Ricci curvature bounds: synthetic versus analytic

Michael Kunzinger
(joint work with Michael Oberguggenberger, James A. Vickers)

Singularities and Curvature in General Relativity

The question, outline

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Two main approaches to Ricci curvature in low regularity:

- Synthetic: Based on methods from Optimal Transport, expresses Ricci bounds in terms of weak displacement convexity of entropy functionals. Extends even to metric measure spaces.
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Plan of the talk

- Distributional curvature bounds and regularization
- Geometry of $C^{1,1}$-(semi-) Riemannian metrics
- OT for $C^{1,1}$-metrics
- Synthetic lower Ricci curvature bounds
- Synthetic from distributional bounds in C^1
- Distributional from synthetic bounds in $C^{1,1}$
- Further results, Outlook

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$\mathrm{Vol}(M)\ldots$ volume bundle, $\Gamma_c^k(M, \mathrm{Vol}(M))$ comp. supp. C^k one-densities.

$$\mathcal{D}'^{(k)}(M) := \Gamma_c^k(M, \mathrm{Vol}(M))'$$

$$\mathcal{D}'^{(k)}\mathcal{T}_s^r(M) := \Gamma_c^k(M, T_r^s \otimes \mathrm{Vol}(M))' \cong \mathcal{D}'^{(k)}(M) \otimes_{C^\infty(M)} \mathcal{T}_s^r(M)$$

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Distributional/$\mathrm{L}^2_{\mathrm{loc}}$ connection:

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathcal{D}'\mathcal{T}_0^1(M) \qquad \text{resp.} \quad \to \mathrm{L}^2_{\mathrm{loc}}\mathcal{T}_0^1(M).$$

Riemann tensor of $\mathrm{L}^2_{\mathrm{loc}}$-connection: $X, Y, Z \in \mathfrak{X}(M)$, $\theta \in \Omega^1(M)$:

$$R(X,Y,Z)(\theta) := (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)(\theta) \in \mathcal{D}'(M).$$

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For $g \in C^1$: unique Levi-Civita connection, R, Ric defined in $\mathcal{D}'^{(1)}$, and locally

$$R^m_{ijk} = \partial_j \Gamma^m_{ik} - \partial_k \Gamma^m_{ij} + \Gamma^m_{js}\Gamma^s_{ik} - \Gamma^m_{ks}\Gamma^s_{ij}$$
$$\mathrm{Ric}_{ij} = R^m_{imj}.$$

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Distributional curvature bound

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Regularization of tensor distributions

Let $T \in \mathcal{D}'\mathcal{T}^r_s(M)$. Atlas (U_α, ψ_α), $\xi_\alpha \in \mathcal{D}(U_\alpha)$ partition of 1, $\chi_\alpha \in \mathcal{D}(U_\alpha)$, $|\chi_\alpha| \leq 1$, $\chi_\alpha \equiv 1$ near $\mathrm{supp}\xi_\alpha$. $\rho \geq 0$ mollifier. Then

$$T *_M \rho_\varepsilon := \sum_\alpha \chi_\alpha \cdot (\psi_\alpha)^*\big(((\psi_\alpha)_*)(\xi_\alpha \cdot T)) * \rho_\varepsilon\big) \in \mathcal{T}^r_s(M).$$

Properties:

- $T *_M \rho_\varepsilon \in C^\infty$
- $T \star_M \rho_\varepsilon \to T$ in $\mathcal{D}'\mathcal{T}^r_s(M)$ (resp. in C^k_{loc} or $W^{k,p}_{\mathrm{loc}}$ if T is contained in these spaces)
- $T \in \mathcal{D}'(M)$, $T \geq 0 \ \Rightarrow\ T \star_M \rho_\varepsilon \geq 0$ in $C^\infty(M)$.

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Consequences for $g \in C^1$

Let $g_\varepsilon := g \star_M \rho_\varepsilon$, $X, Y \in \mathfrak{X}(M)$. then:

- $\mathrm{Ric}_g(X, Y) \star_M \rho_\varepsilon - \mathrm{Ric}_{g_\varepsilon}(X, Y) \to 0$ in $C^0(M)$ as $\varepsilon \to 0$.
- $\mathrm{Ric}[\mathrm{g}] *_M \rho_\varepsilon - \mathrm{Ric}[g_\varepsilon] \to 0$ in C^0.

