Optimal regularity and Uhlenbeck compactness in Lorentzian geometry and beyond

Moritz Reintjes
Assistant Professor
City University of Hong Kong

Singularities and Curvature in General Relativity

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Collaborator: Blake Temple (University of California, Davis)

Preview

Shock wave solutions of Einstein-Euler equations appear singular: Riemann curvature in L^∞, but metric only Lipschitz continuous.

[Groah-Temple, `04]

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Thm: ("Optimal Regularity") [R. & Temple, 2019]
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Thm: ("Optimal Regularity") [R. & Temple, 2019]
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Proof:
Write connection transformation law as solvable system of elliptic PDE's for the regularising transformation.

Optimal Regularity
and
Uhlenbeck Compactness

The setting:

Covariant derivative $\nabla = \partial + \Gamma$

- Connection components: $\Gamma \equiv \Gamma^k_{ij}$ $(k, i, j = 1,\ldots,n)$

E.g.: $\Gamma^k_{ij} = g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ for a metric g_{ij} .

- Their Riemann curvature: $\mathrm{Riem}(\Gamma) = \mathrm{Curl}(\Gamma) + [\Gamma, \Gamma]$

Both defined on an open & bounded set $\Omega \subset \mathbb{R}^n$.

The problem of optimal regularity is local.

- The set $\Omega \subset \mathbb{R}^n$ represents a chart (x, U) on a manifold, $\Omega = x(U)$.

Optimal regularity and coordinate transformations:

$$\Gamma \in W^{1,p}$$

$$\Big\downarrow \frac{\partial}{\partial y}$$

$$\mathrm{Riem}(\Gamma) \in L^p$$

"Optimal Regularity"

- $\Gamma \in L^p$ means $\int |\Gamma|^p \, dx < \infty$ component-wise
- $\Gamma \in W^{1,p}$ means $\Gamma \in L^p$ & $\partial\Gamma \in L^p$

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Optimal regularity and coordinate transformations:

$\mathrm{Riem}(\Gamma) \sim \mathrm{Curl}(\Gamma)$

makes this possible

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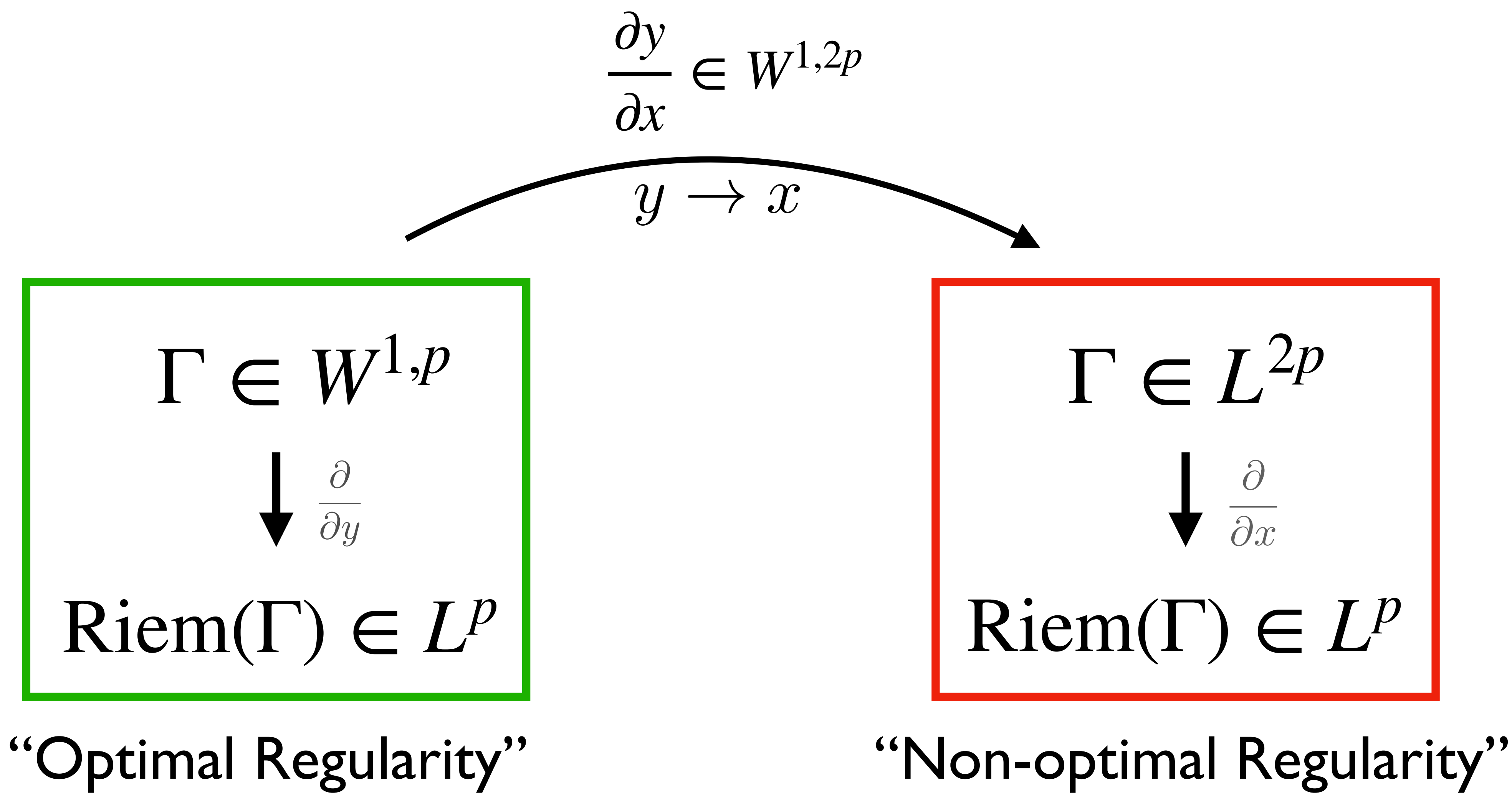

