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Tests of two- and multilevel patched refinement FEM code for nonlinear multilayer interaction

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Abstract

Tests of the developed patched local refinement FEM codes for 3-D simulation of pile systems are presented. The considered benchmarks are derived from the real-life engineering practise of MOST XXI Ltd.

The problem is described by a coupled system of nonlinear elliptic partial differential equations of second order, with strongly varying coefficients. This system is discretized by trilinear finite elements.

The **B**lock-**S**ize **R**eduction **B**lock-ILU (BSR BILU) preconditioning algorithm is implemented for **P**reconditioned **C**ongugate **G**radient (PCG) iterative solution of large-scale linear systems arising from **F**inite **E**lement **M**ethod (FEM) discretization of the 3D elasticity equations. The patched locally refined problems are treated by variants of the **F**ast **A**daptive **C**omposite grid (FAC) algorithm. The *secant method* is applied as *outer* nonlinear iterative procedure. At every outer iteration, the BSR BILU PCG algorithm is used as *inner* iterative procedure.

A lot of figures and tables illustrate the performed numerical tests. They well demonstrate the abilities of the applied methods and algorithms as well as the robustness of the developed codes.

1 Introduction

We consider 3D problem of elasto–plasticity under the general assumptions of small displacements and isotropic material properties. The stressed–strained state is described by a coupled system of three differential equations, which are written in the form

$$\begin{aligned} D\underline{\sigma} + \underline{p} = &\quad 0 \qquad && \forall (x_1, x_2, x_3) \in \Omega \\ \underline{u} = &\quad \underline{u}_D && \forall (x_1, x_2, x_3) \in \Gamma_D \\ \sum_{i=1}^{3} \sigma_{ij} n_i = &\quad p_{Nj} && \forall (x_1, x_2, x_3) \in \Gamma_N, \quad j = 1 \div 3, \end{aligned} \tag{1}$$

where $\underline{u}^T = (u_1, u_2, u_3)$ is the vector of the unknown displacements, Ω is the computational domain (in our considerations Ω is assumed to be a logical cube), and Γ_D and Γ_N are the parts of the boundary of Ω with respectively Dirichlet and Neumann boundary conditions; $\underline{p}$ is the vector of the volume forces, and $\underline{u}_D$ and $\underline{\sigma}_N$ are the given boundary data, respectively the vector of the given displacements on Γ_D and the vector of the given stresses on Γ_N. The strains $\underline{\epsilon}^T = (\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{23}, 2\epsilon_{31})$ and the stresses $\underline{\sigma}^T = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})$ are determined as follows

$$\underline{\epsilon} = D^T \underline{u},$$

$$D = \begin{pmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \end{pmatrix},$$

$$\underline{\sigma} = E^* \begin{pmatrix} (1-\nu) & \nu & \nu & & & \\ \nu & (1-\nu) & \nu & & & \\ \nu & \nu & (1-\nu) & & & \\ & & & (1/2-\nu) & & \\ & & & & (1/2-\nu) & \\ & & & & & (1/2-\nu) \end{pmatrix} \underline{\epsilon},$$

where $\nu = \nu(\epsilon)$ and $E = E(\epsilon)$ stand respectively for the Poisson ratio and for the deformation module, and $E^* = \frac{E}{(1+\nu)(1-2\nu)}$.

The last relations represent the nonlinear nature of the generalized Hooke's law specified by the following additional assumptions:

- The Poisson ratio $\nu \in (0, \frac{1}{2})$ is a constant for a given material (soil layer or constructive element);
- The stress–strain dependence for any nonlinear material is given either as a function $\sigma = \varphi(\epsilon)$ or as a set of experimental data.

These assumptions are natural for the considered in this report engineering problems, where the soil layers are typically nonlinear with their specific elasto–plastic characteristics, while the concrete piles are usually treated as linear elastic solids. The coefficients in the boundary value problem (1) are piece–wise continuous with a (possible strong) jump through the inner boundaries between the different medias.

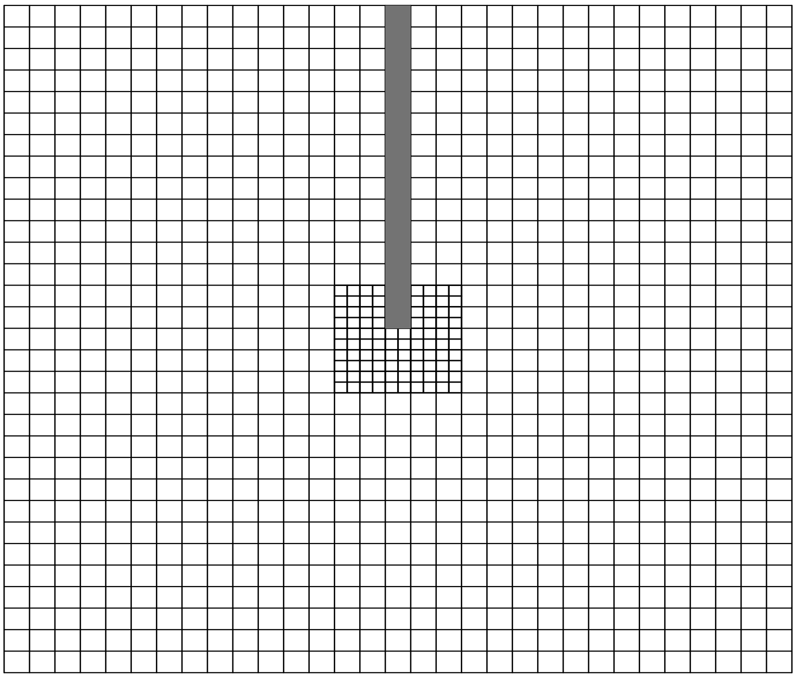
To specify the stressed–strained state of the elasto–plastic media by a scalar function the *strain intensity* ϵ_0 is introduced:

$$\epsilon_0 = \sqrt{\frac{2}{3}} \sqrt{(\epsilon_{11} - \epsilon_{22})^2 + (\epsilon_{22} - \epsilon_{33})^2 + (\epsilon_{33} - \epsilon_{11})^2 + 6(\epsilon_{12}^2 + \epsilon_{23}^2 + \epsilon_{13}^2)}.$$

