The Einstein–Infeld–Hoffmann Legacy in Mathematical Relativity

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Erik Amorim

Outline

Particles as Singularities of the Fields. Part I (GR)

- (1921) **Hermann Weyl**: Elementary particles as singular points in a three-dimensional spacelike section of four-dimensional spacetime with many time-like singularities.

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  (Letter to Max Born, Dec. 4)

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(Letter to Max Born, Dec. 4)
- (1927) **A. Einstein & J. Grommer**:
General Relativity and the Law of Motion,
Sitzungsber. Preuss. Akad., Jan. 6, pp.2-13
“The law of motion is completely determined by the field equations, though ... proven only for the ... equilibrium.”

Particles as Singularities of the Fields. Part I (GR)

- (1938) **A. Einstein, L. Infeld, & B. Hoffmann**: *The Gravitational Equations and the Problem of Motion*, Annals Math. **39**, Jan., pp.65–100
  "(1) By ... method of approximation, ... the gravitational field due to [slowly] moving particles is determined.
  (2) It is shown that for two-dimensional spatial surfaces containing singularities, certain surface integral conditions are valid which determine the motion."

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- (1941) **P. R. Wallace**:
  Relativistic Equations of Motion in Electromagnetic Theory, Amer. J. Math. **63**, pp.729–236.

Particles as Singularities of the Fields. Part I (GR)

The gist of the EIH argument:

- Imagine an electromagnetic Lorentz spacetime $(\mathcal{M}^{1,3}, \mathbf{F})$ with N charged, time-like singularities of infinite extent.
- Away from the singularities:

$$\text{Einstein}: \boldsymbol{R} - \tfrac{1}{2}R\boldsymbol{g} = \tfrac{8\pi G}{c^4}\boldsymbol{T}[\boldsymbol{F},\boldsymbol{g}] \qquad (E)$$

$$\text{Maxwell}: \boldsymbol{dF} = \mathbf{0}; \qquad \boldsymbol{d} * \boldsymbol{F} = \mathbf{0} \qquad (M)$$

- Twice contracted second Bianchi identity:

$$\boldsymbol{\nabla}\cdot\left(\boldsymbol{R} - \tfrac{1}{2}R\boldsymbol{g}\right) = \mathbf{0} \qquad (B)$$

- (E) & (B) $\Longrightarrow$ $\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{T}[\boldsymbol{F},\boldsymbol{g}] = \mathbf{0}}$ $\overset{SIC}{\Longrightarrow}$ $\boxed{\textit{law of motion}}$.

Particles as Singularities of the Fields. Part I (GR)

"We have therefore obtained the Newtonian equations of motion from the field equations alone, without extra assumption such as ... the law of geodetic lines, or by a special choice of an energy impulse tensor.
From the above derivation of the Newtonian equations of motion, the general mechanism becomes apparent by which the Lorentz equations for the motion of electric particles can be obtained. In this case we have to consider the gravitational equations in which the Maxwell energy-momentum tensor appears on the right, and also the Maxwell field equations, and treat the whole set of equations by our approximation method. It is necessary, now, to give each singularity an electric charge e in addition to its mass m. We .... obtain the full Lorentz force together with the relativistic correction to the mass."

Particles as Singularities of the Fields. Part I (GR)

Alas, their Mathematics does not Support their Claim

"It is most convenient to take definite, infinitesimally small spheres whose centers are at the singularities, but in this case infinities of the types

$$\lim const./r^n, \qquad n \text{ a positive integer}, \qquad r \to 0$$

can occur in the values of the partial integrals. Since these must cancel, however, in the final result, we may merely ignore them throughout the calculation of the surface integrals."

(EIH, p. 92)

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(EIH, p. 92)

However, the infinities do not cancel!

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Alas, their Mathematics does not Support their Claim

EIH's surface integral conditions (SIC) correspond to the assumption of a positive bare mass of the timelike singularities.
EIH did not know that this (presumably) implies Black Holes

Particles as Singularities of the Fields. Part II (SR)

Negative Infinite Bare Mass Renormalization to the rescue?