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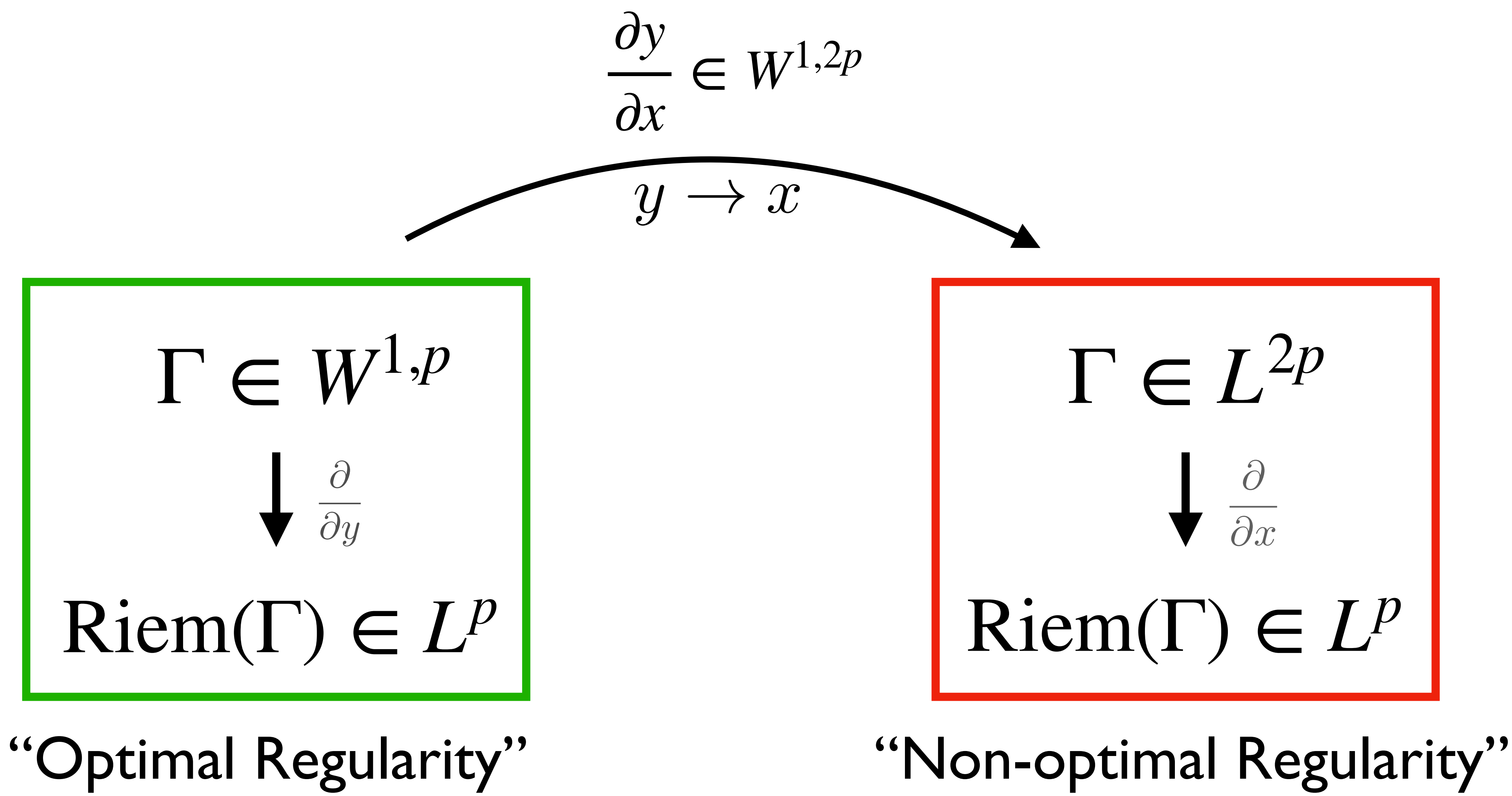

$$\Gamma \to \Gamma + \partial\left(\frac{\partial x}{\partial y}\right)$$

$$\mathrm{Riem}(\Gamma) \to \frac{\partial x}{\partial y} \cdot \mathrm{Riem}(\Gamma)$$

