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Application of BSR BILU preconditioning for nonlinear simulation of pile foundations in 3D.

FEM package. Nonlinear problems. Benchmark 1.

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Abstract

The (B)lock-(S)ize (R)eduction (B)lock-ILU preconditioning algorithm is implemented for 3D simulation of a single pile in nonlinear soil media (Benchmark 1). The problem is described by a coupled system of nonlinear elliptic partial differential equations with strongly varying coefficients. The *secant method* is applied as *outer* nonlinear iterative procedure. At every outer iterative step the BSR BILU PCG algorithm is applied as *inner* iterative procedure. The influence of the inner stopping criterion on the convergence of the nonlinear iterations is discussed. A comparison between linear and nonlinear solution of the benchmark problem is also analyzed.

1 Introduction

We consider the 3D problem of elasto–plasticity under the following general assumptions: the displacements are small; the material properties are isotropic.

Then the stressed–strained state is described by a coupled system of three differential equations, which are written in the form

$$
\begin{aligned}
D\underline{\sigma} + \underline{p} = \quad & 0 \qquad && \forall (x_1, x_2, x_3) \in \Omega \qquad (1)\\
\\
\underline{u} = \quad & \underline{u}_D \qquad && \forall (x_1, x_2, x_3) \in \Gamma_D \\
\sum_{i=1}^{3} \sigma_{ij} n_i = \quad & p_{Nj} \qquad && \forall (x_1, x_2, x_3) \in \Gamma_N, \quad j = 1 \div 3,
\end{aligned}
$$

where $\underline{u}^T = (u_1, u_2, u_3)$ is the vector of the unknown displacements, Ω is the computational domain (in our considerations Ω is assumed to be a logical cube), and Γ_D and Γ_N are the parts of the boundary of Ω with respectively Dirichlet and Neumann boundary conditions; $\underline{p}$ is the vector of the volume forces, and $\underline{u}_D$ and $\underline{\sigma}_N$ are the given boundary data, respectively the vector of the given displacements on Γ_D and the vector of the given stresses on Γ_N. The strains $\underline{\epsilon}^T = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{23}, \epsilon_{31})$ and the stresses $\underline{\sigma}^T = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})$ are determined as follows

$$
D = \begin{pmatrix}
\frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} \\
0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} & 0 \\
0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1}
\end{pmatrix},
$$

$$
\underline{\epsilon} = D^T \underline{u},
$$

$$
\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{pmatrix}
= E^*
\begin{pmatrix}
(1-\nu) & \nu & \nu & & & \\
\nu & (1-\nu) & \nu & & & \\
\nu & \nu & (1-\nu) & & & \\
 & & & (1-2\nu) & & \\
 & & & & (1-2\nu) & \\
 & & & & & (1-2\nu)
\end{pmatrix}
\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{23} \\ \epsilon_{31} \end{pmatrix},
$$

where $\nu = \nu(\epsilon)$ and $E = E(\epsilon)$ stand respectively for the Poisson ratio and for the deformation module, and $E^* = \frac{E}{(1+\nu)(1-2\nu)}$.

The last relations represent the nonlinear nature of the generalized Hooke's law. There are various approaches to describe the nonlinearity of the elasto–plastic materials. Here the generalized Hooke's law is specified by the following additional assumptions:

- The Poisson ratio $\nu \in (0, \frac{1}{2})$ is a constant for a given material (soil layer or constructive element);
- The stress–strain dependence for any nonlinear material is given either as a function $\sigma = \varphi(\epsilon)$ or as a set of experimental data.

These assumptions are natural for the considered engineering problems, where the soil layers are typically nonlinear with their specific elasto–plastic characteristics, while the piles are usually treated as linear elastic solids. Obviously, this means that the coefficients in the boundary value problem (1) are piece–wise continuous

with jumps through the inner boundaries between the different soil layers as well as between the soil and the piles.

To specify the stressed–strained state of the elasto–plastic media by a scalar function the *strain intensity* ϵ_0 is introduced:

$$\epsilon_0 = \sqrt{\frac{2}{3}}\sqrt{(\epsilon_{11}-\epsilon_{22})^2+(\epsilon_{22}-\epsilon_{33})^2+(\epsilon_{33}-\epsilon_{11})^2+6(\epsilon_{12}^2+\epsilon_{23}^2+\epsilon_{13}^2)}.$$

Now if the function φ is known then the deformation module $E = E(\underline{\epsilon})$ is determined by the relation

$$E(\underline{\epsilon}) = \varphi'(\frac{3}{2(1+\nu)}\epsilon_0). \tag{2}$$

It is important that the function $\varphi(\epsilon)$ is strictly increasing which guarantees the ellipticity of the problem (1).

