \tag{13}$$

$$V_h \;=\; \left\{ v \in U_h^d : \quad v \;=\; 0 \quad \text{on} \quad \partial\Omega_R \setminus \Gamma_1 \right\}. \tag{14}$$

The choice of the standard FE bases in V_H and V_h then defines two isomorphisms

$$I_H : R_H \;=\; R^{n_2} \;\rightarrow\; V_H\,, \quad I_h : R_h \;=\; R^{n_3} \;\rightarrow\; V_h\,. \tag{15}$$

Natural embedings $V_H \subset V_{H,h}$ and $\hat{V}_h \subset V_{H,h}$, where $\hat{V}_h$ is created by functions from V_h which are extended by zero to the whole Ω, define two operators

$$I_H^{H,h} : R_H \;\rightarrow\; R_{H,h}\,, \quad I_h^{H,h} : R_h \;\rightarrow\; R_{H,h}\,. \tag{16}$$

With the aid of the identity (10) with $R_{H,h}$ replaced by R_H or R_h we can now define two auxiliary matrices $A_{k,i}^H$ and $A_{k,i}^h$. It also holds that

$$A_{k,i}^H \;=\; \left(I_H^{H,h}\right)^T A_{k,i}\, I_H^{H,h}, \quad A_{k,i}^h \;=\; \left(I_h^{H,h}\right)^T A_{k,i}\, I_h^{H,h}\,. \tag{17}$$

Both $A^H_{k,i}$ and $A^h_{k,i}$ will be regular and we can introduce two other operators

$$B^H_{k,i} = I^{H,h}_H \, (A^H_{k,i})^{-1} \left(I^{H,h}_H\right)^T , \quad B^h_{k,i} = I^{H,h}_h \, (A^h_{k,i})^{-1} \left(I^{H,h}_h\right)^T . \tag{18}$$

The use of $B^H_{k,i}$ and $B^h_{k,i}$ as preconditioners give the FAC method (see [11] and [9]). This method applied to (9) gives

$$\mathbf{w}^{i,0} \qquad \text{given} \tag{19}$$

$$\mathbf{w}^{i,j+1/2} = \mathbf{w}^{i,j} + B^h_{k,i} \left(\mathbf{r}^{\,i} - A_{k,i}\, \mathbf{w}^{i,j}\right) \tag{20}$$

$$\mathbf{w}^{i,j+1} = \mathbf{w}^{i,j+1/2} + B^H_{k,i} \left(\mathbf{r}^{\,i} - A_{k,i}\, \mathbf{w}^{i,j+1/2}\right) \tag{21}$$

When we solve (9) by the conjugate gradient method then a symmetric preconditioner can be constructed by using one and half iteration of the FAC method starting from the zero initial guess. It gives the BEPS preconditioner $B \simeq A^{-1}$, see [6] or [9]. The operation $\mathbf{v} = B\mathbf{r}$ is performed in the following steps

$$\mathbf{v}^{1/2} = B^h_{i,k}\, \mathbf{r} \tag{22}$$

$$\mathbf{v}^{2/2} = \mathbf{v}^{1/2} + B^H_{i,k} \left(\mathbf{r} - A_{k,i}\, \mathbf{v}^{1/2}\right) \tag{23}$$

$$\mathbf{v}^{3/2} = \mathbf{v}^{2/2} + B^h_{i,k} \left(\mathbf{r} - A_{k,i}\, \mathbf{v}^{2/2}\right) \tag{24}$$

$$\mathbf{v} = \mathbf{v}^{3/2} . \tag{25}$$

Note that

- the subproblems with $A^H_{k,i}$ and $A^h_{k,i}$ can be again solved only approximately by some most inner iterative method,
- it is easier to keep the subgrids $\mathcal{T}_H$ and $\mathcal{T}_h$ to be structured and not too much anisotropic which gives advantages of efficient matrix storage, vectorization of matrix by vector product and good convergence of the most inner iterative method.
- both FAC and BEPS methods leads to a problem decomposition during the solution of the linear system (9). It is also possible to use the parallel variants of the FAC and BEPS methods, e.g. AFAC (see [11], [9]) or additive preconditioner $B_{par} = B^h_{i,k} + B^H_{i,k}$.

