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BSR BILU preconditioning for 3D simulation of pile foundation systems in multi-layer nonlinear media. Benchmark 2.

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Abstract

The **B**lock-**S**ize **R**eduction **B**lock-ILU (BSR BILU) preconditioning algorithm is implemented for **P**reconditioned **C**ongugate **G**radient (PCG) iterative solution of large-scale linear systems arising from **F**inite **E**lement **M**ethod (FEM) discretization of the 3D elasticity equations. The general BSR BILU algorithm is applied as a basic solver to the numerical simulation of a pile foundation system in nonlinear multi-layer soil media. The problem is described by a coupled system of nonlinear elliptic partial differential equations of second order, with strongly varying coefficients. This system is discretized by trilinear finite elements. The *secant method* is applied as *outer* nonlinear iterative procedure. At every outer iteration, the BSR BILU PCG algorithm is used as an *inner* iterative procedure. Numerical results illustrating the performance of the preconditioning technique as well as the robustness of the nonlinear iterative solver are presented.

1 Introduction

We consider 3D problem of elasto–plasticity under the following general assumptions: the displacements are small; the material properties are isotropic.

Then the stressed–strained state is described by a coupled system of three differential equations, which are written in the form

$$\begin{aligned} D\underline{\sigma} + \underline{p} = &\quad 0 && \forall (x_1, x_2, x_3) \in \Omega \end{aligned} \qquad (1)$$

$$\begin{aligned} \underline{u} = &\quad \underline{u}_D && \forall (x_1, x_2, x_3) \in \Gamma_D \\ \sum_{i=1}^{3} \sigma_{ij} n_i = &\quad p_{Nj} && \forall (x_1, x_2, x_3) \in \Gamma_N, \quad j = 1 \div 3, \end{aligned}$$

where $\underline{u}^T = (u_1, u_2, u_3)$ is the vector of the unknown displacements, Ω is the computational domain (in our considerations Ω is assumed to be a logical cube), and Γ_D and Γ_N are the parts of the boundary of Ω with respectively Dirichlet and Neumann boundary conditions; $\underline{p}$ is the vector of the volume forces, and $\underline{u}_D$ and $\underline{\sigma}_N$ are the given boundary data, respectively the vector of the given displacements on Γ_D and the vector of the given stresses on Γ_N. The strains $\underline{\epsilon}^T = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{23}, 2\epsilon_{31})$ and the stresses $\underline{\sigma}^T = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})$ are determined as follows

$$\underline{\epsilon} = D^T \underline{u},$$

$$D = \begin{pmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{pmatrix},$$

$$\underline{\sigma} = E^* \begin{pmatrix} (1-\nu) & \nu & \nu & & & \\ \nu & (1-\nu) & \nu & & & \\ \nu & \nu & (1-\nu) & & & \\ & & & (1/2-\nu) & & \\ & & & & (1/2-\nu) & \\ & & & & & (1/2-\nu) \end{pmatrix} \underline{\epsilon},$$

where $\nu = \nu(\epsilon)$ and $E = E(\epsilon)$ stand respectively for the Poisson ratio and for the deformation module, and $E^* = \frac{E}{(1+\nu)(1-2\nu)}$.

The last relations represent the nonlinear nature of the generalized Hooke's law. There are various approaches to describe the nonlinearity of the elasto–plastic materials. Here the generalized Hooke's law is specified by the following additional assumptions:

- The Poisson ratio $\nu \in (0, \frac{1}{2})$ is a constant for a given material (soil layer or constructive element);
- The stress–strain dependence for any nonlinear material is given either as a function $\sigma = \varphi(\epsilon)$ or as a set of experimental data.