2 Benchmark 1

This benchmark problem illustrates some typical requirements to the HIPERGEOS code related to the bridge engineering applications. The problem is described by the differential equations (1). The considered foundation element is a filling pile. It is assumed that the construction technology guarantees ideal contact between the piles and the surrounding soil layers.

The computational domain is the parallelogram $\Omega = (0, x_1^{max}) \times (0, x_2^{max}) \times (0, x_3^{max})$. The boundary conditions are as follows: homogeneous Dirichlet boundary conditions on the bottom side $x_1 = 0$ (the displacements are equal to zero); homogeneous Neumann boundary conditions on the vertical sides of Ω (the stresses are equal to zero); Neumann boundary conditions on the top side $x_3 = x_3^{max}$, where the load (V, H) from the upper construction is applied. The computational domain is determined by $x_1^{max} = 31.0m$, and $x_2^{max} = x_3^{max} = 37.2m$. The cross section of Ω, $\alpha : x_3 = 18.6$, is presented in Fig.1. The cross section size of the pile is $D = 1.2m$. The length of the pile and the size of the active zone under the top of the pile are respectively $L_P = 15m$ and $H_{act} = 12m$. The mechanical characteristics of the pile are as follows: $E_P = 31500MPa$; $\nu_P = 0.2$.

The scheme of loading (the vertical load V and the horizontal load H) of the pile is shown in Fig.1, where $V = 3000kN$ and $H = 0kN$ (for this report only). It is assumed that the load s uniformly distributed on the cross section of the pile.

The stress–strain diagram $\sigma = \varphi(\epsilon)$ and the other needed characteristics of the soil layer are given in Fig.2.

The scheme of discretization is shown in Fig.1. The mesh sizes are $h_{x1} = 1m$ and $h_{x2} = h_{x3} = 1.2m$.

The finite element method with trilinear finite elements is used for the numerical solution of the problem. Then the size of the corresponding discrete problem is $N = 3n_{x_1}n_{x_2}n_{x_3} = 3.32^3 = 98304$.

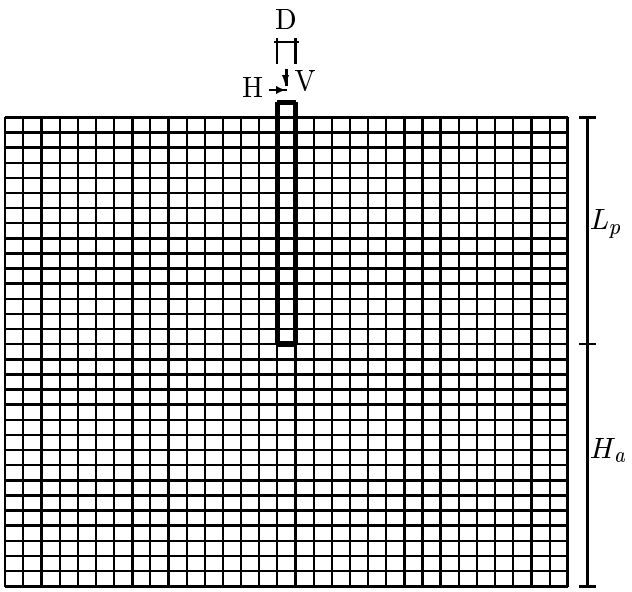
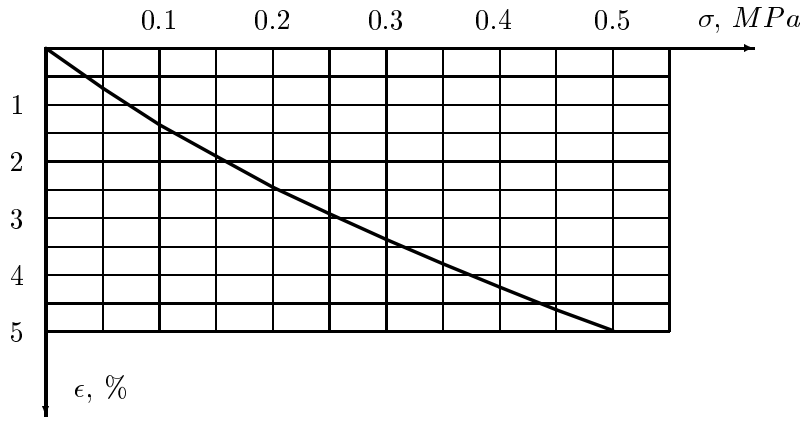


