Optimal order preconditioning of finite difference matrices

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Abstract

A new multilevel preconditioner is proposed for the iterative solution of two dimensional discrete second order elliptic PDEs. It is based a recursive block incomplete factorization of the system matrix partitioned in a two-by-two block form, in which the submatrix related to the first block of unknowns in approximated by a MILU(0) factorization, and the Schur complement computed from a diagonal approximation of the latter submatrix.

It is shown that this technique, combined with a simple W cycle scheme, leads to optimal order preconditioning of five point finite difference matrices. This result holds independently of possible anisotropy or jumps in the PDE coefficients as long as the latter are piecewise constant on the coarsest mesh. Numerical results illustrate the efficiency and the robustness of the proposed method.

keywords: iterative methods for linear systems, acceleration of convergence preconditioning.

AMS CLASSIFICATION : 65F10, 65B99, 65N20.

1 Introduction

This paper deals with the iterative solution of large sparse symmetric positive (nonnegative) definite linear systems

$$A\mathbf{u} = \mathbf{b} \tag{1.1}$$

whose coefficient matrix is an M-matrix arising from the discretization of a two dimensional second order elliptic PDE. In such cases, the conjugate gradient method combined with a suitable preconditioning is a choice method (e.g. [3, 4, 15, 20, 23]). The preconditioner is here a symmetric positive definite matrix B such that solving a system with B is easy whereas the (spectral) condition number of the preconditioned system

$$\kappa(B^{-1}A) \;=\; \frac{\lambda_{\max}(B^{-1}A)}{\lambda_{\min}(B^{-1}A)}\ ,$$

on which depends the convergence rate, has to be as small as possible[1]. In particular, a preconditioner has an optimal order of computational complexity when

- solving a system with B requires $\mathcal{O}(n)$ operations, where n is the number of unknowns;
- $\kappa(B^{-1}A)$ is bounded independently of the grid size and possibly other problem dependent parameters.

In this paper, we propose a method that originates from the algebraic multilevel preconditioning developed in [2, 9, 10, 11, 32]. Like there, the key ingredient is the recursive use of a two-level preconditioner based on a block incomplete factorization of the system matrix partitioned in a two-by-two block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\ , \tag{1.2}$$

where the first block of unknowns corresponds to the fine grid nodes and the second block of unknowns to the coarse grid nodes.

The method proposed here is however original because it directly works for usual finite difference matrices using the standard multilevel partitioning of the unknowns, whereas most previous methods of this type either resort to a further recursive red black like partitioning [5, 6, 9, 29], or originate from the hierarchical basis multigrid method and require some handling to work with standard finite difference matrices [7, 10, 11, 12, 13, 14, 30, 31, 32, 33].

More precisely, we consider

$$B^{(2)} = \begin{pmatrix} P & \\ A_{21} & S \end{pmatrix} \begin{pmatrix} I & P^{-1}A_{12} \\ & I \end{pmatrix}$$

as basic two-level preconditioner where P is a MILU(0) factorization of A_{11} (e.g. [3, 16, 19, 21, 22]) computed with respect to a natural ordering of the concerned nodes, and where $S = A_{22} - A_{21}\, K\, A_{12}$ for some diagonal matrix K (see below).

[1] throughout this paper, $\lambda_{\max}(C)$ and $\lambda_{\min}(C)$ denote respectively the largest and the smallest eigenvalue of C

Since S has similar structure as A, this two-level scheme can be applied recursively, and, on account of its very simple structure, the resulting preconditioner is easy to implement on structured grids using efficient code based on direct addressing.

The sketch of our analysis is as follows. First (Section 2) we show by purely algebraic means that

$$\lambda_{\min}\left(B^{(2)^{-1}} A\right) \geq 1 , \tag{1.3}$$

$$\lambda_{\max}\left(B^{(2)^{-1}} A\right) \leq \lambda_{\max}\left(P^{-1} A_{11}\right) \cdot \lambda_{\max}\left(S^{-1} S_A\right) , \tag{1.4}$$

where $S_A = A_{22} - A_{21}\, A_{11}^{-1} A_{12}$ is the Schur complement of A. Next (Section 3), we consider more particularly five point finite difference matrices and show that $\lambda_{\max}\left(S^{-1} S_A\right) \leq 2$ and $\lambda_{\max}\left(P^{-1} A_{11}\right) \leq \frac{3}{2}$ when the PDE coefficients are piecewise constant on the coarse mesh. Finally (Section 4), we consider the multilevel preconditioner corresponding to the W cycle scheme in which the exact inversion of S required in the two-level method is exchanged for a stationary iteration with the same preconditioner appropriately scaled on the coarser level. We show that the same upper bound (1.4) holds while $\lambda_{\min}$ remains uniformly bounded below.

The method has therefore an optimal order of computational complexity since each preconditioned conjugate gradient iteration is on the other hand less than two times more costly than with a mere ILU(0) like preconditioner. The results of numerical experiments are reported in Section 5.

2 The two-level method

As stated in the introduction, we focus here on discrete second order elliptic PDEs. However, from a purely algebraic point of view, the method is well defined for any symmetric M-matrix with nonnegative row-sum partitioned in a two-by-two block form (1.2).

Let us first recall that an M-matrix has nonpositive offdiagonal entries and that a symmetric matrix with nonpositive offdiagonal entries is an M-matrix if and only if it is nonnegative definite, which in particular occurs when it has nonnegative row-sum [17].

If A is a symmetric M-matrix with nonnegative row-sum, it is then straightforward to show that A_{11} is also a symmetric M-matrix with nonnegative row-sum, entailing that MILU factorizations of A_{11} (e.g. [3, 16, 19, 21, 22]) are well defined and lead to nonnegative definite matrices P satisfying

$$\mathbf{v}_1^t\, P\, \mathbf{v}_1 \leq \mathbf{v}_1^t\, A_{11}\, \mathbf{v}_1 \qquad \forall \mathbf{v}_1 \tag{2.1}$$

and

$$P\, \mathbf{e}_1 = A_{11}\, \mathbf{e}_1 \tag{2.2}$$

where $\mathbf{e}_1 = (1\,1 \ldots 1)^t$.

Let then Δ be the diagonal matrix such that

$$\Delta\, \mathbf{e}_1 = A_{11}\, \mathbf{e}_1 . \tag{2.3}$$

Since A has nonnegative row sum, one has, for all i,

$$(A_{12})_{ij} \neq 0 \ \text{ for some } \ j \quad \Rightarrow \quad \Delta_{ii} > 0 , \tag{2.4}$$

and letting K be the diagonal matrix such that

$$K_{ii} = \begin{cases} \Delta_{ii}^{-1} & \text{if } \Delta_{ii} \neq 0 \\ 0 & \text{otherwise} \end{cases}, \tag{2.5}$$

the matrix

$$S = A_{22} - A_{21}\, K\, A_{12} \tag{2.6}$$

is nothing but the Schur complement of

$$\widetilde{A} = \begin{pmatrix} \Delta & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

in which one has deleted the zero lines and columns.