Theorem Let $g \in C^1$, M compact. TFAE:

(i) $\mathrm{Ric}[g] \geq K$ in $\mathcal{D}'$.

(ii) $\forall \delta > 0\ \exists \varepsilon_0 > 0\ \forall \varepsilon < \varepsilon_0 :\ \mathrm{Ric}[g_\varepsilon] \geq K - \delta$.

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- $D_y^2 := x \mapsto g(\exp_y^{-1} x, \exp_y^{-1} x)$ is $C^{1,1}$, with $T_x D_y^2 = 2g(\dot\sigma(1), \,.\,)$, where $\sigma(t) = \exp_y(t \cdot \exp_y^{-1} x)$ and $(y, x) \mapsto P(y, x) := \dot\sigma(1)$ is the position vector field of x with respect to y.

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- $E : (y, w) \mapsto (y, \exp_y(w))$ is strongly differentiable over the zero section with invertible differential.
- Cost function $\phi(x) := d^2(x, y)/2$ is super-differentiable a.e.

[Min:15], [KSS:14]

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Wasserstein-distance between μ and ν:

$$W_2(\mu,\nu) = \Big[\inf_{\pi \in \mathrm{Cpl}(\mu,\nu)} \int_{X \times X} d_g(x,y)^2 \, d\pi(x,y)\Big]^{\frac{1}{2}}$$

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Alternative interpretation:
Existence and uniqueness of geodesics in $(\mathcal{P}(M), W_2)$ between μ and ν: $t \mapsto \mu_t := (T_t)_\sharp \mu$, where

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Synthetic Ricci curvature bounds (1)

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(X, d) metric space, $f : X \to \bar{\mathbb{R}}$ weakly K-convex if $\forall\, x, y \in X\ \exists$ geodesic $\gamma : [0,1] \to X$ from x to y such that for all $t \in [0,1]$

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- $U \in \mathcal{DC}_\infty$ if $e^\lambda U(e^{-\lambda})$ convex.
- $\lambda_K(U) := \inf_{r>0} K \frac{rU'_+(r) - U(r)}{r}$
- For $U_\infty(r) := r \log(r)$, $\lambda_K(U_\infty) = K$.

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Def. (X, d, ν) has ∞-Ricci bounded below by K if $\forall U \in \mathcal{DC}_\infty$, U_ν is weakly λ_K-convex on $(\mathcal{P}_2(X), W_2)$.

Th.: (Sturm/Lott/Villani) Let (M, g) be a compact C^2-Riemannian manifold. Then (with $\nu_g := d\mathrm{Vol_g}/\mathrm{Vol_g}(M)$)

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Standard proofs for synthetic $\Rightarrow$ pointwise in C^2 rely on

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Strategy of proof for $g \in C^{1,1}$

- Regularize g to g_ε.
- Suppose $\mathrm{Ric}_{g_{\varepsilon_k}}(v_k, v_k) < (K - \delta) g_{\varepsilon_k}(v_k, v_k)$, $v_k \to v \in T_{x_0}M$.
- Construct exceptional Wasserstein geodesics for $g_k \equiv g_{\varepsilon_k}$, show convergence to W-geodesic for g.
- Derive contradiction by inserting measures with support $\to \{x_0\}$.

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Pick $\phi_k : M \to \mathbb{R}$ such that $\nabla^{g_k}\phi_k(x_k) = -v_k$ and $\mathrm{Hess}^{g_k}(\phi_k)(x_k) = 0$, and analogously ϕ for g at x_0. Can have $\phi_k \to \phi$ in nbhd V of x_0. Set $c(x,y) := d_g(x,y)^2/2$, $c_k(x,y) := d_{g_k}(x,y)^2/2$.

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Pick $\phi_k : M \to \mathbb{R}$ such that $\nabla^{g_k}\phi_k(x_k) = -v_k$ and $\mathrm{Hess}^{g_k}(\phi_k)(x_k) = 0$, and analogously ϕ for g at x_0. Can have $\phi_k \to \phi$ in nbhd V of x_0. Set $c(x,y) := d_g(x,y)^2/2$, $c_k(x,y) := d_{g_k}(x,y)^2/2$.