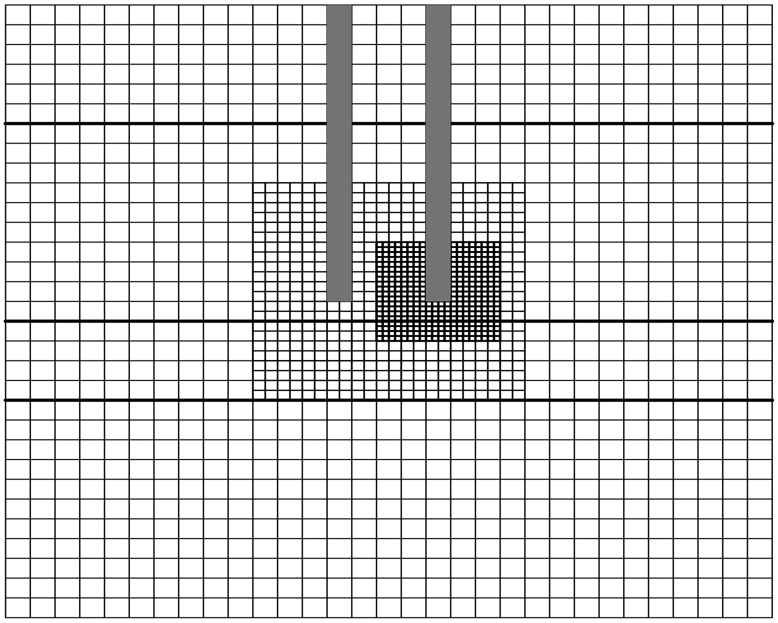
Now if the function φ is known then the deformation module $E = E(\underline{\epsilon})$ is determined by the relation

$$E(\underline{\epsilon}) = \varphi' \left(\frac{3}{2(1+\nu)} \epsilon_0 \right). \qquad (2)$$

It is important to note that the function $\varphi(\epsilon)$ is strictly increasing which guarantees the ellipticity of the problem (1).



(a) Two-level patched local refinement. Benchmark 1.



(b) Three-level patched local refinement. Benchmark 2.

Figure 1: 3-D simulation of pile systems with patched local refinement.

The linearized case of the problem corresponds to the case where $\varphi(\epsilon)$ is a linear function, i.e., the deformation module is a constant for a given soil layer.

The remainder of the report is organized as follows. The benchmark problems are briefly described in the next section. Tests of the FAC FEM code for PCG solution of Benchmark 1 in the linear case is presented in §3. The tests of the multilevel patched refinement FEM code for nonlinear multilayer interaction are presented in the next §4. The last section contains a list of conclusions and remarks.

2 Benchmark problems

Two benchmark problems presented by MOST XXI are considered in this project. They incorporate some typical requirements to the HIPERGEOS code related to bridge engineering applications. The problems are described by the differential equations (1). The considered foundation elements are filling piles that are part of the foundation system of "The Asparuhov Most", one of the biggest bridges in Bulgaria. It is assumed that the construction technology guarantees the ideal contact between the piles and the surrounding soil layers.

The computational domain is the parallelogram $\Omega = (0, x_1^{max}) \times (0, x_2^{max}) \times (0, x_3^{max})$. The boundary conditions are as follows: homogeneous Dirichlet boundary conditions on the bottom side $x_3 = 0$ (the displacements are equal to zero); homogeneous Neumann boundary conditions on the vertical sides of Ω (the stresses are equal to zero); Neumann boundary conditions on the top side $x_3 = x_3^{max}$, where the load (V, H) from the upper construction is applied.

- **Benchmark 1.** Single pile in a homogeneous sandy clay soil layer is considered. The computational domain is determined by $x_1^{max} = x_2^{max} = 37.2m; x_3^{max} = 31.0m$. The vertical cross section $x_2 = 18.6$ (the same as with $x_1 = 18.6$) of Ω is presented in Fig.1(a). The cross section size of the pile is $D = 1.2m$. The length of the pile is $L_P = 15m$. The mechanical characteristics of the pile are as follows: $E_P = 31500MPa$; $\nu_P = 0.2$.

  The external loading includes a vertical force $V = 3000kN$ and a horizontal force $H = 250kN$ oriented along the x_1 direction. It is assumed that the loads are uniformly distributed on the cross section of the pile.

  The stress–strain diagram $\sigma = \varphi(\epsilon)$ and the other needed characteristics of the soil layer are given in Table 1.

  The basic scheme of discretization corresponds to mesh sizes $h_{x1} = h_{x2} = 1.2m$ and $h_{x3} = 1m$. An idea of a composite grid with two-level local mesh refinement is shown also in Fig.1(a). The local refinement area covers the zone of most active interaction between the top of the pile and the surrounding soil media.

  The finite element method with trilinear finite elements is used for the numerical solution of the problem. The size of the discrete problem corresponding to the basic discretization is $N = 98304$.

- **Benchmark 2.** Doubled pile system in a multilayer soil media is considered in this test problem. The computational domain as well as the sizes and the mechanical characteristics of the piles are the same as in the previous benchmark (see Fig.1(b). The distance between the piles is $a = 2.4m$.