S is therefore a symmetric M-matrix [3, Theorem 6.10]. Further, in finite difference applications, it is easy to see that S has the same nonzero pattern as the corresponding difference matrix defined on the coarse grid nodes. This leads us to propose, as basic two-level preconditioner,

$$B^{(2)} = \begin{pmatrix} P & \\ A_{21} & S \end{pmatrix} \begin{pmatrix} I & P^{-1}A_{12} \\ & I \end{pmatrix} \tag{2.7}$$

where P is a MILU factorization of A_{11} and S defined by (2.6), (2.5), (2.3). The basic properties of $B^{(2)}$ are gathered in the following Lemma.

Lemma 2.1 *Let $A, P, K, S, B^{(2)}$ be defined as described above.*

(1) S is a symmetric M-matrix with nonnegative row-sum;

(2) $\mathbf{v}_1^t\, P\, \mathbf{v}_1 \;\leq\; \mathbf{v}_1^t\, A_{11}\, \mathbf{v}_1 \qquad \forall \mathbf{v}_1$ *;*

(3) If P is nonsingular, $\mathbf{v}_2^t\, A_{21}\, P^{-1}\, A_{21}\, \mathbf{v}_2^t \;\leq\; \mathbf{v}_2^t\, A_{21}\, K\, A_{12}\, \mathbf{v}_2 \qquad \forall \mathbf{v}_2$ *.*

Proof To prove (1), first observe that S has nonpositive offdiagonal entries, all three terms $(-A_{21})$, $(-A_{12})$ and K being nonnegative. Next, since $A_{12}\,\mathbf{e}_2 = (A\mathbf{e})_1 - \Delta\,\mathbf{e}_1$ and since $A_{21}\, K\, \Delta\mathbf{e}_1 = A_{21}\,\mathbf{e}_1$ because of (2.5), (2.4), one has

$$\begin{aligned} S\,\mathbf{e}_2 &= A_{22}\,\mathbf{e}_2 - A_{21}\, K\, A_{12}\,\mathbf{e}_2 \\ &= A_{22}\,\mathbf{e}_2 + A_{21}\,\mathbf{e}_1 - A_{21}\, K\, (A\mathbf{e})_1 \,, \end{aligned}$$

which is readily found nonnegative. On the other hand, (2) is a well known property of MILU factorizations (e.g. [16]). Finally, (3) is equivalent to

$$\mathbf{w}_1^t\, P^{-1}\, \mathbf{w}_1 \;\leq\; \mathbf{w}_1^t\, K\, \mathbf{w}_1 \qquad \forall \mathbf{w}_1 : (\mathbf{w}_1)_i = 0 \;\text{ for all }\; i : (A_{12}\,\mathbf{e}_2)_i = 0$$

which, because of (2.4), in particular holds when

$$\mathbf{z}_1^t\, \Delta\, P^{-1}\, \Delta\, \mathbf{z}_1 \;\leq\; \mathbf{z}_1^t\, \Delta\, K\, \Delta\, \mathbf{z}_1 \qquad \forall \mathbf{z}_1 \,.$$

Now, $\Delta\, K\, \Delta = \Delta$ by (2.5) and P^{-1} is nonnegative since P is the product of M-matrices, whereas $\Delta\, P^{-1}\, \Delta\,\mathbf{e}_1 = \Delta\,\mathbf{e}_1$ by (2.2). Thus, $\Delta\, K\, \Delta - \Delta\, P^{-1}\, \Delta$ has nonpositive offdiagonal entries and zero row-sum, i.e. is a symmetric M-matrix, therefore nonnegative definite. ■

Using these basic properties, the following theorem proves that the two-level preconditioner is spectrally equivalent to the system matrix if $\mathcal{O}(1)$ constant β, η exist such that

$$\begin{array}{rcll} \mathbf{v}_1^t\, P\, \mathbf{v}_1 & \geq & \beta\; \mathbf{v}_1^t\, A_{11}\, \mathbf{v}_1 & \forall \mathbf{v}_1\ , \\ \mathbf{v}_2^t\, S\, \mathbf{v}_2 & \geq & \eta\; \mathbf{v}_2^t \left(A_{22} - A_{21}\, A_{11}^{-1}\, A_{12}\right) \mathbf{v}_2 & \forall \mathbf{v}_2\ . \end{array}$$

Lower bounds on η and β are derived in Section 3.

Theorem 2.1 *Let*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

be a symmetric nonnegative definite matrix such that A_{11} is invertible. Let P be a symmetric matrix with same dimensions as A_{11} such that

$$\beta\; \mathbf{v}_1^t\, A_{11}\, \mathbf{v}_1 \;\leq\; \mathbf{v}_1^t\, P\, \mathbf{v}_1 \;\leq\; \mathbf{v}_1^t\, A_{11}\, \mathbf{v}_1 \qquad \forall \mathbf{v}_1 \tag{2.8}$$

for some β such that $0 < \beta \leq 1$. Let

$$S = A_{22} - A_{21}\, K\, A_{12} \tag{2.9}$$

where K is a symmetric matrix such that

$$\mathbf{v}_2^t\, A_{21}\, P^{-1}\, A_{12}\, \mathbf{v}_2 \;\leq\; \mathbf{v}_2^t\, A_{21}\, K\, A_{12}\, \mathbf{v}_2 \qquad \forall \mathbf{v}_2\ , \tag{2.10}$$

and let

$$B^{(2)} = \begin{pmatrix} P & \\ A_{21} & S \end{pmatrix} \begin{pmatrix} I & P^{-1}A_{12} \\ & I \end{pmatrix} .$$

If

$$\mathbf{v}_2^t\, S\, \mathbf{v}_2 \;\geq\; \eta\; \mathbf{v}_2^t \left(A_{22} - A_{21}\, A_{11}^{-1}\, A_{12}\right) \mathbf{v}_2 \qquad \forall \mathbf{v}_2 \tag{2.11}$$

for some η such that $0 < \eta < 1$, then

$$\eta\, \beta\; \mathbf{v}^t\, A\, \mathbf{v} \;\leq\; \mathbf{v}^t\, B^{(2)}\, \mathbf{v} \;\leq\; \mathbf{v}^t\, A\, \mathbf{v} \qquad \forall \mathbf{v}\ . \tag{2.12}$$

Proof $\mathbf{v}^t\, B^{(2)}\, \mathbf{v} \leq \mathbf{v}^t\, A\, \mathbf{v}$ is immediate since

$$A - B^{(2)} = \begin{pmatrix} A_{11} - P & \\ & A_{22} - S - A_{21}\, P^{-1}\, A_{12} \end{pmatrix}$$

is readily found nonnegative definite by (2.8) and (2.9), (2.10).