These assumptions are natural for the considered in this paper engineering problems, where the soil layers are typically nonlinear with their specific elasto–plastic

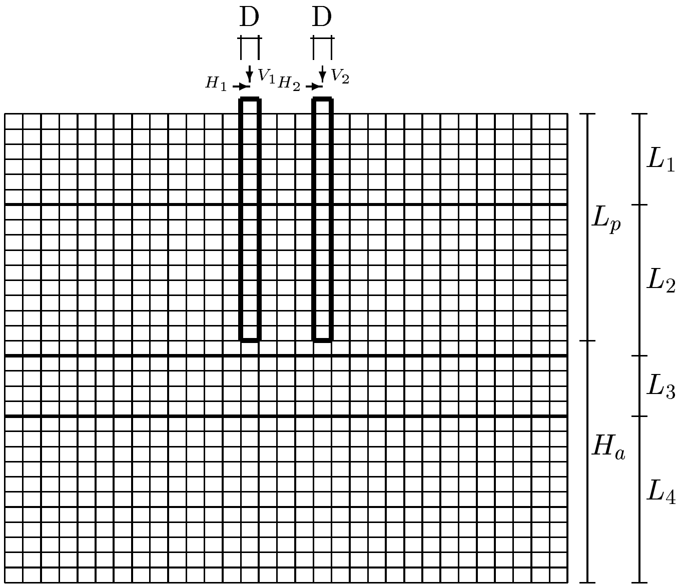

Figure 1: *Cross section* $\alpha : x_3 = 18.6$ *of the computational domain* Ω.

characteristics, while the concrete piles are usually treated as linear elastic solids. The coefficients in the boundary value problem (1) are piece–wise continuous with a (possible strong) jump through the inner boundaries between the different medias.

To specify the stressed–strained state of the elasto–plastic media by a scalar function the *strain intensity* ϵ_0 is introduced:

$$\epsilon_0 = \sqrt{\frac{2}{3}}\sqrt{(\epsilon_{11}-\epsilon_{22})^2+(\epsilon_{22}-\epsilon_{33})^2+(\epsilon_{33}-\epsilon_{11})^2+6(\epsilon_{12}^2+\epsilon_{23}^2+\epsilon_{13}^2)}.$$

Now if the function φ is known then the deformation module $E = E(\underline{\epsilon})$ is determined by the relation

$$E(\underline{\epsilon}) = \varphi'\left(\frac{3}{2(1+\nu)}\epsilon_0\right). \qquad (2)$$

It is important to note that the function $\varphi(\epsilon)$ is strictly increasing which guarantees the ellipticity of the problem (1).

2 Benchmark engineering problem.

This benchmark comes from the bridge engineering. A doubled pile foundation system in a multilayer nonlinear soil media is considered. The computational domain is a parallelogram determined by $x_1^{max} = 31m$; $x_2^{max} = x_3^{max} = 37.2m$. Note, that here x_1 is the coordinate axes in the vertical direction. The boundary conditions are as follows: homogeneous Dirichlet boundary conditions on the bottom side $x_1 = 0$ (all displacements are equal to zero); homogeneous Neumann boundary conditions on the vertical sides of Ω (the stresses are equal to zero); Neumann boundary conditions on the top side $x_1 = x_1^{max}$, where the load (V, H) from the upper construction is applied.

The cross section $x_3 = 18.6$ of Ω is presented in Fig.1. The cross section size of the piles is $D = 1.2m$. The distance between the piles is $a = 2.4m$. The length of the piles, the size of the active zone under the top of the piles, and the mechanical characteristics of the piles are as follows: $L_P = 15m$, $H_{act} = 12m$, $E_P = 31500MPa$ and $\nu_P = 0.2$.

The scheme of the loading (the vertical loads V_1, V_2 and the horizontal load $H = H_1 = H_2$) are shown in Fig.1, where $V_1 = 2000kN$, $V_2 = 4000kN$ and $H = 150kN$.

The concrete pile is assumed to have linear behaviour with constant mechanical characteristics $E_P = 31500MPa$ and $\nu_P = 0.2$. The Poisson ratios for the soil layers are as follows: Layer 1: $L_1 = 6m$; Organic clay; $\nu_{L_1} = 0.4$; Layer 2: $L_2 = 10m$; Softly plastic clay; $\nu_{L_2} = 0.35$; Layer 3: $L_3 = 4m$; Clay sand; $\nu_{L_3} = 0.25$; Layer 4: $L_4 = 14m$; Moderate sand; $\nu_{L_4} = 0.2$.

TABLE 1

Discrete values of the functions $\sigma = \varphi(\epsilon)$, corresponding to the soil layers L_i, $i =\in \{1, 2, 3, 4\}$.

