

A synthetic null energy condition

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In metric(-measure) geometry with positive signature, there are theories of

- sectional curvature bounds based on triangle comparison ([Aleksandrov...](#))
- pointed Gromov-Hausdorff limits of manifolds under lower Ricci and upper dimensional bounds ([Fukaya, Gromov, Cheeger-Colding, ...](#))
- Ricci lower bounds via displacement convexity of entropy ([Bakry-Emery, Lott-Sturm-Villani, Ambrosio-Gigli-Savare, ...](#))

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Can something similar be done in Lorentzian geometry?

- tidal forces ([Kunzinger-Sämman '18](#))
- convergence of spaces ([Müller 22+](#), [Minguzzi-Suhr 22+](#))
- Einstein equation ([M. 20](#), [Mondino-Suhr 18+](#), [Cavalletti-Mondino 20+](#), [Braun 22+](#), ...)

Definition (Time-separation function)

On a set M of events, a *time-separation function* refers to $\ell : M \times M \rightarrow \{-\infty\} \cup [0, \infty)$ satisfying the reverse triangle inequality and antisymmetry: $\forall x, y, z \in M$

$$\ell(x, y) \geq \ell(x, z) + \ell(z, y) \quad (1)$$

$$\min\{\ell(x, y), \ell(y, x)\} > -\infty \Leftrightarrow x = y. \quad (2)$$

Remark: (1) + (2) $\Rightarrow \ell(x, x) = 0$; (2) gives the arrow of time

Example (Minkowski space)

$M = R^{1,3}$ with $\ell(x, y) = L(y - x)$ where

$$L(v) = \begin{cases} |g(v, v)|^{1/2} & \text{if } v \text{ is future-directed} \\ -\infty & \text{else.} \end{cases}$$

Notice $L(v)$ is *concave* (as is $L(v)^q$ for any $0 < q \leq 1$ if $(-\infty)^q := -\infty$).

Example (Causal spaces)

A time-separation function induces $M_{\leq}^2 = \{(x, y) \in M^2 \mid \ell(x, y) \geq 0\}$ a partial order and $M_{\ll}^2 = \{(x, y) \in M^2 \mid \ell(x, y) > 0\}$ a preorder. The triple (M, \leq, \ll) is a special example of what [Kronheimer and Penrose '67](#) call a *causal space*.

Definition (Causal and timelike futures)

We say y lies in the *causal future* of x and write $x \leq y$ if $\ell(x, y) \geq 0$; we say y lies in the *timelike future* of x and write $x \ll y$ if $\ell(x, y) > 0$. Also

$$\begin{aligned} J^+(x) &= \{y \in M \mid \ell(x, y) \geq 0\} & J^+(X) &= \cup_{x \in X} J(x) \\ J^-(y) &= \{x \in M \mid \ell(x, y) \geq 0\} & J^-(Y) &= \cup_{y \in Y} J(y) \\ J(x, y) &= J^+(x) \cap J^-(y) & J(X, Y) &= J^+(X) \cap J^-(Y) \end{aligned}$$

and similarly $I^\pm(z)$ and $I(X, Y)$ but with strict inequalities $\ell > 0$.

Definition (Causal and timelike paths)

A **path** $s \mapsto \sigma(s) \in M$ is called **causal** if and only if $\ell(\sigma(s), \sigma(t)) \geq 0$ for all $s \leq t$, and **timelike** if and only if $\ell(\sigma(s), \sigma(t)) > 0$ for all $s < t$.

Definition (Lorentzian length of a causal path)

The (*negative*) **ℓ -length** of a causal path $\sigma : [a, b] \rightarrow M$ is defined by

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The (*negative*) **ℓ -length** of a causal path $\sigma : [a, b] \rightarrow M$ is defined by

$$\begin{aligned} L_{-\ell}(\sigma) &:= \sup_{k \in \mathbf{N}} \sup_{a=t_0 \leq t_1 \leq \dots \leq t_k=b} - \sum_{i=1}^k \ell(\sigma(t_{i-1}), \sigma(t_i)) \\ &\geq -\ell(\sigma(a), \sigma(b)) \end{aligned}$$

by the triangle inequality.

