HG CP 94–0820–CICT–6–2

ALGEBRAIC TWO-LEVEL PRECONDITIONING OF NON-CONFORMING FEM SYSTEMS

SVETOZAR D. MARGENOV AND PANAYOT S. VASSILEVSKI

Abstract. In this paper we analyze a two-level preconditioner for second order elliptic boundary value problems. Crouzeix-Raviart non-conforming linear finite elements on triangles are implemented to the numerical solution of the problem. A pure algebraic multiplicative two-level preconditioner is constructed using a proper two-level basis. It is important to remind, that the finite element spaces corresponding to two successive levels of mesh refinements are not nested in the case of such non-conforming elements. An uniform estimate with respect to the mesh parameter is derived for the constant in the corresponding strengthened Cauchy-Bunyakowski-Schwartz inequality, as well as, for the condition number of the two-level preconditioner. Numerical tests illustrating the convergence of the proposed method are presented.

Key words: PCG, multilevel preconditioners, non-conforming FEM
AMS subject classifications: 65F10, 65N30

1. Introduction

The recent efforts in development of efficient solution methods for non-conforming finite element (FEM) systems is inspired by their importance for various applications in scientific computations and engineering. In particular, the considered here Crouzeix-Raviart linear finite elements on triangles play a key role in the construction of so called *locking-free* approximations for parameter-dependent problems. The last is one of the leading motivations for the present study.

The algebraic multilevel iteration (AMLI) methods were introduced by Axelsson and Vassilevski (1989,1990) [5, 6] as a quite general approach for construction of preconditioners of optimal complexity for finite element matrices. This requires that the recursive decomposition of the matrix is carried out until a coarse mesh is reached where the problem can be solved with a cost which is proportional (or nearly proportional, see [4]) to the degree of freedom on that mesh.

The optimal order AMLI methods are extensions of the two-level method presented by Bank and Dupont [8] (see also [3, 9]), where a matrix polynomial stabilization of the recursive factorization is used. The uniform estimate of the constant in the strengthened Cauchy-Bunyakowski-Schwartz (C.B.S.) inequality [13] plays a key role in the construction of the algorithm as well as in the convergence analysis of both two-level and multilevel cases (cf., e.g., Eijkhout and Vassilevski [11]).

Optimal order (with respect to the mesh parameter) two-level preconditioners for 2m'th order elliptic finite element problems where recently investigated in a very general setting by Bramble, Pasciak and Zhang [7], where the non-conforming case has been included. The reason for this paper is more specific. We analyze a class of multiplicative algebraic two-level algorithms in a form suitable for a next-step implementation into the framework of the AMLI methods. We need to this purpose a more accurate analysis of the constants in the convergence estimates. In particular, a new uniform estimate for the constant in the strengthened C.B.S. inequality corresponding to the considered two-level non-conforming FEM basis is derived.

Consider the elliptic equation

$$\begin{aligned} -(a(x,y)u_x)_x - (b(x,y)u_y)_y &= f(x,y), \qquad \text{for} \quad (x,y) \in \Omega, \\ u &= 0 \qquad\qquad\; \text{on} \quad \Gamma_D, \\ au_x n_1 + bu_y n_2 &= 0 \qquad\qquad\; \text{on} \quad \Gamma_N, \end{aligned}$$

where the domain Ω is partitioned using triangles $T \in \tau_H$ defined on the coarser mesh ω_H; $n = (n_1, n_2)$ is the outward unit vector normal to the boundary Γ_N; $(.)_x$ and $(.)_y$ denote partial derivatives.

The Galerkin variational formulation of the above problem reads:
given $f \in L^2(\Omega)$ find a function $u \in H_0^1(\Omega) = \{v \in H^1(\Omega;\ \Gamma_D) : v = 0 \text{ on } \Gamma_D\}$, satisfying

$$\mathcal{A}(u,v) = (f,v) \qquad \forall v \in H_0^1(\Omega;\ \Gamma_D), \tag{1}$$

where

$$\mathcal{A}(u,v) = \int_\Omega (au_x v_x + bu_y v_y)\, dx\ dy\cdot$$

The Crouzeix-Raviart non-conforming linear finite elements on triangles are implemented for the numerical solution of (1). The finite element space V_H corresponds to the triangulation τ_H. It consists of piecewise linear functions determined by their values in the midpoints of the sides of the triangles (see Fig. 1-(a)). The nodal basis test functions of V_H have a support of no more than two neighboring triangles where the corresponding node is the midpoint of their common side. The standard computational procedure leads to the linear system of equations

$$A^H \mathbf{u}^H = \mathbf{f}^H, \tag{2}$$

where A^H is the corresponding stiffness matrix.

