# Multi-level Newton methods for nonlinear problems with application to elasticity.

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**Abstract.** We shall investigate the efficiency of two–level and multi–level Newton methods for the solution of nonlinear boundary value problems with special interest in problems of nonlinear elasticity. The investigation uses both theory of the Newton method and theory of approximation by the finite elements.

**Keywords:** inexact damped Newton method, multi–level Newton method, nested iterations, nonlinear elasticity, nonlinear diffusion.

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# 1 Introduction

In this paper, we shall consider numerical techniques for the solution of nonlinear boundary value problems (BVPs) by the finite element (FE) method. Assuming that the nonlinearity is treated by the Newton method, we shall study the efficiency of the *coarse grid approximation*, i.e. the initial approximation for the Newton iterations constructed from the FE solution of the same BVP on a coarser FE grid. The idea of the coarse grid approximation then leads to *two–level* and *multi–level Newton algorithms*.

The starting point for our study is the analysis of the two–level algorithm for the solution of semilinear problems which was performed in [3] and [20].

More exactly, it concerns the semilinear problem

$$\begin{aligned} -\Delta u + f(x,u) &= 0 \qquad \text{in} \quad \Omega \subset R^d & (1.1)\\ u &= 0 \qquad \text{on} \quad \partial\Omega & (1.2) \end{aligned}$$

which can be also formulated variationally in the space $V = H_0^1(\Omega)$. For the two–level Newton algorithm, two FE spaces $V_H$ and $V_h$ are used. It is assumed that $V_H \subset V_h \subset V$. For $f$ we assume that the derivatives $D_u f$, $D_u^2 f$ are continuous and bounded in $\Omega$, $D_u f \leq 0$.

The solution $u \epsilon V$ of the problem (1.1) and (1.2) can be then approximated by $\tilde{u}_h \epsilon V_h$ computed by the following algorithm

**Two–level Newton (TLN) algorithm:**

Step 1: compute the FE solution of the semilinear problem $u_H \epsilon V_H$,

Step 2: use $u_H \epsilon V_H \subset V_h$ as the initial approximation and compute $\tilde{u}_h \epsilon V_h$ by performing *one* Newton iteration.

**Theorem 1.1** *Assume*

- *discrete embedding inequality*

$$\| v_h \|_{0,\infty} \leq C h^{1-\frac{d}{2}-\varepsilon(d)} \| v_h \|_{1,2} \quad \forall v_h \epsilon V_h \tag{1.3}$$

- *FE convergence rates*

$$\| u_h - u \|_{1,2} = 0(h^r) \tag{1.4}$$

$$\| u_h - u \|_{0,2} = 0(h^{r+1}) \tag{1.5}$$

*where $\varepsilon(d)$ is an arbitrarily small positive number for $d = 2$ or zero for $d = 3$, $r \geq 1$.*

*Then the approximate solution $\tilde{u}_h$ computed by the two–level Newton method has the same asymptotical rate of convergence as the FE solution $u_h$,*

$$\| \tilde{u}_h - u \|_{1,2} = 0(h^r) \tag{1.6}$$

*if $H^\alpha = h$ where*

$$\alpha = \frac{2(r+1)}{r - 1 + d/2 + \varepsilon(d)}\,. \tag{1.7}$$

Note that $\| v_h \|_{0,\infty}$, $\| v_h \|_{0,2}$ and $\| v_h \|_{1,2}$ are norms in the spaces $L_\infty(\Omega)$, $L_2(\Omega)$ and $H^1(\Omega)$, respectively. The proof of the discrete embedding inequality can be found in [19]. The FE convergence rates can be shown for sufficiently regular solution of the BVP, see [13] or [12].

The proof of Theorem 1.1 can be found in [3] or [20]. This proof is based on the estimate

$$\| \tilde{u}_h - u_h \|_{1,2} \leq K_h \| u_H - u_h \|_{0,2}^2 \tag{1.8}$$

where $K_h = Ch^{1-\frac{d}{2}-\varepsilon(d)}$.

According to Theorem 1.1, the values $\alpha$ are the following

| | r=1 | r=2 | r $\to \infty$ |
|---|---|---|---|
| d=2 | $\alpha \doteq 4$ | $\alpha \doteq 3$ | $\alpha = 2$ |
| d=3 | $\alpha = \frac{8}{3}$ | $\alpha = \frac{12}{5}$ | $\alpha = 2$ |

This shows that the two–level Newton method is extremely efficient, the effort for solving nonlinear problem is only a little more comparing with corresponding linear problem.

To be not so optimistic, note that the above result is only asymptotic and that it needs some regularity of the solution of the nonlinear boundary value problem (hidden in the FE convergence rate assumptions).