(1938) **Paul Adrien Maurice Dirac**:
Classical theory of radiating electrons,
Proc. Roy. Soc. A **167**, Aug., pp. 148–169

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$$m_{\text{obs}}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\boldsymbol{q} = \boldsymbol{f}^{\text{ext}} + \boldsymbol{f}^{\text{LAUE}}$$

where $\boldsymbol{f}^{\text{ext}} = \frac{e}{c}\boldsymbol{F}^{\text{ext}}(\boldsymbol{q})\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{q}$ is a Lorentz Minkowski-force,

$$\boldsymbol{f}^{\text{LAUE}} = \frac{2e^2}{3c^3}\left(\boldsymbol{g} + \frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{q}\otimes\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{q}\right)\cdot\frac{\mathrm{d}^3}{\mathrm{d}\tau^3}\boldsymbol{q}$$

is von Laue's radiation-reaction Minkowski-force, and ...

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is von Laue's radiation-reaction Minkowski-force, and

$$\boxed{m_{\text{obs}} = \lim_{r \downarrow 0} \left(m_{bare}(r) + \frac{e^2}{2c^2} \frac{1}{r} \right)}$$

defines $m_{bare}(r)$ [N.B.: $m_{bare}(r) \downarrow -\infty$ as $r \downarrow 0$]

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Why should m_{bare} know about r (radius of averaging sphere)?

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How physicists have handled the $\dddot{\mathbf{q}}$ problem

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- To compute $\dddot{\mathbf{q}}$ (FAPP), just take proper time derivative of test-particle law of motion, i.e.

$$\frac{\mathrm{d}^3}{\mathrm{d}\tau^3}\boldsymbol{q} \stackrel{\text{FAPP}}{=} \frac{e}{m_{\text{obs}}c}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\boldsymbol{F}^{\text{ext}}(\boldsymbol{q})\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{q}\right).$$

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- R.h.s. depends only on $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\ddot{\boldsymbol{q}}$.

Mission Accomplished FMPP!

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Fails to deliver in an important special case:

DLL motion along constant E-field is mere test particle motion!

Particles as Singularities of the Fields. Part III (GR)

Sixty years later: DLL still features in the State of Affairs in GR ...

(2011) **Eric Poisson, Adam Pound, & Ian Vega**,
The motion of point particles in curved spacetime,
Living Reviews in Relativity, Sept. (162 pp.)

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$$m_{\text{obs}} \frac{\mathrm{D}}{\mathrm{d}\tau} \boldsymbol{u} = \boldsymbol{f}^{\text{ext}} + \boldsymbol{f}^{\text{LAUE}} + \boldsymbol{f}^{\text{tail}},$$

where $\boldsymbol{u} := \frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{q}$ and $\frac{\mathrm{D}}{\mathrm{d}\tau}\boldsymbol{u} := \frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{u} + \boldsymbol{\Gamma}^{\text{ext}}(\boldsymbol{u},\boldsymbol{u})$, and where

$$\boldsymbol{f}^{\text{ext}} = \frac{e}{c}\boldsymbol{F}^{\text{ext}}(\boldsymbol{q}) \cdot \boldsymbol{u},$$

$$\boldsymbol{f}^{\text{LAUE}} = \frac{2}{3}e^2 \left(\boldsymbol{g} + \frac{1}{c^2}\boldsymbol{u} \otimes \boldsymbol{u}\right) \cdot \left(\frac{1}{6}\boldsymbol{R}^{\text{ext}} \cdot \frac{1}{c}\boldsymbol{u} + \frac{1}{c^3}\frac{\mathrm{D}^2}{\mathrm{d}\tau^2}\boldsymbol{u}\right)$$

$$\boldsymbol{f}^{\text{tail}} = 2e^2 \int_{-\infty}^{\tau} \boldsymbol{H}^{\text{ret}}(\boldsymbol{q}(\tau), \boldsymbol{q}(\tau')) \cdot \boldsymbol{u}(\tau')\mathrm{d}\tau' \cdot \boldsymbol{u}(\tau)$$

and

$$\frac{\mathrm{D}^2}{\mathrm{d}\tau^2}\boldsymbol{u} \overset{\text{FAPP}}{=} \frac{e}{m_{\text{obs}}c}\frac{\mathrm{D}}{\mathrm{d}\tau}\left(\boldsymbol{F}^{\text{ext}}(\boldsymbol{q}) \cdot \boldsymbol{u}\right).$$

Particles as Singularities of the Fields. Part III (GR)

... but this State of Affairs is worse than in SR!

