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"TEST OF TWO– AND MULTILEVEL PATCHED REFINEMENT FEM CODE"

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ABSTRACT. Multilevel preconditioning techniques of BEPS–type for solving elliptic second order boundary value problems in a variational setting on grids with local refinement are described. The problems are discretized on a sequence of nested finite element spaces of conforming piecewise polynomial functions obtained by successive steps of patched local refinement. Two main approaches are presented. One is purely algebraic and the multilevel iterations are of a hybrid V–cycle type. That is, at every group of k_0 steps of simple block elimination of the degrees of freedom at the corresponding refinement level, one performs a correction by certain polynomial approximation based on properly scaled and shifted Chebyshev matrix polynomial. This however might be essential for three space dimensions and mostly for problems with rough (i.e., discontinuous) coefficients. This hybrid multilevel V−cycle BEPS-preconditioner is shown to be of optimal order of complexity for 2–D and 3–D domains. It performs without any restrictions on the shape of the subregions as well as with respect to their number and sizes, and to the way of refining for properly chosen parameters. Other alternative (the second approach) is to use more stable form of the subdomain stiffness matrices (than the recursive use of the two–level hierarchical basis, HB) based on wavelet–like modification of the classical HB. In this case the pure V–cycle BEPS preconditioner is of optimal order. An essential feature here is proper bounded extension operators based on wavelet–like multilevel decomposition of the functions restricted to the interfaces between the fine and coarse regions. Some model numerical experiments in 2–D are presented as well as experiments of FAC preconditioning for Benchmark problem 1 are presented.

1. Introduction

This report is concerned with construction of optimal order preconditioners for solving elliptic problems of second order on grids with local refinement. Earlier results on optimal iterative refinement methods were investigated by Widlund [16], Yserentant [17], Mandel and McCormick [7], McCormick and Thomas [10], Axelsson and Vassilevski [1] and Vassilevski, Petrova and Lazarov [13]. An optimal order multilevel iterative method for second order elliptic boundary value problems with strong anisotropy and discontinuous coefficients were presented in Margenov and Vassilevski [8]. All these methods are mainly based on the domain decomposition technique and can be reduced to solving problems on the subregions and on the interface between them. Hence, these methods might allow effective parallelization.

Two-grid and multilevel iterative refinement algorithms were studied in McCormick [9]. In the recent years the preconditioning technique of Bramble, Ewing, Pasciak, and Schatz [2] was widely used. The algebraic formulation of this domain decomposition algorithm has been given by Ewing, Lazarov, and Vassilevski for two-level [5] and for multilevel cell-centered discretizations in [6]. The preconditioner from [2] is further referred to as a *BEPS-preconditioner.*

In this report we present two approaches of constructing FAC (or BEPS) preconditioners for elliptic finite element equations on grids with multilevel local refinement. The first

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approach, presented in Section 2, is based on the existence of H^1–bounded approximate harmonic extension operators, that extend data given on an interface boundary into the interior of the neighboring subdomain. Such operators based on wavelet–like multilevel decompositions were first proposed in Bramble and Vassilevski [4]. This approach is general and allows for inexact subdomain solvers – an important practical feature of the solution of problems with multilevel local refinement.

The second approach is based on the natural extension of the algebraic multilevel theory proposed in Axelsson and Vassilevski [1] and extended later on in Vassilevski [11]. Versions of algebraic multilevel BEPS-type preconditioners were proposed and investigated in Ewing, Lazarov, and Vassilevski [6]. One of them was of $V-$cycle type, whereas the second one was of pure $W-$cycle type and it was proven in [6] that the $V-$cycle BEPS-preconditioner is nearly optimal but the second one, of $\nu-$fold, $\nu \geq 2$, $V-$ cycle type, is of optimal order. However, in the latter case there was certain restriction on the speed of refining the subregions. We weakened this restriction by allowing the multilevel iteration to be of $V-$cycle type at every k_0 successive steps and only after that to use a $\nu-$fold, $\nu > 1$, $V-$cycle iteration which is based on a certain polynomial acceleration involving properly scaled and shifted Chebyshev polynomials. This approach is purely algebraic and it is based on certain relations between the Schur complements $S^{(k)}$ of $A^{(k)}$ partitioned into the domain decomposition block form and the corresponding Schur complement $\widetilde{S}^{(k)}$ of $\widetilde{A}^{(k)}$. Namely, by a local analysis technique, one can show that $S^{(k)}$ and $\widetilde{S}^{(k)}$ are spectrally equivalent, we estimate the extreme eigenvalues of ${\widetilde{S}^{(k)}}^{-1} S^{(k)}$ and based only on this information the method is constructed. Hence we do not need any regularity of the bilinear form $a(.,.)$. In particular the corresponding relative condition number is bounded independently of possible jumps of the coefficients of the bilinear form if these coefficients are continuous within each element from the initial triangulation τ_1. However, if we assume certain regularity of the refinement our proof can be modified (see [14]) to show that the pure $V-$cycle multilevel BEPS-preconditioner is optimal, similarly to the results of Widlund [16].

