Lorentzian Hausdorff measures, doubling and more

Workshop on
Singularities and Curvature in General Relativity
Nijmegen, The Netherlands

Clemens Sämann
Mathematical Institute
University of Oxford

joint work with Robert McCann

June 21, 2023

Introduction (1/3)

Theorem (Toponogov)

(smooth) Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if $\forall \triangle abc$ (small enough), p, q on the sides of $\triangle abc$

$$d(p,q) \geq \bar{d}(\bar{p},\bar{q}) \qquad (d(p,q) \leq \bar{d}(\bar{p},\bar{q}))$$

Definition

(smooth) semi-Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if *spacelike* sectional curvatures $\geq K$ ($\leq$) and *timelike* sectional curvatures $\leq K$ ($\geq$)

Theorem (Alexander, Bishop 2008)

(smooth) semi-Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if $\forall$ geodesic $\triangle abc$ (small enough), p, q on the sides of $\triangle abc$

$$d_{\text{signed}}(p,q) \geq \bar{d}_{\text{signed}}(\bar{p},\bar{q}) \qquad (d_{\text{signed}}(p,q) \leq \bar{d}_{\text{signed}}(\bar{p},\bar{q}))$$

Introduction (1/3)

Theorem (Toponogov)

(smooth) Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if $\forall \triangle abc$ (small enough), p, q on the sides of $\triangle abc$

$$d(p,q) \geq \bar{d}(\bar{p},\bar{q}) \qquad (d(p,q) \leq \bar{d}(\bar{p},\bar{q}))$$

Definition

(smooth) semi-Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if *spacelike* sectional curvatures $\geq K$ ($\leq$) and *timelike* sectional curvatures $\leq K$ ($\geq$)

Theorem (Alexander, Bishop 2008)

(smooth) semi-Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if $\forall$ geodesic $\triangle abc$ (small enough), p, q on the sides of $\triangle abc$

$$d_{\text{signed}}(p,q) \geq \bar{d}_{\text{signed}}(\bar{p},\bar{q}) \qquad (d_{\text{signed}}(p,q) \leq \bar{d}_{\text{signed}}(\bar{p},\bar{q}))$$

Introduction (1/3)

Theorem (Toponogov)

(smooth) Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if $\forall \triangle abc$ (small enough), p, q on the sides of $\triangle abc$

$$d(p,q) \geq \bar{d}(\bar{p},\bar{q}) \qquad (d(p,q) \leq \bar{d}(\bar{p},\bar{q}))$$

Definition

(smooth) semi-Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if *spacelike* sectional curvatures $\geq K$ ($\leq$) and *timelike* sectional curvatures $\leq K$ ($\geq$)

Theorem (Alexander, Bishop 2008)

(smooth) semi-Riemannian manifold has $Sec(g) \geq K$ ($\leq$) if $\forall$ geodesic $\triangle abc$ (small enough), p, q on the sides of $\triangle abc$

$$d_{\text{signed}}(p,q) \geq \bar{d}_{\text{signed}}(\bar{p},\bar{q}) \qquad (d_{\text{signed}}(p,q) \leq \bar{d}_{\text{signed}}(\bar{p},\bar{q}))$$

Introduction (2/3)

Riemannian manifolds	$\subsetneq$	metric spaces
Lorentzian manifolds / spacetimes	$\subsetneq$	?

analog of metric space in the *Lorentzian setting*?

analog of *Hausdorff measure* and *Hausdorff dimension*?

Introduction (2/3)

Riemannian manifolds	$\subsetneq$	metric spaces
Lorentzian manifolds / spacetimes	$\subsetneq$	?

analog of metric space in the *Lorentzian setting*?

analog of *Hausdorff measure* and *Hausdorff dimension*?

Introduction (2/3)

Riemannian manifolds	$\subsetneq$	metric spaces
Lorentzian manifolds / spacetimes	$\subsetneq$	?

analog of metric space in the *Lorentzian setting*?

analog of *Hausdorff measure* and *Hausdorff dimension*?

Introduction (3/3)

What to do in the *Lorentzian setting*?

want to handle

- spacetimes of low regularity
- no manifold structure
- no metric
- ...

$\rightsquigarrow$ *Lorentzian (pre-)length spaces* (Kunzinger C.S. 2018)
timelike, causal (sectional) *curvature bounds*, *inextendibility*, *warped products*, *singularity theorems*...

Introduction (3/3)

What to do in the *Lorentzian setting*?

want to handle

- spacetimes of low regularity
- no manifold structure
- no metric
- ...

$\rightsquigarrow$ *Lorentzian (pre-)length spaces* (Kunzinger C.S. 2018)
timelike, causal (sectional) *curvature bounds*, *inextendibility*, *warped products*, *singularity theorems*...

Introduction (3/3)

What to do in the *Lorentzian setting*?

want to handle

- spacetimes of low regularity
- no manifold structure
- no metric
- ...

$\rightsquigarrow$ *Lorentzian (pre-)length spaces* (Kunzinger C.S. 2018)
timelike, causal (sectional) *curvature bounds*, *inextendibility*, *warped products*, *singularity theorems*...

Introduction (3/3)

What to do in the *Lorentzian setting*?

want to handle

- spacetimes of low regularity
- no manifold structure
- no metric
- ...

$\rightsquigarrow$ *Lorentzian (pre-)length spaces* (Kunzinger C.S. 2018)
timelike, causal (sectional) *curvature bounds*, *inextendibility*, *warped products*, *singularity theorems*...