Nevertheless, the TLN method seems to be very promising algorithm. Therefore we shall investigate the efficiency of the TLN methods also for problems with nonlinearity located in the main part of the operator, i.e. for quasilinear problems. In Section 2 we shall consider the nonlinear diffusion problem and in Section 3 the nonlinear elasticity problem. Similar investigation can be found in [5] and [21].

Our investigation will be based on an estimate similar to (1.8), namely on the estimate

$$\| \tilde{u}_h - u_h \|_{1,2} \leq \; K_h \; \| u_H - u_h \|_{1,2}^2 \tag{1.9}$$

which is frequently used for proving local quadratic convergence of the Newton method.

We shall show that for quasilinear problems we can not expect so high efficiency of the TLN method. It brings our attention to multi–level Newton algorithms (MLN) with subsequent (adaptive) refinement of the FE grid which provide proper continuation of the approximate solution. The MLN and adaptive MLN methods will be investigated in Sections 4 and 5. Some recent investigation of the MLN methods can be found also in the papers [6], [7] and [4].

# 2 TLN method for nonlinear diffusion

In this Section, we shall investigate the efficiency of the two–level Newton method for the solution of *nonlinear diffusion problems*, i.e. the problems of the form

$$
\begin{aligned}
-\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(k(x,u)\,\frac{\partial u}{\partial x_i}\right) &= f \quad \text{in} \quad \Omega \subset R^d, && (2.1)\\
u &= 0 \quad \text{on} \quad \Gamma_0 \subset \partial\Omega, && (2.2)\\
-k(x,u)\,\frac{\partial u}{\partial \nu} &= \tau \quad \text{on} \quad \Gamma_1 \subset \partial\Omega. && (2.3)
\end{aligned}
$$

We shall assume that $k : \bar{\Omega} \times R \to \langle k_0, k_1\rangle \subset (0,\infty)$ has two bounded derivatives on $\bar{\Omega} \times R$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\Gamma_0 \cup \Gamma_1 = \partial\Omega$ and $\frac{\partial u}{\partial \nu}$ denotes the derivative in the direction of the unit outward normal $\nu$ to $\partial\Omega$. In [14], we can find variational formulation of the problem in the space $V \subset H^1(\Omega)$ as well as the existence and uniqueness results. For the analysis of the two–level Newton method we shall use the following results which can be found again in [14].

**Lemma 2.1** *The FE convergence rates (1.4) and (1.5) are valid for FE with polynomials of order r, regular family of triangulations and sufficiently regular solution of the nonlinear diffusion problem.*

For the proof, see [14], Theorem 1.

**Lemma 2.2** *Let us consider the Newton method applied to the nonlinear systems arising from the FE discretization of the nonlinear diffusion problem (2.1) – (2.3) in the FE space* $V_h$. *For the FE solution* $u_h$, *let us assume that* $\| u_h \|_{1,\infty} \leq C_1$, *where* $C_1$ *is an h–independent constant. Let* $u_h^0$ *and* $u_h^1$ *be the initial guess and the first Newton iteration, respectively. Then there is an h–independent constant* $\delta$ *such that if*

$$\| u_h^0 - u_h \|_{1,2} \leq \delta / s_h \tag{2.4}$$

*then*

$$\| u_h^1 - u_h \|_{1,2} \leq K_h \| u_h^0 - u_h \|_{1,2}^2 \tag{2.5}$$

*where* $K_h = C_2 \, s_h$ *and*

$$s_h = \sup \left\{ \frac{\| v_h \|_{0,\infty}}{\| v_h \|_{1,2}} : v_h \in V_h, \, v_h \neq 0 \right\} \leq C_3 \, h^{1 - \frac{d}{2} - \varepsilon(d)} \tag{2.6}$$

*if we assume that the inequality (1.3) is valid.*

For the proof, see [14], Theorem 2.

**Theorem 2.1** *Consider the solution of the problem (2.1) – (2.3) by the two–level Newton method in the spaces* $V_H, V_h$, $V_H \subset V_h \subset V$. *Let the assumptions (1.3) – (1.5) of Theorem 1.1 are valid.*

*Then the approximate solution* $\tilde{u}_h$ *has the optimal asymptotic rate of convergence, i.e.* $\| \tilde{u}_h - u \|_{1,2} = 0(h^r)$, *if* $H^\alpha = h$ *where*

$$\alpha = \frac{2r}{r - 1 + d/2 + \varepsilon(d)} \, . \tag{2.7}$$

**Proof:** Firstly, we have

$$\| u_H - u_h \|_{1,2} \leq C \left[ h^{r/\alpha} + h^r \right]$$

so that (2.4) will be valid for sufficiently small $h$, $h \leq h_0$ .
Further, from (2.5), we obtain

$$\begin{aligned} \| \tilde{u}_h - u \|_{2,1} &\leq \| u_h - u \|_{2,1} + \| \tilde{u}_h - u_h \|_{2,1} \\ &\leq \quad Ch^r + K_h \| u_H - u_h \|_{2,1}^2 \\ &\leq \quad Ch^r + 2K_h h^{2r} + 2K_h h^{\frac{2r}{\alpha}} \ = \ O(h^r) \end{aligned}$$

for $\alpha$ from (2.7). This finishes the proof.