The remainder of the present report is organized as follows. In Section 2 the approach based on the bounded extension operators is presented. In Section 3 we formulate the so-called *hybrid V-cycle BEPS-preconditioner*, i.e. at every group of k_0 steps of simple block elimination of the nodes on the corresponding refinement level, we perform a correction by certain polynomial approximation based on properly scaled and shifted Chebyshev matrix polynomials. The spectral equivalence of this preconditioner to the composite-grid matrix is proven by Vassilevski, Petrova and Lazarov in [14]. The method can be applied to solve 3–D elasticity problems which is of main interest to our project Copernicus COP-94 00820. A number of test examples that demonstrate the practical behavior of the considered hybrid V-cycle multilevel BEPS-preconditioner for some 2–D model problems are presented in Section 3. These experiments confirm the observation known from the domain decomposition theory that even for problems with discontinuous coefficients there is no need to use more sophisticated than V–cycle preconditioners even for problems with highly discontinuous coefficients in two–space dimensions.

Finally, some preliminary experiments regarding the benchmark problem are presented in the last §5.

2. The multilevel FAC/BEPS-preconditioners based on extension operators

In this section we present a unified approach of constructing FAC (Fast Adaptive Composite–grid) or BEPS preconditioners for elliptic problems on grids with multilevel local refinement. A main attention is paid for the practical computations important inexact subdomain solutions. This is possible due to introducing more stable form of the finite element stiffness matrices based on suitable H^1–bounded approximate harmonic extension operators.

Let $\Omega^{(0)} = \Omega$ be the original given computational domain. We associate with $\Omega^{(0)}$ a regular coarse triangulation $\mathcal{T}_0$ and a corresponding coarse finite element space V_0 with conforming Lagrangian piecewise polynomials. The local refinement of Ω can be described as follows. Given $\Omega^{(k)}$ a polygonal subdomain of $\Omega^{(k-1)}$ refine each element of $\mathcal{T}_{k-1}$ contained in $\Omega^{(k)}$ to generate the regular elements of $\mathcal{T}_k$. These regular elements form a regular (i.e., uniformly refined) triangulation $\mathcal{T}_k^0$ of $\Omega^{(k)}$. We associate with $\mathcal{T}_k^0$ a finite element space V_k^0 of piecewise polynomials that vanish on $\partial\Omega^{(k)}$. Then the global (composite–grid) finite element space V_k is defined by $V_k = V_{k-1} + V_k^0 = V_0 + V_1^0 + \cdots + V_k^0$. Note that each space V_j^0 is supported in $\overline{\Omega}^{(j)}$ and corresponds to a regular triangulation $\mathcal{T}_j^0$.

We will be needing the spaces $W_k \equiv V_k^0 + \cdots + V_J^0$. Here J is the total number of local refinement steps performed.

Let finally, $\Gamma_k = \partial\Omega^{(k)} \setminus \partial\Omega$ and $\Omega_2^{(k-1)} = \Omega^{(k-1)} \setminus \Omega^{(k)}$.

Then the stiffness matrix corresponding to the space W_k admits the following hierarchical block–form:

$$A^{(k)} = \begin{bmatrix} A_2^{(k)} & R^{(k)} \\ L^{(k)} & A^{(k+1)} \end{bmatrix} \begin{matrix} \} & V_k^0\big|_{\Omega_2^{(k)}} \\ \} & W_{k+1} \end{matrix} \quad . \tag{1}$$

Let now $\mathcal{E}_k v_2 \in W_k$ be an extension operator defined for $v_2 \in V_k^0\big|_{\Omega_2^{(k)}}$. We assume that

$$a(\mathcal{E}_k v_2, \mathcal{E}_k v_2) \leq \eta \inf_{v \in W_k:\ v|_{\Omega_2^{(k)}} = v_2} a(v, v) \cdot \tag{2}$$

We remark that the functions $\mathcal{E}_k v_2$ and $v \in W_k$ used in (1) are supported in $\overline{\Omega}^{(k)}$. Hence the bilinear form $a(.,.)$ can actually be restricted to $a_{\Omega^{(k)}}(.,.)$.

Remark 2.1. *For the construction of bounded extension operators $\mathcal{E}_k$ (i.e., satisfying (2)) we refer to Bramble and Vassilevski* [4].

Now, consider the spaces $\widehat{V}_{k,\,2}^0 \equiv \mathcal{E}_k \left(V_k^0\big|_{\Omega_2^{(k)}} \right) \subset V_k^0$ and let the stiffness matrix $\widehat{A}^{(k)}$ correspond to the following space decomposition of $W_k = \widehat{V}_{k,\,2}^0 + W_{k+1}$. Note that this is a direct decomposition. We then have,

$$\widehat{A}^{(k)} = \begin{bmatrix} \widehat{A}_2^{(k)} & \widehat{R}^{(k)} \\ \widehat{L}^{(k)} & A^{(k+1)} \end{bmatrix} \begin{matrix} \} & V_k^0\big|_{\Omega_2^{(k)}} \\ \} & W_{k+1} \end{matrix} \quad . \tag{3}$$

Let $\mathcal{E}_k = \begin{bmatrix} I \\ E_{12} \end{bmatrix}$ and define $E_k = \begin{bmatrix} I & 0 \\ E_{12} & I \end{bmatrix} \begin{matrix} \} & V_k^0\big|_{\Omega_2^{(k)}} \\ \} & W_{k+1} \end{matrix}$. Then, the following relation holds

$$\widehat{A}^{(k)} = E_k^{\star} A^{(k)} E_k \cdot$$

It is clear then that,

$$\begin{aligned} \widehat{A}_2^{(k)} &= \mathcal{E}_k^{\star} A^{(k)} \mathcal{E}_k, \\ \widehat{R}^{(k)} &= R^{(k)} + E_{12}^{\star} A^{(k+1)}, \\ \widehat{L}^{(k)} &= L^{(k)} + A^{(k+1)} E_{12}, \\ E_k^{\star} A^{(k)} E_k\big|_{W_{k+1}} &= A^{(k+1)}. \end{aligned}$$