Lorentzian (pre-)length spaces

X set, $\leq$ preorder on X, $\ll$ transitive relation contained in $\leq$, d metric on X, $\tau\colon X \times X \to [0,\infty]$ lower semicontinuous (with respect to d)

Definition

$(X, d, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z) \qquad (x \leq y \leq z),$$

and $\tau(x,y) = 0$ if $x \not\leq y$ and $\tau(x,y) > 0 \Leftrightarrow x \ll y$;
τ is called *time separation function*

examples

- *smooth spacetimes* (M,g) with usual time separation function $\tau(p,q) := \sup\{L_g(\gamma) : \gamma$ f.d. causal from p to $q\} \cup \{0\}$
- *finite directed graphs*

causal curves and their length, *causality conditions*, τ intrinsic,...

Lorentzian (pre-)length spaces

X set, $\leq$ preorder on X, $\ll$ transitive relation contained in $\leq$, d metric on X, $\tau\colon X \times X \to [0, \infty]$ lower semicontinuous (with respect to d)

Definition

$(X, d, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if

$$\tau(x, z) \geq \tau(x, y) + \tau(y, z) \qquad (x \leq y \leq z),$$

and $\tau(x, y) = 0$ if $x \not\leq y$ and $\tau(x, y) > 0 \Leftrightarrow x \ll y$;
τ is called *time separation function*

examples

- *smooth spacetimes* (M, g) with usual time separation function $\tau(p, q) := \sup\{L_g(\gamma) : \gamma$ f.d. causal from p to $q\} \cup \{0\}$
- *finite directed graphs*

causal curves and their length, *causality conditions*, τ intrinsic,. . .

Lorentzian (pre-)length spaces

X set, $\leq$ preorder on X, $\ll$ transitive relation contained in $\leq$, d metric on X, $\tau\colon X \times X \to [0,\infty]$ lower semicontinuous (with respect to d)

Definition

$(X, d, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z) \qquad\qquad (x \leq y \leq z)\,,$$

and $\tau(x,y) = 0$ if $x \not\leq y$ and $\tau(x,y) > 0 \Leftrightarrow x \ll y$;
τ is called *time separation function*

examples

- *smooth spacetimes* (M,g) with usual time separation function $\tau(p,q) := \sup\{L_g(\gamma) : \gamma$ f.d. causal from p to $q\} \cup \{0\}$
- *finite directed graphs*

causal curves and their length, *causality conditions*, τ intrinsic,. . .

Lorentzian (pre-)length spaces

X set, $\leq$ preorder on X, $\ll$ transitive relation contained in $\leq$, d metric on X, $\tau \colon X \times X \to [0, \infty]$ lower semicontinuous (with respect to d)

Definition

$(X, d, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z) \qquad (x \leq y \leq z)\,,$$

and $\tau(x,y) = 0$ if $x \not\leq y$ and $\tau(x,y) > 0 \Leftrightarrow x \ll y$;
τ is called *time separation function*

examples

- *smooth spacetimes* (M,g) with usual time separation function $\tau(p,q) := \sup\{L_g(\gamma) : \gamma \text{ f.d. causal from } p \text{ to } q\} \cup \{0\}$
- *finite directed graphs*

causal curves and their length, *causality conditions*, τ intrinsic,. . .

Lorentzian (pre-)length spaces

X set, $\leq$ preorder on X, $\ll$ transitive relation contained in $\leq$, d metric on X, $\tau\colon X \times X \to [0,\infty]$ lower semicontinuous (with respect to d)

Definition

$(X, d, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z) \qquad (x \leq y \leq z)\,,$$

and $\tau(x,y) = 0$ if $x \not\leq y$ and $\tau(x,y) > 0 \Leftrightarrow x \ll y$;
τ is called *time separation function*

examples

- *smooth spacetimes* (M,g) with usual time separation function $\tau(p,q) := \sup\{L_g(\gamma) : \gamma$ f.d. causal from p to $q\} \cup \{0\}$
- *finite directed graphs*

causal curves and their length, *causality conditions*, τ intrinsic,. . .

Lorentzian (pre-)length spaces

X set, $\leq$ preorder on X, $\ll$ transitive relation contained in $\leq$, d metric on X, $\tau\colon X \times X \to [0,\infty]$ lower semicontinuous (with respect to d)

Definition

$(X, d, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z) \qquad (x \leq y \leq z)\,,$$

and $\tau(x,y) = 0$ if $x \not\leq y$ and $\tau(x,y) > 0 \Leftrightarrow x \ll y$;
τ is called *time separation function*

examples

- *smooth spacetimes* (M,g) with usual time separation function $\tau(p,q) := \sup\{L_g(\gamma) : \gamma$ f.d. causal from p to $q\} \cup \{0\}$
- *finite directed graphs*

causal curves and their length, *causality conditions*, τ intrinsic,. . .

Lorentzian (pre-)length spaces

X set, $\leq$ preorder on X, $\ll$ transitive relation contained in $\leq$, d metric on X, $\tau\colon X \times X \to [0,\infty]$ lower semicontinuous (with respect to d)

Definition

$(X, d, \ll, \leq, \tau)$ is a *Lorentzian pre-length space* if

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z) \qquad\qquad (x \leq y \leq z)\,,$$

and $\tau(x,y) = 0$ if $x \not\leq y$ and $\tau(x,y) > 0 \Leftrightarrow x \ll y$;
τ is called *time separation function*

examples

- *smooth spacetimes* (M, g) with usual time separation function $\tau(p,q) := \sup\{L_g(\gamma) : \gamma$ f.d. causal from p to $q\} \cup \{0\}$
- *finite directed graphs*

causal curves and their length, *causality conditions*, τ intrinsic,. . .