$\square$

According to Theorem 2.1, the values of $\alpha$ are the following

| | r=1 | r=2 | r $\to \infty$ |
|---|---|---|---|
| d=2 | $\alpha \doteq 2$ | $\alpha \doteq 2$ | $\alpha = 2$ |
| d=3 | $\alpha = \frac{4}{3}$ | $\alpha = \frac{8}{5}$ | $\alpha = 2$ |

It shows that two–level Newton method is still efficient, especially in the two–dimensional case. But the efficiency is not so high as for the semilinear problems.

# 3 TLN method for nonlinear elasticity

In this section, we shall consider the problems of elasticity with nonlinear material behaviour. Such problems arise also from the deformation theory of plasticity and will be of our main interest in this paper. For $\Omega \subset R^d$, such problems can be formulated with the aid of

- the displacement $\quad u : \Omega \to R^{d+1}$,
- the strain vector $\quad \varepsilon : \Omega \to S \subset R^{d^2}$,
- the stress vector $\quad \sigma : \Omega \to S \subset R^{d^2}$ .

For $x \in \Omega$ the vectors $\varepsilon(x) = \varepsilon(u(x))$, $\sigma(x) \in S$ contains properly arranged components of the small strain tensor and the Cauchy stress tensor, respectively. For the details, see [18] or [9].

We shall assume a nonlinear stress–strain relation which can be written in the form

$$\sigma = T(\varepsilon)\,, \quad T : S \to S \tag{3.1}$$

or more precisely

$$\sigma(x) = T(x, \varepsilon(x))\,, \tag{3.2}$$

$$\sigma(x) = 3k(x, \hat{\varepsilon}(x))\,\mathrm{vol}\,(\varepsilon(x)) + 2\mu(x, \bar{\varepsilon}(x))\,\mathrm{dev}\,(\varepsilon(x)) \tag{3.3}$$

where

$$\hat{\varepsilon}(x) = \|\,\mathrm{vol}\,(\varepsilon(x))\,\|^2\,, \quad \bar{\varepsilon}(x)) = \|\,\mathrm{dev}\,(\varepsilon(x))\,\|^2\ . \tag{3.4}$$

Note that $\| \,.\, \|$ is here the Eucledian norm in $R^{d^2}$ and that for any $\eta = (\eta_{ij}) \in S$ ,

$$\mathrm{vol}(\eta) = (\hat{\eta}_{ij})\,,\ \hat{\eta}_{ij} = \frac{1}{d}\,(\eta_{11} + \ldots + \eta_{dd})\,\delta_{ij}\,, \tag{3.5}$$

$$\mathrm{dev}(\eta) \;=\; \eta - \mathrm{vol}(\eta)\,. \tag{3.6}$$

Above, $\delta_{ij}$ denotes the Kronecker's symbol.

The boundary value problem of nonlinear elasticity can be now written in the form

$$\sum_{i=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \;=\; 0 \qquad \text{in} \quad \Omega \subset R^d \quad (i = 1, \ldots, d) \tag{3.7}$$

$$\sigma(x) \;=\; T(x, \varepsilon(u(x))) \qquad \forall\, x \in \Omega \tag{3.8}$$

$$u \;=\; 0 \qquad \text{on} \quad \Gamma_0 \subset \partial\Omega\,, \quad \mathrm{meas}\,(\Gamma_0) \;>\; 0\,, \tag{3.9}$$

$$\sum_{j=1}^{d} \sigma_{ij}\nu_j \;=\; \tau_i \qquad \text{on} \quad \Gamma_1 \subset \partial\Omega \quad (i = 1, \ldots, d)\,. \tag{3.10}$$

This boundary value problem can be formulated variationally in the space $V \subset [H^1(\Omega)]^d$ equipped with the following inner product and norm

$$(u, v)_{1,2} \;=\; (u_1, v_1)_{1,2} + \ldots + (u_d, v_d)_{1,2}, \quad \| \, v \, \|_{1,2} \;=\; (v, v)_{1,2}^{1/2}\,.$$

For the details, see [18] and [9].