Figure 1: *Benchmark 1; Cross section of the computational domain Ω; Basic uniform discretization mesh.*



σ, MPa	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
ϵ, %	0.70	1.35	1.90	2.45	2.92	3.37	3.80	4.21	4.61	4.98

$$\nu = 0.3$$

Figure 2: *Benchmark 1; Homogeneous soil layer; Sandy clay; Stress–strain diagram.*

3 Nonlinear solver – performance analysis

The FEM nonlinear problem can be written in the form

$$A(u)u = f, \tag{3}$$

where $A(u)$ stands for the nonlinear operator obtained after the FEM discretization of (1), f is the right side vector of the discrete problem, and u is the vector of the nodal unknowns. The *secant method* is applied to (3) which leads to the following nonlinear iterative procedure: for a given initial approximation u^0 for $k = 1, 2, \ldots$ until convergence, solve

$$A(u^{(k-1)})u^{(k)} = f. \tag{4}$$

Here $A(u^{(k-1)}$ is a stiffness matrix computed with varying soil deformation module $E(\epsilon)$ (see 2), where the strain intensity $\epsilon_0 = \epsilon_0(u^{(k-1)})$ corresponds to the displacements $u^{(k)}$ from the previous nonlinear iteration step.

TABLE 1
Performance of the 3D S BSR BILU nonlinear solver.
Benchmark 1: Single pile in a nonlinear sandy clay layer;
$n_{x_1} = n_{x_2} = n_{x_3} = 16$*;* $N = 12288$.

i_{NL}	$\epsilon_{PCG}/\epsilon_{NL}$	N_{PCG}	$\|\|u^{(k+1)} - u^{(k)}\|\|_0$
1	$10^{-2}/10^{-2}$	46	1.8975
2		24	0.0919
3		21	0.0361
4		20	0.0274
5		19	0.0215
6		19	0.0177
	N_{it}^{total}	149	
1	$10^{-4}/10^{-2}$	51	1.8974
2		51	0.0840
3		50	0.0279
4		49	0.0233
5		49	0.0173
	N_{it}^{total}	250	
1	$10^{-6}/10^{-2}$	54	1.8974
2		55	0.0840
3		54	0.0279
4		53	0.0233
5		52	0.0173
	N_{it}^{total}	268	

The BSR BILU PCG algorithm (see in [?], [?], and also in [?]) is applied as an inner iterative solver for the linear problem (4).

We call the presented nonlinear solution algorithm S BSR BILU. We analyze in the rest part of this section the performance of the algorithm, varying the problem size, and the iteration stopping parameters ϵ_{NL} and ϵ_{PCG}. The iteration stopping criteria are as follows: (a) $||u^{(k+1)} - u^{(k)}||_0 \leq \epsilon_{NL}||u^0||_0$ for the nonlinear iterative procedure, where $||.||_0$ stands for the discrete L_2-norm, and; (b) $(C^{-1}r^{N_{it}}, r^{N_{it}}) < 10^{-6}$, where r^j stands for the residual at the jth iteration step of the BSR BILU PCG algorithm applied to the linear problem (4).

The numerical test data are presented in two tables, where Table 1 corresponds to a problem size $N = 3n_{x_1}n_{x_2}n_{x_3} = 3.16^3 = 12288$, and Table 2 to $N = 3n_{x_1}n_{x_2}n_{x_3} = 3.32^3 = 98304$.

The number of iterations N_{PCG} (given in the third column) and the residuals $||u^{(k+1)} - u^{(k)}||_0$ (the last column) show a stable behaviour of the global iterative procedure. The best efficient is obtained (see the total number of iterations) for $\epsilon_{PCG} = \epsilon_{NL} = 10^{-2}$.