Now, in order to obtain a more accurate solution of problem (1), an uniform refinement procedure is used to construct the triangulation τ_h. Associated with the triangulation τ_h is the finite element space V_h and the corresponding finite element stiffness matrix A^h, computed using standard nodal basis test functions. Note that V_H is not a subset of V_h, i.e., V_H and V_h are not nested.

The solution of the system of equations corresponding to the refined triangulation τ_h (which is the problem we want to solve, since it better approximates the solution of the differential problem)

$$A^h \mathbf{u} = \mathbf{f} \tag{3}$$

is computed by the preconditioned conjugate gradient (PCG) method. The goal of this paper is to construct an algebraic two-level preconditioner M of the matrix A^h, such that the relative condition number $\kappa(M^{-1}A^h) = O(1)$ uniformly on the mesh parameter.

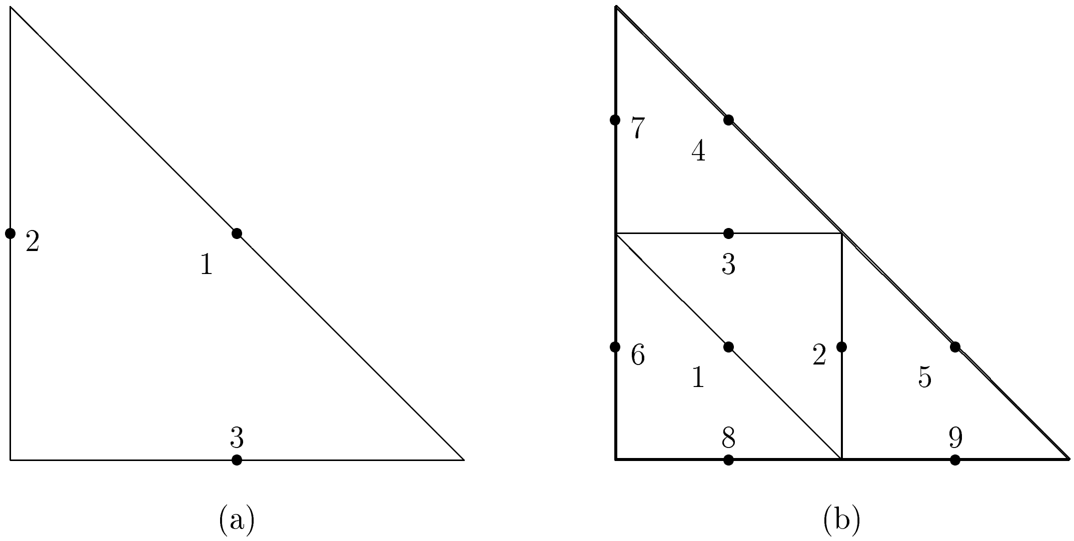

FIGURE 1. *Crouzeix-Raviart linear finite elements: (a) triangle $e \in \tau_H$; (b) refined macro-element $E \in \tau_h$.*

The construction of the two-level preconditioner is presented in the next section. A model analysis of the convergence including the uniform estimate of the constant in the strengthened C.B.S. inequality is presented in §3. The next §4 contains a selected set of numerical tests illustrating the features of the proposed preconditioner. At the end some conclusions and remarks are formulated.

2. Algebraic two-level algorithm

As the finite element spaces V_H and V_h are not nested, a hierarchical two-level basis will be used to define the algebraic two-level preconditioner. Let A_e be the element stiffness matrix corresponding to the triangular element $e \in \tau_H$, and A_E be the macro-element stiffness matrix where the macro-element $E \in \tau_h$ is obtained by a regular bisection refinement of $e \in \tau_H$ into four similar triangles (see Fig. 1). Following the standard FEM assembling procedure we have

$$A^H = assembl\{A_e\}_{e \in \tau_H},$$

$$A^h = assembl\{A_E\}_{E \in \tau_h}.$$

Let us denote by $\phi_E^T = \{\varphi(x,y)_i\}_{i=1}^6$ the macro-element vector of the nodal basis functions. In all local vectors and matrices the numbering of the nodes is as in Fig. 1. We define locally a hierarchical two-level basis $\widetilde{\phi}_E$ by the matrix-vector relation