In the sequel, we shall assume that the material functions $k \;=\; k(x, s)$ and $\mu \;=\; \mu(x, s)$ are continuously differentiable with respect to the variable $s$ and that the following estimates hold for any $(x, s) \in \bar{\Omega} \times \langle 0, \infty)$ :

$$0 \;<\; k_0 \;\leq\; k(x, s) \;\leq\; k_1, \tag{3.11}$$

$$0 \;<\; \mu_0 \;\leq\; \mu(x, s) \;<\; \frac{3}{2} k(x, s) \;\leq\; \mu_1, \tag{3.12}$$

$$0 \;<\; k_0' \;\leq\; \frac{\partial}{\partial s}\left[k(x, s^2)s\right] \;\leq\; k_1', \tag{3.13}$$

$$0 < \mu_0' \leq \frac{\partial}{\partial s}\left[\mu(x, s^2)s\right] \leq \mu_1'. \tag{3.14}$$

Under these assumptions, we can prove that the problem of nonlinear elasticity is potential, monotone and Lipschitz continuous with differentiable nonlinear operator. For the details see [18], [17] and [9].

The analysis of the two–level Newton method can be performed similarly as in Section 2. We shall start with two lemma.

**Lemma 3.1** *The FE convergence rates (1.4) and (1.5) are valid for FE polynomials of order* $r$, *regular family of triangulations and sufficiently regular solution of the BVP of nonlinear elasticity.*

The proof uses monotonicity and Lipschitz continuity together with the approximation properties, cf. [13], Theorem 5.3.4] or [9].

**Lemma 3.2** *Let us consider the Newton method applied to the solution of nonlinear systems arising from the FE discretization of the nonlinear elasticity problem (3.7) – (3.10) in the FE spaces* $V_h$. *Assume that we have quasi regular family of FE subdivisions of the domain* $\Omega$ *fulfilling the inverse assumption and that there is an* $h-$*independent constant* $C_1$ *such that* $\| \varepsilon(u_h) \| \leq C_1$ *for the FE solutions* $u_h \in V_h$.

*Let* $u_h^0$ *and* $u_h^1$ *be the initial guess and the first Newton iteration, respectively. Then*

$$\| u_h^1 - u_h \|_{1,2} \leq K_h \| u_h^0 - u_h \|_{1,2}^2 \tag{3.15}$$

*where*

$$K_h = C_2 h^{-d/2}\,. \tag{3.16}$$

**Proof:** The FE approximation $u_h \in V_h$ of the nonlinear elasticity problem can be described variationally in the form

$$\int\limits_\Omega \langle T(\varepsilon(u_h(x))), \, \varepsilon(v_h(x))\rangle \, dx \;=\; b(v_h) \quad \forall v_h \in V_h \tag{3.17}$$

where

$$b(v_h) \;=\; \int\limits_\Omega \langle f, \, v_h\rangle \, dx \;+\; \int\limits_{\Gamma_1} \langle \tau, \, v_h\rangle \, d\Gamma \, . \tag{3.18}$$

Above, $\langle \, , \, \rangle$ denotes the Eucledian scalar product in $R^n$ , $n = d, d^2$.

We can also use the operator formulation

$$F_h(u_h) \;=\; b_h \tag{3.19}$$

where $F_h : V_h \to V_h$ , $b_h \in V_h$ ,

$$F_h(w_h), \, v_h)_{2,1} \;=\; \int\limits_\Omega \langle T(\varepsilon(w_h)), \, \varepsilon(v_h)\rangle \, dx \tag{3.20}$$

$$(b_h, \, v_h)_{2,1} \;=\; b(v_h) \tag{3.21}$$

for all $w_h, \, v_h \in V_h$ .

In [9] and partially also in [18], [17], we can find proofs of the following properties of $F_h$.

1. $F_h$ has the derivative $F_h'$ ,

$$(F_h'(u_h) \, v_h, \, w_h)_{2,1} \;=\; \int\limits_\Omega \langle T'(\varepsilon(u_h)) \, \varepsilon(v_h), \, \varepsilon(w_h)\rangle \, dx \tag{3.22}$$

for all $u_h, \, v_h, \, w_h \in V_h$ .

2. The derivative $F_h'(u_h)$ is symmetric and uniformly positive definite. For the operator norm associated with the vector norm $\| \, . \, \|_{1,2}$ , we have

$$\| \, [F_h(u_h)]^{-1} \, \|_{1,2} \leq \, C_3 \tag{3.23}$$

where $C_3$ is a constant which does not depend on $u_h$ and $h$.