Next,

$$B^{(2)} - \eta\beta A = \begin{pmatrix} P - \eta\, \beta A_{11} & (1 - \eta\, \beta) A_{12} \\ (1 - \eta\, \beta) A_{21} & S + A_{21}\, P^{-1}\, A_{12} - \eta\, \beta\, A_{22} \end{pmatrix} ,$$

and, since $\eta < 1$, $P - \eta\, \beta\, A_{11}$ is positive definite by (2.8). Hence, $B^{(2)} - \eta\, \beta\, A$ is nonnegative definite if so is its Schur complement

$$S_{B^{(2)}-\eta\beta A} = S - \eta\, \beta\, A_{22} + A_{21} \left(P^{-1} - (1 - \eta\, \beta)^2\, (P - \eta\, \beta\; A_{11})^{-1}\right)\, A_{12}$$

Using (2.11), and noting that it further implies

$$\mathbf{v}_2^t\,A_{22}\,\mathbf{v}_2 \;\geq\; \tfrac{1}{1-\eta}\,\mathbf{v}_2^t\,A_{21}\left(K-\eta\,A_{11}^{-1}\right)A_{12}\,\mathbf{v}_2 \;\geq\; \tfrac{1}{1-\eta}\,\mathbf{v}_2^t\,A_{21}\left(P^{-1}-\eta\,A_{11}^{-1}\right)A_{12}\,\mathbf{v}_2\;,$$

one gets

$$\begin{aligned}
\mathbf{v}_2^t\,S_{B^{(2)}-\eta\beta A}\,\mathbf{v}_2 \;&\geq\; \eta(1-\beta)\,\mathbf{v}_2^t\,A_{22}\,\mathbf{v}_2 \\
&\qquad + \mathbf{v}_2^t\,A_{21}\left(P^{-1}-\eta\,A_{11}^{-1}-(1-\eta\,\beta)^2\left(P-\eta\beta A_{11}\right)^{-1}\right)A_{12}\mathbf{v}_2 \\
&\geq\; \mathbf{v}_2^t A_{21}\left(\tfrac{1-\eta\beta}{1-\eta}P^{-1}-\tfrac{\eta(1-\eta\beta)}{1-\eta}A_{11}^{-1}-(1-\eta\beta)^2\left(P-\eta\beta A_{11}\right)^{-1}\right)A_{12}\mathbf{v}_2 \\
&=\; \tfrac{1-\eta\beta}{1-\eta}\,\mathbf{v}_2^t\,A_{21}\,P^{-\frac{1}{2}}f\left(P^{-\frac{1}{2}}A_{11}P^{-\frac{1}{2}}\right)P^{-\frac{1}{2}}\,A_{12}\,\mathbf{v}_2\;,
\end{aligned}$$

where $f(X)=I-\eta\,X^{-1}-(1-\eta\beta)(1-\eta)(I-\eta\beta X)^{-1}$.

Since $P^{-\frac{1}{2}}\,A_{11}\,P^{-\frac{1}{2}}$ is symmetric positive definite, $S_{B^{(2)}-\eta\beta A}$ is then proved nonnegative definite if $f(\lambda)\geq 0 \quad \forall\lambda\in\sigma\left(P^{-\frac{1}{2}}\,A_{11}\,P^{-\frac{1}{2}}\right)\subset[1,\beta^{-1}]$. This is easily checked because

$$\begin{aligned}
f(\lambda) \;&=\; (1-\eta\beta\lambda)^{-1}\left((1-\eta\beta\lambda)(1-\eta\lambda^{-1})-(1-\eta\beta)(1-\eta)\right) \\
&=\; (1-\eta\beta\lambda)^{-1}\;\eta(1-\lambda^{-1})(1-\beta\lambda).
\end{aligned}$$

■

3 Auxiliary results for finite difference matrices

Here we analyze η and β in the case A arises from the five point finite difference discretization of

$$-\partial_x\,a_x\,\partial_x\,u-\partial_y\,a_y\,\partial_y\,u=f \tag{3.1}$$

on a arbitrary "coarse" mesh to which one has added one level of uniform refinement. We further assume that a_x, a_y are piecewise constant on this coarse mesh and that, in case of variable coefficients, one uses the point mesh box integration scheme [25], which, for the PDE (3.1) is in fact equivalent to a linear finite element scheme with right triangles.

Concerning η, we show in subsection 3.1 below that our assumptions imply $S=\frac{1}{2}A_c$, where A_c is the matrix associated to the discretization on the coarse mesh. Since

$$\mathbf{v}_2^t\left(A_{22}-A_{21}\,A_{11}^{-1}\,A_{12}\right)\mathbf{v}_2 \;\leq\; \mathbf{v}_2^t\,A_c\,\mathbf{v}_2 \qquad \forall\mathbf{v}_2 \tag{3.2}$$

[30], it follows that $\eta\geq\frac{1}{2}$.

About β, we note that A_{11} has condition number $\mathcal{O}(1)$ [1, 7, 13], leading to expect a nice value. However, an accurate analysis is difficult because the incomplete factorization is essentially recursive whereas a general bound should be based only on local estimations. Nevertheless, we show in subsection 3.2 that β is not smaller than $\frac{2}{3}$. As our lower bound on η, this result holds independently of possible anisotropy in the PDE coefficients. We further conjecture that the actual value of β is still better in practice since we never observed in our numerical experiment that $\lambda_{\max}\left(P^{-1}A_{11}\right)$ could be larger than $\frac{4}{3}$.

3.1 Proof that $S = \frac{1}{2}\,A_c$

We first carry out the proof in the case all boundary conditions are of the Neumann type. Then, A and A_c are singular M-matrix with zero row-sum. Therefore, $\Delta\,\mathbf{e}_1 = A_{11}\,\mathbf{e}_1 = -A_{12}\,\mathbf{e}_2$ and

$$\begin{aligned} S\,\mathbf{e}_2 &= A_{22}\,\mathbf{e}_2 - A_{21}\,K\,A_{12}\,\mathbf{e}_2 \\ &= A_{22}\,\mathbf{e}_2 + A_{21}\,\mathbf{e}_1 \\ &= 0\;. \end{aligned}$$

It suffices thus to check that $S_{ij} = \frac{1}{2}\,(A_c)_{ij}$ for $i \neq j$.

As noted in Section 1, S is the Schur complement of

$$\widetilde{A} = \begin{pmatrix} \Delta & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\;.$$

in which zero lines and columns have been removed.

Since A_{22} is diagonal, this means that the offdiagonal entries in S are those generated by the pointwise elimination of fine grid nodes in $\widetilde{A}$. Now, for the considered discretization scheme, a fine grid node i is either not connected to any coarse grid nodes, or connected to two of them, say j_1, j_2, with then $\Delta_{ii} = -\left(a_{ij_1} + a_{ij_2}\right)$. In the latter case, the created connection writes

$$S_{j_1\,j_2} = -\frac{a_{ij_1}\,a_{ij_2}}{a_{ij_1} + a_{ij_2}}\;.$$

and $S_{j_1\,j_2} = \frac{1}{2}\,(A_c)_{j_1\,j_2}$ follows from the assumption that the PDE coefficients are piecewise constant on the coarse mesh, which implies $a_{ij_1} = a_{ij_2} = (A_c)_{j_1\,j_2}$.