Definition (ℓ -path)

A path $\sigma : [0, 1] \rightarrow M$ is called an ℓ -path if and only if

$$\ell(\sigma(s), \sigma(t)) = (t - s)\ell(\sigma(0), \sigma(1)) > 0 \quad \forall 0 \leq s < t \leq 1.$$

We denote the set of ℓ -paths by $\text{TPath}^\ell(M)$.

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Definition

We call (M, ℓ) a *timelike ℓ -path space* if each timelike related pair of events $x \ll y$ are connected by an ℓ -path.

- Kunzinger and Sämann's (*regular*) *globally hyperbolic Lorentzian length spaces* provide a rich class of examples of timelike ℓ -path spaces
- to achieve this, they need a (metrizable) topology

a variation on Kunzinger & Sämman (hereafter K-S)

Definition (Metric spacetime)

A metric space (M, d) equipped with its metric topology and a time-separation function ℓ is called a *metric spacetime*

Definition (Causal curve)

A nonconstant causal *path* is called a causal *curve* if it is *d -Lipschitz*.

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Definition (Non-totally imprisoning)

A metric spacetime (M, d, ℓ) is *non-totally imprisoning* if each compact $K \subset M$ admits a bound $B < \infty$ such that all causal curves σ in K (i.e. $\sigma : I \subset \mathbb{R} \rightarrow K$ with $\sigma(I) \subset K$) have d -length $L_d(\sigma) \leq B$.

Definition (Globally hyperbolic)

A metric spacetime (M, d, ℓ) is *globally hyperbolic* if it is non-totally imprisoning and the causal diamond $J(x, y)$ is compact for each $x, y \in M$.

Definition (Timelike curve-connected; Lorentzian geodesic space)

A metric spacetime is *timelike curve-connected* iff each $x \ll y$ are connected by a timelike curve; it is a *Lorentzian geodesic space* iff each $x < y$ are connected by a causal curve σ with $L_{-\ell}(\sigma) = -\ell(\sigma(0), \sigma(1))$.

Without global hyperbolicity, K-S's definition of a *Lorentzian length space (LLS)* is involved. With global hyperbolicity it can be defined via:

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Theorem ((M. 23+) Characterizing Lorentzian length spaces)

Assuming globally hyperbolicity, a metric spacetime (M, d, ℓ) is an *LLS* iff it is (a) a *timelike curve-connected* (b) *Lorentzian geodesic space*; (c) $I^{\pm}(x)$ both nonempty $\forall x \in M$; (d) ℓ is lower semicontinuous and (e) $\ell_+ = \max\{\ell, 0\}$ is continuous.

- In such spaces, K-S showed that metric topology coincides with the order topology induced by ℓ ; this implies *gh LLS's are independent of d !*
- Burtscher & Garcia-Hevelling 21+ characterize global hyperbolicity of an LLS via existence of Cauchy time functions (and surfaces)

- Unfortunately, it's not clear that all ℓ -paths are **continuous**!

Definition (Regular(ly localizable))

An LLS is **regular** (or **regularly localizable**) if for any $L_{-\ell}$ -minimizing causal curve, $L_{-\ell}(\sigma|_{[a,b]}) = 0$ with $\sigma|_{[a,b]}$ **non-constant** implies $L_{-\ell}(\sigma) = 0$.

Lemma (M. 23+)

*In a **globally hyperbolic regular LLS**, each ℓ -path is continuous.*

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Lemma (M. 23+)

*In a **globally hyperbolic regular LLS**, each ℓ -path is continuous.*

Corollary (Relation of ℓ -paths to $L_{-\ell}$ -extremizers)

In a globally hyperbolic regularly localizable Lorentzian length space:

- Every ℓ -path becomes a d -Lipschitz $L_{-\ell}$ -minimizing curve after a continuous increasing (not necessarily Lipschitz) reparameterization.*
- K-S**: Conversely, every $L_{-\ell}$ -minimizing curve with timelike separated endpoints becomes an ℓ -path after a similar reparameterization.*

(a) resolves an awkward gap in the literature.