3. For $u_h, \, z_h \in V_h$, we obtain

$$\begin{aligned} & \| \, F_h'(u_h) - F_h'(z_h) \, \|_{1,2} \leq \\ & \leq \; C_4 \, \rho \, (\| \, \varepsilon(u_h) \, \|_{0,\infty} + \| \, \varepsilon(z_h) \, \|_{0,\infty}) \, \| \, \varepsilon(u_h) - \varepsilon(z_h) \, \|_{0,\infty} \\ & \leq \; C_4 \, \rho(2C_1) \, h^{-d/2} \, \| \, u_h - z_h \, \|_{1,2} \; . \end{aligned} \tag{3.24}$$

Above, $\rho : \langle 0, \, \infty) \to \langle 0, \infty)$ is an increasing function, $\rho(0) \, = \, 0$. The inverse estimate, see [13], Theorem 3.2.6 or [12], Theorem 4.5.11 , gives

$$\| \, \varepsilon(v_h) \, \|_{0,\infty} \leq \| \, v_h \, \|_{1,\infty} \leq \, h^{-d/2} \, \| \, v_h \, \|_{1,2} \qquad \forall v_h \in V_h \, . \tag{3.25}$$

Now, we can investigate the Newton method. Exploiting the above properties, we obtain

$$\begin{aligned} F_h'(u_h^0)(u_h^1 - u_h^0) \; &= \; F_h(u_h) - F_h(u_h^0) \; = \\ &= \int_0^1 F'(u_h^0 + t(u_h - u_h^0)) \, (u_h - u_h^0) dt \end{aligned}$$

$$\begin{aligned} & F_h'(u_h^0)(u_h^1 - u_h) \; = \\ & = \int_0^1 \left[ F'(u_h^0 + t(u_h - u_h^0)) \; - \; F'(u_h^0) \right] (u_h - u_h^0) dt \end{aligned}$$

This imply the estimate

$$\| u_h^1 - u_h \|_{1,2} \leq \frac{1}{2} C_3 C_4 \, \rho(2C_1) \, h^{-d/2} \ \| u_h^0 - u_h \|_{1,2}^2$$

which finishes the proof.

□

**Theorem 3.1** *Consider solution of the nonlinear elasticity problem (3.7) – (3.10) by the two–level Newton method with the FE spaces $V_H, V_h$, $V_H \subset V_h \subset V$. Let (1.4) and (1.5), (3.11) – (3.14) and the assumptions of Lemma 3.2 are valid.*

*Then the approximate solution $\tilde{u}_h$ has the same asymptotic rate of convergence as the FE solution $u_h$, i.e.*

$$\| \tilde{u}_h - u \|_{1,2} = O(h^r)$$

*if $H^\alpha = h$ where*

$$\alpha = \frac{2r}{r + d/2} \, . \tag{3.26}$$

The proof of Theorem 3.1 is similar to the proof of Theorem 2.1.

According to (3.26), the high efficiency of the two–level algorithm can be expected only in cases of $r > 1$ which demands higher order FE and higher regularity of the solution of the BVP.

# 4 Adaptive multi–level Newton method

In Section 3, we could see that the efficiency of the two–level Newton method depends too much on the regularity of the solution of the solved BVP and on the quadratic (superlinear) convergence of the Newton method. The obtained results indicates that the two–level Newton method have not to be sufficiently efficient for the solution of quasilinear problems.

A remedy can be found in the multi-level Newton method with more gradual continuation in the mesh size, cf. [8], [16], [6], [7], [4]. For multi-level methods, we can still expect that a lot of the computational work will be done on lower levels, thus, we can expect efficiency of theses methods. This efficiency will be further enhanced if we exploit the coarser grids for additional purposes as e.g. for adaptive construction of the final FE grid or multigrid solution of the arising linear systems.

**Adaptive multi–level Newton (MLN) algorithm:**

*Step 0:*
- *given* the initial coarse grid,
- *construct* the corresponding FE space $V_{h_0}$,
- *solve* the nonlinear FE system by the (damped inexact) Newton method,
- *stop* the Newton iterations if the approximation $\tilde{u}_{h_0}$ is sufficiently close to the FE solution $u_{h_0} \in V_{h_0}$.

*Step $l$ $(l > 0)$ :*
- *given* the FE space $V_{h_{l-1}}$ and the approximation $\tilde{u}_{h_{l-1}}$,
- *refine* the FE grid,
- *construct* the new FE space $V_{h_l}$, $V_{h_{l-1}} \subset V_{h_l}$,
- *solve* the nonlinear FE system in $V_{h_l}$ by one or few iterations of the (inexact damped) Newton method,
- *stop* the Newton iterations if $\tilde{u}_{h_l}$ is sufficiently close to $u_{h_l} \in V_{h_l}$.

Note that the inexact damped Newton method is described e.g. in [15] or [2].

The multilevel Newton method is controlled by several parameters which have to be tuned out for an efficient implementation. These parameters are the following.

1. *Accuracy parameters* $(\alpha_l)$ for stopping the Newton iterations by the condition $\| \tilde{u}_{h_l} - u_{h_l} \| \leq \alpha$. We can choice $\alpha_l$ related to the dis-

cretization errors $\| u_{h_l} - u \|$ . We can also use more risky strategy with a given small number of Newton iterations on the intermediate levels $l = 1, \ldots, l_{max} - 1$. On the level $l = 0$, we perform more iterations to catch the nonlinearity, on the finest level, we solve the problem to the prescribed accuracy. Further discussions can be found in [8], [2], [6] and [16].