$$\widetilde{\phi}_E = J_E \phi_E, \tag{4}$$

$$J_E = \frac{1}{2}\begin{pmatrix} 2 & & & & & & & & \\ & 2 & & & & & & & \\ & & 2 & & & & & & \\ & & & 1 & -1 & & & & \\ & & & & & 1 & -1 & & \\ & & & & & & & 1 & -1 \\ & & & 1 & 1 & & & & \\ & & & & & 1 & 1 & & \\ & & & & & & & 1 & 1 \end{pmatrix}.$$

We will mark all stiffness matrices and vectors related to the above introduced hierarchical two-level basis by tilde. Then the global two–level HB (hierarchical basis) stiffness matrix reads:

$$\widetilde{A}^h = assembl\{\widetilde{A}_E\}_{E\in\tau_h},$$

where

$$\widetilde{A}_E = JAJ^T.$$

We split now the two-level stiffness matrix $\widetilde{A}^h$ into 2×2 block form

$$\widetilde{A}^h = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{pmatrix}. \tag{5}$$

The block $\widetilde{A}_{11}$ corresponds to interior nodal unknowns with respect to the macro-elements $E \in \tau_h$. The first step of the two-level preconditioning algorithm is to eliminate these unknowns. For this purpose we factor $\widetilde{A}^h$, i.e.,

$$\widetilde{A}^h = \begin{pmatrix} \widetilde{A}_{11} & 0 \\ \widetilde{A}_{21} & \widetilde{B} \end{pmatrix} \begin{pmatrix} I_1 & \widetilde{A}_{11}^{-1}\widetilde{A}_{12} \\ 0 & I_2 \end{pmatrix}, \tag{6}$$

where $\widetilde{B} = \widetilde{A}_{22} - \widetilde{A}_{21}\widetilde{A}_{11}^{-1}\widetilde{A}_{12}$ stands for the Schur complement of this elimination step. It is important to notice that $\widetilde{A}_{11}$ is a block-diagonal matrix with blocks which are 3×3 matrices. In other words, this elimination is local, and can be performed macro-element by macro-element.

At the next step of the construction of the preconditioner we approximate the matrix $\widetilde{B}$ written again in 2×2 block form

$$\widetilde{B} = \begin{pmatrix} \widetilde{B}_{11} & \widetilde{B}_{12} \\ \widetilde{B}_{21} & \widetilde{B}_{22} \end{pmatrix} = \begin{pmatrix} \widetilde{B}_{11} & 0 \\ \widetilde{B}_{21} & \widetilde{S} \end{pmatrix} \begin{pmatrix} I_1 & \widetilde{B}_{11}^{-1}\widetilde{B}_{12} \\ 0 & I_2 \end{pmatrix}, \tag{7}$$

where the first pivot block $\widetilde{B}_{11}$ corresponds to the two-level basis functions which are defined as half–differences of nodal basis functions (see (4)), and $\widetilde{S} = \widetilde{B}_{22} - \widetilde{B}_{21}B_{11}^{-1}\widetilde{B}_{12}$ is the current Schur complement.

Now we are in a position to define the two-level preconditioner M:

Definition 1. *(multiplicative or block Gauss-Seidel preconditioning scheme)*

$$M = \begin{pmatrix} \widetilde{A}_{11} & 0 \\ \widetilde{A}_{21} & M_{\widetilde{B}} \end{pmatrix} \begin{pmatrix} I & \widetilde{A}_{11}^{-1}\widetilde{A}_{12} \\ 0 & I \end{pmatrix}, \tag{8}$$

where

$$M_{\widetilde{B}} = \begin{pmatrix} \widetilde{D}_{11} & 0 \\ \widetilde{B}_{21} & \widetilde{B}_{22} \end{pmatrix} \begin{pmatrix} I & (\widetilde{D}_{11})^{-1}\widetilde{B}_{12} \\ 0 & I \end{pmatrix}, \tag{9}$$

and where: $\widetilde{D}_{11} = \omega \widetilde{B}_{11}^{(d)}$; $\widetilde{B}_{11}^{(d)}$ *stands for the diagonal part of* $\widetilde{B}_{11}$*; and* $\omega > 0$ *is a parameter.*

The following theorem is a straightforward conclusion from the classical result by Axelsson and Gustafsson [3] (see also [1, 17]).