Hausdorff measures and dimension

Definition

(X, d) metric space, $A \subseteq X$, $\delta > 0$, $N \in [0, \infty)$

$$\mathcal{H}^N_\delta(A) := \inf\{c_N \sum_i \mathrm{diam}(A_i)^N : A \subseteq \bigcup_i A_i, \mathrm{diam}(A_i) \leq \delta\}$$

N-dimensional Hausdorff measure $\mathcal{H}^N(A) := \sup_{\delta>0} \mathcal{H}^N_\delta(A)$

Definition

Hausdorff dimension $\dim^H(A) := \inf\{N \geq 0 : \mathcal{H}^N(A) = 0\}$

Hausdorff measures and dimension

Definition

(X, d) metric space, $A \subseteq X$, $\delta > 0$, $N \in [0, \infty)$

$$\mathcal{H}^N_\delta(A) := \inf\{c_N \sum_i \operatorname{diam}(A_i)^N : A \subseteq \bigcup_i A_i, \operatorname{diam}(A_i) \leq \delta\}$$

N-dimensional Hausdorff measure $\mathcal{H}^N(A) := \sup_{\delta>0} \mathcal{H}^N_\delta(A)$

Definition

Hausdorff dimension $\dim^H(A) := \inf\{N \geq 0 : \mathcal{H}^N(A) = 0\}$

Lorentzian analog of Hausdorff measures

Definition

X Lorentzian pre-length space, $J(x,y) := J^+(x) \cap J^-(y)$

$$\rho^N(J(x,y)) := \omega_N \tau(x,y)^N$$

$\omega_N := \frac{\pi^{\frac{N-1}{2}}}{N\,\Gamma(\frac{N+1}{2})2^{N-1}}$, Γ Euler's gamma function, $N \in [0,\infty)$

$\mathbb{N} \ni N \geq 2$: *$\rho^N(J(x,y))$ = vol. CD* in N-dim Minkowski w eq. time-sep.

Definition

X Lorentzian pre-length space, $A \subseteq X$, $\delta > 0$, $N \in [0,\infty)$

$$\mathcal{V}^N_\delta(A) := \inf\{\sum_i \rho^N(J_i) : A \subseteq \bigcup_i J_i, \mathrm{diam}(J_i) \leq \delta, J_i = J(x_i,y_i)\}$$

N-dimensional Lorentzian measure $\mathcal{V}^N(A) := \sup_{\delta>0} \mathcal{V}^N_\delta(A)$

Lorentzian analog of Hausdorff measures

Definition

X Lorentzian pre-length space, $J(x,y) := J^+(x) \cap J^-(y)$

$$\rho^N(J(x,y)) := \omega_N \tau(x,y)^N$$

$\omega_N := \frac{\pi^{\frac{N-1}{2}}}{N\,\Gamma(\frac{N+1}{2})2^{N-1}}$, Γ Euler's gamma function, $N \in [0,\infty)$

$\mathbb{N} \ni N \geq 2$: $\rho^N(J(x,y)) =$ *vol. CD* in N-dim Minkowski w eq. time-sep.

Definition

X Lorentzian pre-length space, $A \subseteq X$, $\delta > 0$, $N \in [0,\infty)$

$$\mathcal{V}^N_\delta(A) := \inf\{\sum_i \rho^N(J_i) : A \subseteq \bigcup_i J_i, \mathrm{diam}(J_i) \leq \delta, J_i = J(x_i,y_i)\}$$

N-dimensional Lorentzian measure $\mathcal{V}^N(A) := \sup_{\delta>0} \mathcal{V}^N_\delta(A)$

Lorentzian analog of Hausdorff measures

Definition

X Lorentzian pre-length space, $J(x,y) := J^+(x) \cap J^-(y)$

$$\rho^N(J(x,y)) := \omega_N \tau(x,y)^N$$

$\omega_N := \frac{\pi^{\frac{N-1}{2}}}{N\,\Gamma(\frac{N+1}{2})2^{N-1}}$, Γ Euler's gamma function, $N \in [0,\infty)$

$\mathbb{N} \ni N \geq 2$: $\rho^N(J(x,y)) =$ *vol. CD* in N-dim Minkowski w eq. time-sep.

Definition

X Lorentzian pre-length space, $A \subseteq X$, $\delta > 0$, $N \in [0,\infty)$

$$\mathcal{V}^N_\delta(A) := \inf\{\sum_i \rho^N(J_i) : A \subseteq \bigcup_i J_i, \mathrm{diam}(J_i) \leq \delta, J_i = J(x_i,y_i)\}$$

N-dimensional Lorentzian measure $\mathcal{V}^N(A) := \sup_{\delta>0} \mathcal{V}^N_\delta(A)$

Synthetic dimension

Definition

$(X, d, \ll, \leq, \tau)$ Lorentzian pre-length space, $A \subseteq X$, the *synthetic dimension of A* is

$$\dim^\tau(A) := \inf\{N \geq 0 : \mathcal{V}^N(A) < \infty\}$$

Proposition

X locally d-uniform ($\tau = o(1)$) Lorentzian pre-length space, $A \subseteq X$
$N = \dim^\tau(A)$ if and only if $\forall k < N < K$: $\mathcal{V}^k(A) = \infty$, $\mathcal{V}^K(A) = 0$; thus

$$\dim^\tau(A) = \sup\{N \geq 0 : \mathcal{V}^N(A) = \infty\}$$

Synthetic dimension

Definition

$(X, d, \ll, \leq, \tau)$ Lorentzian pre-length space, $A \subseteq X$, the *synthetic dimension of A* is

$$\dim^\tau(A) := \inf\{N \geq 0 : \mathcal{V}^N(A) < \infty\}$$

Proposition

X locally d-uniform ($\tau = o(1)$) Lorentzian pre-length space, $A \subseteq X$
$N = \dim^\tau(A)$ if and only if $\forall k < N < K$: $\mathcal{V}^k(A) = \infty$, $\mathcal{V}^K(A) = 0$; thus

$$\dim^\tau(A) = \sup\{N \geq 0 : \mathcal{V}^N(A) = \infty\}$$

One-dimensional measure versus length

Null curves are zero-dimensional

$\gamma\colon [a,b] \to X$ future directed *null* curve in strongly causal Lorentzian pre-length space: $\dim^\tau(\gamma([a,b])) = 0$