2. *Damping parameters* $(\tau_l)$ ensuring the convergence of the Newton methods. The damping with $\tau_l < 1$ can be expected especially on the coarsest level. For an adaptive damping, see e.g. [16].

3. *Accuracy for inner iterations* $(\eta_l)$ can be defined as lowest accuracy which do not deteriorate the convergence of the Newton method. For an adaptive choice, see [10], [16].

4. *Refinement parameters* $(\rho_l)$. Based on some error indicators, see e.g.[1], we can choice the elements which will be refined and the size of the element reduction $\rho_l$. We must compromise two controversial demands for improving the approximation properties of the FE space and providing good initial guess for the Newton process on the newly constructed FE grid.

For implementation of the multi–level Newton method it is important that nearly all the above parameters can be *adaptively controlled by the convergence of the Newton method.*

# 5 Convergence theory

We start our investigation of the multi-level Newton method in a more general setting. To this end, we shall assume that

A1: $V$ is a Hilbert space equipped with the inner product $( \ , \ )$ and the norm $\| \, . \, \|$,

A2: $\phi : V \to R^1$ is a twice Gateaux differentiable functional,

A3: $F : V \to V$ is a nonlinear operator defined by the identity,

$$(F(u), v) \;=\; D\phi(u, v) \qquad \forall u, v \in V\,, \tag{5.1}$$

A4: there are two positive constants $m_0$, $m_1$ such that

$$m_0 \parallel v \parallel^2 \leq\; D^2\phi(u, v, v) \;\leq\; m_1 \parallel v \parallel^2 \qquad \forall u, v \in V\,, \tag{5.2}$$

A5: $u$ denotes the unique solution of the equation

$$F(u) \;=\; 0. \tag{5.3}$$

The equation (5.3) can be then solved by the multi-level Newton method which produce a sequence of iterations $\tilde{u}_0$, $\tilde{u}_1\,, \ldots$ . We assume that

A6: $\tilde{u}_{\,i} \in V_{\,i}$, where $V_i$ is a subspace of $V$,

A7: $V_0 \subset V_1 \subset \ldots \subset V$ and

$$\vartheta_{\,i} \;=\; \inf_{v \in V_i} \parallel u - v \parallel \to 0 \qquad \text{for} \quad i \to \infty, \tag{5.4}$$

A8: $\tilde{u}_{\,i+1} \;=\; \tilde{u}_{\,i} - \omega\, \Delta_{\,i+1}\,, \quad \omega \in \Big(0, 2\frac{m_0}{m_1}\Big)\,,$

A9: $\Delta_{\,i+1}$ is computed from the equation

$$G_{\,i+1}(\tilde{u}_{\,i})\Delta_{\,i+1} \;=\; F_{i+1}(\tilde{u}_{\,i}) \tag{5.5}$$

A10: where $F_i : V_i \to \; V_i$ is given by the identity

$$(F_i(u),\, v) \;=\; D\phi(u,\, v) \qquad \forall u, v \in V_i, \tag{5.6}$$

A11: for any $u \in V_{\,i}\,,\; G_{\,i}(u) : V_1 \to V_i$ is a symmetric linear operator given by the identity

$$(G_{\,i}(u)\, v,\, w) \;=\; D^2\phi(u,\, v,\, w) \qquad \forall u, v, w \in V_i. \tag{5.7}$$

**Theorem 5.1** *The multi-level Newton method described above is convergent. Moreover, we have*

$$\phi\,(\tilde{u}_0) \;\geq\; \phi(\tilde{u}_1) \;\geq\; \ldots \geq \phi(u), \tag{5.8}$$

$$\|\, \tilde{u}_{\,i-1} - u \,\| \leq \frac{m_1}{m_0}\, q(\omega)^{-1/2}\, [\phi(\tilde{u}_{\,i-1}) - \phi(\tilde{u}_{\,i})]^{1/2} + c\,\vartheta_{\,i} \tag{5.9}$$

*where*

$$q(\omega) \;=\; m_0\,\omega - \frac{1}{2} m_1\,\omega^2 \;>\; 0\,. \tag{5.10}$$

**Proof:** Using (5.2) and the mean value theorem, we can prove that the operators $F_i$ are strongly monotone and Lipschitz continuous. Therefore, there are unique solutions $u_{\,i} \in V_i$ of the equations $F_i(u_{\,i}) \;=\; 0,\; i = 0, 1, \ldots$ .