5 Convergence of the Newton–FAC/BEPS methods

In this Section, we firstly remember the convergence results for FAC and BEPS method applied to elasticity problems. Then, we shall use these results for analysis of the convergence of the "simplest" Newton–FAC/BEPS method based on the initial stiffness method for solving the nonlinear system (6) and FAC or BEPS method for solving the auxiliary linear systems (9).

Let us consider an elasticity problem with the corresponding bilinear form $a(u,v) = a_{k,i}(u,v)$ defined in (11) with the elasticity matrix $D_{ki} = D$ depending on the Lamè moduli λ, μ or alternatively on the elasticity modulus E and the Poisson's ratio ν. Assume that λ, μ, E and ν are constant on the elements of $\mathcal{T}_H$.

Assume for simplicity that $\mathcal{T}_H$ arises by dividing the computational domain Ω into squares or cubes which are subsequently divided into two triangles or six tetrahedra, see Figure 5. $\mathcal{T}_h$ then arises by dividing some squares into four squares or some cubes into eight cubes which are again subsequently divided in the same manner as before, see Figure 6.

Then there are a constant $\gamma < 1$, which does not depend on the density of $\mathcal{T}_H$ (we shall call it $h-$independent), such that

$$| \, a(u,v) \, | \leq \gamma \sqrt{a(u,u)} \, \sqrt{a(v,v)} \quad \text{for all} \quad u \in V_H^{\perp}, \, v \in V_h^{\perp} \tag{26}$$

where $V_H^{\perp}$, $V_h^{\perp}$ are $a-$orthogonal complements of V_H, V_h in $V_{H,h}$. Note that more information concerning the value of γ can be found in [10] and [7].

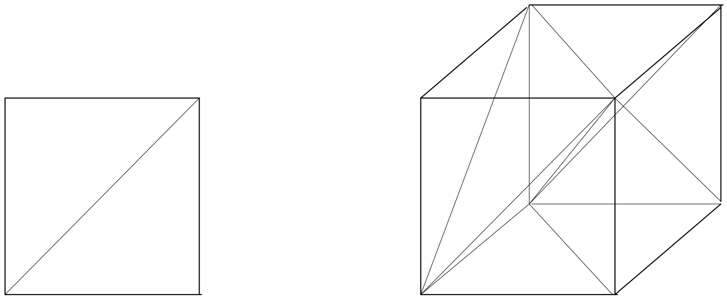

Figure 5: Type of division of a square and cube.

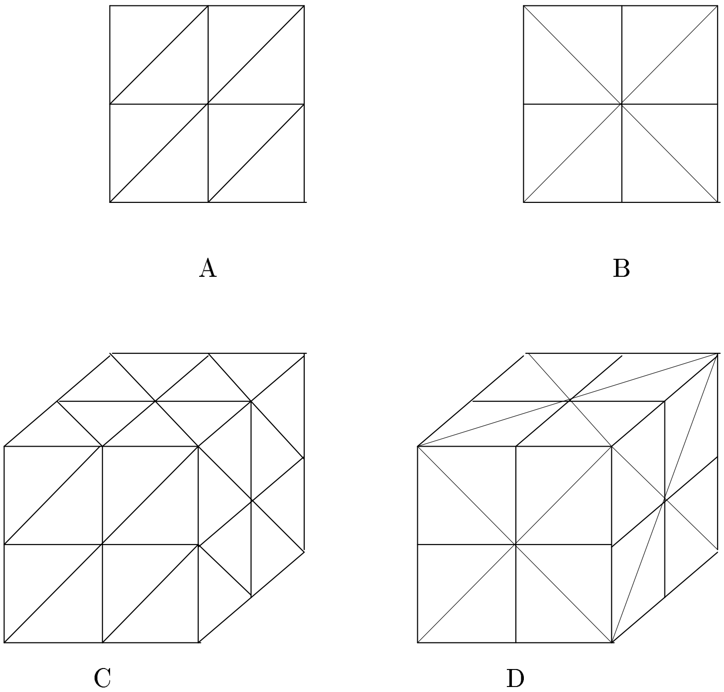

Figure 6: Type of refinement in a square or cube.