The scheme of the loading of the piles is given by $V_1 = 2000kN$, $V_2 = 4000kN$ and $H_1 = H_2 = 150kN$. The indexes of the forces correspond to the number of the pile where the left one is the first. The soil media consists of four horizontal layers numbered towards up-to-down direction.

The stress–strain diagrams $\sigma = \varphi(\epsilon)$ and the other needed characteristics of the different soil layers are given in Table 1.

The basic scheme of discretization is the same as in Benchmark 1. A composite grid for three-level local mesh refinement is shown in Fig.1(b).

TABLE 1.

Stress–strain diagrams of the soil layers for Benchmarks 1-2.

σ, MPa	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
Benchmark 1.: Homogeneous sandy clay layer, $\nu = 0.3$.										
ϵ, %	0.70	1.35	1.90	2.45	2.92	3.37	3.80	4.21	4.61	4.98
Benchmark 2.: Layer 1, organic clay, $\nu = 0.4$.										
ϵ, %	2.00	3.40	4.55	5.60	6.60	7.45	8.25	9.00	9.70	10.4
Benchmark 2.: Layer 2, softly plastic clay, $\nu = 0.35$.										
ϵ, %	0.90	1.65	2.35	2.95	3.50	4.00	4.48	4.92	5.32	5.71
Benchmark 2.: Layer 3, clay sand, $\nu = 0.25$.										
ϵ, %	0.50	1.00	1.45	1.84	2.21	2.55	2.88	3.20	3.48	3.71
Benchmark 2.: Layer 4, clay sand, $\nu = 0.20$.										
ϵ, %	0.30	0.65	0.95	1.23	1.49	1.71	1.91	2.10	2.27	2.40

3 Tests of FAC patched refinement FEM code

The Benchmark 1 in linear formulation is considered in this section. The deformation module of the homogeneous sandy clay layer is taken as $E_{soil} = 1.00kN/cm^2$ averaging the corresponding stress-strain diagram from Table 1.

The developed FAC patched refinement FEM code realizes the symmetric version of the multiplicative BEPS algorithm. The construction of the optimal order preconditioner for solving elliptic problems of second order on grids with local refinement is presented in a more detailed form in Technical Report [11]. Some first tests for Benchmark 1. are given also in the same report.

The convergence rate of the FAC PCG solver is examined here, varying the refinement factor

$$\rho = \frac{H_{coarse}}{h_{fine}} \in \{2, 4\}.$$

The results for $\rho = 2$ and $\rho = 4$ are presented respectively in Table 2 and Table 3.

The behaviour of the residuals shown in the second column is stable. The number of iterations does not depend on ρ, which corresponds to the theory of the method if the regular grid subproblems are solved exactly. What is important to point on is, that in our tests, inexact solution of the regular grid problems is used, where the number of the performed BSR BILU inner iterations is limited to six.

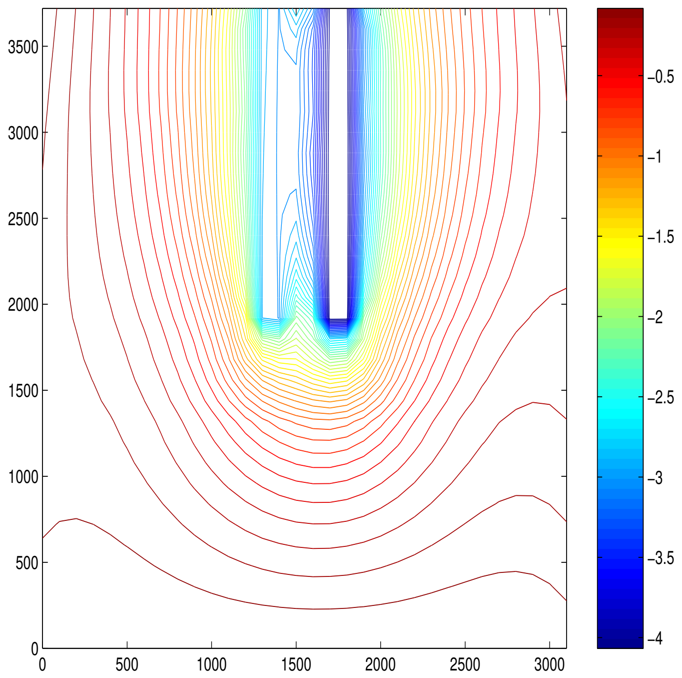
The observed small differences between pseudoresiduals for $\rho = 2$ and $\rho = 4$ (Table 2 and Table 3) can be explained by the relatively small size of the locally refined subdomain.

TABLE 2
Iterations in the FAC patched local refinement PCG algorithm.
Benchmark 1.
$\rho = H_{coarse}/h_{fine} = 2.$