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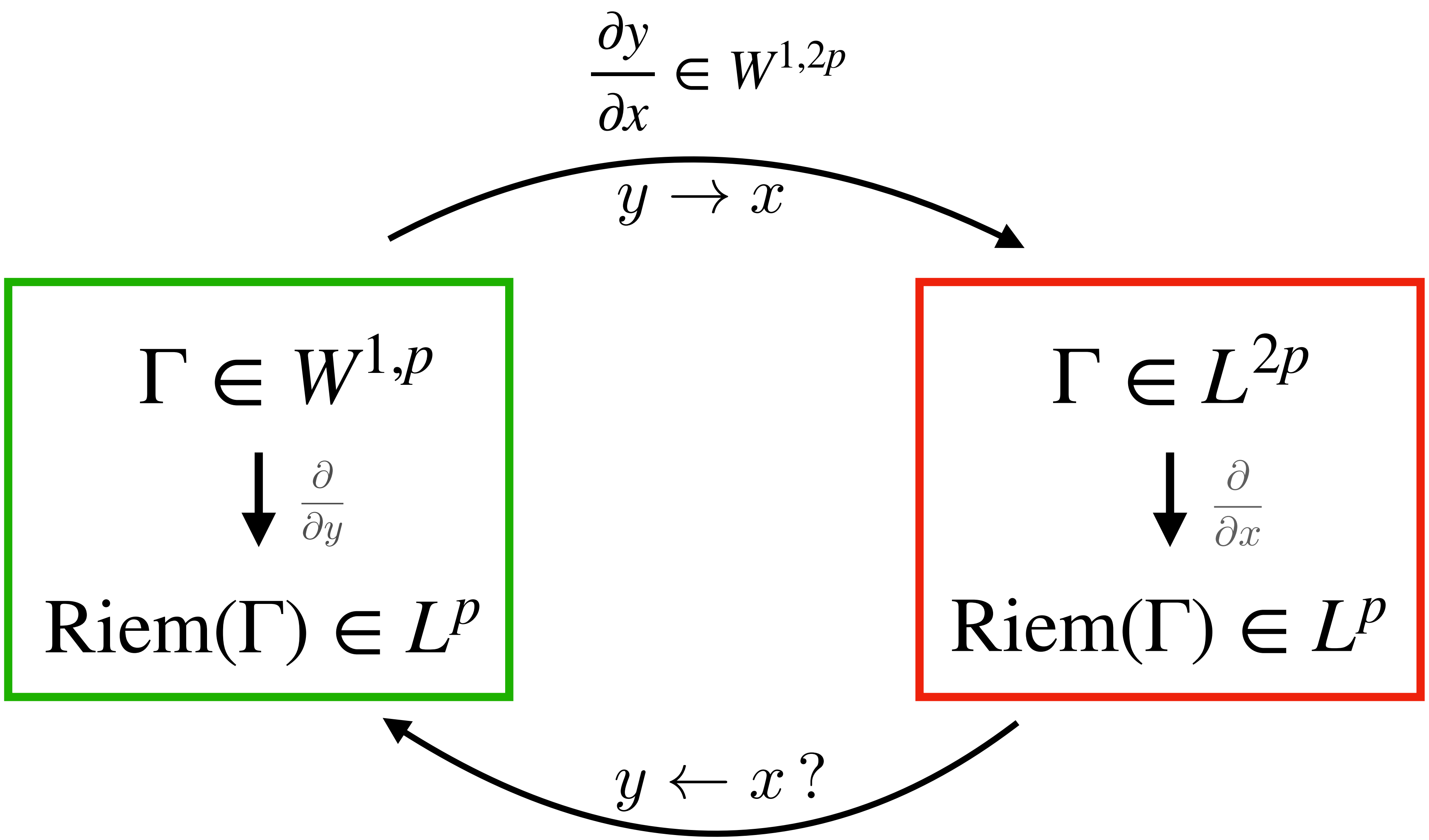

Question: $\exists\, x \to y\,?$

Optimal regularity and coordinate transformations:

Typical, when solving
Einstein equations.
E.g.: GR shock waves.

$$\Gamma \in W^{1,p}$$
$$\Big\downarrow \frac{\partial}{\partial y}$$
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- Riemannian metrics (pos. def.). [Kazdan-DeTurck, 1981]