Theorem 1. *Let the parameter* ω *from Definition 1. is chosen so that*

$$\mathbf{v}_1^T \widetilde{B}_{11}\mathbf{v} \le \mathbf{v}_1^T \widetilde{D}_{11}\mathbf{v} \le \delta \mathbf{v}_1^T \widetilde{B}_{11}\mathbf{v}. \tag{10}$$

Then the relative condition number of the two-level preconditioner (8) satisfies the estimate

$$\kappa(M^{-1}A^h) \le \frac{\delta}{1-\gamma^2}, \tag{11}$$

where γ *is the constant in the strengthened C.B.S. inequality corresponding to the* 2×2 *block-presentation (7) of the matrix* $\widetilde{B}$ *(7).*

■

3. Model analysis of the parameters of the two-level algorithm

We show in this section how the parameter ω from (9) can be chosen, and derive uniform estimates for δ, and γ, the constant in the strengthened C.B.S. inequality, from Theorem 1. The model analysis is for the isotropic problem

$$\begin{aligned} -(a(x,y)u_x)_x - (a(x,y)u_y)_y &= f(x,y), && \text{for} \quad (x,y) \in \Omega, \\ u &= 0 && \text{on} \quad \Gamma_D, \\ au_x n_1 + au_y n_2 &= 0 && \text{on} \quad \Gamma_N, \end{aligned}$$

where the coarser triangulation ω_H consist of isosceles triangles defined on an uniform mesh with a mesh size H, and where $a(x,y) = a_T = a|_T$ is a constant in each element $T \in \tau_H$. In this case

$$A^h = assembl\{A_E\}_{E \in \tau_h},$$

where

$$
(12)\qquad A_E = 2a_E \begin{pmatrix}
4 & -1 & -1 & & & -1 & & -1 & \\
-1 & 2 & & & -1 & & & & \\
-1 & & 2 & -1 & & & & & \\
 & & -1 & 2 & & & -1 & & \\
 & -1 & & & 2 & & & & -1 \\
-1 & & & & & 1 & & & \\
 & & & -1 & & & 1 & & \\
-1 & & & & & & & 1 & \\
 & & & & -1 & & & & 1
\end{pmatrix}.
$$

The local numbering in (12) is as in Fig. 1. Since the transformation of the stiffness matrix A^h to the two-level matrix $\widetilde{A}^h$ as well as the first step elimination are local, the matrix $\widetilde{B^h}$ can be assembled by the related macro-element matrices, i.e.,

$$
\widetilde{B}^h = assembl\{\widetilde{B}_E\}_{E\in\tau_h},
$$

where

$$
(13)\qquad \widetilde{B}_E = \frac{a_E}{6}\begin{pmatrix}
9 & 3 & -3 & 0 & -3 & 3 \\
3 & 5 & -1 & 2 & -1 & -1 \\
-3 & -1 & 5 & 2 & -1 & -1 \\
0 & 2 & 2 & 8 & -4 & -4 \\
-3 & -1 & -1 & -4 & 5 & -1 \\
3 & -1 & -1 & -4 & -1 & 5
\end{pmatrix}.
$$

Remark 1. *The case $E\cap\Gamma_D \neq \emptyset$ requires corresponding modifications in the formulas for A_E and $\widetilde{B}_E$ but we omit these details here as they do not influence the validity of the fore-coming considerations.*

To determine the parameter ω the following generalized eigenvalue problem is considered.

$$
(14)\qquad \widetilde{B}_{E;11}\mathbf{v} = \lambda \widetilde{B}^{(d)}_{E;11}\mathbf{v}.
$$

Here $\widetilde{B}_{E;11}$ is the macro-element contribution to $\widetilde{B}_{11}$ (see (7)) and $\widetilde{B}^{(d)}_{E;11}$ is its diagonal part. Solving the characteristic equation we find the eigenvalues of (14), i.e.:

$$
\begin{vmatrix}
9(1-\lambda) & 3 & -3 \\
3 & 5(1-\lambda) & -1 \\
-3 & -1 & 5(1-\lambda)
\end{vmatrix} = 0,
$$

$$
25\lambda^3 - 75\lambda^2 + 64\lambda - 16 = 0,
$$

$$
(5\lambda - 4)(5\lambda^2 - 11\lambda + 4) = 0,
$$

and

$$
\lambda_1 = \frac{11-\sqrt{41}}{10}, \qquad \lambda_2 = 0.8, \qquad \lambda_3 = \frac{11+\sqrt{41}}{10}.
$$

We apply now the results from the local spectral analysis to prove the next lemma.