Note that the constant γ generally depends on the Poisson's ratio ν but in some cases, we can found also an upper estimate $\bar{\gamma} < 1$ which is independent on the material parameters (which are constant on the coarse grid elements from $\mathcal{T}_H$).

For example, for the refinement from Figure 6, we obtain the following values

case	γ for $\nu = 0.25$	$\bar{\gamma}$
A	~ 0.799	$\sqrt{3/4}$
B	~ 0.874	1
C	~ 0.889	$\sqrt{0.9}$
D	~ 0.917	1

The above results about the constant in the strengthened Cauchy–Schwarz inequality can be exploited for the analysis of convergence of the FAC or BEPS methods. Note that similar results can be proved also for a more general grids $\mathcal{T}_H,\ \mathcal{T}_h,$ see [10] and [7].

Theorem 1 *Let us consider an elasticity problem as above which is discretized by the FE method using some composite grid. Let us assume that the strengthened Cauchy–Schwarz inequality (26) holds with an h – independent constant $\gamma < 1$.*

Then the FAC and BEPS iterative methods for the solution of the linear system arising from the composite grid FE method converge independently on h. More exactly, for any $\varepsilon > 0$ there is an h-independent number $n = n(\varepsilon)$ such that

$$\| \mathbf{e}^n \|_E \leq \varepsilon \| \mathbf{e}^0 \|_E \tag{27}$$

where $\mathbf{e}^i \in R_{H,h}$ is the error of the $i - th$ FAC or BEPS iteration and for $\mathbf{u} \in R_{H,h}$ the norm $\| \mathbf{u} \|_E$ is defined as

$$\| \mathbf{u} \|_E = \sqrt{a(I_{H,h}\mathbf{u},\, I_{H,h}\mathbf{u})}\,. \tag{28}$$

Proof. The proof follows from the results of [9]. Let us only remember the main steps of this proof.

1. $e^{i+1} = P_{\hat{R}_H^\perp} P_{\hat{R}_h^\perp} e^i$

   where $\hat{R}_H = \text{Range}\left(I_H^{H,h}\right)$, $\hat{R}_h = \text{Range}\left(I_h^{H,h}\right)$, $\hat{R}_H^\perp$ and $\hat{R}_h^\perp$ are the complements of $\hat{R}_H$ and $\hat{R}_h$ in $R_{H,h}$ which are orthogonal with respect to the scalar product induced by the bilinear form a, $P_{\hat{R}_H^\perp}$ and $P_{\hat{R}_h^\perp}$ denote the corresponding orthogonal projections to $\hat{R}_H^\perp$ and $\hat{R}_h^\perp$.

2. $\rho\left(P_{\hat{R}_H^\perp} P_{\hat{R}_h^\perp}\right) = \| P_{\hat{R}_H^\perp} P_{\hat{R}_h^\perp} P_{\hat{R}_H^\perp} \|_E = \| P_{\hat{R}_h^\perp} P_{\hat{R}_H^\perp} P_{\hat{R}_h^\perp} \|_E \leq \gamma^2$

3. for the FAC method $\| \mathbf{e}^{i+3} \|_E \leq \gamma^4 \| \mathbf{e}^i \|_E$

4. for the BEPS method $\text{cond}\left(B^{-1}A\right) \leq 1/(1-\gamma^2)$,

$$\| \mathbf{e}^i \|_E \leq 2 \left[\frac{1-\sqrt{1-\gamma^2}}{1+\sqrt{1-\gamma^2}} \right]^i \| e^0 \|_E \ .$$

□

The results of Theorem 1 in combination with the results from [2] concerning the convergence of the inexact initial stiffness method give the following theorem.