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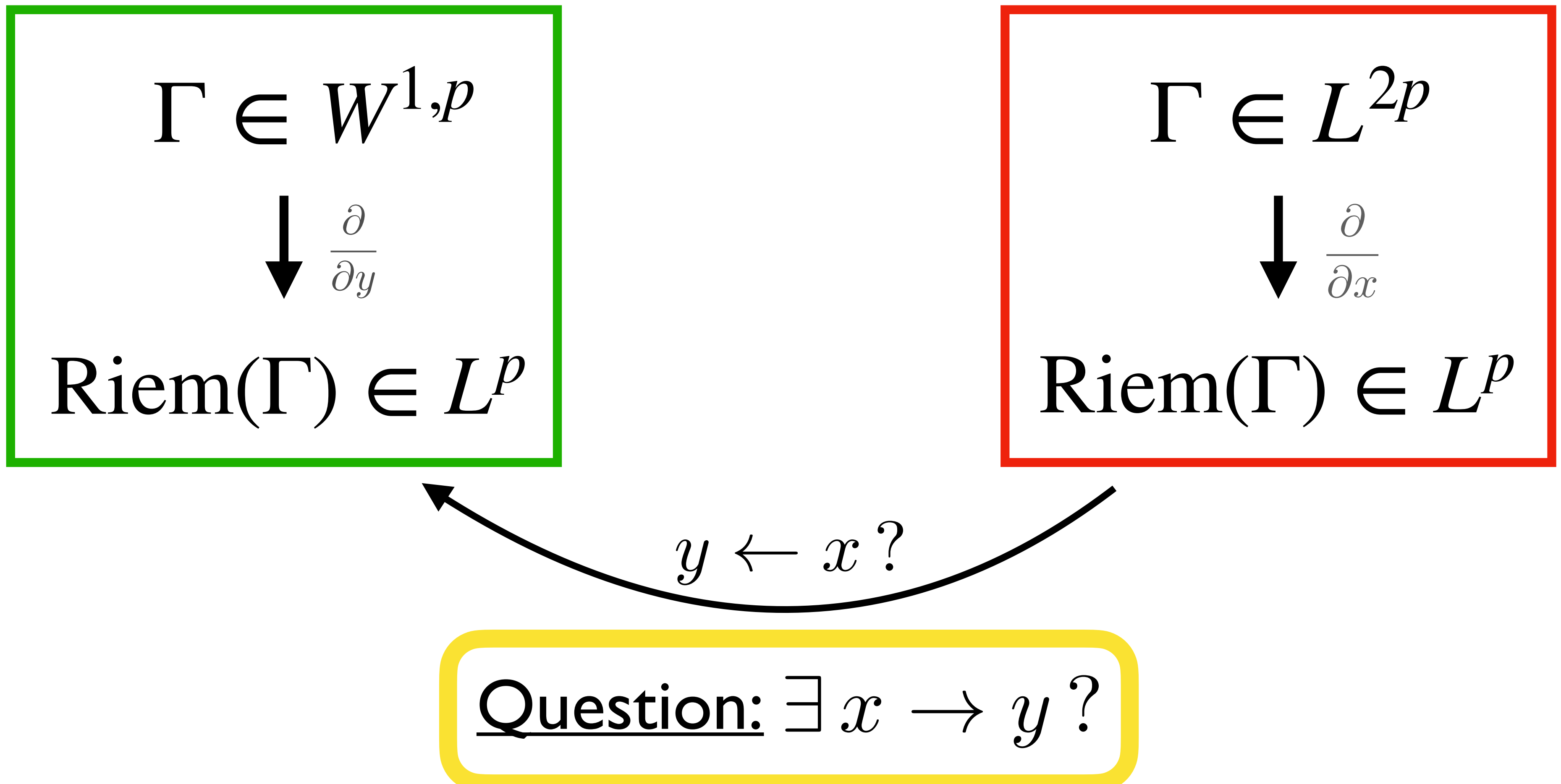

Prior Results: Yes, smoothing transformation exists for…

- Riemannian metrics (pos. def.). [Kazdan-DeTurck, 1981]
- Lorentzian metrics, (L^∞), under restrictive conditions, ruling out shock waves. [Anderson, 2002] and [LeFloch & Chen, 2008]
- Lorentzian metrics, (L^∞), across single shock surfaces. [Israel, 1966]
- Lorentzian metrics, (L^∞), across spherical shock interactions. [R. & Temple, 2014]

Optimal regularity and coordinate transformations:

$$\Gamma \in W^{1,p} \xrightarrow{\ \frac{\partial}{\partial y}\ } \mathrm{Riem}(\Gamma) \in L^p$$

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Question: $\exists\, x \to y\,?$

Thm 1: Yes, smoothing transformation exists… for any affine connection, ($p > n/2$)!

Thm 1: ("Optimal Regularity") [R. & Temple, 2019/2021]
Let $n/2 < p < \infty$. Assume that in x-coordinates

$$\|\Gamma_x\|_{L^{2p}} + \|\mathrm{Riem}(\Gamma_x)\|_{L^p} \leq M.$$

Then, locally there exists a coordinate transformation $x \to y$ to a connection of optimal regularity, $\Gamma_y \in W^{1,p}$, such that

$$\|\Gamma_y\|_{W^{1,p}} + \|J\|_{W^{1,2p}} \leq C(M),$$

where $J \equiv \frac{\partial y}{\partial x}$ and $C(M) > 0$ depends only on Ω, n, p & $M > 0$.

Norms are taken component-wise in fixed x-coordinates.

E.g.: $$\|\Gamma\|_{L^p} \equiv \sum_{k,i,j} \|\Gamma^k_{ij}\|_{L^p} = \sum_{k,i,j} \Big(\int_\Omega |\Gamma^k_{ij}|^p dx \Big)^{\frac{1}{p}}$$

$$\|\Gamma\|_{W^{1,p}} \equiv \|\Gamma\|_{L^p} + \|D\Gamma\|_{L^p}$$

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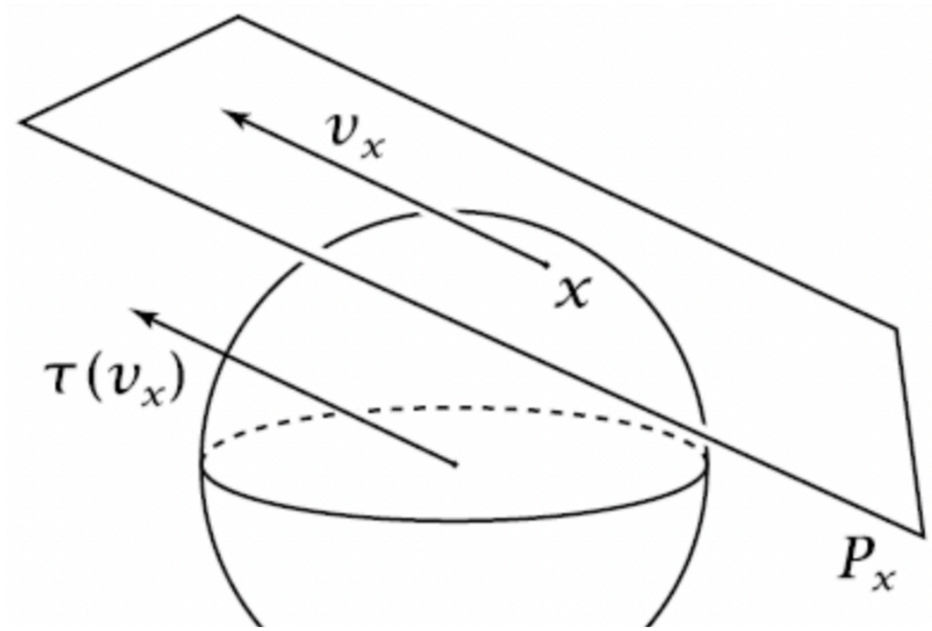
- **Higher levels of optimal regularity** [R. & Temple, 2018]:

$$\Gamma_x, \mathrm{Riem}(\Gamma_x) \in W^{m,p} \longrightarrow \Gamma_y \in W^{m+1,p}, \qquad (m \geq 1, p > n).$$

Our results extends from tangent bundles to vector bundles:



Tangent bundle. (General Relativity)

Connection: Γ

Transformation group: Jacobians $J = \frac{\partial y}{\partial x}$

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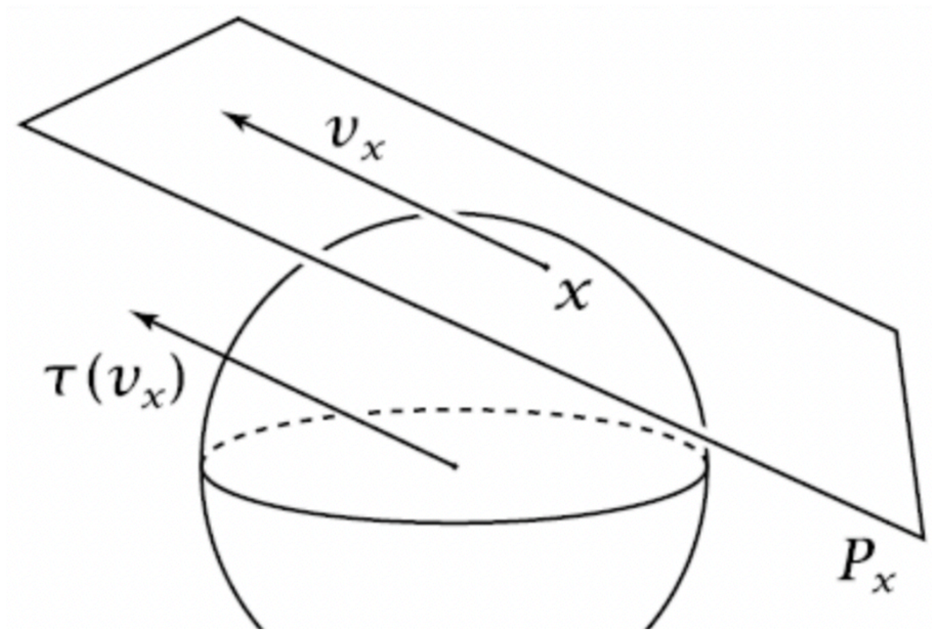

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Vector bundle. (Yang-Mills Theory)

Connection: $(\Gamma, \mathbf{A})$

Transformation group: $SO(r, s)$

Signature of metric η in orthogonality condition $U^T \eta U = \eta$.

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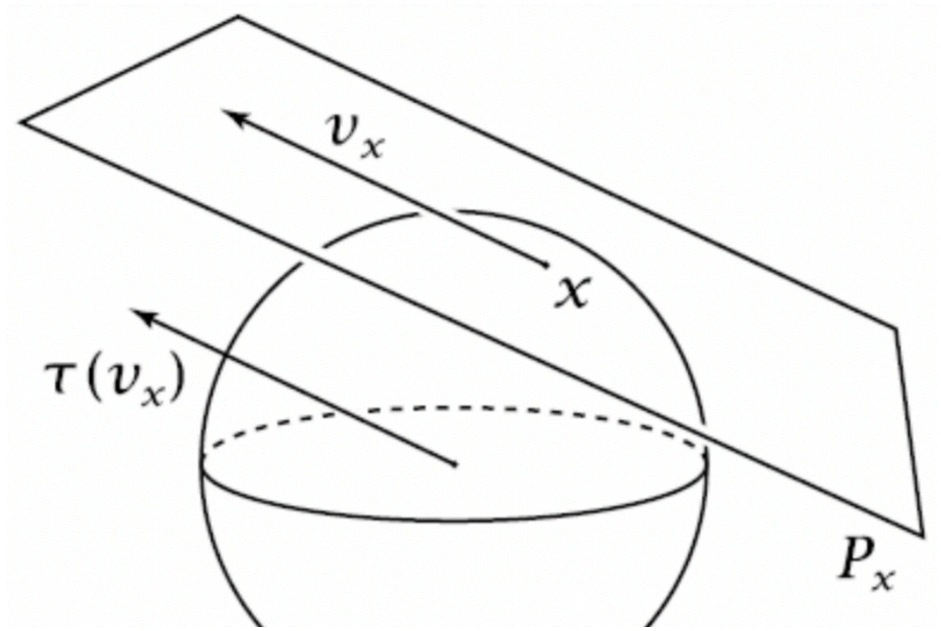

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Thm 1: ("Optimal regularity") [R. & Temple, 2021]

Assume $\|(\Gamma_x, \mathbf{A}_\mathsf{a})\|_{L^{2p}} + \|\mathrm{Riem}(\Gamma_x, \mathbf{A}_\mathsf{a})\|_{L^p} \leq M,\quad (p > n/2).$

Then, locally there exists a coord./gauge transformation, $x \to y, U \in SO(r,s)$,

to a connection of optimal regularity, $(\Gamma_y, \mathbf{A}_\mathsf{b}) \in W^{1,p}(\Omega)$, $\mathsf{b} = U \cdot \mathsf{a}, J \equiv \frac{\partial y}{\partial x}$

with $\|(\Gamma_y, \mathbf{A}_\mathsf{b})\|_{W^{1,p}} \leq C(M)$ and $\|(J, U)\|_{W^{1,2p}} \leq C(M)$.