Proof of lemma:

- For convenience, we deal only with metric spacetimes (M, d, ℓ) which are **closed Lorentzian geodesic subsets** of **globally hyperbolic regular Lorentzian length spaces** (g.h.r. LLS).

Now that timelike geodesics exist:

- given a triple $x \ll z \ll y$ of timelike related events, we can compare the Lorentzian length of a bisector to that of the Minkowski triangle with the same Lorentzian sidelengths
- and similarly for generalized bisectors (i.e. ratios other than 1 : 1)

- K-S define $T\text{-sec}(M, d, \ell) \geq 0$ if our generalized bisector is longer (and $T\text{-sec}(M, d, \ell) \leq 0$ if it is shorter) for all such timelike triangles
- they define $\pm T\text{-sec}(M, d, \ell) \geq k \in \mathbb{R}$ analogously by comparing to timelike triangles in constant curvature Lorentzian spaces
- they also give causal sectional curvature bounds and show such bounds prevent branching of ℓ -geodesics:

Definition (timelike nonbranching)

(M, ℓ) *timelike nonbranching* if for all $\tilde{\sigma}, \sigma \in \text{TPath}^\ell$ with $\sigma|_{[\frac{1}{3}, \frac{2}{3}]} = \tilde{\sigma}|_{[\frac{1}{3}, \frac{2}{3}]}$ then $\tilde{\sigma} = \sigma$;

- Alexander-Bishop '08 shows **consistency** of these definitions with smooth timelike sectional curvature bounds on Lorentzian manifolds
- Minguzzi-Suhr '22+ show **stability** of a similar bound
- Beran-Ohanyan-Rott-Solis '22+: $T\text{-sec}(M, d, \ell) \geq 0$ and existence of a timelike line implies **geometric splitting** of (M, d, ℓ)

To pass from sectional to Ricci curvature / Einstein eq requires averaging:

Definition (Optimal transport distance between measures)

- Given metric spaces (M^\pm, d^\pm) , let $\mathcal{P}(M)$ denote the Borel probability measures on M and $\mathcal{P}_c(M)$ those with compact support.
- *Push-forward*: given $G : M^- \rightarrow M^+$ Borel and $\mu^- \in \mathcal{P}(M^-)$, define $\mu^+ = G_{\#}\mu^- \in \mathcal{P}(M^+)$ by $\mu^+(B) = \mu^-(G^{-1}(B))$ for all $B \subset M^+$.
- Letting $\pi^\mp(x^-, x^+) = x^\mp$ denote the projection from $M^- \times M^+$ onto its left and right factors, set $\Gamma(\mu^-, \mu^+) = \{\gamma \in \mathcal{P}(M^- \times M^+) \mid \pi_{\#}^\pm \gamma = \mu^\pm\}$.

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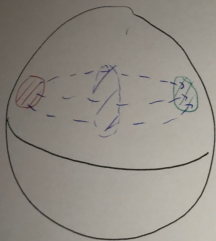
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- Given $p \in [1, \infty)$ and $M = M^\pm$, the *p-Kantorovich-Rubinstein-Wasserstein distance* d_p between $\mu^\pm \in \mathcal{P}(M)$ defined by

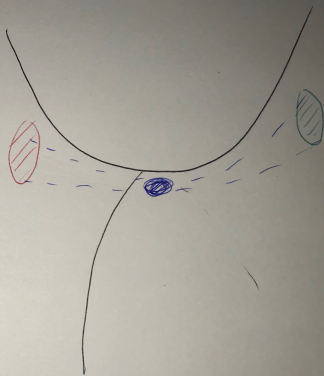
$$d_p(\mu^-, \mu^+) := \inf_{\gamma \in \Gamma(\mu^+, \mu^-)} \left(\int_{M^2} d(x, y)^p d\gamma(x, y) \right)^{1/p} \quad (3)$$

is well-known to metrize convergence against functions growing no faster than $d(x, \cdot)^p$ provided (M, d) is *Polish* (i.e. complete and separable), in which case the inf is attained.

- If (M, d) is a geodesic space so is $(\mathcal{P}_c(M), d_p)$.