Concerning the general cases where the boundary conditions may be of the Dirichlet type, we note that the matrices A, A_c are equal to the matrices $A^{(N)}, A_c^{(N)}$ of the corresponding Neumann problem in which some rows and columns have been symmetrically deleted. Thus,

$$A^{(N)} = \left(\begin{array}{cc|cc} A_{11} & F & A_{12} & * \\ E & * & \dagger & * \\ \hline A_{21} & \dagger & A_{22} & 0 \\ * & * & 0 & * \end{array}\right)$$

$$A_c^{(N)} = \begin{pmatrix} A_c & * \\ * & * \end{pmatrix}\,,$$

where the blocks marked $*$ are not relevant for our purpose, and where the blocks noted $\dagger$ are in fact zero because whenever a fine grid node belongs to some (Dirichlet) boundary, then the two adjacent coarse grid nodes necessarily belong to the same boundary.

Further, a fine grid node i (not belonging to a Dirichlet boundary) that is connected to two coarse grids nodes j_1, j_2 may *not* be additionally connected to fine grid boundary nodes. Indeed, its possible fine grid neighbors lie in the center of boxes delimited by four coarse grid nodes. Therefore, the corresponding row in F is zero for such nodes, entailing $\Delta_{ii} = (A_{11}\,\mathbf{e}_1)_i = \left(A_{11}^{(N)}\,\mathbf{e}_1\right)_i = \Delta_{ii}^{(N)}$. It is then easy to see that

$$S^{(N)} = \begin{pmatrix} S & * \\ * & * \end{pmatrix}\,,$$

i.e. $S^{(N)} = \frac{1}{2} A_c^{(N)}$ implies $S = \frac{1}{2} A_c$.

3.2 Proof that $\beta \geq \frac{2}{3}$

We first note that $P - \beta A_{11}$ is nonnegative definite if and only if $A_{11} - \frac{1}{1-\beta} R$ is nonnegative definite, where $R = A_{11} - P$ is the error matrix that has zero row-sum by (2.2).

A typical finite difference grid is depicted on Fig.1 where the nonzero offdiagonal entries in A_{11} and R are represented by respectively straight and dot lines. One sees that a MILU(0) factorization of A_{11} leads to the rejection of all fill-in, i.e. letting $A_{11} = D - L - L^t$ where D is diagonal and L strictly lower triangular, one has $P = (P_p - L)\, P_p^{-1}\, (P_p - L^t)$ where P_p is diagonal.

One can further observe that, to the natural decomposition $A_{11} = \Sigma_e A_{11}^{(\ell)}$, one may associate a decomposition $R = \Sigma_e R^{(\ell)}$, where a typical elementary matrix $A_{11}^{(\ell)}$ and the corresponding $R^{(\ell)}$ are shown on Fig.2. Because a_x and a_y are piecewise constant on the coarse grid, one has, using the local numbering indicated on Fig.2 ,

$$A_{11}^{(\ell)} = \begin{pmatrix} a_x^{(\ell)} + a_y^{(\ell)} & 0 & -a_y^{(\ell)} & 0 & 0 \\ 0 & a_x^{(\ell)} + a_y^{(\ell)} & -a_x^{(\ell)} & 0 & 0 \\ -a_y^{(\ell)} & -a_x^{(\ell)} & 2\left(a_x^{(\ell)} + a_y^{(\ell)}\right) & -a_x^{(\ell)} & -a_y^{(\ell)} \\ 0 & 0 & -a_x^{(\ell)} & a_x^{(\ell)} + a_y^{(\ell)} & 0 \\ 0 & 0 & -a_y^{(\ell)} & 0 & a_x^{(\ell)} + a_y^{(\ell)} \end{pmatrix}$$

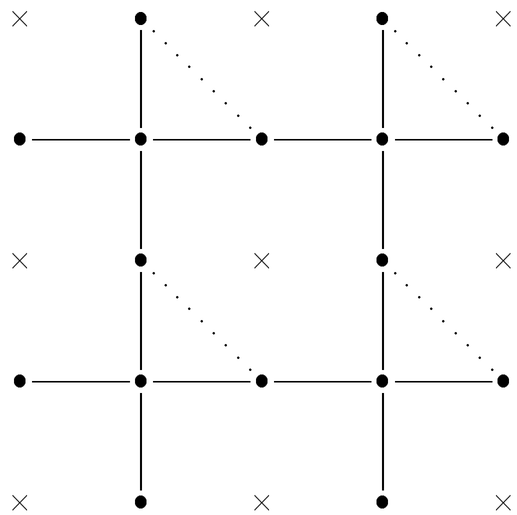

Figure 1: Typical finite difference grid where the nodes marked $\times$ and $\bullet$ are respectively the coarse and the fine grid nodes; nonzero offdiagonal entries in A_{11} are represented by straight lines and nonzero offdiagonal entries in R by dot lines

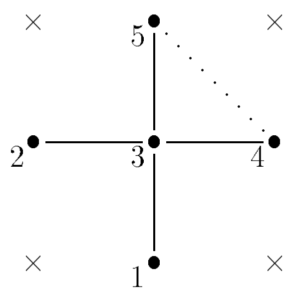

Figure 2: Elementary box of the finite difference grid; offdiagonal entries in $A_{11}^{(\ell)}$ and in the corresponding $R^{(\ell)}$ are represented as on Fig. 1.

and

$$R^{(\ell)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{a_x^{(\ell)}\, a_y^{(\ell)}}{p_3^{(\ell)}} & -\frac{a_x^{(\ell)}\, a_y^{(\ell)}}{p_3^{(\ell)}} \\ 0 & 0 & 0 & -\frac{a_x^{(\ell)}\, a_y^{(\ell)}}{p_3^{(\ell)}} & \frac{a_x^{(\ell)}\, a_y^{(\ell)}}{p_3^{(\ell)}} \end{pmatrix} , \qquad (3.3)$$

where $p_3^{(\ell)}$ is the diagonal entry of P_p at the node with local number 3.

Since nodes with local numbers 1 and 2 are eliminated exactly, there holds

$$p_3^{(\ell)} = 2\left(a_x^{(\ell)} + a_y^{(\ell)}\right) - \frac{\left(a_y^{(\ell)}\right)^2}{p_1^{(\ell)}} - \frac{\left(a_x^{(\ell)}\right)^2}{p_2^{(\ell)}} \ .$$

On the other hand, (2.2) implies (see e.g. [16])

$$P_p\, \mathbf{e}_1 \geq L^t\, \mathbf{e}_1 + A_{11}\, \mathbf{e}_1 \ ,$$

whence

$$p_1^{(\ell)} \geq a_y^{(\ell)} + a_x^{(\ell)} \qquad , \qquad p_2^{(\ell)} \geq a_x^{(\ell)} + a_y^{(\ell)} \ .$$

Therefore,

$$p_3^{(\ell)} \ \geq \ 2\left(a_x^{(\ell)} + a_y^{(\ell)}\right) - \frac{\left(a_x^{(\ell)}\right)^2 + \left(a_y^{(\ell)}\right)^2}{a_x^{(\ell)} + a_y^{(\ell)}} \ = \ \frac{\left(a_x^{(\ell)}\right)^2 + \left(a_y^{(\ell)}\right)^2 + 4\, a_x^{(\ell)}\, a_y^{(\ell)}}{a_x^{(\ell)} + a_y^{(\ell)}} \ ,$$

and using this lower bound instead of the exact value in (3.3), one gets a matrix $\widetilde{R}^{(\ell)}$ that is not smaller than $R^{(\ell)}$ in the positive definite sense; $\mathbf{v}_1^t(P - \beta\, A_{11})\mathbf{v}_1 \geq 0$ can therefore be proved by checking that $A^{(\ell)} - \frac{1}{1-\beta}\, \widetilde{R}^{(\ell)}$ is nonnegative definite.