Definition 2.1. *Let $B^{(J)}$ be a spectrally equivalent preconditioner to $A^{(J)}$. Let also $\widehat{B}_2^{(k)}$ be spectrally equivalent preconditioners to the blocks $\widehat{A}_2^{(k)} = \mathcal{E}_k^{\star} A^{(k)} \mathcal{E}_k$. Then,*

$$\widehat{B}^{(k)} = \begin{bmatrix} \widehat{B}_2^{(k)} & 0 \\ \widehat{L}^{(k)} & B^{(k+1)} \end{bmatrix} \begin{bmatrix} I & \widehat{B}_2^{(k)^{-1}} \widehat{R}^{(k)} \\ 0 & I \end{bmatrix} \begin{matrix} \} & V_k^0\big|_{\Omega_2^{(k)}} \\ \} & W_{k+1} \end{matrix} \quad . \tag{4}$$

The FAC–preconditioner $B^{(k)}$ for $A^{(k)}$ is obtained from $\widehat{B}^{(k)}$ using the back–transformation, i.e., $B^{(k)} = E_k^{-\star} \widehat{B}^{(k)} E_k^{-1}$.

Remark 2.2. *Note that the actions of* $\widehat{A}_2^{(k)} = \mathcal{E}_k^\star A^{(k)} \mathcal{E}_k$ *are readily available through the actions of* $A^{(k)}$ *and the actions of* $\mathcal{E}_k^\star$ *and* $\mathcal{E}_k$. *Extension operators with computable actions of* $E_{12}^\star$ *and* E_{12} *are found in* [4]. *One can also define preconditioner* $\widehat{B}_2^{(k)}$ *for* $\widehat{A}_2^{(k)}$ *based on any available preconditioner* $B_0^{(k)}$ *for* $A_0^{(k)}$. *Here,* $A_0^{(k)}$ *is the finite element operator corresponding to the regular finite element space* V_k^0 *(i.e., on the regular triangulation* $\mathcal{T}_k^0$*). One possible definition (cf. Vassilevski* [12]*) is as follows:*

$$\widehat{B}_2^{(k)^{-1}} = [I,\ 0] B_0^{(k)^{-1}} \begin{bmatrix} I \\ 0 \end{bmatrix} \cdot$$

2.1. **Analysis of the FAC–preconditioner.** In this subsection we analyze the multilevel FAC–preconditioner defined in Definition 2.1. We assume:

$$\begin{aligned} (\widehat{A}_2^{(k)} v_2, v_2) &\le (\widehat{B}_2^{(k)} v_2, v_2) \le (1+\beta)(\widehat{A}_2^{(k)} v_2, v_2) \quad \text{for all } v_2 \in V_k^0\big|_{\Omega_2^{(k)}} \\ (A^{(J)} w, w) &\le (B^{(J)} w, w) \le (1+\beta)(A^{(J)} w, w) \quad \text{for all } w \in W_J \cdot \end{aligned} \tag{5}$$

By recursion for any $v \in W_k$, set $v^{(k)} = v$ and for $j = k, \ldots, J$ define $v_2^{(j)} = v^{(j)}\big|_{\Omega_2^{(j)}}$ and $v^{(j+1)} = v^{(j)} - \mathcal{E}_j v_2^{(j)} \in W_{j+1}$. Denoting $\mathcal{R}_j \psi = \psi|_{\Omega_2^{(j)}}$, we get

$$v^{(j+1)} = (I - \mathcal{E}_j \mathcal{R}_j) v^{(j)} = \prod_{r=j,j-1,\ldots,k} (I - \mathcal{E}_r \mathcal{R}_r) v^{(k)} \cdot$$

This product is estimated, as in Bramble, Pasciak, Wang and Xu [3]. One has, for any $k \le j \le J$:

$$a(v^{(j)}, v^{(j)}) \le \eta\, a(v,v) \quad \text{for all } v \in W_k \cdot \tag{6}$$

By construction, we have

$$((B^{(k)} - A^{(k)}) v, v) \ge 0 \quad \text{for all } v \in W_k.$$

By recursion, one can also show the following inequalities:

$$\begin{aligned} ((B^{(k)} - A^{(k)})v, v) &= ((\widehat{B}_2^{(k)} - \widehat{A}_2^{(k)}) v_2^{(k)}, v_2^{(k)}) + ((B^{(k+1)} - A^{(k+1)}) v^{(k+1)}, v^{(k+1)}) \\ &\quad + (\widehat{L}^{(k)} \widehat{B}_2^{(k)^{-1}} \widehat{R}^{(k)} v^{(k+1)}, v^{(k+1)}) \\ &\le \beta (A^{(k)} \mathcal{E}_k \mathcal{R}_k v^{(k)}, \mathcal{E}_k \mathcal{R}_k v^{(k)}) + ((B^{(k+1)} - A^{(k+1)}) v^{(k+1)}, v^{(k+1)}) \\ &\quad + (A^{(k+1)} v^{(k+1)}, v^{(k+1)}) \\ &\le \beta \sum_{j=k}^{J-1} (A^{(j)} \mathcal{E}_j \mathcal{R}_j v^{(j)}, \mathcal{E}_j \mathcal{R}_j v^{(j)}) + \beta (A^{(J)} v^{(J)}, v^{(J)}) \\ &\quad + \sum_{j=k+1}^{J} (A^{(j)} v^{(j)}, v^{(j)}) \\ &\le (1 + \beta\eta) \sum_{j=k}^{J} (A^{(j)} v^{(j)}, v^{(j)}) \\ &\le (J - k + 1)(1 + \beta\eta)(A^{(k)} v^{(k)}, v^{(k)}) \cdot \end{aligned}$$