	σ, MPa	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
Layer 1	ϵ, %	2.00	3.40	4.55	5.60	6.60	7.45	8.25	9.00	9.70	10.4
Layer 2	ϵ, %	0.90	1.65	2.35	2.95	3.50	4.00	4.48	4.92	5.32	5.71
Layer 3	ϵ, %	0.50	1.00	1.45	1.84	2.21	2.55	2.88	3.20	3.48	3.71
Layer 4	ϵ, %	0.30	0.65	0.95	1.23	1.49	1.71	1.91	2.10	2.27	2.40

The stress-strain diagram for the different soil layers are given in a discrete form in Table 1. The finite element method with trilinear finite elements is used for the numerical solution of the problem (see, e.g., in [1]). The size of the discrete problem corresponding to the mesh shown in Fig.1 is $N = 98304$.

3 Block–size reduction BILU algorithm

The finite element formulation of the considered 3D problem leads to a symmetric positive definite block system and therefore is amenable to iterative solution by block schemes ([1], [2], [3], [7]). In this section we briefly present the BSR BILU method proposed by Chan and Vassilevski in [5](see also in [4] and [6]).

The method is defined for any given block–tridiagonal matrix A with positive definite symmetric part. Note that this is a much larger class than the class of M–matrices for which the more classical block–ILU methods have proven existence.

The next theorem characterizes the convergence rate of the BSR BILU algorithm.

Theorem 1. The BSR BILU preconditioner C_{BSR} has relative condition number with respect to the stiffness matrix A of order $\kappa(C_{BSR}^{-1}A) = O((\frac{H}{h})^2)$, where $\frac{H}{h} = \max\{\frac{N_{x1}}{m_p}, \frac{N_{x2}}{m_l}\}$ is the maximal size reduction factor.

More precisely, in the construction of the BSR BILU preconditioner m_p stands for the coarse–block size with respect to the *plane* blocks as well as m_l is the coarse–block size related to the *line* blocks. The proof follows from the general theory as presented in [6], and from the Korn's inequality (see a detailed proof in the 2D case in [4]). The application of the BSR BILU algorithm for 3D simulation of a single pile in the linear case was considered in [7].

Let us consider now the linearization of the benchmark problem with averaged deformation modules for the soil layers as follows: $E_{L_1} = 5.2MPa$, $E_{L_2} = 9.4MPa$, $E_{L_3} = 14.0MPa$ and $E_{L_4} = 21.4MPa$. We analyze the performance rate of the PCG method for this linear problem, varying the parameters (m_p, m_l) of the BSR BILU algorithm. The goal of this consideration is to determine the BSR BILU parameters to be used in the nonlinear solver.

We show in Table 2 the number of iterations as a measure of the convergence rate of the preconditioner. The iteration stopping criterion is $(Cr^{N_{it}}, r^{N_{it}}) < 10^{-6}$, where r^j stands for the residual at the jth iteration step of the PCG method and N_{it} stands for the number of iterations needed to achieve the required accuracy.

TABLE 2

Performance of the 3D BSR BILU linear solver.
Doubled pile system in a multilayer media; $N_{x_1} = N_{x_2} = N_{x_3} = 32$; $N = 98304$.

m_p	m_l	number of iterations	user time in minutes
1	1	221	37
1	2	162	30
2	1	217	39
2	2	151	30
1	3	177	37
3	1	216	41
2	3	167	37
3	2	149	32
3	3	168	39
1	4	169	43
4	1	216	45

In general, the presented data are in a good agreement with the theoretical estimate. They clearly illustrate the flexibility of the iterative algorithm. We observe a stronger influence of the reduction parameter m_l on the number of iterations. The number of the iterations is relatively large, that is influenced by the strong coefficients jumps.

Finally, we examine the *real time* performance, measured by the *user time*. The numerical data show, that the user time slightly vary when $1 \le m_p \le 3$ and $1 \le m_l \le 3$. More precisely, the minimal user time is achieved for the pair $(m_p, m_l) = (1, 2)$, which are used in the nonlinear solver presented in the next section.

4 Nonlinear solver – performance analysis

The FEM nonlinear problem can be written in the form

$$A(\mathbf{u})\mathbf{u} = \mathbf{f}, \tag{3}$$

where $A(u)$ stands for the nonlinear operator obtained after the FEM discretization of (1), $\mathbf{f}$ is the right side vector of the discrete problem, and $\mathbf{u}$ is the vector of the nodal unknowns.