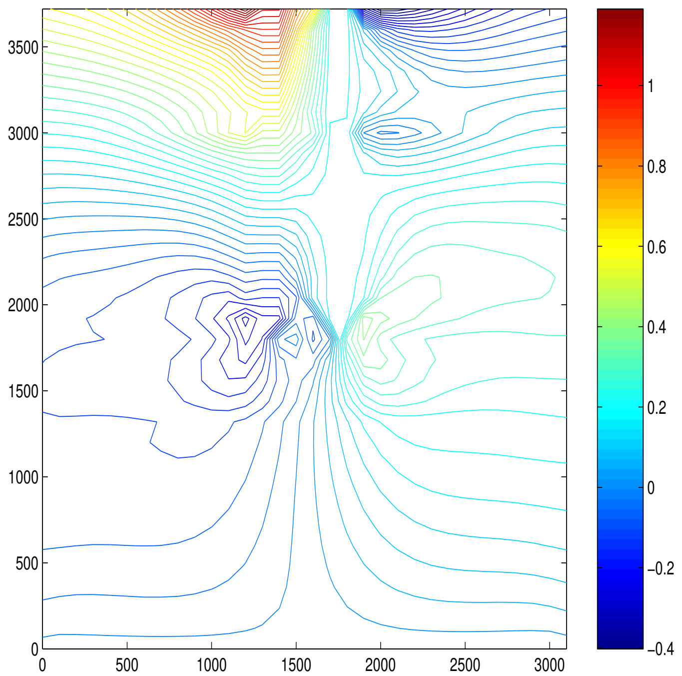
iteration	$(C^{-1}r^j, r^j)^{1/2}$
1	0.555651027741785
2	0.284304969270632
3	6.366803797052252E-002
4	1.414685241755766E-002
5	8.856740780740230E-003
6	5.710598498175179E-003
7	2.783122241915364E-003
8	1.720606130706399E-003
9	1.474086893936963E-003
10	1.558045542229898E-003
11	1.130009923830911E-003
12	4.739055886288881E-004

TABLE 3
Iterations in the FAC patched local refinement PCG algorithm.
Benchmark 1.
$\rho = H_{coarse}/h_{fine} = 4.$

iteration	$(C^{-1}r^j, r^j)^{1/2}$
1	0.555616743803494
2	0.284284933299905
3	6.366786235812413E-002
4	1.414565247893816E-002
5	8.856611957577958E-003
6	5.710848197148091E-003
7	2.783469744019926E-003
8	1.720666216304354E-003
9	1.474235652778188E-003
10	1.558253923083739E-003
11	1.130387807263348E-003
12	4.740947998297675E-004

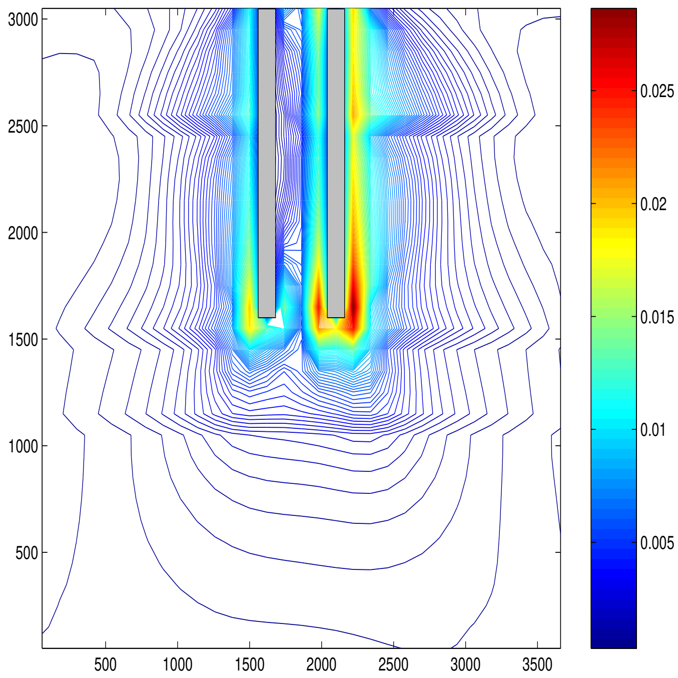


(a) Isolines of the vertical displacements w in cm

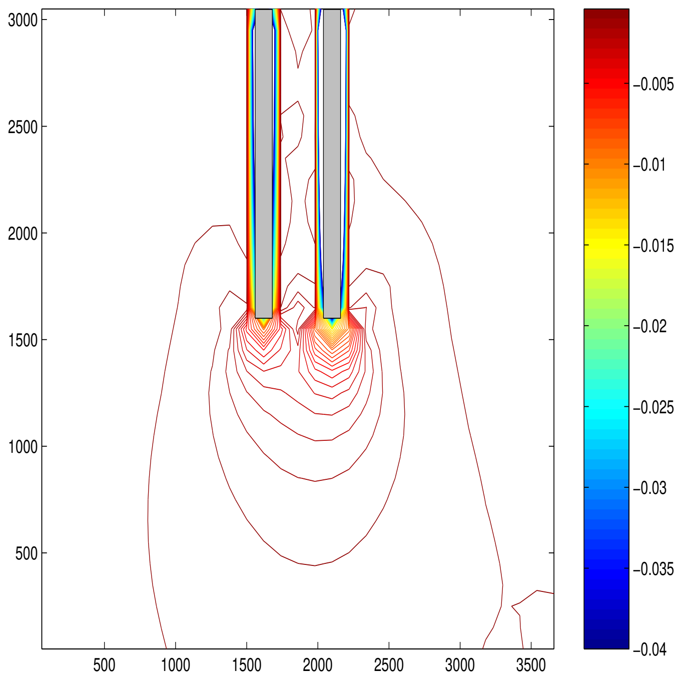


(b) Isolines of the horizontal displacements v in cm

Figure 2: 3-D simulation of a doubled pile system in a nonlinear multilayer media; patched local refinement. Refinement Level 1.