A more refined analysis, based on a strengthened Cauchy–Schwarz inequality can show a level–independent upper bound, i.e., the following bound can be proved:

$$\sum_{j=k}^{J-1} (\widehat{L}^{(j)} \widehat{A}_2^{(j)^{-1}} \widehat{R}^{(j)} v^{(j+1)}, v^{(j+1)}) \le \sigma (A^{(k)} v^{(k)}, v^{(k)}) \cdot$$

3. The algebraic multilevel BEPS-preconditioners

In this section we describe an algebraic way of constructing multilevel BEPS preconditioners. The form of the preconditioner used here does not allow for inexact subdomain solvers.

Consider a sequence of finite element stiffness matrices $\{A^{(k)}\}_{k=1}^{J}$, that is

$$A^{(k)} = \left(a(\phi_i^{(k)}, \phi_j^{(k)})\right)_{x_i, x_j \in N^{(k)}} .$$

$\left\{\phi_i^{(k)}\right\}$ are nodal basis functions in the finite element spaces V_k, $V_1 \subset V_2 \subset \ldots \subset V_J$, where J is the last discretization level. The bilinear form

$$a(u, \phi) = \int_{\Omega} \sum k_{i,j}(x, y) \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_j}\, dx$$

is supposed symmetric, elliptic and bounded.

Here $N^{(k)}$ is the set of the nodes of the corresponding triangulation τ_k of the considered domain Ω, a two dimensional polygon or a 3–diimensional parallelepiped. The set $\{\tau_k\}$ is obtained by successive local refinement. We have at every discretization level the following partitioning of the nodes $N^{(k)} = N_1^{(k)} \bigcup N_2^{(k)}$, where $N_1^{(k)}$ corresponds to the nodes, vertices, of the elements to be refined at the next level $k+1$.

Then any matrix $A^{(k)}$ admits the following domain decomposition 2×2 block form

$$A^{(k)} = \left[\begin{array}{cc} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{array} \right] \begin{array}{l} \} \, N_1^{(k)} \\ \} \, N_2^{(k)} \end{array} . \tag{7}$$

Note that $A_{11}^{(k)}$ corresponds to the discretization of the considered problem on a grid which is obtained by uniform refinement, that is on a regular grid. We denote by $\widetilde{A}^{(k)}$ the matrix $A^{(k-1)}$ but partitioned into the following block form

$$\widetilde{A}^{(k)} = \left[\begin{array}{cc} \widetilde{A}_{11}^{(k)} & \widetilde{A}_{12}^{(k)} \\ \widetilde{A}_{21}^{(k)} & \widetilde{A}_{22}^{(k)} \end{array} \right] \begin{array}{l} \} \, N^{(k-1)} \setminus N_2^{(k)} \\ \} \, N_2^{(k)} \end{array} . \tag{8}$$

Recall that by construction $N_2^{(k)} \subset N^{(k-1)}$ ($N_2^{(k)}$ is the set of nodes on the level k, which corresponds to vertices of elements which are not refined, that is they are also nodes from $N^{(k-1)}$).

At the k-th level of discretization using the algebraic formulation of the BEPS-preconditioner we have

$$B^{(k)} = \left[\begin{array}{cc} A_{11}^{(k)} & 0 \\ A_{21}^{(k)} & \widetilde{S}^{(k)} \end{array} \right] \left[\begin{array}{cc} I & {A_{11}^{(k)}}^{-1} A_{12}^{(k)} \\ 0 & I \end{array} \right] \tag{9}$$

where $\widetilde{S}^{(k)} = \widetilde{A}_{22}^{(k)} - \widetilde{A}_{21}^{(k)} {\widetilde{A}_{11}^{(k)}}^{-1} \widetilde{A}_{12}^{(k)}$ is the Schur complement of $\widetilde{A}^{(k)}$ from (8).

When solving systems with $B^{(k)}$ one needs two solutions with $A_{11}^{(k)}$, that is on a regular grid and one solving with $\widetilde{S}^{(k)}$, which is based on a solving with $\widetilde{A}^{(k)}$, that is on a coarser grid.