Now, with $\frac{a_x^{(\ell)}}{a_y^{(\ell)}} = r$ and $\beta = \frac{2}{3}$, one has

$$A^{(\ell)} - 3\,\widetilde{R}^{(\ell)} = a_y^{(\ell)} \begin{pmatrix} 1+r & 0 & -1 & 0 & 0 \\ 0 & 1+r & -r & 0 & 0 \\ -1 & -r & 2(1+r) & -r & -1 \\ 0 & 0 & -r & \frac{(1+r)(1+r+r^2)}{1+r^2+4r} & \frac{3r(1+r)}{1+r^2+4r} \\ 0 & 0 & -1 & \frac{3r(1+r)}{1+r^2+4r} & \frac{(1+r)(1+r+r^2)}{1+r^2+4r} \end{pmatrix} .$$

Obviously, the 2×2 top left block is positive definite. The nonnegative definiteness of $A^{(\ell)} - 3\,\widetilde{R}^{(\ell)}$ may therefore be checked on the Schur complement

$$a_y^{(\ell)} \begin{pmatrix} \frac{1+r^2+4r}{1+r} & -r & -1 \\ -r & \frac{(1+r)(1+r+r^2)}{1+r^2+4r} & \frac{3r\,(1+r)}{1+r^2+4r} \\ -1 & \frac{3r\,(1+r)}{1+r^2+4r} & \frac{(1+r)(1+r+r^2)}{1+r^2+4r} \end{pmatrix} .$$

Computing again the Schur complement resulting from the elimination of the first node, one gets

$$a_y^{(\ell)}\,\frac{1+r}{1+r^2+4r} \begin{pmatrix} 1+r & 2r \\ 2r & r+r^2 \end{pmatrix} ,$$

which is readily found nonnegative definite since $r(1+r)^2 - 4\,r^2 = r(1-r)^2 \geq 0$ for all positive r.

4 The multilevel method

In this section we consider the multilevel method resulting form the recursive application of the two-level preconditioner (2.7). This means that the needed inversion of S is exchanged for a multiplication by

$$M = \mathcal{P}_m \left({B^{(S)}}^{-1}\, S \right)\; S^{-1} \tag{4.1}$$

where $\mathcal{P}_m$ is a polynomial such that $\mathcal{P}_m(0) = 0$, and where $B^{(S)}$ is the preconditioner on the coarser level.

From the general theory of polynomial acceleration in [2, 10, 11], and from our estimates for the two-level method, we conclude that second degree polynomials are here sufficient to stabilize the condition number.

The shifted Chebyshev polynomials considered in [2, 10, 11] lead then to

$$M = 2\,\tau\;\; {B^{(S)}}^{-1} - \tau^2\;\; {B^{(S)}}^{-1}\;\; S\;\; {B^{(S)}}^{-1} \;, \tag{4.2}$$

where τ is some parameter. Note that

$$(I - MS) = \left(I - \tau\;\; {B^{(S)}}^{-1}\;\; S \right)^2 \;, \tag{4.3}$$

i.e. this choice corresponds to a stationary iteration with the preconditioner appropriately scaled on the coarser level.

Defining η, β as in Section 2, assuming τ such that $\frac{1}{2} < \tau < 2\eta\,\beta$ and letting

$$\alpha = \min\left(\frac{2\tau - 1}{\tau^2} \ , \ \frac{2\tau}{\eta\,\beta} - \frac{\tau^2}{(\eta\,\beta)^2} \right) \ ,$$

we prove in Theorem 4.1 below that if

$$\lambda_{\min}\left(B^{(S)\,-1}\, S \right) \ \geq \ \alpha \ , \tag{4.4}$$

$$\lambda_{\max}\left(B^{(S)\,-1}\, S \right) \ \leq \ (\eta\,\beta)^{-1} \ , \tag{4.5}$$

then

$$\lambda_{\min}\left(B^{-1}\, A \right) \ \geq \ \alpha \ , \tag{4.6}$$

$$\lambda_{\max}\left(B^{-1}\, A \right) \ \leq \ (\eta\,\beta)^{-1} \ . \tag{4.7}$$

A simple induction argument (bottom to top) shows then that (4.6), (4.7) are eigenvalue bounds for the full multilevel method if β^{-1} and η^{-1} are upper bounds on respectively $\lambda_{\max}\left(P^{-1}\,A_{11}\right)$ and $\lambda_{\max}\left(S^{-1}\,S_A\right)$ that are valid on all successively visited levels. This is in particular true for the estimates derived in Section 3 as long as the PDE coefficients are piecewise constant on the coarsest level on which S is inverted exactly.

Concerning τ, the optimal choice corresponds to $\tau = \sqrt{\eta\,\beta}$ for which $\frac{2\tau-1}{\tau^2} = \frac{2\tau}{\eta\,\beta} - \frac{\tau^2}{(\eta\,\beta)^2}$. With $\eta\,\beta \geq \frac{1}{3}$ as proved in Section 3, this leads to $\tau = \sqrt{\frac{1}{3}}$ and thus $\alpha = 0.46$. We however recommend to use $\tau = \sqrt{\frac{3}{8}}$ instead. Indeed, as written in Section 3, we always observed in practice that $\beta \geq \frac{3}{4}$ and thus $\eta\,\beta \geq \frac{3}{8}$; using $\tau = \sqrt{\frac{3}{8}}$ improves then α from 0.48 to 0.60. On the other hand, in case one would have nevertheless $\eta\,\beta = \frac{1}{3}$, one has still $\alpha = \frac{2\tau}{\eta\,\beta} - \frac{\tau^2}{(\eta\,\beta)^2} = 0.30$, that is only 35% less then with $\tau = \sqrt{\frac{1}{3}}$.