Proposition

$\gamma\colon [a,b] \to X$ f.d. *causal* curve, X strongly causal: $\mathcal{V}^1(\gamma([a,b])) \leq L_\tau(\gamma)$; all causal diamonds $J(x,y)$ *closed* (e.g. X is globally hyperbolic), then

$$\mathcal{V}^1(\gamma([a,b])) = L_\tau(\gamma)$$

Countable sets are zero dimensional and measured by their cardinality

X strongly causal, $N \in [0,\infty)$; additionally in case $N > 0$ assume $\forall x \in X$, $\forall U$ nhd. of x $\exists x^\pm \in U$ s.t. $x^- < x < x^+$, $x^- \not\ll x \not\ll x^+$: $A \subseteq X$ *countable*, then $\mathcal{V}^N(A) = 0$ for $N > 0$; and $A \subseteq X$ *arbitrary* then $\mathcal{V}^0(A) = |A|$ (cardinality of A)

One-dimensional measure versus length

Null curves are zero-dimensional

$\gamma\colon [a,b] \to X$ future directed *null* curve in strongly causal Lorentzian pre-length space: $\dim^\tau(\gamma([a,b])) = 0$

Proposition

$\gamma\colon [a,b] \to X$ f.d. *causal* curve, X strongly causal: $\mathcal{V}^1(\gamma([a,b])) \leq L_\tau(\gamma)$; all causal diamonds $J(x,y)$ *closed* (e.g. X is globally hyperbolic), then

$$\mathcal{V}^1(\gamma([a,b])) = L_\tau(\gamma)$$

Countable sets are zero dimensional and measured by their cardinality

X strongly causal, $N \in [0,\infty)$; additionally in case $N > 0$ assume $\forall x \in X$, $\forall U$ nhd. of x $\exists x^\pm \in U$ s.t. $x^- < x < x^+$, $x^- \not\ll x \not\ll x^+$: $A \subseteq X$ *countable*, then $\mathcal{V}^N(A) = 0$ for $N > 0$; and $A \subseteq X$ *arbitrary* then $\mathcal{V}^0(A) = |A|$ (cardinality of A)

One-dimensional measure versus length

Null curves are zero-dimensional

$\gamma\colon [a,b] \to X$ future directed *null* curve in strongly causal Lorentzian pre-length space: $\dim^\tau(\gamma([a,b])) = 0$

Proposition

$\gamma\colon [a,b] \to X$ f.d. *causal* curve, X strongly causal: $\mathcal{V}^1(\gamma([a,b])) \leq L_\tau(\gamma)$; all causal diamonds $J(x,y)$ *closed* (e.g. X is globally hyperbolic), then

$$\mathcal{V}^1(\gamma([a,b])) = L_\tau(\gamma)$$

Countable sets are zero dimensional and measured by their cardinality

X strongly causal, $N \in [0,\infty)$; additionally in case $N > 0$ assume $\forall x \in X$, $\forall U$ nhd. of x $\exists x^\pm \in U$ s.t. $x^- < x < x^+$, $x^- \not\ll x \not\ll x^+$: $A \subseteq X$ *countable*, then $\mathcal{V}^N(A) = 0$ for $N > 0$; and $A \subseteq X$ *arbitrary* then $\mathcal{V}^0(A) = |A|$ (cardinality of A)

One-dimensional measure versus length

Null curves are zero-dimensional

$\gamma\colon [a,b] \to X$ future directed *null* curve in strongly causal Lorentzian pre-length space: $\dim^\tau(\gamma([a,b])) = 0$

Proposition

$\gamma\colon [a,b] \to X$ f.d. *causal* curve, X strongly causal: $\mathcal{V}^1(\gamma([a,b])) \leq L_\tau(\gamma)$; all causal diamonds $J(x,y)$ *closed* (e.g. X is globally hyperbolic), then

$$\mathcal{V}^1(\gamma([a,b])) = L_\tau(\gamma)$$

Countable sets are zero dimensional and measured by their cardinality

X strongly causal, $N \in [0,\infty)$; additionally in case $N > 0$ assume $\forall x \in X$, $\forall U$ nhd. of x $\exists x^\pm \in U$ s.t. $x^- < x < x^+$, $x^- \not\ll x \not\ll x^+$: $A \subseteq X$ *countable*, then $\mathcal{V}^N(A) = 0$ for $N > 0$; and $A \subseteq X$ *arbitrary* then $\mathcal{V}^0(A) = |A|$ (cardinality of A)

Dimension and measure of Minkowski subspaces (1/2)

Lemma

restriction of $\mathcal{V}^k$ to spacelike subspace of Minkowski spacetime $\mathbb{R}^n_1$ with algebraic dimension k is *positive multiple of Hausdorff measure* $\mathcal{H}^k$

Linear *null* hypersurfaces have geometric *codimension two*

Lemma

$n \geq 2$, $S \subset \mathbb{R}^n_1$ *null* subspace of algebraic dimension $k \neq n$, then $\dim^\tau(S) = k - 1$ and Lorentzian measure splits as $\mathcal{V}^{k-1} = c\,\mathcal{H}^{k-1} \times \mathcal{H}^0$ on $S = R \times \mathbb{R}\nu$, where R spacelike subspace of S, $\nu \in S$ null vector