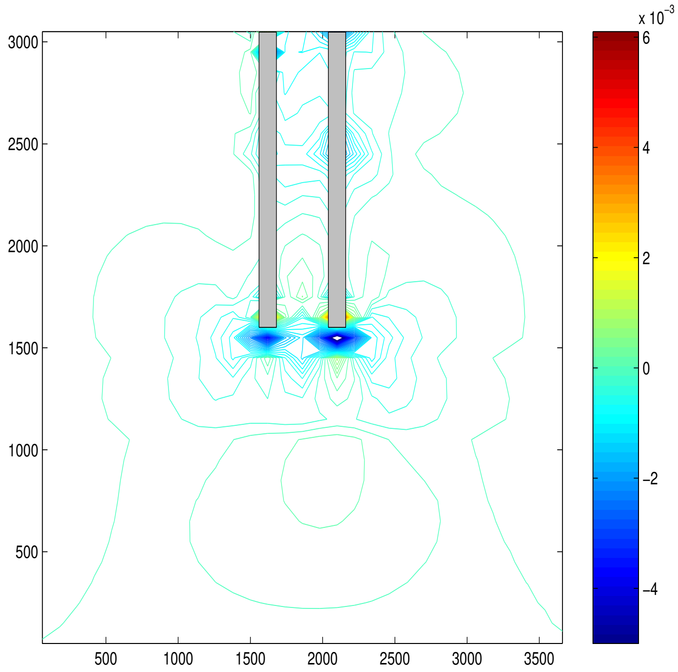


(a) Isolines of the strain intensity ϵ_0

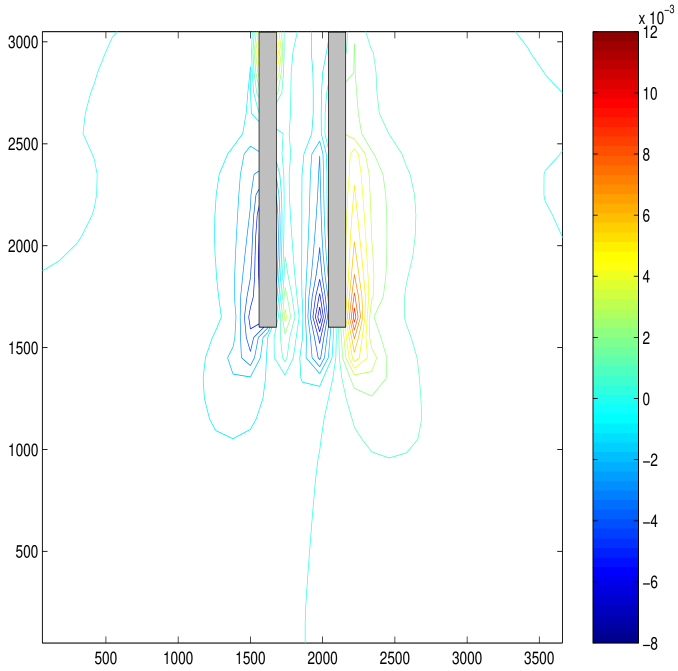


(b) Isolines of the vertical stresses σ_{33} in kN/cm^2

Figure 3: 3-D simulation of a doubled pile system in a nonlinear multilayer media. Patched local refinement. Refinement Level 1.

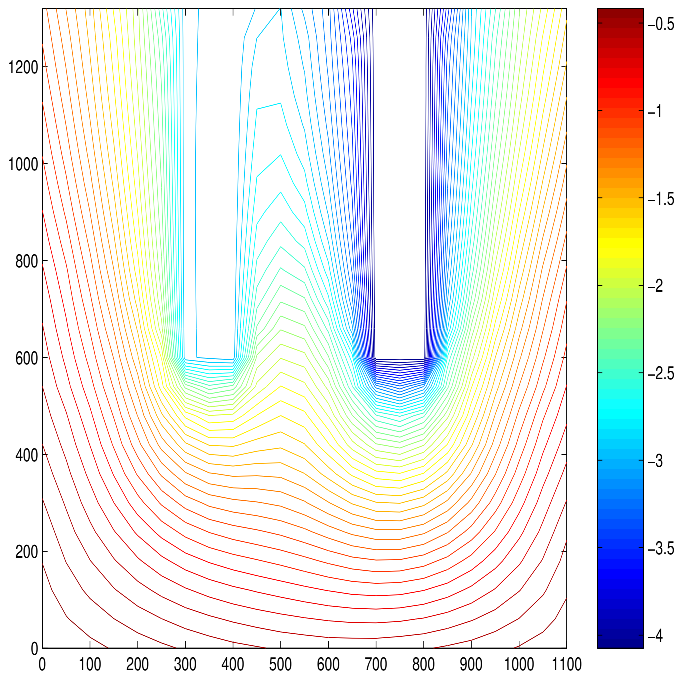


(a) Isolines of the horizontal stresses σ_{11} in kN/cm^2

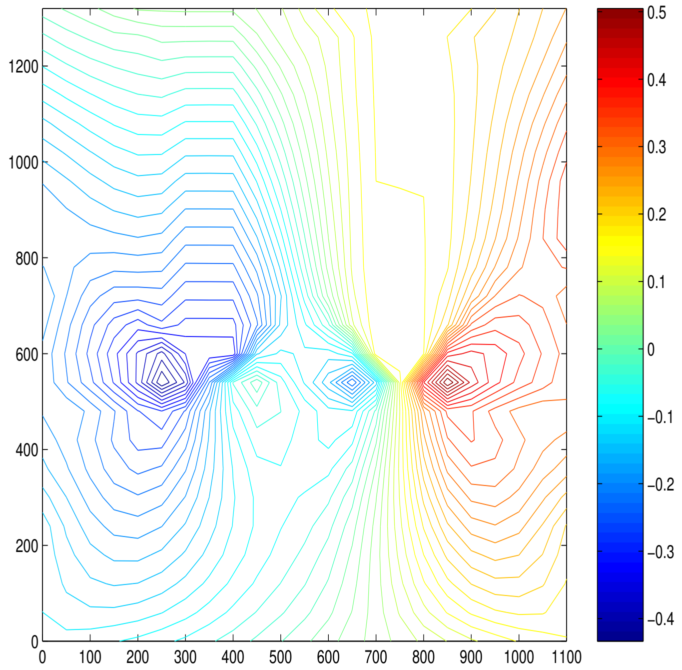


(b) Isolines of the shear stresses σ_{12} in kN/cm^2

Figure 4: 3-D simulation of a doubled pile system in a nonlinear multilayer media. Patched local refinement. Refinement Level 1.