Theorem 4.1 *Let A, P, S, K be defined as in Theorem 2.1 and satisfy (2.8)-(2.11) for some β, η such that $0 < \beta \leq 1$, $0 < \eta < 1$ and $\eta\,\beta > \frac{1}{4}$.*

Let $B^{(S)}$ be a symmetric positive definite matrix with same dimensions as A_{22}, and let

$$B = \begin{pmatrix} P & \\ A_{21} & M^{-1} \end{pmatrix} \begin{pmatrix} I & P^{-1}A_{12} \\ & I \end{pmatrix} \ , \tag{4.8}$$

where

$$M = \tau\, B^{(S)\,-1} - \tau^2\, B^{(S)\,-1}\ S\ \ B^{(S)\,-1}$$

for some τ such that $\frac{1}{2} < \tau < 2\eta\,\beta$.

If

$$\eta\beta\ \mathbf{v}_2^t\, S\,\mathbf{v}_2 \ \leq \ \mathbf{v}_2^t\, B^{(S)}\,\mathbf{v}_2 \ \leq \ \alpha^{-1}\ \mathbf{v}_2^t\, S\,\mathbf{v}_2 \qquad \forall \mathbf{v}_2 \ ,$$

where

$$\alpha = \min\left(\frac{2\tau - 1}{\tau^2} \ , \ \frac{2\tau}{\eta\,\beta} - \frac{\tau^2}{(\eta\,\beta)^2} \right) \ , \tag{4.9}$$

then

$$\eta\beta\ \mathbf{v}^t\, A\,\mathbf{v} \ \leq \ \mathbf{v}^t\, B\,\mathbf{v} \ \leq \ \alpha^{-1}\ \mathbf{v}^t\, A\,\mathbf{v} \qquad \forall \mathbf{v} \ . \tag{4.10}$$

Proof We first show that

$$\lambda_{\min}(MS) \;\geq\; \alpha \;, \tag{4.11}$$
$$\lambda_{\max}(MS) \;\leq\; 1 \;. \tag{4.12}$$

The upper bound (4.12) is easily deduced from the already checked identity (4.3). On the other hand, (4.11) is equivalent to $\alpha \leq \min_{\lambda\in\sigma\left(B^{(S)\,-1} S\right)} 2\tau\,\lambda - (\tau\,\lambda)^2$, which a fortiori holds when

$$\alpha \;\leq\; \min_{\lambda\in[\alpha,(\eta\,\beta)^{-1}]} 2\tau\,\lambda - (\tau\,\lambda)^2 \;=\; \min\left(2\tau\,\alpha - (\tau\,\alpha)^2\;,\; \frac{2\tau}{\eta\,\beta} - \frac{\tau^2}{(\eta\,\beta)^2}\right)$$

Now , $\alpha \leq \frac{2\tau}{\eta\,\beta} - \frac{\tau^2}{(\eta\,\beta)^2}$ holds by (4.9) whereas $2\tau\,\alpha - (\tau\,\alpha)^2 - \alpha = \alpha\left((2\tau-1) - \tau^2\alpha\right)$ is nonnegative by (4.9) too, whence (4.11).

From (4.12), it follows that

$$\mathbf{v}^t\,B\,\mathbf{v} \;\geq\; \mathbf{v}^t \begin{pmatrix} P & A_{12} \\ A_{21} & S + A_{21}P^{-1}\,A_{12} \end{pmatrix} \mathbf{v} \qquad \forall\mathbf{v}\;.$$

The left inequality (4.10) is then proved by application of Theorem 2.1. To prove the right one, consider

$$A - \alpha\,B = \begin{pmatrix} A_{11} - \alpha\,P & (1-\alpha)A_{12} \\ (1-\alpha)A_{21} & A_{22} - \alpha\,M^{-1} - \alpha\,A_{21}\,P^{-1}\,A_{12} \end{pmatrix} \;.$$

Since $\alpha < 1$, $A_{11} - \alpha\,P$ is positive definite by (2.8). Hence, $A - \alpha\,B$ is nonnegative definite if so is its Schur complement

$$S_{A-\alpha\,B} = A_{22} - \alpha\,M^{-1} - \alpha\,A_{21}\left(P^{-1} + (1-\alpha)^2\;\left(A_{11} - \alpha\,P\right)^{-1}\right)\;A_{12}.$$

Now, (4.11) implies $\mathbf{v}_2^t\,\alpha\,M^{-1}\,\mathbf{v}_2 \leq \mathbf{v}_2^t\,S\,\mathbf{v}_2 \quad \forall\mathbf{v}_2$, whereas

$$\mathbf{v}_2^t\,(A_{22} - S)\,\mathbf{v}_2 \;=\; \mathbf{v}_2^t\,A_{21}\,K\,A_{12}\,\mathbf{v}_2 \;\geq\; \mathbf{v}_2^t\,A_{21}\,P^{-1}\,A_{12}\,\mathbf{v}_2$$

by (2.10). Therefore,

$$\mathbf{v}_2^t\,S_{A-\alpha\,B}\,\mathbf{v}_2 \;\geq\; (1-\alpha)\;\mathbf{v}_2^t\,A_{21}\left(P^{-1} - {\scriptstyle(1-\alpha)}\left(A_{11} - \alpha\,P\right)^{-1}\right)A_{12}\,\mathbf{v}_2$$

while $P^{-1} - {\scriptstyle(1-\alpha)}\left(A_{11} - \alpha\,P\right)^{-1}$ is nonnegative definite if and only if

$$(1-\alpha)\;\mathbf{v}_1^t\,P\,\mathbf{v}_1 \;\leq\; \mathbf{v}_1^t(A_{11} - \alpha\,P)\,\mathbf{v}_1 \qquad \forall\mathbf{v}_1\;,$$

which is true by (2.8).

∎

Computational complexity of one iteration step

Let n be the order of the system (1.1), n_1 the number of nodes in block 1 and n_2 the number of nodes in block 2. Let $nz(C)$ denote the number of nonzero entries in (sub)matrix C, and a *flop* be a basic ($+$ or $*$) arithmetic operation.

One application of the preconditioner (4.8) requires

- two solves with P, each of them involving $2\,nz(A_{11})$ *flops*;
- one multiplication by A_{12} ($2\,nz(A_{12})$ *flops*) and one multiplication by A_{12} ($2\,nz(A_{21})$ *flops*);
- one multiplication by M that involves one scaling and one saxpy operation on vectors of length n_2 ($3\,n_2$ *flops*), one multiplication by S (($2\,nz(S) - n_2$) *flops*) and two applications of the preconditioner on the coarser level with n_2 nodes.