Now, we can split the error in the following way

$$\begin{aligned} \|\, \tilde{u}_{\,i-1} - u \,\| \;&\leq\; \|\, \tilde{u}_{\,i-1} - u_{\,i} \,\| + \|\, u_{\,i} - u \,\| \\ &\leq\; \|\, \tilde{u}_{\,i-1} - u_{\,i} \,\| + c\,\vartheta_{\,i}\,. \end{aligned} \tag{5.11}$$

The last term above comes from a generalization of the Cea's lemma, see [13], Theorem 5.3.4 or [9]. Applying again the mean value theorem, we obtain

$$\phi(\tilde{u}_i) \;=\; \phi(\tilde{u}_{\,i-1}) - (F_i(\tilde{u}_{\,i-1})\,,\, \omega\Delta_i) \;+\; \frac{1}{2}(G_{\,i}(\tilde{u}_{\,i-1} - \theta\omega\Delta_{\,i})\,\omega\Delta_{\,i}\,, \omega\Delta_{\,i})$$

where $\theta \in (0, 1)$ and

$$(F_i(\tilde{u}_{\,i-1})\,,\, \Delta_i) \;=\; (G_{\,i}(\tilde{u}_{\,i-1})\,\Delta_{\,i}\,,\, \Delta_i) \;\geq\; m_0 \,\|\, \Delta_{\,i} \,\|^2,$$

$$(G_{\,i}(\tilde{u}_{\,i-1} - \theta\omega\Delta_{\,i})\,\Delta_{\,i}\,, \Delta_{\,i}) \;\leq\; m_1 \,\|\, \Delta_{\,i} \,\|^2\,.$$

Combining the above results, we obtain

$$\phi(\tilde{u}_i) \leq \phi(\tilde{u}_{i-1}) - \omega\, m_0 \parallel \Delta_i \parallel^2 + \frac{1}{2}\omega^2\, m_1 \parallel \Delta_i \parallel^2$$

and therefore

$$q(\omega) \parallel \Delta_i \parallel^2 \leq \phi(\tilde{u}_{i-1}) - \phi(\tilde{u}_i) \tag{5.12}$$

for $q(\omega)$ introduced in (5.10). This proves (5.8).

From (5.5), (5.7) and (5.2), we can conclude that

$$\parallel F_i(\tilde{u}_{i-1}) \parallel \leq \parallel G_i(\tilde{u}_{i-1}) \parallel \parallel \Delta_i \parallel \leq m_1 \parallel \Delta_i \parallel .$$

Thus,

$$\parallel F_i(\tilde{u}_{i-1}) \parallel \leq m_1\, q(\omega)^{-\frac{1}{2}} \left[\phi(\tilde{u}_{i-1}) - \phi(\tilde{u}_i)\right]^{1/2} \tag{5.13}$$

where the right hand side go to zero for $i$ growing to the infinity. Moreover

$$m_0 \parallel \tilde{u}_{i-1} - u_i \parallel^2 \leq (F_i(\tilde{u}_{i-1}) - F_i(u_i), \tilde{u}_{i-1} - u_i)$$

$$\leq \parallel F_i(\tilde{u}_{i-1}) \parallel \parallel \tilde{u}_{i-1} - u_i \parallel$$

which implies that

$$\parallel \tilde{u}_{i-1} - u_i \parallel \leq \frac{1}{m_0} \parallel F_i(\tilde{u}_{i-1}) \parallel . \tag{5.14}$$

Estimates (5.13) and (5.14) together with (5.11) give the inequality (5.9). It completes the proof.

□

**Note 5.1:** Theorem 5.1 is based on the investigation of the Newton method which can be found in [17], see also [9].

**Note 5.2:** Theorem 5.1 can be further generalized to include inexactly computed Newton iterations, for the technical details, see [9].

**Theorem 5.2** *Let us consider numerical solution of the nonlinear elasticity problem (3.7) – (3.10) in* $\Omega \subset R^d$ *by the multi-level Newton method with one or more damped Newton iterations on each level.*

*Let us assume that (3.11) – (3.14) hold. Let the nonlinear elasticity problem is weakly formulated in the space* $V \subset [H^1(\Omega)]^d$ *and let the FE spaces* $V_h$ *, exploited for the multi-level Newton method, fulfill the condition* $\inf\{\parallel u - v_{h_i} \parallel : v_{h_i} \in V_{h_i}\} \to 0$ *for* $i \to \infty$ *and any* $u \in V$.

*Then the multilevel Newton method converges.*

**Proof.** We apply Theorem 5.1 with $V \subset [H^1(\Omega)]^d$, $\parallel . \parallel = \parallel . \parallel_{1,2}$ and $V_j = V_{h_i}$. Note that $V_i = V_{h_i}$ if only one Newton iteration is performed on each level, $V_j = V_{j+1} = \ldots = V_{j+k} = V_{h_i}$ if $k$ Newton iterations are performed on the level $i$.

The verification of the assumptions of Theorem 5.1 can be found e.g. in [9], see also [18] and [17].