The Green function in the integral “defining” the tail force $\boldsymbol{f}^{\text{tail}}$,

$$\int_{-\infty}^{\tau} \boldsymbol{H}^{\text{ret}}(\boldsymbol{q}(\tau), \boldsymbol{q}(\tau')) \cdot \boldsymbol{u}(\tau')\mathrm{d}\tau',$$

diverges at the upper limit of integration!

Particles as Singularities of the Fields. Part III (GR)

but another part of this State of Affairs is worse than in SR!

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On p. 104, PPV have this to say about their “tail force integral:” “We recall that the integration [from $-\infty$ to τ] must be cut short at $\tau^- := \tau - 0^+$ to avoid the singular behaviour of the retarded Green’s function at coincidence [$\tau' = \tau$].”

Is the tail-force integral convergent? Even if, there is a problem: The PPV equation of motion does not pose a Cauchy problem!

Particles as Singularities of the Fields. Part III (GR)

And for the "observable mass" m_{obs} PPV have this to say:

On p. 103, PPV tell the reader that

$$\delta m = \lim_{r \downarrow 0} \frac{2e^2}{3c^2} \frac{1}{r}$$

"is formally a divergent quantity, but .... m denoting the (also formally divergent) bare mass of the particle, m and δm combine to form the particle's observable mass

$$m_{\text{obs}} = m + \delta m,$$

which is finite."

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But why?

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"confess that ... our expression for δm is admittedly incorrect,"

But why?

Because ... "we are wrong by a factor of $4/3$."

Particles as Singularities of the Fields. Part IV (SR)

As to that “factor $4/3$”

Particles as Singularities of the Fields. Part IV (SR)

As to that "factor $4/3$"

(1960s) **Richard Feynman** (Lectures of physics):
"... and when Einstein showed it always had to be 1 instead of $4/3$, there was great confusion."

Particles as Singularities of the Fields. Part IV (SR)

As to that "factor $4/3$"

(1960s) **Richard Feynman** (Lectures of physics):
"... and when Einstein showed it always had to be 1 instead of $4/3$, there was great confusion."

(1922) **Enrico Fermi**
Concerning a contradiction between electrodynamic and the relativistic theory of electromagnetic mass,
Zeitschrift f. Physik, Nuovo Cimento, and yet one more!

"Classical electron theory" (Abraham, Lorentz) predicted a value of $m = (4/3)(E/c^2)$ for the electromagnetic mass m by spherical averaging in the laboratory frame. Fermi showed that spherical averaging in the particle rest frame gave $m = E/c^2$.

Particles as Singularities of the Fields. Part V (GR)

As to averaging:

(2011) **E. Poisson, A. Pound, & I. Vega** (p. 143)

Enunciation of an “axiom”:

“The force on the particle arises from the piece of the field that survives angle averaging.”

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Convexity theory has this to say:

Theorem: "Averaging a piecewise continuous function over the neighborhood of a discontinuity can produce any value between the extreme values through a suitable choice of averaging procedure."

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So much for that "axiom."

Particles as Singularities of the Fields. Part V (GR)

The PPV treatment of the problem is a sad state of affairs!

This is a rare opportunity for mathematical physicists!

EIH when $G = 0$: Electrodynamics with point charges

- Let's analyze the EIH argument in the formal limit $G \to 0$ where $\boldsymbol{g} \to \boldsymbol{\eta}$ (Minkowski metric).