Letting $c(n)$ be the number of *flops* required for one application of the preconditioner, one has thus, since $nz(A_{22}) = n_2$,

$$\begin{aligned} c(n) &= 4\,nz(A_{11}) + 2\,(nz(A_{12}) + nz(A_{21}) + n_2 + nz(S)) + 2\,c(n_2) \\ &= 2\,(nz(A_{11}) + nz(A) + nz(S) + c(n_2))\,. \end{aligned}$$

For five point difference matrices, $nz(A) \approx 5n$, $nz(A_{11}) \approx n_1 + 2n \approx \frac{11}{4}n$, $nz(S) \approx 5n_2 \approx \frac{5}{4}n$ and $n_2 \approx \frac{1}{4}n$. Therefore,

$$c(n) \;\approx\; 18n + 2c\left(\frac{n}{4}\right) \;.$$

Applying this formula recursively on j levels one gets

$$c(n) \;\approx\; 18n \cdot \left(\sum_{k=0}^{j-1}\left(\frac{1}{2}\right)^k\right) \;+\; (2)^j\;\tilde{c}\left(\frac{n}{4^j}\right) \;,$$

where $\tilde{c}\left(\frac{n}{4^j}\right)$ is the cost of the needed exact inversion of S on the last level. Since $\tilde{c}(\tilde{n}) = \mathcal{O}\left(\tilde{n}^{5/4}\right)$, $(2)^j\,\tilde{c}\left(\frac{n}{4^j}\right)$ is negligible for j sufficiently large whereas $\sum_{k=0}^{j-1}\left(\frac{1}{2}\right)^k \le \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k = 2$. Hence

$$c(n) \;\approx\; 36n \;,$$

which completes the proof that the preconditioner has optimal order of computational complexity.

Since the conjugate gradient algorithm involves additionally two inner products ($4n$ *flops*), three saxpy operations ($6n$ *flops*) and one multiplication by A (about $9n$ *flops*), the total cost of each preconditioned conjugate gradient iteration is about $55n$ *flops*. By comparison, it is $29n$ *flops* with a ILU(0) like preconditioner and $33n$ *flops* with a block ILU like preconditioner (see e.g. [8]).

5 Numerical results

We have tested our multilevel preconditioner on linear systems resulting from the five point finite difference discretization of

$$-\partial_x\, a_x\, \partial_x\, u - \partial_y\, a_y\, \partial_y\, u = \quad \text{in} \quad \Omega = (0,1)\times(0,1)$$

$$\begin{cases} u &= 0 \quad \text{on } \Gamma_0 \subset \partial\,\Omega \\ \dfrac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma_1 = \partial\,\Omega\backslash\Gamma_0 \;. \end{cases}$$

We use the point mesh box integration scheme [25] with uniform mesh size h in both directions, except on the problem *Stretched* that has been retrieved from [18].

In each experiment, the reported numbers of iterations (# it.) and numbers of floating point operations per unknown ($\frac{\#\ flops}{n}$) are those necessary to reduce the relative residual error $\frac{\|r_k\|}{\|b\|}$ below 10^{-7} when using the preconditioned conjugate gradient algorithm with the zero vector as initial approximation. The reported *flop* counts exclude preprocessing, that is the operations needed to compute the successive incomplete factorizations of A_{11} and approximate Schur complements.

In each case, the multilevel preconditioner is defined as described in Section 4 by a recursive use of (4.8) with $\tau = \sqrt{\frac{3}{8}}$. We stop the recursion when the number of unknowns in the second block of the two-by-two partitioning is less or equal to 81, and use at this stage the two-level preconditioner (2.7) with an exact LU factorization of the approximate Schur complement S.

We first consider the model Poisson problem ($a_x = a_y = f \equiv 1$, $\Gamma_1 = \partial\Omega$) with $h = 2^{-m}$ for m integer. The results are reported in Table 1 for the multilevel method, which is further compared on Fig. 3 with some classical (block) ILU type preconditioners[2]. One sees that the multilevel method has grid independence convergence and already outmatches standard preconditioning techniques for fairly moderate problem sizes.

We next consider the following test problems, where d is a positive parameter.

Problem 1

$$\begin{aligned} a_x &= a_y = \begin{cases} d & \text{in } \left(\frac14,\frac34\right)\times\left(\frac14,\frac34\right) \\ 1 & \text{elsewhere ,} \end{cases} \\ f &= \begin{cases} 100 & \text{in } \left(\frac14,\frac34\right)\times\left(\frac14,\frac34\right) \\ 0 & \text{elsewhere ,} \end{cases} \\ \Gamma_0 &= \{(x,y) \mid 0 \le x \le 1,\ y = 0\} \;. \end{aligned}$$

Problem 2

$$a_x = \begin{cases} d & \text{in } \left(\frac14,\frac12\right)\times\left(\frac14,\frac12\right) \\ 1 & \text{elsewhere ,} \end{cases}$$

[2]these are defined with respect to a natural ordering of the unknowns; block preconditioners BILU, MBILU have been computed as in [8, 24], i.e. are identical to respectively INV(1) and MINV(1) in [21]).

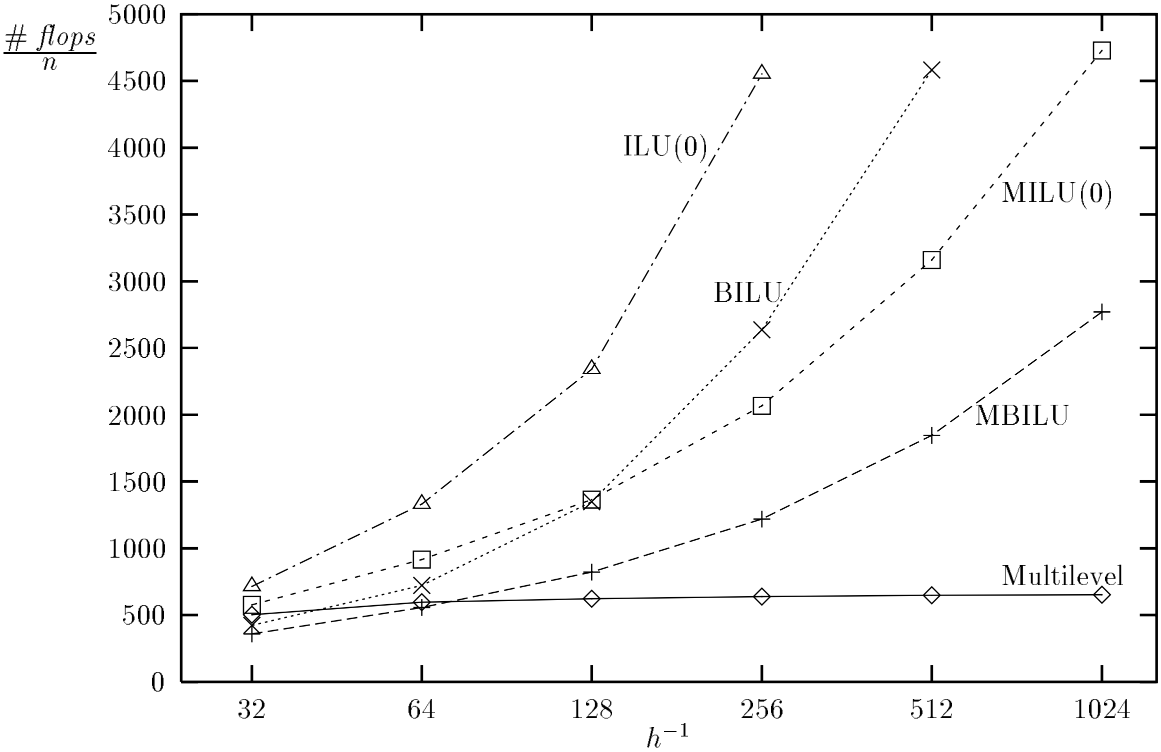

Figure 3: Poisson problem: comparison of the multilevel method with (block) ILU preconditioning.

h^{-1}	$\lambda_{\min}$	$\lambda_{\max}$	κ	# it.	$\frac{\#\ flops}{n}$
32	.86	1.97	2.30	11	502.
64	.85	2.00	2.34	12	596.
128	.85	2.00	2.35	12	624.
256	.85	2.00	2.36	12	639.
512	.85	2.00	2.36	12	649.
1024	.85	2.00	2.36	12	654.