Let $\widehat{B}^{(k)}$ be an approximation to the stiffness matrix $A^{(k)}$ so that

$$\lambda \left[{A^{(k)}}^{-1} \widehat{B}^{(k)} \right] \in [1, \widehat{\lambda}_k]. \tag{10}$$

Let $k_0 \geq 1$ be a given integer parameter. Then redefine, $B^{(k)} = \widehat{B}^{(k)}$ and for $s = k+1, k+2, \ldots, k+k_0$

$$B^{(s)} = \begin{bmatrix} A_{11}^{(s)} & 0 \\ A_{21}^{(s)} & \widetilde{C}^{(s)} \end{bmatrix} \begin{bmatrix} I & A_{11}^{(s)^{-1}} A_{12}^{(s)} \\ 0 & I \end{bmatrix}, \tag{11}$$

where $\widetilde{C}^{(s)}$ is the Schur complement of $\widetilde{B}^{(s)}$, that is the Schur complement of $B^{(s-1)}$ but partitioned into the block form,

$$\widetilde{B}^{(s)} = \begin{bmatrix} \widetilde{A}_{11}^{(s)} & \widetilde{A}_{12}^{(s)} \\ \widetilde{A}_{21}^{(s)} & \widetilde{C}^{(s)} + \widetilde{A}_{21}^{(s)} \widetilde{A}_{11}^{(s)^{-1}} \widetilde{A}_{12}^{(s)} \end{bmatrix} \begin{array}{l} \} \, N^{(s-1)} \setminus N_2^{(s)} \\ \} \, N_2^{(s)} \end{array} . \tag{12}$$

The blocks $\left\{A_{i,j}^{(s)}\right\}_{i,j=1}^2$, $\left\{\widetilde{A}_{i,j}^{(s)}\right\}_{i,j=1}^2$ are from the block partitionings (7) and (8) of $A^{(s)}$ and $\widetilde{A}^{(s)}$, respectively.

Now we define the *hybrid $V-$cycle algebraic multilevel BEPS-preconditioner* which combines the two versions (V- and W-cycle methods) proposed in [6]. See also a corresponding definition in the hierarchical multilevel case, [11].

Let $p_\nu = p_\nu(t)$ be a given polynomial of degree $\nu \geq 1$ such that

$$p_\nu(0) = 1, \qquad 0 \leq p_\nu(t) < 1, \qquad t \in (0,1]. \tag{13}$$

For a given integer $k_0 \geq 1$ we define,

(1) $B^{(1)} = A^{(1)}$,

(2) for $k = 2, 3, \ldots, J$ set

$$B^{(k)} = \begin{bmatrix} A_{11}^{(k)} & 0 \\ A_{21}^{(k)} & C^{(k)} \end{bmatrix} \begin{bmatrix} I & A_{11}^{(k)^{-1}} A_{12}^{(k)} \\ 0 & I \end{bmatrix},$$

where $C^{(k)} = \widetilde{C}^{(k)}$ for $k \neq sk_0 + 1$ and $\widetilde{C}^{(k)}$ is the Schur complement of $\widetilde{B}^{(k)}$, i.e. the Schur complement of $B^{(k-1)}$ partitioned into the block form (12). For $k = sk_0 + 1$, $s = 1, 2, \ldots, J/k_0 - 1$ we first correct $\widetilde{B}^{(k)}$ to $\widehat{B}^{(k-1)}$ as follows

$$\widehat{B}^{(k-1)^{-1}} = \left[I - p_\nu\left(\widetilde{B}^{(k)^{-1}} \widetilde{A}^{(k)}\right)\right] \widetilde{A}^{(k)^{-1}} \tag{14}$$

and then $C^{(k)}$ is defined as the Schur complement of $\widehat{B}^{(k)}$, partitioned into 2×2 block form with respect to the partitioning of the nodes $N^{(k-1)} = (N^{(k-1)} \setminus N_2^{(k)}) \bigcup N_2^{(k)}$.

Since the condition number of $B^{(s)^{-1}} A^{(s)}$, for $s = k+1, k+2, \ldots, k+k_0$ can increase unacceptably, we make a correction in order to stabilize its possible growth.

We choose the following polynomial

$$p_\nu(t) = \frac{1 + T_\nu\left(\frac{1+\alpha-2t}{1-\alpha}\right)}{1 + T_\nu\left(\frac{1+\alpha}{1-\alpha}\right)}, \qquad t \in [\alpha, 1]. \tag{15}$$

T_ν is the Chebyshev polynomial of degree ν, that is $T_0(x) = 1$, $T_1(x) = x$, $T_{s+1}(x) = 2xT_s(x) - T_{s-1}(x)$, $s = 1, 2, \ldots$ and $\alpha \in (0,1)$ is a properly chosen parameter.

From (14) it is seen that the following estimate for $\widehat{\lambda}_{k-1} = \lambda_{\max}\left[\widetilde{A}^{(k)^{-1}} \widehat{B}^{(k-1)}\right]$ is valid for any polynomial $p_\nu(t)$ satisfying (13)

$$\widehat{\lambda}_{k-1} = \sup\left\{\frac{1}{1 - p_\nu(t)},\ t \in [1/\lambda_{k-1}, 1]\right\}, \text{ where } \lambda_{k-1} = \lambda_{\max}\left[\widetilde{A}^{(k)^{-1}} \widetilde{B}^{(k)}\right].$$

Let for a given $k = sk_0$ the parameter α satisfy

$$0 < \alpha \leq 1/\lambda_{k-1}.$$

Then by (15) we get the following improvement of $\widehat{\lambda}_{k-1}$

$$
\begin{aligned}
\widehat{\lambda}_{k-1} &\leq \sup\left\{\frac{1}{1-p_\nu(t)}, \qquad t\in[\alpha,1]\right\}, \qquad (\alpha\leq 1/\lambda_{k-1})\\
&= \frac{1}{1-\frac{2}{1+T_\nu\left(\frac{1+\alpha}{1-\alpha}\right)}} = \frac{T_\nu\left(\frac{1+\alpha}{1-\alpha}\right)+1}{T_\nu\left(\frac{1+\alpha}{1-\alpha}\right)-1}\\
&= \frac{(1+q^\nu)^2}{(1-q^\nu)^2}, \text{ where } q=\frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}\\
&= \left[\frac{(1+\sqrt{\alpha})^\nu+(1-\sqrt{\alpha})^\nu}{2\sqrt{\alpha}\sum\limits_{s=1}^{\nu}(1+\sqrt{\alpha})^{\nu-s}(1-\sqrt{\alpha})^{s-1}}\right]^2. \qquad (16)
\end{aligned}
$$