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Banach-Alaoglu Theorem

Uhlenbeck compactness

for general connections

on vector bundles.

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Assume $\|(\Gamma_x, \mathbf{A}_{\mathsf{a}})\|_{L^{2p}} + \|\mathrm{Riem}(\Gamma_x, \mathbf{A}_{\mathsf{a}})\|_{L^p} \leq M$, $(p > n/2)$.

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↓ Banach-Alaoglu Theorem

Thm 2: ("Uhlenbeck compactness") [R. & Temple, 2021]

Let $(\Gamma_i, \mathbf{A}_i) \in L^\infty$ be a sequence of connections on $SO(r,s)$ vector bundle in fixed gauge and x-coord's.

Assume $\|(\Gamma_i, \mathbf{A}_i)\|_{L^\infty} + \|\mathrm{Riem}(\Gamma_i, \mathbf{A}_i)\|_{L^p} \leq M$ for $p > n$.

Then, in coord's/gauges (y_i, b_i) of optimal regularity, a subsequence of $(\Gamma_{y_i}, \mathbf{A}_{\mathsf{b}_i})$ converges weakly in $W^{1,p}$ and strongly in L^p.

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K. Uhlenbeck's original compactness theorem: ['82; Abel & Steele Prize]

Assumes

- a fixed smooth Riemannian metric g on the base manifold, $(\Gamma_i \equiv \Gamma_g)$;
- connections $\mathbf{A}_i \in W^{1,p}$ of optimal regularity (on fibre); $p \geq \frac{n}{2}$;
- $\|\mathrm{Riem}(\mathbf{A}_i)\|_{L^p} \leq M$. (invariant uniform bound)
- compact gauge group $\mathscr{G} \subset SO(n)$;

Asserts convergence of subsequence $\mathbf{A}_i$ weakly in $W^{1,p}$, strongly in L^p.

The RT-equations

Proof of Main Theorem

The coordinate/gauge transformations which regularise a connection $(\Gamma, \mathbf{A})$ to optimal regularity, are solutions of the *Regularity Transformation (RT-) equations*:

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$$\begin{cases} \Delta\tilde{\Gamma} = \delta d\Gamma - \delta(dJ^{-1} \wedge dJ) + d(J^{-1}B) \\ \Delta J = \delta(J\Gamma) - \langle dJ; \tilde{\Gamma}\rangle - B \\ d\overrightarrow{B} = \overrightarrow{\mathrm{div}}(dJ \wedge \Gamma) + \overrightarrow{\mathrm{div}}(Jd\Gamma) - d(\langle dJ; \tilde{\Gamma}\rangle) \\ \delta\overrightarrow{B} = v \end{cases}$$

Regularises Γ by J

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$\Longrightarrow$ ellipticity & non-invariance of RT-equations

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Independent of each other!

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Derivation of affine RT-equations:

$\tilde{\Gamma} \equiv J^{-1}JJ \cdot \Gamma_y$

Connection Transfo. Law: $\tilde{\Gamma} = \Gamma_x - J^{-1}dJ$

Optimal

$\Gamma_y \sim \tilde{\Gamma} \in W^{1,p}$

Non-optimal

$\Gamma_x \in L^{2p}$ & $d\Gamma_x \in L^p$

M.R. & B. Temple, "The Regularity Transformation Equations…", Adv. Theor. Math. Phys. 24.5 (2020).

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$$\Delta J = \langle dJ; (\Gamma_x - \tilde{\Gamma})\rangle + J\big(\delta\Gamma_x - \delta\tilde{\Gamma}\big)$$

$\Delta \equiv d\delta + \delta d$

Laplacian in $\mathbb{R}^n$

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Introduce unknown B by $B \equiv J\,\delta\tilde{\Gamma}$

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Motivated by "Riemann-flat condition":

Coord's of optimal regularity exist

$\iff \quad \exists\tilde{\Gamma}$ with $\mathrm{Riem}(\Gamma - \tilde{\Gamma}) = 0$

$\iff \quad d\tilde{\Gamma} = \ldots$

R. & Temple, "… Riemann-flat condition…", Arch. Rat. Mech. Anal. 235.

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Impose $d\overrightarrow{J} \equiv \mathrm{Curl}(J) = 0$ on B. $\Rightarrow$ J integrable to coord's

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Impose $d\vec{J} \equiv \mathrm{Curl}(J) = 0$ on B. $\Rightarrow J$ integrable to coord's

$d\overrightarrow{B} = \overrightarrow{\mathrm{div}}(dJ \wedge \Gamma) + \overrightarrow{\mathrm{div}}(J d\Gamma) - d\Big(\overrightarrow{\langle dJ; \tilde{\Gamma}\rangle}\Big)$ $\Rightarrow B \in L^p$