Lemma 1. *If* $\omega = \lambda_3$ *then the condition*

$$\mathbf{v}_1^T \widetilde{B}_{11} \mathbf{v}_1 \le \mathbf{v}_1^T \widetilde{D}_{11} \mathbf{v}_1 \le \delta \mathbf{v}_1^T \widetilde{B}_{11} \mathbf{v}_1$$

from Theorem 1 holds with

$$\delta = \frac{\lambda_3}{\lambda_1} = \frac{11 + \sqrt{41}}{11 - \sqrt{41}} \approx 3.78.$$

Proof.

$$\mathbf{v}_1^T \widetilde{B}_{11} \mathbf{v}_1 = \sum_{E \in \tau_h} \mathbf{v}_{E;1}^T \widetilde{B}_{E;11} \mathbf{v}_{E;1} \le \lambda_3 \sum_{E \in \tau_h} \mathbf{v}_{E;1}^T \widetilde{B}_{E;11}^{(d)} \mathbf{v}_{E;1} = \mathbf{v}_1^T \widetilde{D}_{11} \mathbf{v}_1$$

$$\mathbf{v}_1^T \widetilde{D}_{11} \mathbf{v}_1 = \lambda_3 \sum_{E \in \tau_h} \mathbf{v}_{E;1}^T \widetilde{B}_{E;11}^{(d)} \mathbf{v}_{E;1} \le \frac{\lambda_3}{\lambda_1} \sum_{E \in \tau_h} \mathbf{v}_{E;1}^T \widetilde{B}_{E;11} \mathbf{v}_{E;1} = \delta \mathbf{v}_1^T \widetilde{B}_{11} \mathbf{v}_1$$

□

The second step of this analysis is to estimate the constant $\gamma \in [0, 1)$ in the strengthened C.B.S. inequality corresponding to the two-level splitting of the matrix $\widetilde{B}$ (7). The next classical lemma (see e.g. [2, 11]) will be used to this purpose.

Lemma 2. *The constant* $\gamma \in [0, 1)$ *in the strengthened C.B.S. inequality corresponding to the two-level splitting of the matrix* $\widetilde{B}$ *satisfies the estimate*

$$\gamma^2 \le \gamma_E^2 = 1 - \mu_E^{(1)}, \tag{15}$$

where $\mu_E^{(1)}$ *is the smallest eigenvalue of the generalized eigenvalue problem*

$$\widetilde{S}_E \mathbf{v} = \mu_E \widetilde{B}_{E;22} \mathbf{v}, \qquad \mathbf{v} \ne \textit{const}\ \mathbf{e}, \quad \mathbf{e} = (1, 1, 1)^T, \tag{16}$$

where $\widetilde{S}_E = \widetilde{B}_{E;22} - \widetilde{B}_{E;21} \widetilde{B}_{E;11}^{-1} \widetilde{B}_{E;12}$.

■

In the case under consideration

$$\widetilde{S}_E = \frac{a_E}{2} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \qquad \widetilde{B}_{E;22} = \frac{a_E}{6} \begin{pmatrix} 8 & -4 & -4 \\ -4 & 5 & -1 \\ -4 & -1 & 5 \end{pmatrix},$$

and the eigenvalues of (16) are the roots of the characteristic equation

$$\begin{vmatrix} 6 - 8\mu_E & -3 + 4\mu_E \\ -3 + 4\mu_E & 3 - 5\mu_E \end{vmatrix} = 0,$$

that is

$$\mu_E^{(1)} = \frac{1}{2} \qquad \text{and} \qquad \mu_E^{(2)} = \frac{3}{4}$$

At the end of the present section, we summarize in the next theorem the results from Theorem 1, Lemma 1 and Lemma 2, with $\mu_E^{(1)} = \frac{1}{2}$.