Theorem 2 *Let us consider the plasticity problem of Section 2 discretized by the incremental FE method with the stress operator T_σ defined on the basis of the elasto–plastic tangent (see Section 3). Let us assume that the incremental FE method uses some composite grid FE space which fulfills the assumptions of Theorem 1.*

For the solution of the nonlinear systems (6), the Newton–FAC/BEPS method is assumed to be used which combines the initial stiffness method (i.e. $A_{k,i} = A_e$) and the solution of the arising linear systems (9) by the FAC or BEPS method up to the inner accuracy ε_1, i.e. we compute an approximation $\mathbf{w}^{i,j}$ to $\mathbf{w}^i = A_{k,i}^{-1}\mathbf{r}^i$ such that

$$\| \mathbf{w}^{i,j} - \mathbf{w}^i \|_E \leq \varepsilon_1 \| \mathbf{w}^i \|_E \,. \tag{29}$$

Then for any $\varepsilon_1 < (\nu_0 - 1)(\nu_0 + 1)^{-1}$ with ν_0 from (5), the Newton–FAC/BEPS methods converge independently on h. This means that for any $\varepsilon > 0$ and for any $\varepsilon_1 < (\nu_0 - 1)(\nu_0 + 1)^{-1}$ there exist h-independent numbers $n_0 = n_0(\varepsilon)$ and $n_1 = n_1(\varepsilon_1)$, such that after performing n_0 outer Newton type iteration with n_1 inner FAC or BEPS iterations within each of the outer iterations, we get the solution of the nonlinear system with the accuracy

$$\| \mathbf{e}^{k,n_0} \|_E \leq \varepsilon \| \mathbf{e}^{k,0} \|_E \tag{30}$$

where $\mathbf{e}^{k,i}$ is now the error of the $i-th$ composite Newton – FAC/BEPS iteration.

Proof. From [2], we obtain the following:

- the inexact initial stiffness method converges if the auxiliary linear systems are solved within the accuracy ε_1, $\varepsilon_1 < (1-\nu_0)(1+\nu_0)^{-1}$. According to Theorem 1, such inner accuracy is guaranteed by performing $n_1(\varepsilon_1)$ inner iterations where $n_1(\varepsilon_1)$ is independent on h.
- the convergence of the outer iterations is then characterized by the estimate

$$\| \mathbf{r}^{k,i+1} \|_{-E} \leq \xi \ \| \mathbf{r}^{k,i} \|_{-E}, \quad \xi = \nu_0 + \varepsilon_1 + \nu_0 \varepsilon_1 \, .$$

where $\| \mathbf{v} \|_{-E} = \| A_e^{-1} \mathbf{v} \|_E$.

From the above estimate, we obtain that

$$\| \mathbf{e}^{k,i+1} \|_E \leq \xi^{\,i+1} \frac{1+\nu_0}{1-\nu_0} \ \| e^{k,0} \|_E \ .$$

This imply that there is an $h-$ independent number $n_0(\varepsilon)$ which guarantees (30).

□

6 Concluding remarks

The paper has two aims. Firstly, it outlines the structure and components of an adaptive incremental FE algorithm which updates the FE grids during the incremental loading. Secondly, it describes the implementation of the composite grid FE technique within the incremental FE scheme.

Concerning the composite grid technique, let us note that

- fully structured composite grids can be efficient in many applications with gradually developing plastic zones. Nevertheless, they may be not be sufficiently flexible for other applications. In that cases, it may be useful to deal with semi-structured composite grids which consist from structured coarse grid and unstructured fine grid created by clusters of refined elements.
- convergence results similar to those of Theorem 2 are expected also for other types of the Newton–FAC/BEPS methods and other types of discretization procedures. Nevertheless, the proof of $h-$ independent convergence results may encounter the following difficulties
    1. the matrices A_{k_i} correspond to the bilinear forms $a_{k,i}$ whose coefficients may vary inside the coarse grid elements. To obtain an analogy of Theorem 1, we can then use spectral equivalence of $a_{k,i}$ to the bilinear form of elasticity a or to some form $\bar{a}_{k,i}$ with locally averaged coefficients.
    2. it is an open problem yet if the convergence of the Newton–type solvers is $h-$independent in the case of the the return mapping incremental FE method, cf. [3].

References

[1] J. Alberty, C. Carstensen and D. Zarrabi: *Adaptive numerical analysis in primal elastoplasticity with hardening.* Preprint 98–13, Christian - Albrechts - Universitat, Kiel, 1998-Marz.