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- Turning Gravity off, the central claim of EIH reduces to: *Away from the singularities, with*

$$\text{Maxwell}: \boldsymbol{dF} = \mathbf{0}; \qquad \boldsymbol{d} * \boldsymbol{F} = \mathbf{0}$$

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- Formally equivalent to a distributional formulation on $\mathbb{R}^{1,3}$:
  1. $\boldsymbol{T}[\boldsymbol{F}, \eta] \longrightarrow \boldsymbol{T}[\boldsymbol{F}, \eta] + \boldsymbol{T}[\delta, \eta]$
  2. $\boldsymbol{d} * \boldsymbol{F} = \mathbf{0} \longrightarrow \boldsymbol{d} * \boldsymbol{F} = \delta$

Letting $G \to 0$: Electrodynamics with point charges

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  2. $\boldsymbol{d} * \boldsymbol{F} = \mathbf{0} \longrightarrow \boldsymbol{d} * \boldsymbol{F} = \delta$
  3. The above setup symbolically leads to Lorentz electrodynamics $\leftarrow$ ill-defined!
  4. However, this is not the end of the story! After one modification the setup yields a well-posed law of motion!
  5. $\boldsymbol{d} * \boldsymbol{F} = \mathbf{0} \longrightarrow \boldsymbol{dM} = \delta$ with suitable law of the electromagnetic vacuum $\boldsymbol{F} \leftrightarrow \boldsymbol{M}$

Pre-metric Maxwell–Lorentz field equations

- Minkowski spacetime threaded by N timelike world-lines
- Lorentz frame with space vector $\mathbf{s} \in \mathbb{R}^3$ and time $t \in \mathbb{R}$
- The evolution equations for the **B**, **D** fields

$$\partial_t \mathbf{B}(t, \mathbf{s}) = -\nabla \times \mathbf{E}(t, \mathbf{s})$$
$$\partial_t \mathbf{D}(t, \mathbf{s}) = +\nabla \times \mathbf{H}(t, \mathbf{s}) - 4\pi \sum_{k=1}^{N} e_k \dot{\mathbf{q}}_k(t) \delta_{\mathbf{q}_k(t)}(s)$$

- The constraint equations for the **B**, **D** fields

$$\nabla \cdot \mathbf{B}(t, \mathbf{s}) = 0$$
$$\nabla \cdot \mathbf{D}(t, \mathbf{s}) = 4\pi \sum_{k=1}^{N} e_k \delta_{\mathbf{q}_k(t)}(s)$$

- The constraint for the sources: subluminal velocities

$$|\dot{\mathbf{q}}_k(t)| < 1$$

Need Electromagnetic Vacuum Law: $(\mathbf{B},\mathbf{D}) \leftrightarrow (\mathbf{H},\mathbf{E})$

- Maxwell(-Lorentz)'s law

$$\mathbf{H} = \mathbf{B}$$
$$\mathbf{E} = \mathbf{D}$$

- Born-Infeld's law

$$\mathbf{H} = \frac{\mathbf{B} - \frac{1}{b^2}\mathbf{B}\times(\mathbf{B}\times\mathbf{D})}{\sqrt{1+\frac{1}{b^2}(|\mathbf{B}|^2+|\mathbf{D}|^2)+\frac{1}{b^4}|\mathbf{B}\times\mathbf{D}|^2}}$$
$$\mathbf{E} = \frac{\mathbf{D} - \frac{1}{b^2}\mathbf{D}\times(\mathbf{D}\times\mathbf{B})}{\sqrt{1+\frac{1}{b^2}(|\mathbf{B}|^2+|\mathbf{D}|^2)+\frac{1}{b^4}|\mathbf{B}\times\mathbf{D}|^2}}$$

- Bopp-Landé-Thomas(-Podolsky) law (N.B.: $\Box := \partial_t^2 - \Delta$)

$$\mathbf{H}(t,\mathbf{s}) = \left(1 + \varkappa^{-2}\Box\right)\mathbf{B}(t,\mathbf{s})$$
$$\mathbf{D}(t,\mathbf{s}) = \left(1 + \varkappa^{-2}\Box\right)\mathbf{E}(t,\mathbf{s})\,.$$

Rigorous Results on the Field Cauchy Problems

- ML field Cauchy problem (standard):
  Global well-posedness (weak) with "arbitrary" data.
- MBLTP field Cauchy problem (standard):
  Global well-posedness (weak) with "arbitrary" data.
- MBI field Cauchy problem:
  - Global well-posedness (classical) with small data (no charges!) (J. Speck; F. Pasqualotto)
  - Finite-time blow up with certain plane wave data (no charges!) (Y. Brenier; cf. D. Serre)
  - Existence and Uniqueness of static finite-energy solutions with N fixed point charges; real analyticity away from point charges (M.K.; cf. Bonheure et al.)