Since $q^\nu \longrightarrow 0$, when $\nu\longrightarrow\infty$, it is seen that $\widehat{\lambda}_{k-1}$ can be done arbitrary close to 1 if ν is sufficiently large.

To estimate the condition number of ${A^{(s)}}^{-1}B^{(s)}$, $s=k+1,\ldots,k+k_0$, we need the following auxiliary fact used in [11]. Detailed proofs of the theorems below can be found out in [14].

Lemma 3.1. *There exists an increasing function $\eta=\eta(k_0)$ such that*

$$
\mathbf{v}_2^t A^{(k)}\mathbf{v}_2 \leq \eta(k_0)\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{bmatrix}^t A^{(k+k_0)}\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{bmatrix}, \qquad (17)
$$

where $\begin{bmatrix}\mathbf{v}_1\\ \mathbf{v}_2\end{bmatrix}\begin{array}{l}\}\,N^{(k+k_0)}\setminus N^{(k)}\\ \}\,N^{(k)}\end{array}$, *all* $\mathbf{v}_1,\ \mathbf{v}_2$, *and* $k=1,2,\ldots,J$. $\square$

Remark 3.1. *For two–dimensional problems we can set $\eta(k_0)=C\,k_0$, $C=const$ and for three dimensional problem $\eta(k_0)=C\,\mu^{k_0}$, where the constant $\mu\geq 2$ is an upper bound of the ratio of the mesh sizes h_k and h_{k+1} of two consecutive grids, i .e. $\mu\geq\max_{1\leq k\leq l-1}h_k/h_{k+1}$.* $\square$

The spectral equivalence between the Schur complement of the composite-grid matrix at the s-th level and the coarse-grid matrix at the same level is given by the following theorem

Theorem 3.1. *The following inequalities are valid*

$$
\frac{1}{\eta(1)}\mathbf{v}_2^t\widetilde{S}^{(s)}\mathbf{v}_2\leq\mathbf{v}_2^tS^{(s)}\mathbf{v}_2\leq\mathbf{v}_2^t\widetilde{S}^{(s)}\mathbf{v}_2,\ \ all\ \mathbf{v}_2,
$$

i.e. the Schur complements $\widetilde{S}^{(s)}$ and $S^{(s)}$ of $\widetilde{A}^{(s)}$ and $A^{(s)}$, respectively, see (8) and (7), are spectrally equivalent. ($\eta(1)$ is from (17)). $\square$

Remark 3.2. *In the two-level case we have that $\eta(1)=1/(1-\gamma^2)$, where γ is the constant from the strengthened Cauchy–Bunyakowski–Schwarz inequality.* $\square$

Now the main result on estimation of the relative condition number of $B^{(k)}$ with respect to $A^{(k)}$ is summarized in the following theorem

Theorem 3.2. *Let $\nu>\sqrt{\eta(k_0)}$, see (17) and $\alpha\in(0,1)$ satisfy the inequality*

$$
\frac{(1-\alpha)^\nu}{\alpha\left[\sum_{s=1}^{\nu}(1+\sqrt{\alpha})^{\nu-s}(1-\sqrt{\alpha})^{s-1}\right]^2}\leq\left[1/\alpha-\sum_{s=0}^{k_0}\eta(s)\right]/\eta(k_0). \qquad (18)
$$

Then the algebraic multilevel preconditioner $B^{(k)}$ of hybrid $V-$cycle type (1), (2), with the properly scaled and shifted Chebyshev polynomials, (15), are spectrally equivalent to the corresponding stiffness matrices $A^{(k)}$ and the following estimate is valid

$$\lambda \left[{A^{(k)}}^{-1} B^{(k)} \right] \in [1, 1/\alpha]. \square$$

The theory for multilevel splitting can be applied in the design of the finite element solution of 3–D elasticity problems on grids with regular patched local refinement. The solution of such kind of problems is of essential interest to our project Copernicus COP-94 00820.

The two-level case combined with some preliminary numerical results for 3–D elasticity equations defined on a model problem with boundary conditions of mixed type was shown in Vassilevski and Zikatanov [15]. After the discretization, the elasticity equation leads to a second order coupled system of elliptic partial differential equations for the displacements in each coordinate direction. The preconditioned conjugate gradient (PCG) method with the BEPS-preconditioner is used for the solution of the composite grid problem. The preconditioner requires solutions on regular grids (the global coarse grid and the fine grid in the subdomain with local patched refinement). The values of the functions on the so-called *slave nodes* are obtained by interpolation, which comes from the natural embedding of the coarse finite element space in the fine finite element space defined in the subdomain which is locally refined.