$Re \geq 0$



$Re \leq 0$

Definition (Causal and timelike measures)

In a Polish g.h.r LLS (M, d, ℓ) , given $\mu, \nu \in \mathcal{P}(M)$ and $q \in (0, 1]$ set

$$\Gamma_{\leq}(\mu, \nu) := \{\gamma \in \Gamma(\mu, \nu) \mid \gamma[M_{\leq}^2] = 1\} = \{\text{causal measures}\}$$

$$\Gamma_{\ll}(\mu, \nu) := \{ \quad " \quad \mid \gamma[M_{\ll}^2] = 1\} = \{\text{timelike measures}\}$$

Lemma (Lift time-separation from events to measures)

$$\ell_q(\mu, \nu) := \max_{\gamma \in \Gamma_{\leq}(\mu, \nu)} \left(\int_{M^2} \ell(x, y)^q d\gamma(x, y) \right)^{1/q} \quad (4)$$

makes $(\mathcal{P}_c(M), \ell_q)$ into a *timelike ℓ_q -path space*. Not all such ℓ_q -paths are d_1 -continuous;

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makes $(\mathcal{P}_c(M), \ell_q)$ into a **timelike ℓ_q -path space**. Not all such ℓ_q -paths are d_1 -continuous; one (all) will be if (μ, ν) is (**strongly**) timelike q -dualizable:

Definition (timelike q -dualizability)

Let $\Gamma^q = \Gamma^q(\mu, \nu)$ denote the set of maximizers. Then

- (μ, ν) are **timelike q -dualizable** if $\Gamma_{\ll}^q := \Gamma^q \cap \Gamma_{\ll}(\mu, \nu)$ is non-empty and $\exists u \oplus v \in L^1(\mu \times \nu)$ which dominates ℓ^q on $\text{spt}(\mu \times \nu) \cap M_{\leq}^2$.
- (μ, ν) are **strongly** timelike q -dualizable if, in addition, $\Gamma^q \subset \Gamma_{\ll}(\mu, \nu)$

Definition (Polish / proper metric-measure spacetime)

A *metric-measure spacetime* refers to a Lorentzian geodesic closed subset (M, d, ℓ) of a g.h.r. LLS, equipped with a Borel measure $m \geq 0$, finite on bounded sets, satisfying $M = \text{spt } m$. It's called *Polish* if complete and separable, and *proper* if all bounded subsets $X \subset M$ are compact.

Example (Smooth metric-measure spacetimes)

Any smooth, connected, Hausdorff, time-oriented, n -dimensional Lorentzian manifold (M^n, g) of signature $(+ - \dots -)$ is second-countable (Ozeki-Nomizu '61) and its topology comes from a complete Riemannian metric \tilde{g} (Geroch '68). With the distance $d_{\tilde{g}}$ and time-separation function ℓ_g induced by \tilde{g} and g respectively, is a *proper g.h.r. LLS* provided it has no closed causal curves and causal diamonds $J(x, y)$ are compact. Letting $V \in C^\infty(M)$ and vol_g denote its Lorentzian volume, setting $dm = e^{-V} d\text{vol}_g$ makes it a proper metric-measure spacetime. We call such spaces *smooth metric-measure spacetimes*.

Synthetic timelike Ricci bounds

Desiderata:

- consistency (with the analogous smooth bounds)
- stability (preservation under suitable limits)
- consequences (e.g. Hawking-type singularity theorem)

Definition (Entropy)

We define the relative *entropy* by

$$H(\mu | m) := \begin{cases} \int_M \rho \log \rho dm & \text{if } \mu \in \mathcal{P}_c^{ac}(M) \text{ and } \rho := \frac{d\mu}{dm}, \\ +\infty & \text{if } \mu \in \mathcal{P}_c(M) \setminus \mathcal{P}_c^{ac}(M). \end{cases}$$

- our sign convention is opposite to that of the physicists' entropy

Entropic **weak timelike curvature-dimension** conditions

Definition (TCD versus **w**TCD; e.g. $K = 0 = 1/N$)