Controlled in L^p

$d\Gamma \in L^p \Leftrightarrow \mathrm{Riem}(\Gamma) \in L^p$

for $\Gamma \in L^{2p}$

By cancellation of $\delta\Gamma$-terms

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Impose $d\overrightarrow{J} \equiv \mathrm{Curl}(J) = 0$ on B. $\Rightarrow$ J integrable to coord's

$d\overrightarrow{B} = \overrightarrow{\mathrm{div}}(dJ \wedge \Gamma) + \overrightarrow{\mathrm{div}}(J d\Gamma) - d\Big(\overrightarrow{\langle dJ; \tilde{\Gamma} \rangle} \Big)$ $\Rightarrow B \in L^p$

Set $\delta \overrightarrow{B} = v$, for v free vector field

$$\begin{cases} \Delta \tilde{\Gamma} = \delta d\Gamma - \delta(dJ^{-1} \wedge dJ) + d(J^{-1}B) \\ \Delta J = \delta(J\Gamma) - \langle dJ; \tilde{\Gamma} \rangle - B \\ d\overrightarrow{B} = \overrightarrow{\mathrm{div}}(dJ \wedge \Gamma) + \overrightarrow{\mathrm{div}}(J d\Gamma) - d(\langle dJ; \tilde{\Gamma} \rangle) \\ \delta \overrightarrow{B} = v \end{cases}$$

the RT-equations. ∎

M.R. & B. Temple, "The Regularity Transformation Equations…", Adv. Theor. Math. Phys. 24.5 (2020).

Conversely:

If $(J, \tilde{\Gamma}, B)$ solves the RT-equations,

then J is a Jacobian, integrable to coordinates,

which regularises Γ to optimal regularity.

Proof and existence theory require careful analysis…

Existence theory for affine RT-equations:

Obstacle!

$$\begin{cases} \Delta\tilde{\Gamma} = \delta d\Gamma - \delta\big(dJ^{-1}\wedge dJ\big) + d(J^{-1}A), \\ \Delta J = \delta(J\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \\ d\vec{A} = \overrightarrow{\mathrm{div}}\big(dJ\wedge\Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\,d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \\ \delta\vec{A} = v, \end{cases}$$

Existence theory for affine RT-equations:

Loss of regularity in iteration:
$dJ, dJ^{-1} \in L^{2p}$, but $dJ^{-1} \wedge dJ \notin L^{2p}$.

$$\begin{cases} \Delta\tilde{\Gamma} = \delta d\Gamma - \delta\big(dJ^{-1} \wedge dJ\big) + d(J^{-1}A), \\ \Delta J = \delta(J\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \\ d\vec{A} = \overrightarrow{\mathrm{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\mathrm{div}}\big(J\, d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \\ \delta\vec{A} = v, \end{cases}$$

Existence theory for affine RT-equations:

Resolution: Remove $dJ^{-1} \wedge dJ$ from iteration, using "gauge-type" freedom.

$$\begin{cases} \Delta\tilde{\Gamma} = \delta d\Gamma - \delta\big(dJ^{-1} \wedge dJ\big) + d(J^{-1}A), \\ \Delta J = \delta(J\Gamma) - \langle dJ;\tilde{\Gamma}\rangle - A, \\ d\vec{A} = \overrightarrow{\text{div}}\big(dJ \wedge \Gamma\big) + \overrightarrow{\text{div}}\big(J\, d\Gamma\big) - d\big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\big), \\ \delta\vec{A} = v, \end{cases}$$

"Gauge-type" freedom

Existence theory for affine RT-equations:

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$$B \equiv A + \langle dJ;\tilde{\Gamma}\rangle$$

$$\Downarrow$$

$$w = v + \delta\Big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\Big)$$

Existence theory for affine RT-equations:

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$$B \equiv A + \langle dJ;\tilde{\Gamma}\rangle \qquad w = v + \delta\Big(\overrightarrow{\langle dJ;\tilde{\Gamma}\rangle}\Big)$$

$$\Delta\tilde{\Gamma} = \delta d\Gamma - \delta\big(dJ^{-1}\wedge dJ\big) + d(J^{-1}A)$$

$$\Delta J = \delta(J\Gamma) - B$$

$$d\vec{B} = \overrightarrow{\mathrm{div}}(dJ\wedge\Gamma) + \overrightarrow{\mathrm{div}}(Jd\Gamma)$$

$$\delta\vec{B} = w$$

Decoupling!

Free to choose!

$$\Delta J = \delta(J\Gamma) - B$$
$$d\vec{B} = \overrightarrow{\mathrm{div}}(dJ \wedge \Gamma) + \overrightarrow{\mathrm{div}}(J d\Gamma)$$
$$\delta\vec{B} = w$$

"Reduced RT-equations"

Linear & independent of $\tilde{\Gamma}$!

How RT-equations give transformation to optimal regularity:

- Integrability of J to coordinates:

J- & $\overrightarrow{B}$-eqn's $\Longrightarrow \Delta(d\overrightarrow{J}) = 0$, $\overset{\partial-data}{\Longrightarrow} d\overrightarrow{J} \equiv \mathrm{Curl}(J) = 0$ when $d\overrightarrow{J}|_{\partial\Omega} = 0$.