The multilevel methods applied for 3–D elasticity problems, with a recursively constructed *BEPS preconditioner*, is a topic of a forthcoming research. The inexact subdomain solvers can be further applied when the hierarchical basis functions in the finite element spaces instead of the standard Lagrangian (nodal) basis functions are used. In this case, the stiffness matrices which correspond to the added nodes in the hierarchical decomposition at each discretization level, are symmetric, positive definite and well-conditioned, i.e., the condition numbers are of order $O(1)$ and are independent of the discretization parameters. This allows to find simple approximations as preconditioners for these matrices, e.g. to use their diagonal part, or more general, to use some incomplete (block-) factorizations. We emphasize that the constants of the spectral equivalence of the preconditioned matrices need not be known explicitly.

4. Numerical Experiments for elliptic problems

Here we present numerical results corresponding to the following bilinear form

$$a(u, \phi) = \int_{\Omega} k(x, y) \bigtriangledown u . \bigtriangledown \phi \, dx \, dy, \tag{19}$$

where Ω is the unit square and the finite element spaces consist of piecewise linear functions on isosceles triangulations τ_k, so that at each level $k = 2, 3, \dots, J = 7$ the locally refined subregion $\Omega^{(k)}$ is the north-east quarter of $\Omega^{(k-1)}$.

We refine the coarse-grid triangles in the subregions into a number of congruent ones, so that $h_k = h_{k-1}/n_0$, where $n_0 = 2, 4, 8$ and $k = 2, 3, \dots, 7$.

The corresponding boundary conditions are of Neumann type on the boundaries $x = 1$ and $y = 1$. On $x = 0$ and $y = 0$ Dirichlet boundary conditions are imposed.

We test the sequences of stiffness matrices $\left\{A^{(k)}\right\}_{k=1}^{J}$, corresponding to the bilinear form from (19) with the following coefficient $k = k(x, y)$:

Problem 1: (a smooth coefficient)

$$k(x, y) = 1 + x^2 + y^2$$

Problem 2: (a discontinuous coefficient)

$$k(x,y)=\begin{cases} 10^{-6}, & x<1/2 \text{ or } y<1/2 \\ 10^{-3}, & 1/2<x<3/4,\ y>3/4 \text{ or } 1/2<x<1,\ 1/2<y<3/4 \\ 1,\ x>3/4 \text{ and } y>3/4 & \end{cases}$$

We consider the pure $V-$cycle multilevel BEPS-preconditioner, i.e. at each level k, $k=2,3,\dots,7$ the polynomials p_ν from (15) are linear, i.e. $p_\nu(t)=1-t,\ \nu=1$.

We estimate the spectrum of $B^{(k)^{-1}}A^{(k)}$ using the Lanczos method where the action of $B^{(k)^{-1}}$ on certain vectors is computed by the algorithm AMLI from [11]. The blocks $A_{11}^{(k)}$ of the matrix at discretization level k correspond to the new nodes added at refinement level k. The subregion problems are solved by a preconditioned conjugate gradient method with a high accuracy. The stopping criterion for these problems was

$$r^t r<\varepsilon, \qquad \varepsilon=10^{-24},$$

where r is the residual vector.

The results are listed in Tables 1, 2 corresponding to the Problems 1, 2 respectively. They illustrate that the pure $V-$cycle multilevel BEPS- preconditioner is of optimal order. Since we solve the problems in the subregions exactly we have that $\lambda_{\min}\left(A^{(k)^{-1}}B^{(k)}\right)$ is always 1. Our experiments are done on uniform grids with meshsize on the initial level $h_1=1/N,\ N=4$ and the number of levels equals to 7.

For Problem 2 we see from Table 2 that the eigenvalues are independent of the jump of the diffusion coefficient. In this case the pure $V-$cycle multilevel BEPS-preconditioner turned out to be also optimal.

In the second test we varied the initial coarse meshsize $h_1=1/N,\ N=8,16,32$ whereas $\frac{h_{k-1}}{h_k}=2$ for both test problems, 1 and 2. The corresponding largest eigenvalues $\lambda_k=\lambda_{\max}\left(A^{(k)^{-1}}B^{(k)}\right)$ are shown in Tables 3-4. Here we also observe that the pure $V-$cycle multilevel BEPS-preconditioner is of optimal order with respect to the coarse-grid size.

Table 1.

Largest eigenvalue λ_k of $A^{(k)^{-1}}B^{(k)}$ for Problem 1.

n_0	k	2	3	4	5	6	7
2	λ_k	1.2698	1.2716	1.2601	1.2712	1.2709	1.2710
4	λ_k	1.4077	1.4079	1.4080	1.4082	1.4083	1.4084
8	λ_k	1.4602	1.4603	1.4605	1.4606	1.4608	1.4609

Table 2.

Largest eigenvalue λ_k of $A^{(k)^{-1}}B^{(k)}$ for Problem 2.

n_0	k	2	3	4	5	6	7
2	λ_k	1.7394	1.7396	1.7398	1.7399	1.7401	1.7403
4	λ_k	2.3217	2.3219	2.3222	2.3224	2.3226	2.3229
8	λ_k	2.6076	2.6079	2.6082	2.6084	2.6087	2.6089

Table 3.

Largest eigenvalue λ_k of $A^{(k)^{-1}}B^{(k)}$ for Problem 1, $n_0=2$.

N	k	2	3	4	5	6	7
8	λ_k	1.2579	1.2529	1.2521	1.2515	1.2510	1.2549
16	λ_k	1.2385	1.2370	1.2361	1.2354	1.2360	1.2358
32	λ_k	1.2368	1.2359	1.2353	1.2348	1.2345	1.2342

Table 4.