Dimension and measure of Minkowski subspaces (1/2)

Lemma

restriction of $\mathcal{V}^k$ to spacelike subspace of Minkowski spacetime $\mathbb{R}^n_1$ with algebraic dimension k is *positive multiple of Hausdorff measure* $\mathcal{H}^k$

Linear *null* hypersurfaces have geometric *codimension two*

Lemma

$n \geq 2$, $S \subset \mathbb{R}^n_1$ *null* subspace of algebraic dimension $k \neq n$, then $\dim^\tau(S) = k - 1$ and Lorentzian measure splits as $\mathcal{V}^{k-1} = c\,\mathcal{H}^{k-1} \times \mathcal{H}^0$ on $S = R \times \mathbb{R}\nu$, where R spacelike subspace of S, $\nu \in S$ null vector

Dimension and measure of Minkowski subspaces (1/2)

Lemma

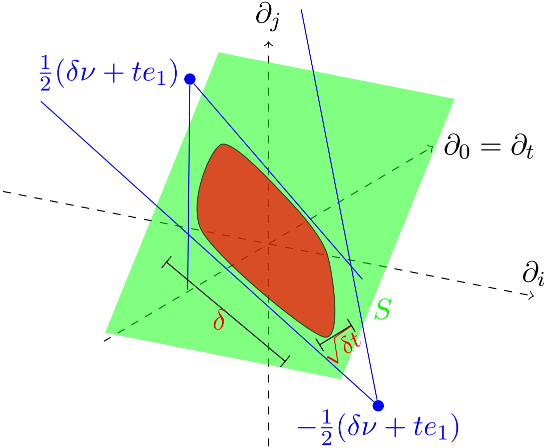
restriction of $\mathcal{V}^k$ to spacelike subspace of Minkowski spacetime $\mathbb{R}^n_1$ with algebraic dimension k is *positive multiple of Hausdorff measure* $\mathcal{H}^k$

Linear *null* hypersurfaces have geometric *codimension two*

Lemma

$n \geq 2$, $S \subset \mathbb{R}^n_1$ *null* subspace of algebraic dimension $k \neq n$, then $\dim^\tau(S) = k - 1$ and Lorentzian measure splits as $\mathcal{V}^{k-1} = c\,\mathcal{H}^{k-1} \times \mathcal{H}^0$ on $S = R \times \mathbb{R}\nu$, where R spacelike subspace of S, $\nu \in S$ null vector

Dimension and measure of Minkowski subspaces (2/2)



The intersection (in red) of the causal cones $J^{\pm}(\mp(\delta\nu + te_1))$ (in blue) with the null subspace S (in green)

Compatibility for continuous spacetimes

Theorem

(M, g) continuous, strongly causal, causally plain spacetime of dim n

- $\mathcal{V}^n = \mathrm{vol}^g$
- $\dim^\tau(M) = n$

- use appropriate *cylindrical neighborhoods*
- machinery of *Federer: Geometric measure theory 1969*
- *doubling* of causal diamonds and doubling of vol^g

Compatibility for continuous spacetimes

Theorem

(M, g) continuous, strongly causal, causally plain spacetime of dim n

- $\mathcal{V}^n = \mathrm{vol}^g$
- $\dim^\tau(M) = n$

- use appropriate *cylindrical neighborhoods*
- machinery of *Federer: Geometric measure theory 1969*
- *doubling* of causal diamonds and doubling of vol^g

Compatibility for continuous spacetimes

Theorem

(M, g) continuous, strongly causal, causally plain spacetime of dim n

- $\mathcal{V}^n = \mathrm{vol}^g$
- $\dim^\tau(M) = n$

- use appropriate *cylindrical neighborhoods*
- machinery of *Federer: Geometric measure theory 1969*
- *doubling* of causal diamonds and doubling of vol^g

Compatibility for continuous spacetimes

Theorem

(M, g) continuous, strongly causal, causally plain spacetime of dim n

- $\mathcal{V}^n = \mathrm{vol}^g$
- $\dim^\tau(M) = n$

- use appropriate *cylindrical neighborhoods*
- machinery of *Federer: Geometric measure theory 1969*
- *doubling* of causal diamonds and doubling of vol^g

Compatibility for continuous spacetimes

Theorem

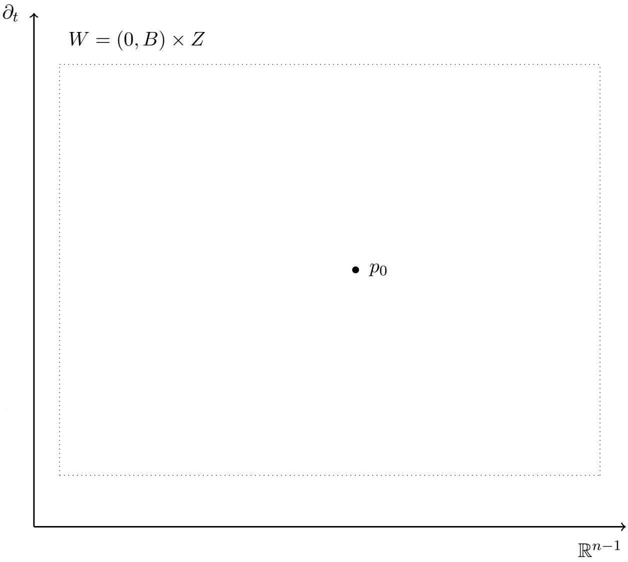
(M, g) continuous, strongly causal, causally plain spacetime of dim n