Lorentz electrodynamics: Field equations

- The evolution equations for the fields,

$$\partial_t \mathbf{B}(t,\mathbf{s}) = -\nabla \times \mathbf{E}(t,\mathbf{s})$$
$$\partial_t \mathbf{E}(t,\mathbf{s}) = +\nabla \times \mathbf{B}(t,\mathbf{s}) - 4\pi \sum_k e_k \dot{\mathbf{q}}_k(t)\delta_{\mathbf{q}_k(t)}(s),$$

- The constraint equations for the fields,

$$\nabla \cdot \mathbf{B}(t,\mathbf{s}) = 0$$
$$\nabla \cdot \mathbf{E}(t,\mathbf{s}) = 4\pi \sum_k e_k \delta_{\mathbf{q}_k(t)}(s)$$

- N.B.: Constraint equations restrict field data only.

Lorentz electrodynamics: Equations of Motion

- Einstein–Lorentz–Poincaré velocity-momentum relation

$$\dot{\mathbf{q}}_k(t) = \frac{1}{m_k}\frac{\mathbf{p}_k(t)}{\sqrt{1+\frac{|\mathbf{p}_k(t)|^2}{m_k^2}}}; \quad m_k \neq 0$$

- Newton's law for the rate of change of momentum

$$\dot{\mathbf{p}}_k(t) = \mathbf{f}_k(t)$$

- Lorentz' law for the electromagnetic force

$$\mathbf{f}_k^{\mathrm{Lor}}(t) = e_k\left[\mathbf{E}(t,\mathbf{q}_k(t)) + \dot{\mathbf{q}}_k(t)\times\mathbf{B}(t,\mathbf{q}_k(t))\right]$$

Lorentz Electrodynamics is *not well-definable*!

- Symbolically the equations of Lorentz Electrodynamics seem to pose a joint Cauchy problem for positions $\mathbf{q}_k(t)$ and momenta $\mathbf{p}_k(t)$, and for the fields $\mathbf{B}(t, \mathbf{s})$ and $\mathbf{E}(t, \mathbf{s})$, with initial data constrained by the divergence equations.
- However, this Cauchy problem is rigorously ill defined!
- Reason: $\mathbf{E}(t, \mathbf{q}_k(t))$ and $\mathbf{B}(t, \mathbf{q}_k(t))$ "infinite in all directions"
- $\mathbf{f}_k^{\text{Lor}}(t)$ can be "defined" through averaging (very popular!), but result depends on how the averaging is done.
- Also, fields too strongly divergent at particle world lines $\longrightarrow$ No meaningful energy-momentum conservation law!
- Deckert and Hartenstein: Singularities on initial light cones.

Beyond Lorentz Electrodynamics: Finite field momenta

- Field momentum density: $\mathbf{\Pi}$
- For ML and for MBI field equations

$$4\pi\mathbf{\Pi} = \mathbf{D} \times \mathbf{B}$$

- For MBLTP field equations

$$4\pi\mathbf{\Pi} = \mathbf{D} \times \mathbf{B} + \mathbf{E} \times \mathbf{H} - \mathbf{E} \times \mathbf{B} - \varkappa^{-2}(\nabla \cdot \mathbf{E})(\nabla \times \mathbf{B} - \varkappa \dot{\mathbf{E}})$$

- The fields $\mathbf{B}, \mathbf{D}, \mathbf{E}, \dot{\mathbf{E}}$ (and $\mathbf{H}$) at $(t, \mathbf{s})$ depend on their initial data and on $\mathbf{q}(\cdot)$, $\mathbf{p}(\cdot)$, and $\mathbf{D}$& $\mathbf{H}$ also on $\mathbf{a}(\cdot)$.
N.B.: $(\mathbf{B}, \mathbf{D})_0 \mapsto (\mathbf{E}, \dot{\mathbf{E}})_0$ feasible!
- $\mathbf{\Pi}(t, \mathbf{s})$ is $L^1_{loc}(\mathbb{R}^3)$ about each $\mathbf{q}(t)$ for MBLTP fields (KTZ), perhaps also for MBI fields, but surely NOT for ML fields.