Theorem 2. *Let the two-level preconditioner for the considered model problem is defined by (8) with parameter* $\omega = (11+\sqrt{41})/10$. *Then the convergence of the preconditioner is independent of the size of* A^h *and the relative condition number is uniformly bounded by*

$$\kappa(M^{-1}A^h) \le \frac{81+11\sqrt{41}}{20} \approx 7.57$$

■

4. Numerical tests

The numerical tests presented in this section illustrate the convergence rate of the two-level algorithm studied in the previous sections. The computations are done with double precision on a SUN SPARC Station 2. The iteration stopping criterion is $(M^{-1}r^{N_{it}}, r^{N_{it}})/(M^{-1}r^0, r^0) < 10^{-6}$, where r^j stands for the residual at the jth iteration step of the preconditioned conjugate gradient method. The number of iterations are presented and analyzed as a measure of the convergence rate, where the mesh size parameter h is varied.

Three test problems (TP) are considered for the isotropic problem

$$\begin{aligned} -(a(x,y)u_x)_x - (a(x,y)u_y)_y &= f(x,y), \qquad \text{for} \quad (x,y) \in \Omega = (0,\pi)\times(0,\pi), \\ u &= 0 \qquad\qquad\quad \text{on} \quad \Gamma_D = \partial\Omega, \end{aligned}$$

where the coefficient $a(x,y)$ is varied as follows:

TP1: $a(x,y) = 1$

TP2: $a(x,y) = \begin{cases} 10^{-1} & if \qquad x \le \frac{\pi}{2} \;\; and \;\; y \le \frac{\pi}{2} \\ 10 & if \qquad x \ge \frac{\pi}{2} \;\; and \;\; y \ge \frac{\pi}{2} \\ 1 & otherwise \end{cases}$

TP3: $a(x,y) = 1 + \frac{1}{2}\sin(2(x+y))$.

The mesh parameter is also varied by $h = \pi 2^{-\ell}$ where $\ell = 2,3,\ldots,7$. The results are presented in Table 1. Here $\omega = \lambda_3 \approx 1.7403$ is the value of the parameter determined in Lemma 1. The level of refinement ℓ and the size of the discrete problem N are given in the first two columns. First two test problems, TP1 and TP2, satisfy the conditions of the model analysis from the previous section. The constant number of iterations for these test problems (see column 3 and 4) confirms the result from Theorem 1. It is interesting to note, that the estimate for the number of iterations that can be computed if substitute the approximate value of $\kappa(M^{-1}A^h) \approx 7.57$ in the general estimate of the preconditioned conjugate gradient method, gives $N^{it} = 10$. The next observation we would point out is that the constant behavior of N_{it} holds also for the third test problem TP3, where the conditions of the model analysis are not satisfied and the coefficient of the differential operator $a(x,y) = 1+\frac{1}{2}\sin(2(x+y))$ is oscillating. This seems to be an important (unproved) feature of the pure algebraic approach in the construction of the two-level preconditioner M. Obviously, for this test problem the parameter λ_3 is not a solution of the corresponding eigenvalue problem (16).

TABLE 1
Number of iterations N^{it}.
Two-level preconditioning of non-conforming FEM system.
The refinement level ℓ is varied, $\omega = \lambda_3$.

ℓ	N	TP1	TP2	TP3
2	56	7	7	9
3	208	8	7	10
4	800	8	8	10
5	3136	8	8	10
6	12416	8	8	10
7	49408	8	8	10

TABLE 2
Number of iterations N^{it}.
Two-level preconditioning of non-conforming FEM system.
The parameter ω is varied, $\ell = 4$.

ω	20	10	8	6	4	2	λ_3	1	0.8	0.6	0.4	0.2
N^{it}_{TP1}	*	45	26	17	11	8	8	9	12	16	25	73
N^{it}_{TP3}	34	23	19	16	13	10	10	11	13	17	31	*

The sensitivity of the convergence rate of the two-level PCG iterative algorithm with respect to the parameter ω is studied in the remaining part of this section. The obtained number of iterations for refinement level $\ell = 4$ are presented in Table 2. Here ω is varied in the interval (0.2, 20). One row corresponds to every of the considered test problems TP1 and TP2. The general conclusion is that the number of iterations has more or less stable behavior when ω is near λ_3 (or more precisely, when in particular $0.8 \leq \omega \leq 4$). We observe that the convergence deteriorates quickly for values of ω smaller than 0.4 and larger than 10. Here "*" stands for the case when the relative stopping criteria is not reached for 200 iterations.