- $\mathcal{V}^n = \mathrm{vol}^g$
- $\dim^\tau(M) = n$

- use appropriate *cylindrical neighborhoods*
- machinery of *Federer: Geometric measure theory 1969*
- *doubling* of causal diamonds and doubling of vol^g

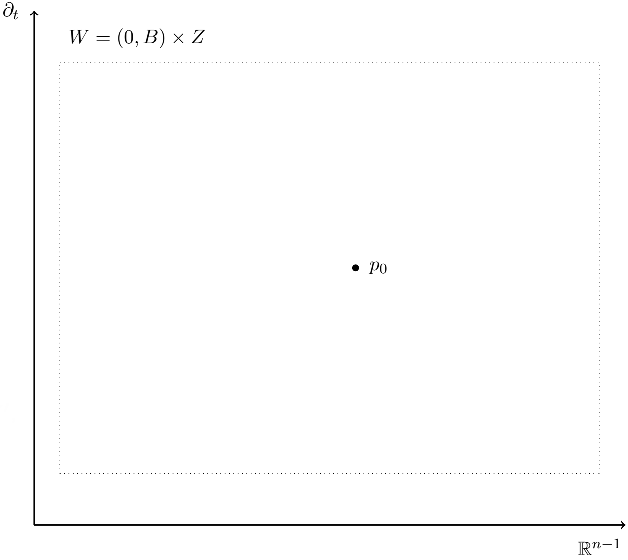
Doubling of causal diamonds in cont. spacetimes (1/2)

1. W open, conn., rel. comp. chart
2. $W = (0, B) \times Z$
3. $\partial_t = \partial_{x^0}$ unif. TL
4. $W' \subseteq W$ open, caus. conv. in W s.t. $p = (t, x), q = (s, x) \in W'$: $\hat{p} = (t - \lambda(s - t), x)$, $\hat{q} = (s + \lambda(s - t), x) \in W$
5. $p = (t, x) \ll q = (s, x)$, $u = (r, y) \ll v = (l, y) \in W'$, $l - r \leq 2(s - t)$, $I(p, q) \cap I(u, v) \neq \emptyset$ $\Rightarrow I(u, v) \subseteq I(\hat{p}, \hat{q}, W)$
6. W arb. small, inside g.h. nhd.



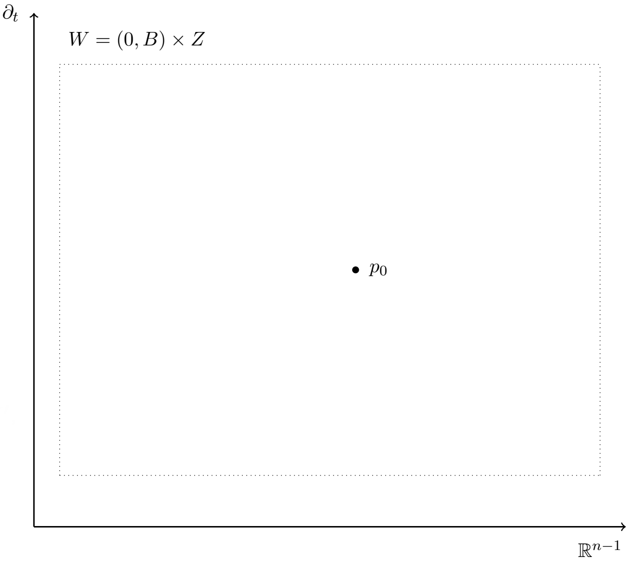
Doubling of causal diamonds in cont. spacetimes (1/2)

1. W open, conn., rel. comp. chart
2. $W = (0, B) \times Z$
3. $\partial_t = \partial_{x^0}$ unif. TL
4. $W' \subseteq W$ open, caus. conv. in W s.t. $p = (t, x), q = (s, x) \in W'$: $\hat{p} = (t - \lambda(s - t), x)$, $\hat{q} = (s + \lambda(s - t), x) \in W$
5. $p = (t, x) \ll q = (s, x)$, $u = (r, y) \ll v = (l, y) \in W'$, $l - r \leq 2(s - t)$, $J(p, q) \cap J(u, v) \neq \emptyset$ $\Rightarrow J(u, v) \subseteq J(\hat{p}, \hat{q}, W)$
6. W arb. small, inside g.h. nhd.



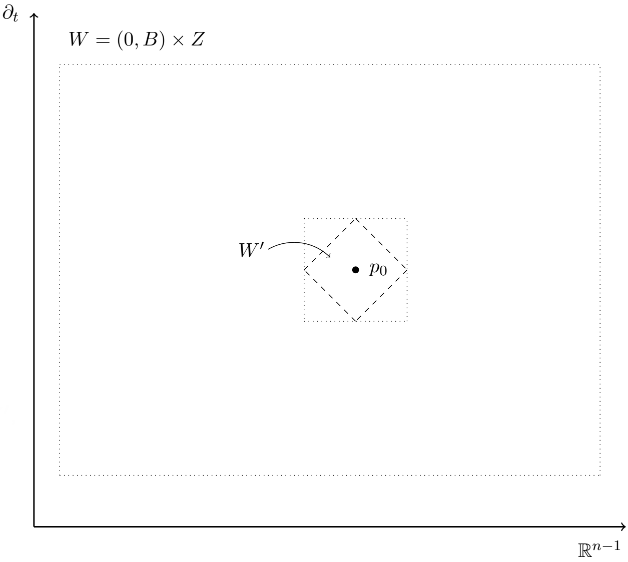
Doubling of causal diamonds in cont. spacetimes (1/2)

1. W open, conn., rel. comp. chart
2. $W = (0, B) \times Z$
3. $\partial_t = \partial_{x^0}$ unif. TL
4. $W' \subseteq W$ open, caus. conv. in W s.t. $p = (t, x), q = (s, x) \in W'$: $\hat{p} = (t - \lambda(s - t), x)$, $\hat{q} = (s + \lambda(s - t), x) \in W$
5. $p = (t, x) \ll q = (s, x)$, $u = (r, y) \ll v = (l, y) \in W'$, $l - r \leq 2(s - t)$, $I(p, q) \cap I(u, v) \neq \emptyset$ $\Rightarrow I(u, v) \subseteq I(\hat{p}, \hat{q}, W')$
6. W arb. small, inside g.h. nhd.