Momentum Conservation $\longrightarrow$ Equations of Motion

- Conservation of total momentum (here: 1 pt charge):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\mathbf{\Pi}(t,\mathbf{s})d^3s$$

With BLTP law: Volterra integral equation for $\mathbf{a} = \mathbf{a}[\mathbf{q},\mathbf{p}]$

- This leads to the fixed point equations

$$\mathbf{q}(t) = \mathbf{q}(0) + \frac{1}{m}\int_0^t \frac{\mathbf{p}}{\sqrt{1+\frac{|\mathbf{p}|^2}{m^2}}}(\tilde{t})d\tilde{t} \qquad =: Q_t(\mathbf{q}(\cdot),\mathbf{p}(\cdot))$$

$$\mathbf{p}(t) = \mathbf{p}(0) - \int_{\mathbb{R}^3}\left(\mathbf{\Pi}(t,\mathbf{s}) - \mathbf{\Pi}(0,\mathbf{s})\right)d^3s =: P_t(\mathbf{q}(\cdot),\mathbf{p}(\cdot))$$

- Well-posedness b/c $(Q_\bullet, P_\bullet)(\,\cdot\,,\,\cdot\,)$ is a Lipschitz Map.

BLTP Electrodynamics as Initial Value Problem exists!

- MBLTP field + N-point-charge Cauchy problem (KTZ)
  - Local well-posedness for admissible initial data & $m \neq 0$.
  - Global well-posedness if in a finite time:
    (a) no particle reaches the speed of light,
    (b) no particle reaches infinite acceleration,
    (c) no two particles reach the same location.
- Energy-Momentum conservation rigorously true.
- “Self”-force analyzed rigorously.
- MBLTP oddities:
  (a) longitudinal electrical waves;
  (b) subluminal transversal electromagnetic wave modes;
  (c) energy functional unbounded below.

The Volterra equation for the acceleration

The key proposition

Proposition (KTZ) *Given $C^{0,1}$ maps $t \mapsto \mathbf{q}(t)$ and $t \mapsto \mathbf{p}(t)$, with $Lip(\mathbf{q}) = v$, $Lip(\mathbf{v}) = a$, and $|\mathbf{v}(t)| \leq v < 1$, the Volterra equation as a fixed point map has a unique C^0 solution $t \mapsto \mathbf{a}(t) = \alpha[\mathbf{q}(\,\cdot\,), \mathbf{p}(\,\cdot\,)](t)$. Moreover, the solution depends Lipschitz continuously on the maps $t \mapsto \mathbf{q}(t)$ and $t \mapsto \mathbf{p}(t)$.*

The proof takes several dozen pages of careful estimates, but at the end of the day it all pans out! The well-posedness result for the joint initial value problem of MBLTP fields and their point charge sources is a corollary of the above Proposition.

Motion along a constant electric capacitor field

This problem is a litmus test

Abraham–Lorentz–Dirac & Landau–Lifshitz & Eliezer equations of motion
fail to yield radiation-reaction.

In BLTP electrodynamics, radiation-reaction exists.

Expansion in powers of $\varkappa$ up to 3rd order included needed to see a non-vanishing term;
joint work with Holly Carley (in press, 2023).

But one really wants to study the large-$\varkappa$ regime, and that is non-perturbative. (Still ongoing.)

Publications

M.K.-H.K. and Tahvildar-Zadeh, A.S.,
"Bopp–Landé–Thomas–Podolsky electrodynamics as initial value problem,"
(in preparation, 2023)

A summary appeared in:
M.K.-H.K., "Force on a point charge source of the classical electromagnetic field," Phys. Rev. D **100**, 065012 (2019); "Erratum," ibid. **101**, 109901(E) (2020).