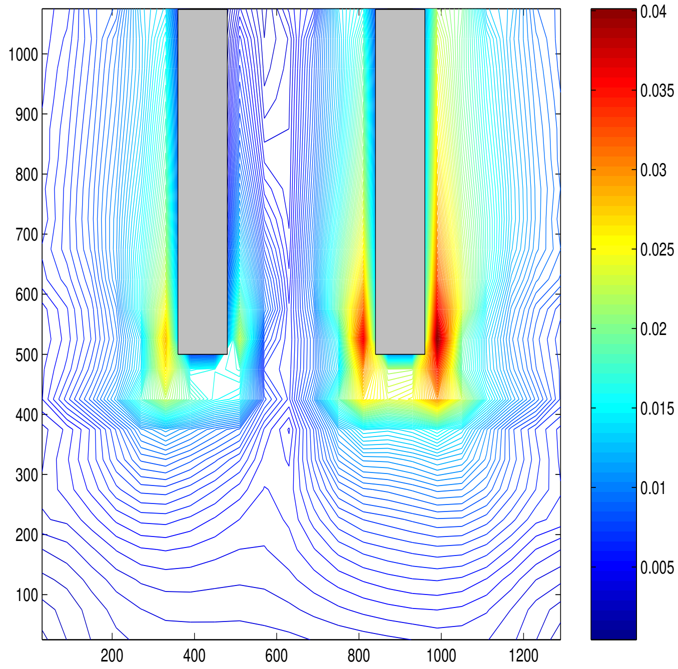


(a) Isolines of the vertical displacements w in cm

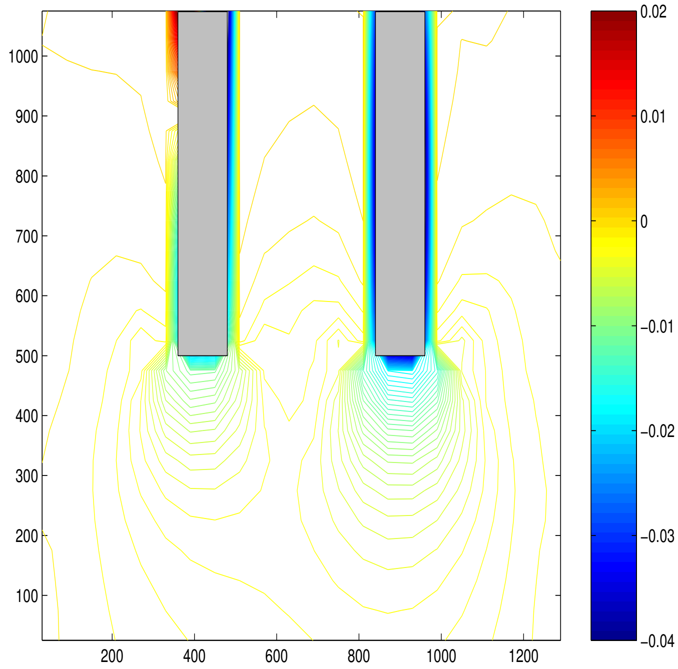


(b) Isolines of the horizontal displacements v in cm

Figure 5: 3-D simulation of a doubled pile system in a nonlinear multilayer media. Patched local refinement. Refinement Level 2.