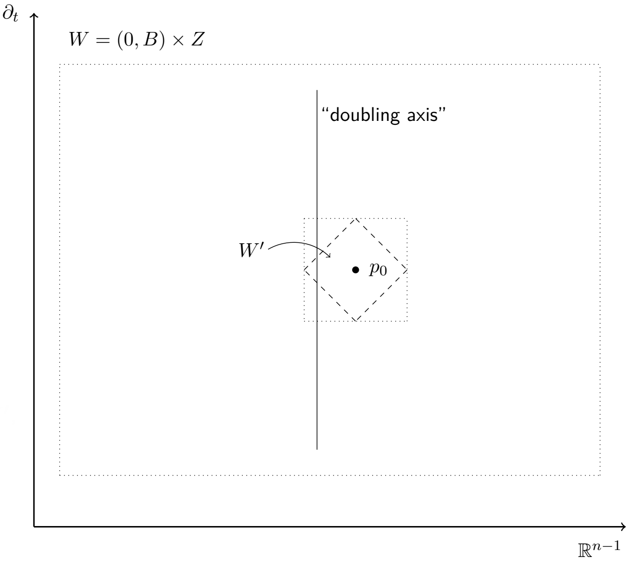
Doubling of causal diamonds in cont. spacetimes (1/2)

1. W open, conn., rel. comp. chart
2. $W = (0, B) \times Z$
3. $\partial_t = \partial_{x^0}$ unif. TL
4. $W' \subseteq W$ open, caus. conv. in W s.t. $p = (t, x), q = (s, x) \in W'$: $\hat{p} = (t - \lambda(s - t), x)$, $\hat{q} = (s + \lambda(s - t), x) \in W$
5. $p = (t, x) \ll q = (s, x)$, $u = (r, y) \ll v = (l, y) \in W'$, $l - r \leq 2(s - t)$, $J(p, q) \cap J(u, v) \neq \emptyset$ $\Rightarrow J(u, v) \subseteq I(\hat{p}, \hat{q}, W)$
6. W arb. small, inside g.h. nhd.



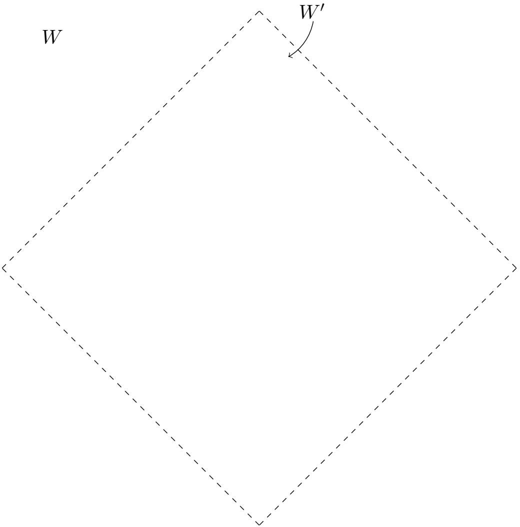
Doubling of causal diamonds in cont. spacetimes (1/2)

1. W open, conn., rel. comp. chart
2. $W = (0, B) \times Z$
3. $\partial_t = \partial_{x^0}$ unif. TL
4. $W' \subseteq W$ open, caus. conv. in W s.t. $p = (t, x), q = (s, x) \in W'$: $\hat{p} = (t - \lambda(s - t), x)$, $\hat{q} = (s + \lambda(s - t), x) \in W$
5. $p = (t, x) \ll q = (s, x)$,
6. W arb. small, inside g.h. nhd.



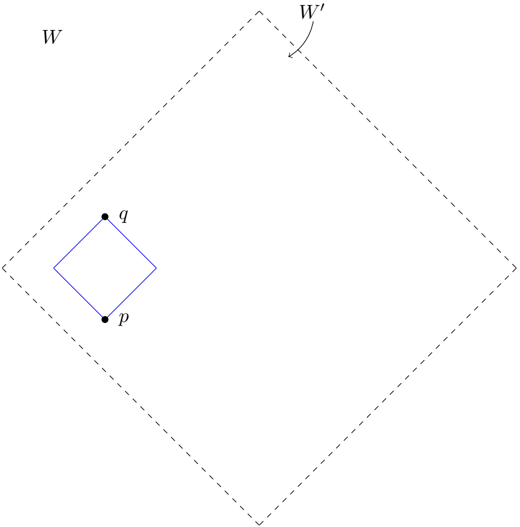
Doubling of causal diamonds in cont. spacetimes (1/2)

1. W open, conn., rel. comp. chart
2. $W = (0, B) \times Z$
3. $\partial_t = \partial_{x^0}$ unif. TL
4. $W' \subseteq W$ open, caus. conv. in W s.t. $p = (t, x), q = (s, x) \in W'$: $\hat{p} = (t - \lambda(s - t), x)$, $\hat{q} = (s + \lambda(s - t), x) \in W$
5. $p = (t, x) \ll q = (s, x)$, $u = (r, y) \ll v = (l, y) \in W'$, $l - r \leq 2(s - t)$, $J(p, q) \cap J(u, v) \neq \emptyset$ $\Rightarrow J(u, v) \subseteq J(\hat{p}, \hat{q}, W)$
6. W arb. small, inside g.h. nhd.