The global well-posedness of the scattering problem of a single point charge in BLTP electrodynamics (for a fixed external, compactly supported potential) is shown in:
Vu Hoang, Maria Radosz, Angel Harb, Aaron DeLeon, and Alan Baza, "Radiation reaction in higher-order electrodynamics," J. Math. Phys. **62**, 072901 (2021).

Switching $G > 0$ on again

WANTED:

A well-posed *generally covariant* joint initial value problem for *the motion of massive charged point particles and the evolution of the electromagnetic and gravitational fields they generate*.

- Extension of the SR results to GR requires:
  - electromagnetic Maxwell equations are equipped with nonlinear or higher-order linear electromagnetic vacuum laws to guarantee integrable field-energy and -momentum densities and mild curvature singularities,
  - $\exists$no Black Holes! $\rightarrow$ Energy-momentum-stress tensor of each particle has negative bare mass.
  - 2^{nd} Bianchi identity holds in weak form and forces $\nabla \cdot \mathbf{T} = 0$ to supply equations of motion (← Big !)
- No well-posedness results for joint Cauchy problem yet.

The Genie is out of the bottle again!

Babysteps toward the gravitational radiation-reaction problem:
A. Burtscher, M.K.-H.K., and A.S. Tahvildar-Zadeh,
"The weak second Bianchi identity for static spherically symmetric spacetimes with a timelike singularity,"
Class. Quantum Grav. **38**, 185001 (2021).

We state conditions on the metric of the spacetime under which the weak second Bianchi identity

$$\int_{\mathcal{M}} \left(R^{\mu}{}_{\nu} - \tfrac{1}{2}Rg^{\mu}{}_{\nu}\right) \nabla_{\mu}\psi^{\nu}\, d\,\mathrm{vol}_{\mathbf{g}} = 0 \qquad (1)$$

holds for all smooth, compactly supported vector fields ψ defined on the spacetime. Technical ingredients

We use spatially conformally flat coordinates that map the point singularity to Bray's sphere of zero area (ZAS).
For regular ZAS, the Bray mass equals their Hawking mass, equals the bare mass of the singularity.

Hoffmann's Electromagnetic Spacetime (1935)

Banesh Hoffmann found an $SO(3)$-symmetric solution family to the Einstein-Maxwell-Born-Infeld field equations with single point charge. For electron / proton parameters (e, M) we can be either in the **black hole** or in the naked singularity sector; borderline case occurs for $b = \frac{M^2}{e^6(\frac{1}{6}B(\frac{1}{4},\frac{1}{4}))^2} \equiv b_{\mathrm{B}}$.

- $g^{(H)}_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$

  $$f(r) \equiv 1 - \frac{2G}{r}\underbrace{\left(M - b^2\int_r^\infty\left(\sqrt{1+\frac{e^2}{b^2s^4}}-1\right)s^2ds\right)}$$

  POSITIVE quasi-local MASS: > 0 for all $r > 0$ if $b = b_B$.
- M =Electrostatic Field Energy of point charge iff $b = b_B$.
- Mild Conical Curvature Singularity at $r = 0$ iff $b = b_B$ (T.-Z.)
- Bianchi identity holds iff $b_{\mathrm{B}} \leq b < \infty$ (NOT RWN) (BKTZ)

Amorim's EMBLTP spacetime

- Erik Amorim proved constructively the existence of an electro-static, spherically symmetric, asymptotically flat solution to the Einstein-Maxwell-BLTP system.
- We don't know whether weak Bianchi holds for this spacetime, but suspect it does.

The problem of motion for other singularities

Weyl and EIH naturally thought of particles as point singularities in spacelike slices of spacetime.
But GR has more to offer! E.g., the ring singularity of the zGKN spacetime, a double-sheeted locally flat spacetime with Zipoy topology and Appell–Sommerfeld electromagnetic fields.

Question: Suppose one succeeds in the EIH-like quest for point singularities; what does it take to generalize to spacetimes with zGKN-like ring singularities?

Fin!

THANK YOU FOR LISTENING!