For $(K, N, q) \in \mathbb{R} \times (0, \infty] \times (0, 1]$ write $(M, d, \ell, m) \in \mathbf{wTCD}_q^e(K, N)$ if and only if every **strongly timelike** q -dualizable finite entropy pair $\mu_0, \mu_1 \in \mathcal{P}_c(M)$ admit a maximizer $\gamma \in \Gamma_{\ll}^q$ and **corresponding** ℓ_q -path $(\mu_t)_{t \in [0,1]}$ along which the entropy $t \in [0, 1] \mapsto h(t) := H(\mu_t \mid m)$ is upper-semicontinuous and distributionally solves the semiconvexity inequality

$$h''(t) \geq \frac{h'(t)^2}{N} + K \|\ell\|_{L^2(\gamma)}^2.$$

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$$h''(t) \geq \frac{h'(t)^2}{N} + K \|\ell\|_{L^2(\gamma)}^2.$$

Cavalletti-Mondino '20 prove all **limits** of $\mathbf{TCD}_q^e(K, N)$ space in a suitable (pointed measured weak) sense lie in $\mathbf{wTCD}_q^e(K, N)$ if $N < \infty$; they also display remarkable similarities to smooth spacetimes (such as a Hawking singularity theorem)

c.f. **Burtscher-Ketterer-M.-Woolgar '20** analogous sharp Riemannian injectivity radius bound; characterizes $\mathbf{RCD}(K, N)$ spaces which attain it

Pointed measured weak convergence [CM20+]

Fixing $x_j \in M_j = \text{spt } m_j$ where m_j is a Radon measure, we say $(M_j, d_j, \ell_j, m_j, x_j) \rightarrow_{pmGL} (M_\infty, d_\infty, \ell_\infty, m_\infty, x_\infty)$ iff all $(M_j, d_j, \ell_j, m_j, x_j)$ embed d -continuously and ℓ -isometrically into a single proper g.h.r. LLS (X, d, ℓ) and after this embedding, $d(x_j, x_\infty) \rightarrow 0$ and the measures $m_j \rightarrow m_\infty$ converge weakly against continuous compactly supported test functions: i.e.

$$\lim_{j \rightarrow \infty} \int_X \phi dm_j = \int_X \phi dm_\infty \quad \forall \phi \in C_c(X).$$

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- although the limit of $TCD_q^e(K, N)$ spaces is only $wTCD_q^e(K, N)$, Braun '22+ shows (q -essentially) **timelike nonbranching** $wTCD_q^e(K, N)$ spaces are $TCD_q^e(K, N)$. Hence a limit of timelike nonbranching $wTCD_q^e(K, N)$ spaces is $wTCD_q^e(K, N)$.
- OPEN QUESTION: unlike in positive signature, it is not known whether some version of timelike nonbranchingness survives the preceding limits

Positive energy \Leftrightarrow displacement convexity of entropy

DEF (N -Bakry-Emery modified Ricci tensor; cf. [Erbar-Kuwada-Sturm'15](#))

Given $N \neq n$ and $V \in C^\infty(M^n)$ define

$$R_{ij}^{(N,V)} := R_{ij} + \nabla_i \nabla_j V - \frac{1}{N-n} (\nabla_i V)(\nabla_j V)$$

THM ([M '20 Consistency](#)) Fix $(K, N, q) \in \mathbb{R} \times (0, \infty] \times (0, 1)$ and a smooth metric-measure spacetime (M^n, g) with $dm = e^{-V} d\text{vol}_g$. Then $(M, d_{\tilde{g}}, \ell_g, m) \in (w)TCD_q^e(K, N)$ if and only if either

- (a) $N = n$, $V = \text{const}$ and $R_{ij} v^i v^j \geq K$ for all unit timelike $(v, x) \in TM$,
- (b) $N > n$ and $R_{ij}^{(N,V)} v^i v^j \geq K$ for all unit timelike vectors $(v, x) \in TM$.