[2] O. Axelsson, R. Blaheta and R. Kohut: *Inexact Newton solvers in plasticity: Theory and experiments*, Numerical Linear Algebra with Applications 4(1997), 133-152 (special issue on plasticity)

[3] R. Blaheta, *Convergence of Newton type solvers in incremental return mapping analysis of elasto–plastic problems,* Comp. Meth. Appl. Mech. Engrg. 147(1997), 167 - 185

[4] R. Blaheta: *Numerical methods in elasto-plasticity*, special issue of Documenta Geonica, in progress

[5] R. Blaheta and O. Axelsson, *Convergence of inexact Newton-like iterations in incremental finite element analysis of elasto-plastic problems.* Comp. Meth. Appl. Mech. Engrg. 141(1997) 281 - 295

[6] J.H. Bramble, D.E. Ewing, J.E. Pasciak, A.H. Schatz: *A preconditioning technique for the efficient solution of problems with local grid refinement*, Comp. Meth. Appl. Mech. Engrg. 67(1988), 149–159

[7] M. Jung, J.F. Maitre: *Some remarks on the constant in the strengthened C.B.S. inequality: Applications to h- and p-hierarchical finite element discretization of elasticity problems*, Preprint SFB393/97-15, TU Chemnitz - Zwickau, 1997

[8] V.G. Korneev and U. Langer, *Approximate Solution of Plastic Flow Theory Problems.* Teubner, Leipzig 1984 .

[9] Jan Mandel and Steve McCormick: *Iterative Solutin of Elliptic Equations with Refinement: The Two–Level Case*, in Domain Decomposition Methods, Edited by Tony F. Chan, Roland Glowinski, Jacques Periaux, Olof B. Widlund, SIAM Philadelphia 1988

[10] S.D. Margenov, *Upper bound of the constant in the strengthened C.B.S. inequality for FEM 2D elasticity equations*, Numer. Linear Algebra Appl. 1(1994), 65-74

[11] Steve McCormick: *Multilevel Adaptive Methods for Partial Differential Equations*, Frontiers in Applied Mathematics, Vol. 6, SIAM, Philadelphia 1989

[12] D. Perić, Ch. Hochard, M. Dutko, D.R.J. Owen: *Transfer operators for evolving meshes in small strain elasto–plasticity.* Comp. Methods Appl. Mech. Engrg. 137(1996) 331–344.

[13] D. Perić, J, Yu, D.R.J. Owen: *On error estimates and adaptivity in elastoplastic solids: Applications to the numerical simulation of strain localization in classical and Cosserat continua.* Int. Journal for Numerical Methods in Engineering, vol. 37, 1351–1379 (1994).

[14] R. Rannacher and F.T. Stuttmaier: *A Posteriori Error Control and Mesh Adaptation for FE models in Elasticity and Elasto-Plasticity*, Preprint 1998-32, University of Heidelberg, 1998.

[15] S. Repin: *A priori and a posteriori error estimates for approximate solutions of plasticity problems*, Modelling'98, Prague July 1998

[16] J.C. Simo and R.L. Taylor, *Consistent tangent operators for rate–independent elastoplasticity,* Comp. Meth. Appl. Mech. Engrg. 48(1985), 101–118.

[17] S. W. Sloan, *Substepping schemes for the numerical integration of elasto-plastic stress-strain relations,* Int. J. Numer. Meth. Engrg. 24(1987), 893-911.

[18] Ch. Wieners: *Parallel multigrid methods for Prandtl–Reuss-plasticity.* IMMB'98, KUN Nijmegen, September 1998.

[19] Ch. Wieners: *An abstract interface for the description of material laws for the simulation of elastoplastic problems.* Preprint SFB 404, University of Stuttgart, 1997

[20] G. Wittum: *Parallel and Adaptive Simulations of Elasto-Plastic Structures*, Modelling'98, Prague July 1998

[21] O. C. Zienkiewicz and R. L. Taylor, *The Finite Element Method, fourth edition.* Mc Graw-Hill, London 1991.