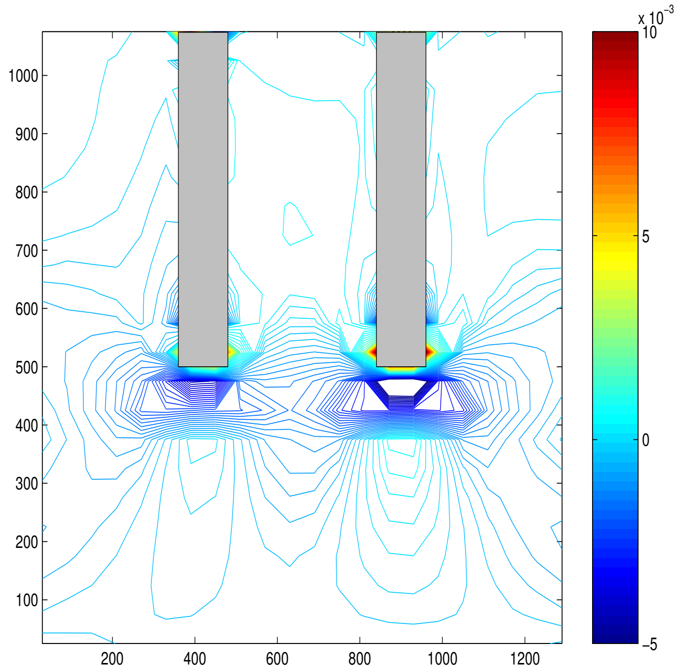


(a) Isolines of the strain intensity ϵ_0

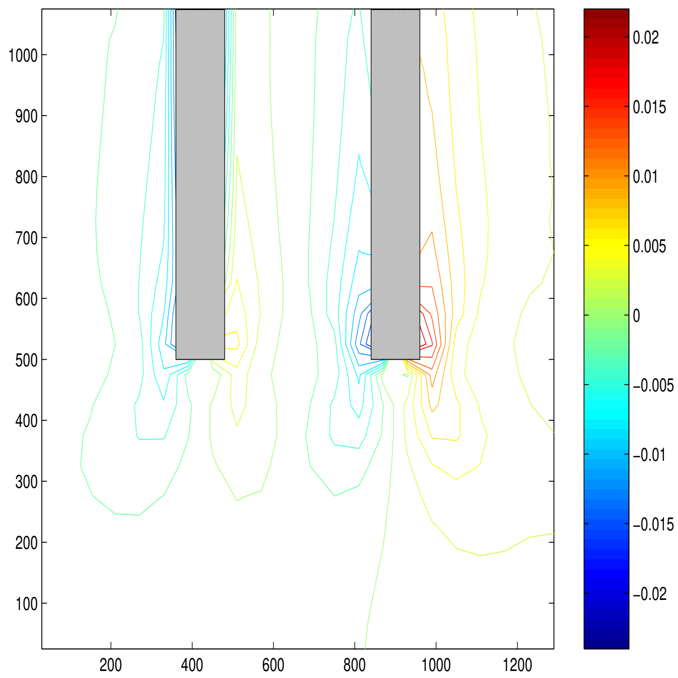


(b) Isolines of the vertical stresses σ_{33} in kN/cm^2

Figure 6: 3-D simulation of a doubled pile system in a nonlinear multilayer media. Patched local refinement. Refinement Level 2.

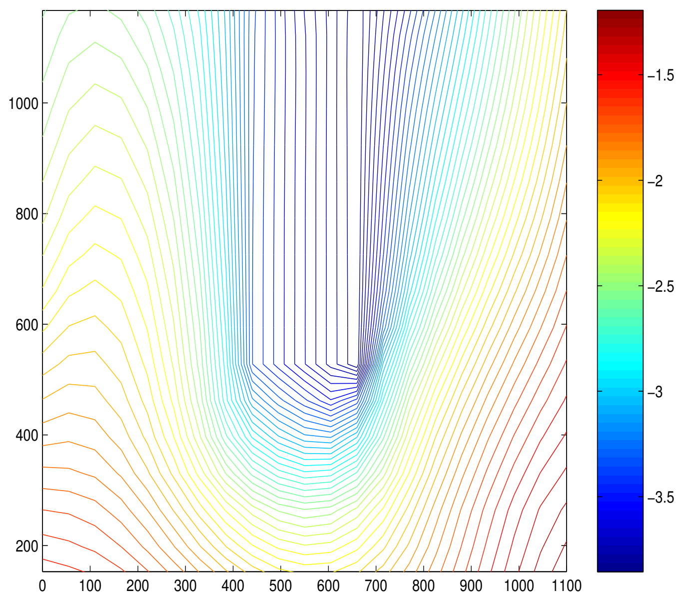


(a) Isolines of the horizontal stresses σ_{11} in kN/cm^2

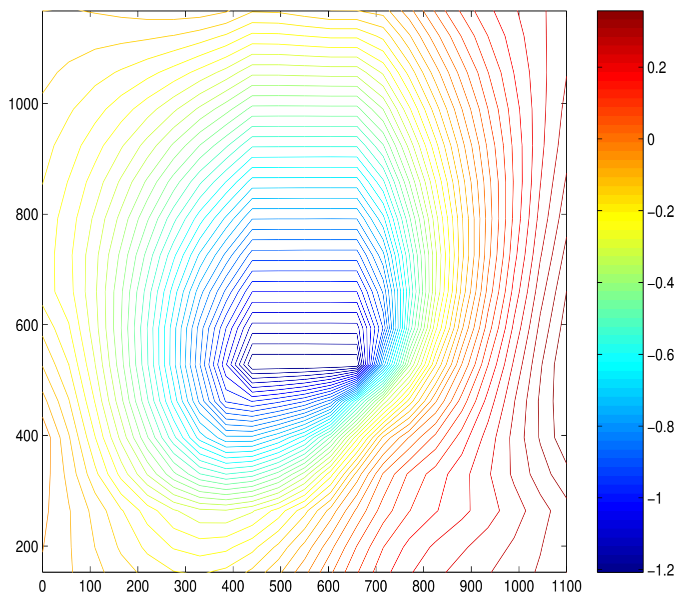


(b) Isolines of the shear stresses σ_{12} in kN/cm^2

Figure 7: 3-D simulation of a doubled pile system in a nonlinear multilayer media. Patched local refinement. Refinement Level 2.