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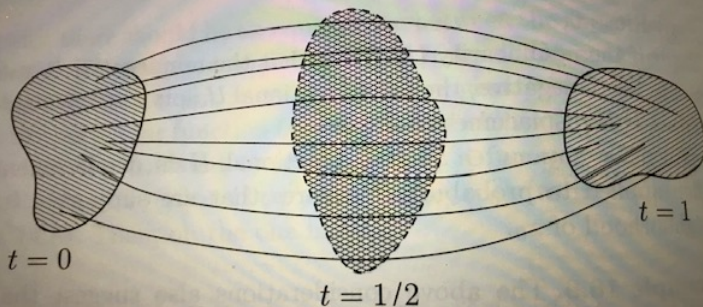
[Mondino-Suhr '18+](#) Use entropic convexity to say also when equality holds, giving a weak (but unstable) solution concept for Einstein field equation.

[Akdemir-Cavalletti-Colinet-M.-Santarcangelo '21](#)

$CD_p(K, N) \cap \{\text{nonbranching}\}$ is independent of $p > 1$

Lazy Gas Experiment (M. 94, Villani 09)

16 Displacement convexity 1



Action minimizing paths satisfy pressureless Euler equation.

Braun 22+:

- $N = \infty$
- alternative definitions of $(w)TCD_q^{(*)}(K, N)$ based on convexity properties of a power-law entropy (instead of $H(\mu | m)$) along ℓ_q -paths

$$S_N(\mu) := -N \int_M \left(\frac{d\mu}{dm}\right)^{1-\frac{1}{N}} dm$$

- equivalence of most of these various definitions to $TCD_q^e(K, N)$ assuming (q -essential) timelike nonbranching

Cavalletti-Mondino '22:

- asked for a synthetic formulation of the null energy condition (NEC)
- stronger physical motivation; more widely satisfied
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Braun 22+:

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On a Lorentzian manifold, smoothness need not imply a local lower bound on Ricci curvature!

Theorem (M' 23+)

Fix a smooth spacetime (M^n, g) with signature $(+ - \dots -)$ and symmetric 2-tensor field Q . Then

$$Q(v, v) \geq 0 \quad \forall (v, x) \in TM \text{ with } g(v, v) = 0$$

holds if and only if each compact subdomain $X \subset M^n$ admits a timelike lower bound $K = K_X$ for its Bakry-Emery N -Ricci tensor:

$$Q(v, v) \geq Kg(v, v) \quad \forall (v, x) \in TX \text{ with } g(v, v) > 0$$

Taking $Q = \text{Ric}^{(N, V)}$ (or $Q_{ab} = 8\pi T_{ab}$ if Einstein holds) motivates

Definition (A synthetic null energy-dimension condition)

Given $(N, q) \in (0, \infty] \times (0, 1)$, a metric-measure spacetime (M, d, ℓ, m) satisfies $wNEC_q^{(e)}(N)$ if and only if each compact subset $X \subset M$ admits a bound $K = K_X \in \mathbb{R}$ such that $J(X, X) \in wTCD_q^{(e)}(K, N)$.

- in other words, the null energy condition is equivalent to a variable lower (semicontinuous) bound $k(x)$ on the timelike Ricci curvature
- Consistency with smooth (NC) + $(n \leq N)$: follows from theorem above
- for (q -essentially) timelike nonbranching spaces $wNC_q^e(N) = NC_q^*(N)$
- Consequences: many of [Cavalletti & Mondino](#)'s nice properties (timelike Bishop-Gromov and Brunn-Minkowski inequalities, needle decomposition, etc) of nonsmooth $wTCD_q^{(e)}(K, N)$ spacetimes are therefore inherited directly by $wNEC_q^{(e)}(N)$ spacetimes; c.f. [Braun-M.](#) (in progress)
- OPEN: it is natural to wonder if a Penrose singularity theorem can hold in this nonsmooth setting? (c.f. [Graf '20](#) on $g \in C^1$ spacetimes (M^n, g) , [Ketterer '23+](#) on $g \in C^\infty$ spacetimes)
- (In)stability: on the other hand, any stability result appears hopeless unless we are will to assume some **uniformity in j** of the lower bound $k(\cdot)$ along the sequence $(M_j, d_j, \ell_j, m_j, x_j)$

A few references

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THANK YOU!