TABLE 3
Performance of the 3D BSR BILU nonlinear solver.

i_{nl}	$\varepsilon_{PCG}/\varepsilon_{nl}$	N_{PCG}	i_{nl}	$\varepsilon_{PCG}/\varepsilon_{nl}$	N_{PCG}	$\lVert\mathbf{e}^{(k)}\rVert_0$
1	$10^{-2}/10^{-4}$	61	1	$10^{-2}/10^{-4}$	61	0.265655
2		24	2		24	0.008350
3		4	3		4	0.001208
4		3	4		3	0.000777
	N_{it}^{total}	92		N_{it}^{total}	92	
1	$10^{-4}/10^{-4}$	74	1	$10^{-4}/10^{-4}$	74	0.266228
2		66	2		66	0.006067
3		24	3		24	0.001015
4		41	4		41	0.000783
5		3	5		3	0.000059
6		1	6		1	0.000019
	N_{it}^{total}	209		N_{it}^{total}	209	
1	$10^{-6}/10^{-4}$	85	1	$10^{-6}/10^{-4}$	85	0.266224
2		78	2		78	0.006070
3		69	3		69	0.000737
4		64	4		64	0.000267
5		19	5		19	0.000053
6		15	6		15	0.000020
			7		2	0.000020
	N_{it}^{total}	330		N_{it}^{total}	332	

The well known *secant method* is applied to (3): for a given initial approximation $\mathbf{u}^0$ for $k = 1, 2, \ldots$ until convergence, solve

$$A(\mathbf{u}^{(k-1)})\mathbf{u}^{(k)} = \mathbf{f}. \tag{4}$$

Here $A(u^{(k-1)})$ is a stiffness matrix computed with soil deformation modules determined by the formula (2)

$$E(\underline{\epsilon}) = \varphi'\left(\frac{3}{2(1+\nu)}\epsilon_0\right),$$

where the strain intensities $\epsilon_0 = \epsilon_0(\mathbf{u^{(k-1)}})$ correspond to the displacements from the previous nonlinear iteration step. The stress–strain dependence functions $\varphi(\epsilon)$

corresponding to the different soil layers are obtained by a monotone interpolation of the data from Table 1. A piece-wise linear approximation for the derivative $\varphi'(\epsilon)$ is used.

The BSR BILU PCG algorithm is applied as an inner iterative solver for the linear problem (4). Here, we analyze the performance of the algorithm, varying the iteration stopping parameters ε_{nl} and ε_{PCG} corresponding to the flowing stopping criteria: (a) $||u^{(k)} - u^{(k-1)}||_0 \leq \varepsilon_{nl}||u^1||_0$ for the nonlinear iterative procedure, where $||.||_0$ stands for the discrete L_2-norm and $u^{(0)} = 0$, and; (b) $(C^{-1}r^{N_{it}}, r^{N_{it}}) < 10^{-6}$, where r^j stands for the residual at the jth iteration step of the BSR BILU PCG algorithm applied to the linear problem (4).

The obtained results are presented in Table 3. The number of iterations N_{PCG} (given in the third column) and the residuals $||\mathbf{e}^{(k)}|| = ||\mathbf{u}^{(k+1)} - \mathbf{u}^{(k)}||_0$ (the last column) show a stable behaviour of the global iterative procedure. Here i_{nl} stands for the number of the current nonlinear iteration. We see that the total number of iterations N_{it}^{total} remains practically the same when ε_{nl} changes from 10^{-4} to 10^{-6}. Such a behaviour seems to be natural for the considered weakly nonlinear problems, where the influence of ε_{PCG} is dominating.

5 Engineering analysis of the nonlinear solution

We briefly analyze in this section some engineering aspects of the performed numerical tests. Note that the considered benchmark corresponds to a real upper bridge structure, that transfers the external loads to the foundation bed soil, through a pile system.

The computed vertical displacements u_3 related to the cross section $\alpha : x_3 = 18.6$ are presented by isolines in Fig. 2. The scale of the isolines is 0.2 cm. The analysis of the field of the displacements confirms that the size of the computational domain is well determined having in mind the considered loads. This means that the boundaries of the domain Ω are far enough from the zone of active interactions between the piles and the surrounding soil layers.