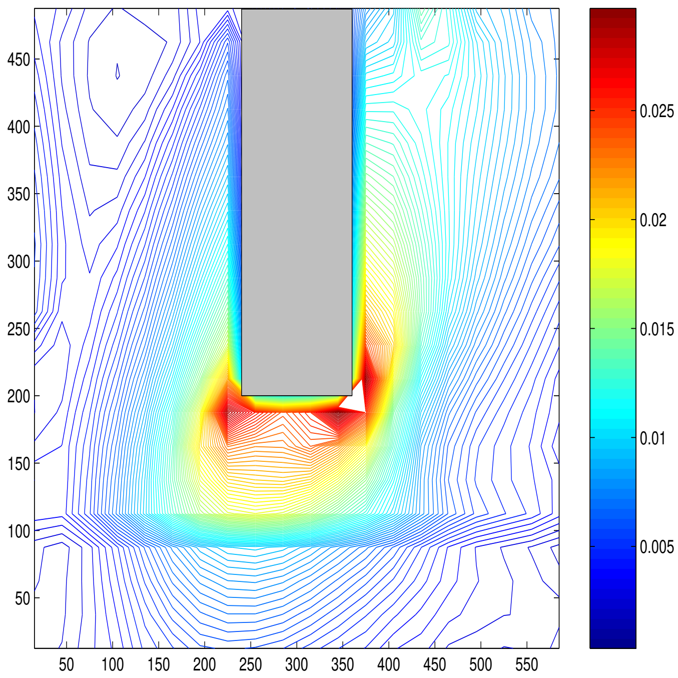


(a) Isolines of the vertical displacements w in cm

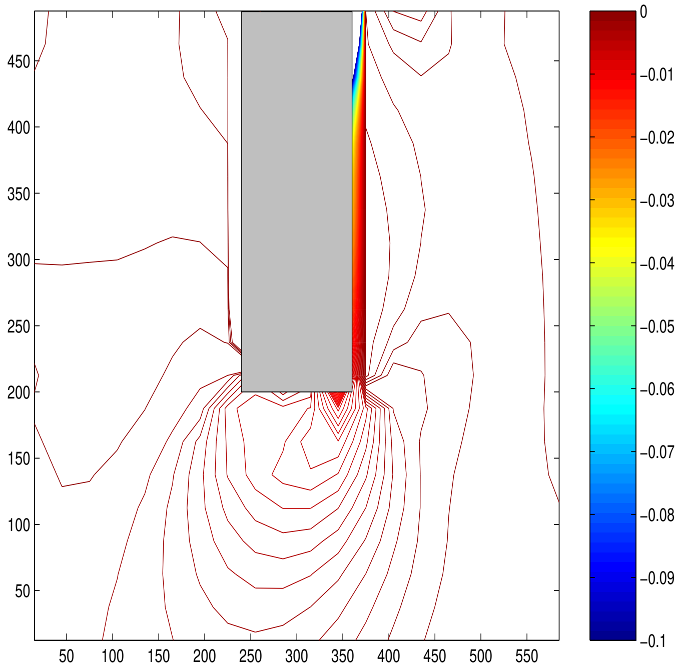


(b) Isolines of the horizontal displacements v in cm

Figure 8: 3-D simulation of a doubled pile system in a nonlinear multilayer media. Patched local refinement. Refinement Level 3.

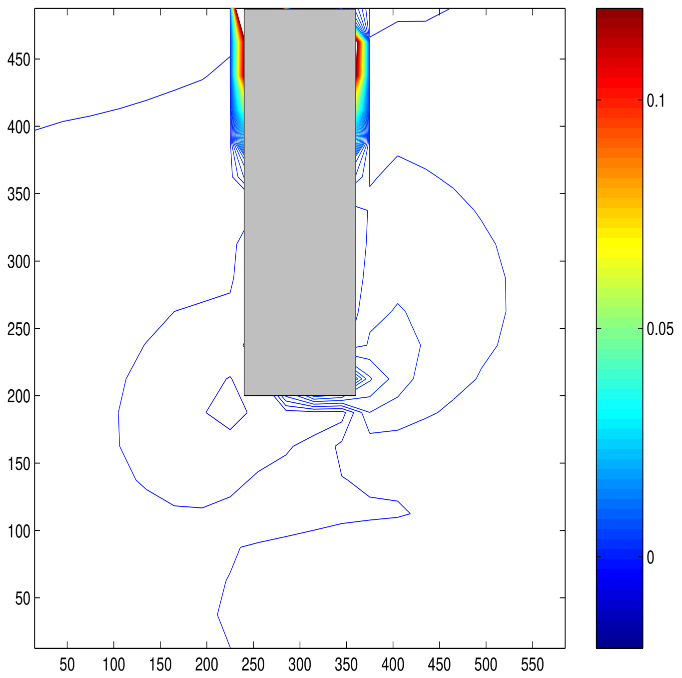


(a) Isolines of the strain intensity ϵ_0

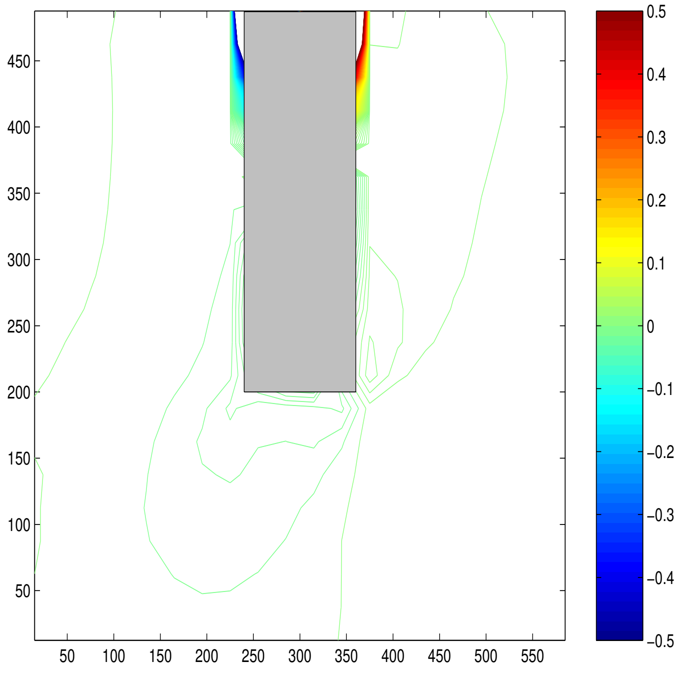


(b) Isolines of the vertical stresses σ_{33} in kN/cm^2

Figure 9: 3-D simulation of a doubled pile system in a nonlinear multilayer media. Patched local refinement. Refinement Level 3.



(a) Isolines of the horizontal stresses σ_{11} in kN/cm^2



(b) Isolines of the shear stresses σ_{12} in kN/cm^2

Figure 10: 3-D simulation of a doubled pile system in a nonlinear multilayer media. Patched local refinement. Refinement Level 3.

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