Th.: ([Gl:19]) (M,g) compact RMF with sectional curvature bounded above by $K \geq 0$. Then $\exists\ C_* := C_*(\mathrm{inj}(M), K, \mathrm{diam}(M)) > 0$ s.t., $\forall \varepsilon > 0$, if $\phi \in C^2(M,\mathbb{R})$ satisfies

$$\|\nabla\phi\|_\infty \leq \min\left(\frac{\varepsilon}{3K\mathrm{diam}(M)}, C_*\right) \qquad \text{and} \qquad \mathrm{Hess}(\phi) \leq (1-\varepsilon)g,$$

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$g \in C^{1,1} \Rightarrow$ can have all ϕ_k c_k-concave $\Rightarrow$ push-forward under $F_t^{(k)}(y) := \exp_y^{g_k}(-t\nabla^{g_k}\phi_k)$ induces OT.

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- Density of $(F_t)_\#(\xi_0 d\mathrm{vol}_g)$:

$$x \mapsto \xi_0(F_t^{-1}(x)) \, \frac{1}{\det DF_t(y)}\bigg|_{y=F_t^{-1}(x)}.$$

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$$J''(t) + K(t)J(t) = 0, \quad J(0) = I_n, \quad J'(0) = \mathrm{Hess}^{g_k}(\phi_k)_y.$$

with $K_{ij}(t) = \langle R^{g_k}(e_i(t), \dot\gamma(t))\dot\gamma(t), e_j(t)\rangle_{g_k(\gamma(t))}$.

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- For dominated convergence, need additional assumption:
  There exists a (Lebesgue-) null set $N \subseteq M$ such that, for each $y \in M \setminus N$,

$$DF_t^{(k)}(y) \to DF_t(y),$$

  uniformly for $t \in [0,1]$.

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- Consequently, $\mu_t^{(k)} = (F_t^{(k)})_\# \mu_0^{(k)} \to (F_t)_\# \mu_0$, as well as $\mu_t^{(k)} \to \chi_t = (H_t)_\# \mu_0$, where $H_t = y \mapsto \exp_y(-t\nabla\psi(y))$

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- Hence, for any μ_0, $(F_t)_\# \mu_0$ is a Wasserstein geodesic.
- W-geodesics are unique, so can use this and $U_\infty(r) = r \log r$ in Def. of ∞-Ricci bound:

$$U_\nu(\mu_t) \le t U_\nu(\mu_1) + (1-t) U_\nu(\mu_0) - \frac{1}{2} K t(1-t) W_2(\mu_0, \mu_1)^2,$$

Here,

$$U_\nu(\mu_t) = \int_M U\Big(\mathrm{vol}_g(M) \cdot \frac{\eta_0(y)}{\det(DF_t)(y)}\Big) \det(DF_t)(y) \frac{d\mathrm{vol}_g(y)}{\mathrm{vol}_g(M)}.$$

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- Standard comparison arguments then give (for k large) the contradiction:

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Th.

Let M be a compact connected mf with $C^{1,1}$-RM g s.t. (M,d_g,ν_g) has ∞-Ricci curvature $\geq K$. Assume that some subsequence of $g \star_M \rho_\varepsilon$, satisfies the convergence condition. Then also $\mathrm{Ric}_g \geq Kg$ in the distributional sense.

Further investigations

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Braun/Calisti (2022): Lorentzian setting

- (M, g) globally hyperbolic with timelike Ric_g bounded below in $\mathcal{D}'$.
- $g \in C^1 \Rightarrow M$ has timelike measure-contraction property TMCP.
- $g \in C^{1,1} \Rightarrow M$ has timelike curvature-dimension property TCD.
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Compatibility of singularity theorems

- Generalizations of Hawking/Penrose singularity theorems to $g \in C^1$ using $\mathcal{D}'$ methods.
- Cavalletti/Mondino: Synthetic Hawking theorem assuming TMCP and synth. mean curvature condition.
- C/M (+ TL-nonbranching) implies C^1-Hawking: $\mathcal{D}'$-assumptions imply mean curvature cond., and Braun/Calisti$\Rightarrow$ TMCP.

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