Largest eigenvalue λ_k of $A^{(k)^{-1}}B^{(k)}$ for Problem 2, $n_0=2$.

N	k	2	3	4	5	6	7
8	λ_k	1.7394	1.7396	1.7398	1.7399	1.7401	1.7403
16	λ_k	1.6446	1.6447	1.6449	1.6451	1.6452	1.6454
32	λ_k	1.6436	1.6438	1.6440	1.6443	1.6445	1.6446

5. Numerical Experiments for the benchmark 1 problem

The Benchmark 1 considers a single pile in a homogeneous sandy clay soil layer. The computational domain is determined by $x_1^{max} = x_2^{max} = 37.2m; x_3^{max} = 31.0m$. The vertical cross section $x_2 = 18.6$ (the same as with $x_1 = 18.6$) of Ω is presented in Fig.1. The cross section size of the pile is $D = 1.2m$. The length of the pile and the size of the active zone under the top of the pile are respectively $L_P = 15m$ and $H_{act} = 12m$. The mechanical characteristics of the pile and the surrounding soil media are as follows: $E_{pile} = 31500MPa$; $\nu_{pile} = 0.2$; $E_{soil} = 7.1MPa$; $\nu_{soil} = 0.4$. The linear formulation of the problem is treated here. The scheme of loading (the vertical load V and the horizontal load H oriented along the x_1 direction) of the pile is shown in Fig.1, where $V = 3000kN$ and $H = 250kN$. It is assumed that the loads are uniformly distributed on the cross section of the pile.

The coarse FEM grid is presented in Fig.1. The corresponding discrete problem has a size $N = 98304$. Two-level patched refinement is used to obtain a more accurate solution in the zone of strongest interaction between the pile and the soil layer. The locally refined grid is shown in Fig.2.

The developed FAC preconditioner is tested for iterative solution of the corresponding composite grid linear system. The convergence rate of the PCG solver is presented in Table 5. The behavior of the residuals shown in the second column is stable and corresponds to the theory of the method.

Table 5
Iterations in the FAC patched local refinement PCG algorithm for Benchmark 1.

iteration	$(C^{-1}r^j, r^j)^{1/2}$
1	0.555651027741785
2	0.284304969270632
3	6.366803797052252E-002
4	1.414685241755766E-002
5	8.856740780740230E-003
6	5.710598498175179E-003
7	2.783122241915364E-003
8	1.720606130706399E-003
9	1.474086893936963E-003
10	1.558045542229898E-003
11	1.130009923830911E-003
12	4.739055886288881E-004

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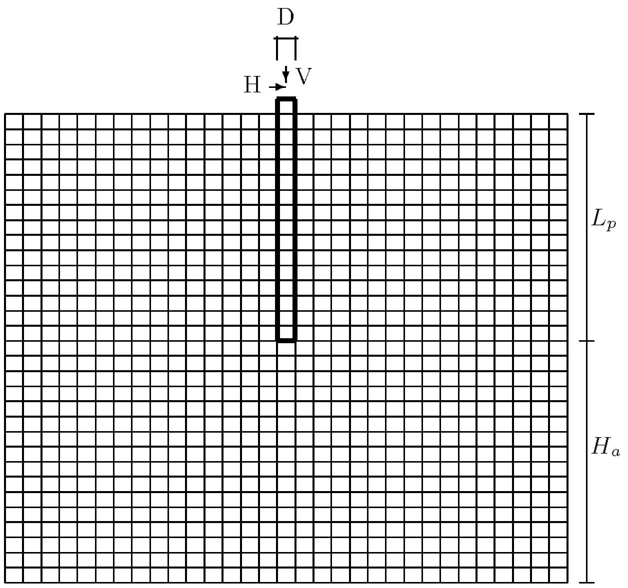

FIGURE 1. *Benchmark 1; Cross section of the computational domain Ω; Basic uniform discretization mesh.*

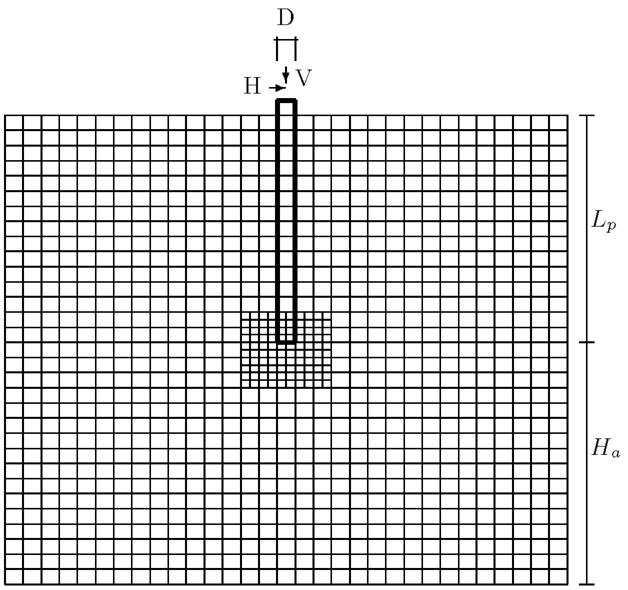

FIGURE 2. *Benchmark 1; Cross section of the computational domain Ω; One level local refinement composite grid.*

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