5. Conclusions and remarks

An optimal order two-level preconditioning algorithm for non-conforming finite element systems is studied. The proposed construction is pure algebraic. The presented numerical tests confirm the theoretical estimates obtained for a model isotropic elliptic problem and for uniform coarse triangulation τ_H.

The derived estimate of the constant γ in the strengthened C.B.S. inequality, namely $\gamma^2 \leq 0.5$, is a basic result for the next step investigation of optimal order multilevel algorithms following the framework of the AMLI methods.

Another important next step in the development of robust algebraic multilevel preconditioners for non-conforming finite elements is to estimate the constant γ, uniformly on the the shape of the triangles from τ_H, i.e., uniformly on the angles of the macroelements $E \in \tau_h$.

Remark 2. *Another way to handle non–conforming discretization is to algebraically generate coarse–grid matrices that corresponds to an algebraic way of defining coarse–element spaces, that are nested; therefore, standard multilevel/multigrid procedures are then straightforward to apply. This approach was taken in Petrova, Tobiska and Vassilevski* [14] *for constructing multigrid schemes for non–conforming spaces on rectangular elements (introduced in Rannacher and Turek* [15]*).*

References

[1] O. Axelsson, Iterative Solution Methods, *Cambridge, University press* (1996).

[2] O. Axelsson, V.A. Barker, Finite Element Solution of Boundary Value Problems. Theory and Computation, *Orlando, FL: Academic*(1984).

[3] O. Axelsson, I. Gustafsson, Preconditioning and two-level multigrid methods of arbitrary degree of approximation, *Math. Comp.*, **40** (1983), 219-242.

[4] O. Axelsson, M. Neytcheva, Scalable parallel algorithms in CFD computations, *Comput. Fluid. Dyn. Rev.(M. Hafez and K. Oshima, eds: Chichester: Wiley)*, **1**(1985), 837-857.

[5] O. Axelsson and P.S. Vassilevski,Algebraic multilevel preconditioning methods, I, *Numer. Math*, **56** (1989), 157-177.

[6] O. Axelsson and P.S. Vassilevski, Algebraic multilevel preconditioning methods, II, *SIAM J. Numer. Anal.*, **27** (1990), 1569-1590.

[7] J.H. Bramble, J.E. Pasciak and X. Zhang, Two-level preconditioners for 2m'th order elliptic finite element problems, *East-West J. Numer. Math*, Vol.4, No.2 (1996), 99-120.

[8] R. Bank, T. Dupont, Analysis of a two-level scheme for solving finite element equations, *Report CNA-159, University of Texas at Austin* (1980).

[9] D. Braess, The contraction number of a multigrid method for solving Poisson equation, *Numer. Math.*, **37**(1981), 384-404.

[10] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, Springer-Verlag (1994).

[11] V. Eijkhout and P.S. Vassilevski, The role of the strengthened C.B.S. inequality in multilevel methods, *SIAM Review*, **33** (1991), 405-419.

[12] Y.A. Kuznetsov, Algebraic multigrid domain decomposition methods, *Soviet J. Numer. Methods and Math. Modelling*, **4** (1989), 351-380.

[13] J.F. Maitre and F. Musy, The contraction number of a class of two-level methods; an exact evaluation for some finite element subspaces and model problems, *Multigrid Methods, Proceedings, Koln-Porz 1981, (W. Hackbusch and U. Trottenberg, eds.), Lect. Notes in Math. Springer-Verlag*, **960** (1982), 201-216.

[14] S. I. Petrova, L. Tobiska and P. S. Vassilevski, *Algebraic multigrid methods for streamline-diffusion non–conforming finite element discretization of convection–diffusion problems*, Otto–von–Guericke Universität Magdeburg, Fakultät für Mathematik, Preprint Nr. 30, 1997.

[15] R. Rannacher and S. Turek, *Simple nonconforming quadrilateral Stokes element*, Numer. Meth. for Partial Differential Equations **8** (1992), 97-111.

[16] P.S Vassilevski, Hybrid V-cycle algebraic multilevel preconditioners, *Math. Comp.*, **58** (1992), 489-512.

[17] P.S Vassilevski, On two ways of stabilizing the hierarchical basis multilevel methods, *SIAM Review*, **39** (1997), 18-53.