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- Optimal regularity is obtain as follows:

The reduced RT-equations induce cancellation of terms involving $\delta\Gamma$, which implies $\tilde{\Gamma}' \equiv \Gamma - J^{-1}dJ$ solves the gauge transformed first RT-equation

$$\Delta\tilde{\Gamma}' = \delta d\Gamma - \delta(dJ^{-1} \wedge dJ) + d(J^{-1}B')$$

$B' \equiv B - \langle dJ; \tilde{\Gamma}' \rangle$

$$\Longrightarrow \ \|\tilde{\Gamma}'\|_{W^{1,p}} \leq C(M)$$

$\|\Gamma_x\|_{L^{2p}} + \|d\Gamma_x\|_{L^p} \leq M$

$$\Longrightarrow \ \|\Gamma_y\|_{W^{1,p}} \leq C(M)$$

$(\Gamma_y)^\gamma_{\alpha\beta} \equiv J^\gamma_k (J^{-1})^i_\alpha (J^{-1})^j_\beta (\tilde{\Gamma}')^k_{ij}$

■

Thm: ("Existence") (R. & Temple, 2019/2021)

Assume $\|\Gamma_x\|_{L^{2p}} + \|\mathrm{Riem}(\Gamma_x)\|_{L^p} \leq M$ in x-coordinates, $(n/2 < p < \infty)$.

Then, locally, there exists a solution $(J, B) \in W^{1,2p} \times L^{2p}$ of the reduced RT-eqn's with $\mathrm{Curl}(J) = 0$, J invertible, and

$$\|I - J\|_{W^{1,2p}} + \|I - J^{-1}\|_{W^{1,2p}} + \|B\|_{L^{2p}} \leq C(M)$$

for some constant $C(M) > 0$ only depending on M, Ω, n, p.

Proof:

- Iteration via Poisson & Cauchy-Riemann equations with $W^{-1,p}$-sources.
- Augment reduced RT-eqn's by elliptic PDE's to replace $d\overrightarrow{J} = 0$ with Dirichlet data $J = dy$:

$$d\Psi_{k+1} = \overrightarrow{\delta(J_k \cdot \Gamma)} - \overrightarrow{B_{k+1}}, \quad \Delta y_{k+1} = \Psi_{k+1}, \qquad \Longrightarrow \Delta(J - dy) = 0 \ldots$$

- Introduce ϵ-rescaling of equations by domain restriction. $\Rightarrow$ Convergence.
- Extend existence theory for Cauchy-Riemann eqn's to $W^{-1,p}$-sources.

Possible, since A-eqn comes without ∂-data.

■

Applications

Applications:

- Uhlenbeck compactness in Lorentzian geometry.

  Non-optimal connections on fibre and tangent; non-compact groups $SO(r,s)$.

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Corollary: (GR Shock Waves)
The L^∞ metric connections of GR shock waves are regularised to $\Gamma_y \in W^{1,p}$, i.e., to Hölder continuity $(p > n)$.

➡ Geodesic curves exist. (Particle trajectories)

➡ Locally inertial coordinates exist. (Newtonian limit)

➡ Metrics in $C^{0,1} \simeq W^{1,\infty}$ are regularised to $W^{2,p} \simeq C^{1,\alpha}$.

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  Newtonian limit, locally inertial coordinates & geodesics exist.

- Existence and uniqueness of geodesics for affine L^p connections with bounded curvature. [arXiv:2306.04868]
  - Existence requires $\mathrm{Riem}(\Gamma) \in L^p$.
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  - Zero-mollification limit to weak solution.

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  - Geometric notion of "weak solution" in non-optimal coordinates.
  - Zero-mollification limit to weak solution.
  - Applicable to general second order ODE's without underlying geometry:

$$\ddot{c} + \Gamma(c)\dot{c}\dot{c} = K(t, c, \dot{c})$$

$\Gamma(c)$: L^p; $K(t,c,\dot{c})$: $C^{0,1}$

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  $\mathrm{Riem}(\Gamma) \in L^p \implies$ existence; $\mathrm{Riem}(\Gamma) \in W^{1,p} \implies$ uniqueness.

- Strong Cosmic Censorship with bounded curvature. [arXiv:2304.04444]

  Assume a family of maximal Cauchy developments of generic data is inextendable as Lorentzian manifolds with metrics uniformly bounded in $W^{2,p}$, (some $p < \infty$).
  Then it is inextendable as Lorentzian manifolds with Lipschitz continuous metrics uniformly bounded in $C^{0,1}$ with curvature uniformly bounded in L^p.

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Metrics with $g \notin C^{0,1}$ and $\mathrm{Riem}(g) \notin L^p$ might be unphysical… [Crusciel-Grant, '12]

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  Inextendability with $C^{0,1}$ metrics with L^p Riemann curvature.

- Penrose Singularity Thm for $C^{0,1}$ metrics of bounded curvature. [to come]
  - Assumes covering of coordinate patches with metrics uniformly bounded in $C^{0,1}$ and curvature uniformly bounded in L^p_{loc}.
  - Builds heavily on work by Graf ['20], and Kunzinger, Steinbauer, … ['15, '18, '22]
  - Spacetimes violating this assumptions are quite singular to begin with… [Crusciel-Grant, '12]

Conclusion:

Curvature always controls the derivative of a connection,
regardless of metric and metric signature,
as a consequence of the connection transformation law,
expressed as the elliptic RT-equations.

- M. R. & B. Temple, "Optimal regularity and Uhlenbeck compactness for General Relativity and Yang-Mills Theory", (2022), Proc. Roy. Soc. A 479: 20220444. [arXiv:2202.09535]
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Thank you very much for your attention!

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