The isolines of the strain intensity ϵ_0 and of the vertical stress σ_{11} are given in Figs. 3-4. The scales of the isolines for these figures are as follows: a) for the strain intensity: 0.002; b) for the stresses: 0.001 kN/cm^2. The strains and the stresses are computed in the centers of the finite elements. The stresses only in the soil media are shown in Fig. 4. The zones with a big density of isolines in Fig. 3 are these where the nonlinear redistribution of the stresses is considerable. A more detailed analysis shows that approximately 12% of the vertical external load is transfered to the soil spots under the top of the piles. The isolines of the shear stresses σ_{12} in a scale of 0.001 kN/cm^2 are presented in Fig. 5..

The figures show also that the zones around the down parts of the piles require a local mesh refinement to achieve a better accuracy of the strains/stresses.

Let us compare now the results produced by our code with these calculated by analytical engineering formulae in the simple case of a single pile in a homogeneous soil media loaded vertically only with $V = 3000kN$. The maximal value of the vertical

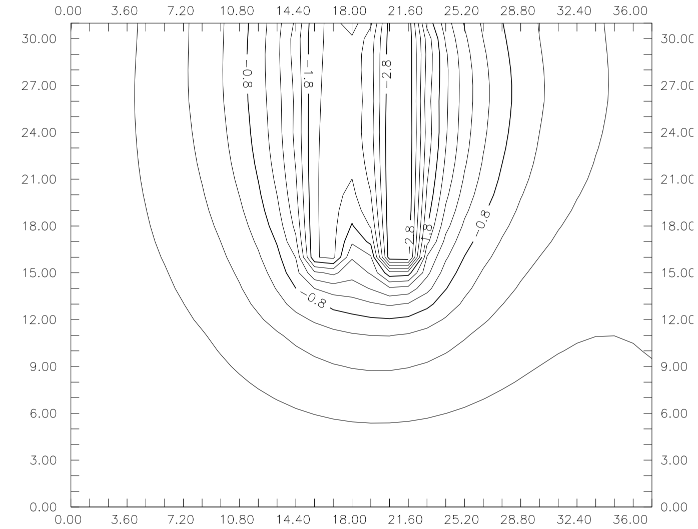

Figure 2: *3D simulation of a doubled pile system in a nonlinear multilayer media; Cross section* $\alpha : x_3 = 18.6$ *of the computational domain* Ω*; Isolines of the vertical displacements* u_1 *in* $[cm]$.

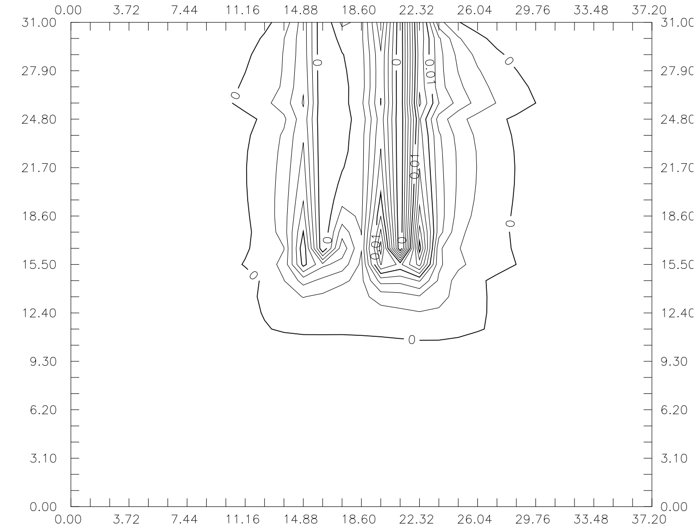

Figure 3: *3D simulation of a doubled pile system in a nonlinear multilayer media; Cross section* $\alpha : x_3 = 18.6$ *of the computational domain* Ω*; Isolines of the strain intensity* ϵ_0.