Table 1: results for the Poisson problem

$$a_y = \begin{cases} d & \text{in } \left(\frac{1}{2},\frac{3}{4}\right)\times\left(\frac{1}{2},\frac{3}{4}\right) \\ 1 & \text{elsewhere}, \end{cases}$$

$$f = \begin{cases} 1 & \text{in } \left(\frac{1}{4},\frac{3}{4}\right)\times\left(\frac{1}{4},\frac{3}{4}\right) \\ 0 & \text{elsewhere}, \end{cases}$$

$$\Gamma_0 = \{(x,y) \mid 0 \le x \le 1,\ y=0\}\ .$$

Problem *Stretched* [18]

This problem solves the Laplace equation with Neumann boundary conditions ($\Gamma_1 = \partial\,\Omega$) on a non uniform Cartesian $(M+1)\times(M+1)$ grid, $M = 128, 256, 512$. In the neighborhood of the boundaries the grid is refined in such a way that the ratio of maximum mesh size to minimum mesh size is equal to 1000 and the ratio of subsequent mesh sizes is kept constant. The right hand side is $\mathbf{b} = A\,\mathbf{u}$, where $\mathbf{u}$ is the function

$$u(x,y) = (x\,y(1-x)(1-y))^2 \;\; e^{x^2 y}$$

evaluated at the grid points.

Problem *Discontinuous* [18]

$$\begin{aligned} a_x &= a_y = \begin{cases} 10^5 & \text{in} \quad (0.3, 0.8) \times (0.3, 0.8) \\ 1 & \text{elsewhere} \ , \end{cases} \\ \Gamma_1 &= \partial\Omega \ , \\ f^{-1} &= 400 \ , \end{aligned}$$

same r.h.s. as in problem *Stretched.*

The last two problems are especially interesting because they do not result from successive refinement steps applied to a problem defined on a coarse grid. However, as pointed out in Section 1, our multilevel method is applicable to any symmetric M-matrix if some hierarchy of nodes if provided. Further, the algebraic results in Sections 2 and 4 are general. Only the analysis of η and β in Section 3 is restricted to a particular context, but one may guess that the range of problems for which η and β take similar values is much larger in practice.

For the problem *Stretched*, it was natural to consider the same hierarchy of nodes as on the topologically equivalent $(N+1)\times(N+1)$ grid that results when using a uniform mesh size $h = \frac{1}{N}$. For the problem *Discontinuous*, we used the hierarchy of nodes that is induced by considering the actual 401×401 grid as part of a larger 513×512 grid; the node at $(0,0)$ is then on the coarsest grid, but the nodes on the lines $x = 0.3$ and $y = 0.3$ are fine grid nodes after three levels of recursion.

Note that due to the Neumann boundary conditions, the solution of the last two problems is determined up to a constant, hence the coefficient matrix is singular. The multilevel preconditioner is then singular too, but admissible because it has the same kernel as A [26]. In practice, only the last pivot of the LU factorization of the Schur complement on the coarsest level is zero. It suffices then to exchange it for an arbitrary positive value to use the preconditioner trouble free as a regular one [27]. Besides, all direction vectors in the preconditioned conjugate gradient algorithm are projected onto the range of A. This improves the stability at the price of two more *flops* per unknown and per iteration [28].

The results are reported in Tables 2 and 3. Comparing with the results for the model problem, the method appears nearly insensitive to jumps in the PDE coefficients. Looking more particularly at the results for the problem *discontinuous*, it seems that this robustness even extends to problems where these jumps do not only occur across the edges of the coarsest mesh.

Considering, on the other hand the results for Problem 2 and Problem *Stretched*, the method appears slightly more sensitive to anisotropy or grid stretching. However, it

Problem 1

d	$\lambda_{\min}$	$\lambda_{\max}$	κ	# it.	$\frac{\#\ flops}{n}$
			$h^{-1} = 128$		
10^{-4}	.85	1.99	2.34	13	692.
10^{-2}	.85	2.00	2.35	13	692.
1	.85	2.00	2.35	12	639.
10^{2}	.85	2.00	2.34	13	692.
10^{4}	.86	2.00	2.33	13	692.
			$h^{-1} = 512$		
10^{-4}	.85	2.01	2.36	13	709.
10^{-2}	.85	2.01	2.36	13	709.
1	.85	2.01	2.36	12	655.
10^{2}	.85	2.01	2.35	13	709.
10^{4}	.85	2.01	2.35	13	709.

Problem *Discontinuous*

	$\lambda_{\min}$	$\lambda_{\max}$	κ	# it.	$\frac{\#\ flops}{n}$
	.85	2.01	2.36	11	623.

Table 2: results for Problem 1 and Problem *Discontinuous*

remains of optimal order as proved theoretically for problems like Problem 2. Further, we obtain practically grid independence convergence for the problem *Stretched*, which has to be considered as very difficult in this respect [18].

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Problem 2

d	$\lambda_{\min}$	$\lambda_{\max}$	κ	# it.	$\frac{\#\ flops}{n}$
		$h^{-1} = 128$			
10^{-4}	.73	2.00	2.76	14	745.
10^{-3}	.73	2.00	2.74	14	745.
10^{-2}	.78	2.00	2.55	13	692.
10^{-1}	.85	2.05	2.41	12	639.
1	.85	2.00	2.35	12	639.
10^{1}	.85	2.09	2.45	13	692.
10^{2}	.79	2.18	2.77	15	799.
10^{3}	.74	2.22	3.02	16	852.
10^{4}	.72	2.23	3.08	17	905.
		$h^{-1} = 512$			
10^{-4}	.68	2.01	2.96	15	818.
10^{-3}	.72	2.01	2.80	14	764.
10^{-2}	.78	2.20	2.81	14	764.
10^{-1}	.85	2.09	2.46	13	709.
1	.85	2.01	2.36	12	655.
10^{1}	.85	2.10	2.46	13	709.
10^{2}	.79	2.20	2.80	15	818.
10^{3}	.72	2.29	3.17	17	928.
10^{4}	.68	2.31	3.40	18	982.

Problem *Stretched*

h^{-1}	$\lambda_{\min}$	$\lambda_{\max}$	κ	# it.	$\frac{\#\ flops}{n}$
128	.73	2.16	2.94	13	723.
256	.72	2.20	3.04	13	733.
512	.71	2.26	3.18	13	738.

Table 3: results for Problem 2 and Problem *Stretched*

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