□

Now, we shall consider the adaptive multi-level Newton method in the case of sufficiently regular solution of the BVP.

**Theorem 5.3** *Let us consider the numerical solution of the nonlinear elasticity problem in* $\Omega \subset R^d$. *Assume sufficiently regular solution of this BVP problem and higher order FEs so that the FE convergence rate is* $O(h^r)$ *with* $r > d/2$, *i.e.*

$$\parallel u - u_h \parallel_{1,2} \leq C_1 h^r \, , \; r > d/2 \, . \tag{5.15}$$

*Assume also that (3.15) and (3.16) are valid.*

*Now consider the adaptive multi-level Newton method with one Newton iteration on each level. Having sufficiently good approximation* $\tilde{u}_{h\,i}$ *, we shall construct* $V_{h\,i+1}$ *in such way that*

$$C_1 h^r_{i+1} = \| \, u - \tilde{u}_{h\,i} \, \|^{1+\alpha}_{1,2} , \quad h_{\,i+1} \, \leq \, h_0 \tag{5.16}$$

*where*

$$\alpha = \frac{2r-d}{2r+d} \in (0,1). \tag{5.17}$$

*Then there exists a constant* $\bar{c}$*, depending only on* $C_1$ *,* $C_2$ *from (3.16) ,* $r$*,* $d$ *and* $h_0$ *, such that*

$$\| \, u - \tilde{u}_{h\,i+1} \, \|_{1,2} \leq \, \bar{c} \, \| \, u - \tilde{u}_{h\,i} \, \|^{1+\alpha}_{1,2} \; . \tag{5.18}$$

**Note 5.3:** Theorem 5.3 shows *superlinear convergence* of the multi-level Newton method in case of sufficiently regular solution. For increasing $r$ , we can obtain nearly quadratic convergence. It is an interesting result, because the superlinear convergence of the Newton method in the infinite dimensional space is otherwise an open question, see [17], [18].

**Proof.** We shall split the error similarly as in Theorem 5.1 and use (5.16) and (3.15). It gives

$$\begin{aligned}
\| \, u - \tilde{u}_{h\,i+1} \, \|_{1,2} &\leq \| \, u - u_{h\,i+1} \, \|_{1,2} + \| \, u_{h\,i+1} - \tilde{u}_{h\,i+1} \, \|_{1,2} \\
&\leq \; \| \, u - \tilde{u}_{h\,i} \, \|^{1+\alpha}_{1,2} + K_{h\,i+1} \, \| \, u_{h\,i+1} - \tilde{u}_{h\,i} \, \|^2_{1,2} \\
&\leq \; \| \, u - \tilde{u}_{h\,i} \, \|^{1+\alpha}_{1,2} + 2K_{h\,i+1} \, \| \, u_{h\,i+1} - u \, \|^2_{1,2} \, + 2K_{h\,i+1} \, \| \, u - \tilde{u}_{h\,i} \, \|^2_{1,2} \\
&\leq \; \| \, u - \tilde{u}_{h\,i} \, \|^{1+\alpha}_{1,2} + 2 \, C_2 \, h^{-d/2}_{i+1} C_1 h^{2r}_{i+1} \\
&\quad + 2 \, C_2 \, h^{-d/2}_{i+1} \, \| \, u - \tilde{u}_{h\,i} \, \|^{1-\alpha}_{1,2} \| \, u - \tilde{u}_{h\,i} \, \|^{1+\alpha}_{1,2} \, .
\end{aligned}$$

Using the assumptions (5.16) and (5.17), we obtain (5.17) with

$$C \;=\; 1 + 2\, C_1 C_2 \; h_{i+1}^{r-d/2} + 2\, C_2 h_{i+1}^{-d/2} \; C_1^{\frac{1-\alpha}{1+\alpha}} \; h_{i+1}^{r\,\frac{1-\alpha}{1+\alpha}}$$

$$\leq\; 1 + 2\, C_1 C_2\, h_0^{r-d/2} + 2\, C_1^{\frac{1-\alpha}{1+\alpha}}\, C_2 \;=\; \bar{c}\,.$$

□

# 6 Conclusions

In this paper we provided a theoretical investigation of efficiency of the two-level and multi-level Newton methods, especially in the case of solution of the BVPs of nonlinear elasticity. The obtained results are based mainly on the theory of the Newton method and the properties of the finite element method. Similar results can be obtained for the application of the two-level and multi-level methods to the solution of other problems, e.g. the problems of nonlinear magnetostatics.

Unfortunately, only a restricted number of numerical experiments have been performed yet. These experiments, reported in [11], confirm the smaller efficiency of the coarse grid correction for a model nonlinear elasticity problem. Further research in this respect will be necessary, especially for the case of less regular solution and adaptive FE techniques.

Another research directions are indicated in the papers [6], [7] and [4].

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