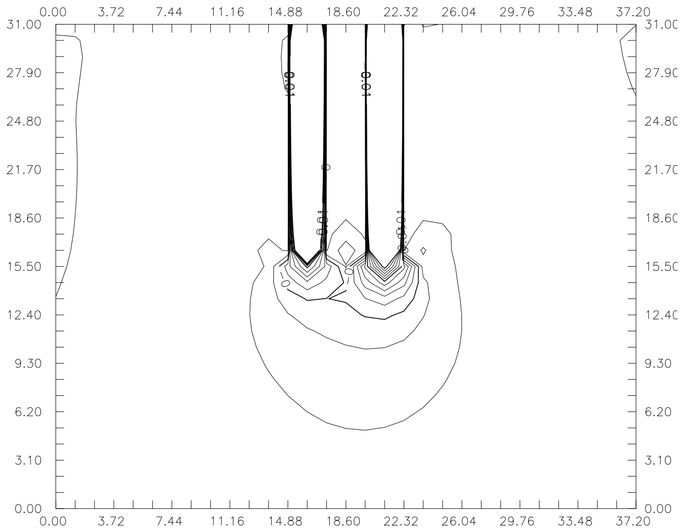

Figure 4: *3D simulation of a doubled pile system in a nonlinear multilayer media; Cross section* $\alpha : x_3 = 18.6$ *of the computational domain* Ω*; Isolines of the vertical stresses* σ_{11} *in* $[\frac{kN}{sm^2}]$.

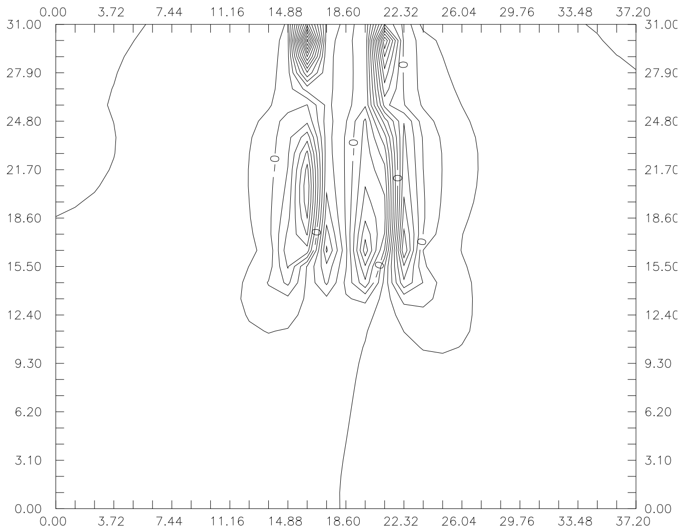

Figure 5: *3D simulation of a doubled pile system in a nonlinear multilayer media; Cross section* $\alpha : x_3 = 18.6$ *of the computational domain* Ω*; Isolines of the shear stresses* σ_{12} *in* $[\frac{kN}{sm^2}]$.

displacements u_1 and the vertical normal stresses σ_{11} corresponding to both linear ($E_{soil} = 1.00kN/cm^2$) and nonlinear cases are given in Table 4.
These results well illustrate the good accuracy of the FEM model (note, that the engineering formulae are based on completely different assumptions, where the computational domain is the whole semi-space). The differences are small and in the expected directions. The maximal nonlinear displacements (the maximal nonlinear normal stresses) are $32 - 33\%$ ($6 - 20\%$) larger then the linear ones. This clearly demonstrates the importance of the nonlinear consideration of the problem.

TABLE 4
Engineering analysis; vertical displacements and vertical stresses.
Single pile in a nonlinear sandy clay layer.

	linear model $[cm]$	nonlin. model $[cm]$	linear model $[kN/cm^2]$	nonlin. model $[kN/cm^2]$
FEM code	$u_1^{max} = 1.92$	$u_1^{max} = 2.55$	$\sigma_{11}^{max} = 0.0094$	$\sigma_{11}^{max} = 0.0113$
analytically	$u_1^{max} = 2.00$	$u_1^{max} = 2.64$	$\sigma_{11}^{max} = 0.0103$	$\sigma_{11}^{max} = 0.0109$

6 Conclussions and remarks

A new robust iterative algorithm for large-scale nonlinear 3D elasticity problems is presented. The numerical tests confirm the convergence theory of the BSR BILU preconditioners for symmetric and positive definite block-tridiagonal matrices that might be not M-matrices. The engineering analysis shows that the created code can be used for numerical simulation of pile foundation systems in nonlinear media.

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