TABLE 2
Performance of the 3D S BSR BILU nonlinear solver.
Benchmark 1: Single pile in a nonlinear sandy clay layer;
$n_{x_1} = n_{x_2} = n_{x_3} = 32$; $N = 98304$.

i_{NL}	$\epsilon_{PCG}/\epsilon_{NL}$	N_{PCG}	$\|\|u^{(k+1)} - u^{(k)}\|\|_0$
1	$10^{-2}/10^{-2}$	61	0.2656
2		23	0.0076
3		5	0.0020
	N_{it}^{total}	89	
1	$10^{-4}/10^{-2}$	74	0.2662
2		66	0.0051
3		24	0.0022
	N_{it}^{total}	164	
1	$10^{-6}/10^{-2}$	85	0.2662
2		78	0.0051
3		67	0.0023
	N_{it}^{total}	210	

4 Engineering analysis of the nonlinear solution

We analyze in this section the nonlinear behaviour of considered benchmark problem. The presented test data confirm that the nonlinearity of the soil layer is relatively weak.

The isolines of the strain intensity ϵ_0 are given in Figs. 3. The scale of these isolines is 0.001. The strains are computed in the centers of the finite elements. The maximal strain intensity is $\epsilon_0^{max} = 0.01581$. This means that the obtained solution corresponds to soil layer deformation module nonlinearly taking value in the interval $E_{soil} \in [E_{soil}^{min}, E_{soil}^{max}] = [0.7143, 0.9091]$. One can see that this interval is not too small. The density of the isolines indicates the zone of the soil layer, where the nonlinear model arouses a more considerable redistribution of the stresses as e result from the deformation module changes.

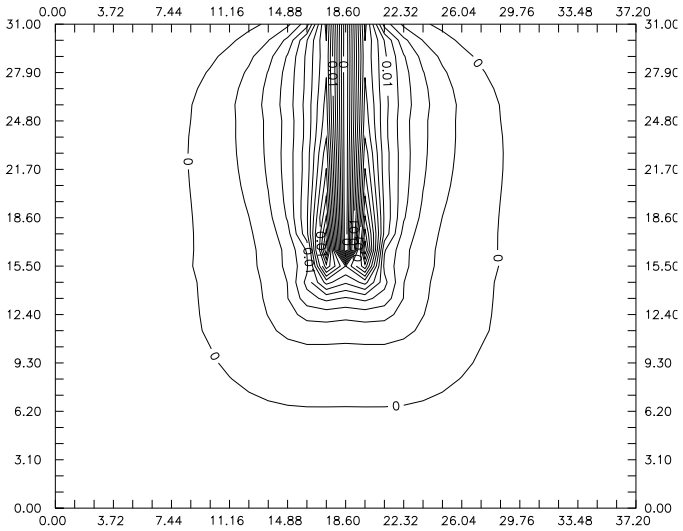


Figure 3: *3D simulation of a single pile in a nonlinear sandy clay layer; Cross section of the computational domain* Ω $\alpha : x_3 = 18.6$*; Isolines of the strain intensity* ϵ_0.

At the end of this section, we compare the results produced by our code to these calculated by standard engineering formulae. Note that such formulae exist for some model problems only, what is, e.g., Benchmark 1 in the case $H = 0$. The maximal value of the vertical displacements u_1 and the vertical normal stresses σ_{11} corresponding to both linear ($E_{soil} = 1.00 kN/cm^2$) and nonlinear cases are given in Table 3.

TABLE 3
Engineering analysis; vertical displacements and vertical stresses.
Benchmark 1: Single pile in a nonlinear sandy clay layer.

	linear model $[cm]$	nonlin. model $[cm]$	linear model $[kN/cm^2]$	nonlin. model $[kN/cm^2]$
FEM code	$u_1^{max} = 1.92$	$u_1^{max} = 2.55$	$\sigma_{11}^{max} = 0.0094$	$\sigma_{11}^{max} = 0.0113$
eng. formulae	$u_1^{max} = 2.00$	$u_1^{max} = 2.64$	$\sigma_{11}^{max} = 0.0103$	$\sigma_{11}^{max} = 0.0109$

The test data well illustrate the accuracy of the FEM model (note, that the engineering formulae are based on completely different assumptions related the computational

domain). The differences are small and in the natural directions. The maximal nonlinear displacements (the maximal nonlinear normal stresses) are $32-33\%$ $(6-20\%)$ larger then the linear ones. This clearly demonstrates the importance of the nonlinear consideration of the problem.

The analysis of the numerical results leads to the following conclusions:

- The developed HIPERGEOS code can be used for computer simulation of pile systems with application to bridge engineering;
- The computer simulation is the only way for solution of such real life problems. Only strongly simplified cases can be treated by traditional engineering formulae.
- The opportunity for nonlinear simulation of pile systems in multilayer soil media provokes new requirements to the quality of the used geological and mechanical characteristics.

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