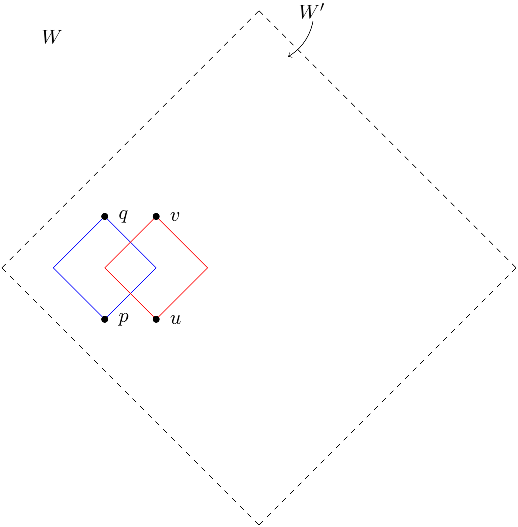
Doubling of causal diamonds in cont. spacetimes (1/2)

1. W open, conn., rel. comp. chart
2. $W = (0, B) \times Z$
3. $\partial_t = \partial_{x^0}$ unif. TL
4. $W' \subseteq W$ open, caus. conv. in W s.t. $p = (t, x), q = (s, x) \in W'$: $\hat{p} = (t - \lambda(s - t), x)$, $\hat{q} = (s + \lambda(s - t), x) \in W$
5. $p = (t, x) \ll q = (s, x)$, $u = (r, y) \ll v = (l, y) \in W'$, $l - r \leq 2(s - t)$, $J(p, q) \cap J(u, v) \neq \emptyset$ $\Rightarrow J(u, v) \subseteq J(\hat{p}, \hat{q}, W)$
6. W arb. small, inside g.h. nhd.



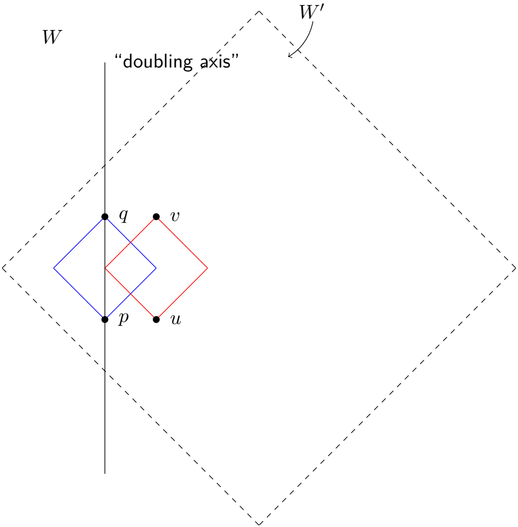
Doubling of causal diamonds in cont. spacetimes (1/2)

1. W open, conn., rel. comp. chart
2. $W = (0, B) \times Z$
3. $\partial_t = \partial_{x^0}$ unif. TL
4. $W' \subseteq W$ open, caus. conv. in W s.t. $p = (t, x), q = (s, x) \in W'$: $\hat{p} = (t - \lambda(s - t), x)$, $\hat{q} = (s + \lambda(s - t), x) \in W$
5. $p = (t, x) \ll q = (s, x)$, $u = (r, y) \ll v = (l, y) \in W'$, $l - r \leq 2(s - t)$, $J(p, q) \cap J(u, v) \neq \emptyset$ $\Rightarrow J(u, v) \subseteq J(\hat{p}, \hat{q}, W)$
6. W arb. small, inside g.h. nhd.



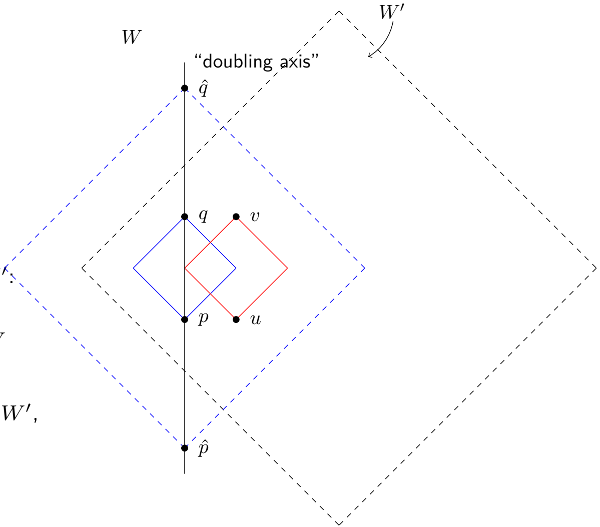
Doubling of causal diamonds in cont. spacetimes (1/2)

1. W open, conn., rel. comp. chart
2. $W = (0, B) \times Z$
3. $\partial_t = \partial_{x^0}$ unif. TL
4. $W' \subseteq W$ open, caus. conv. in W s.t. $p = (t, x), q = (s, x) \in W'$: $\hat{p} = (t - \lambda(s-t), x)$, $\hat{q} = (s + \lambda(s-t), x) \in W$
5. $p = (t, x) \ll q = (s, x)$, $u = (r, y) \ll v = (l, y) \in W'$, $l - r \leq 2(s-t)$, $J(p, q) \cap J(u, v) \neq \emptyset$ $\Rightarrow J(u, v) \subseteq J(\hat{p}, \hat{q}, W)$
6. W arb. small, inside g.h. nhd.



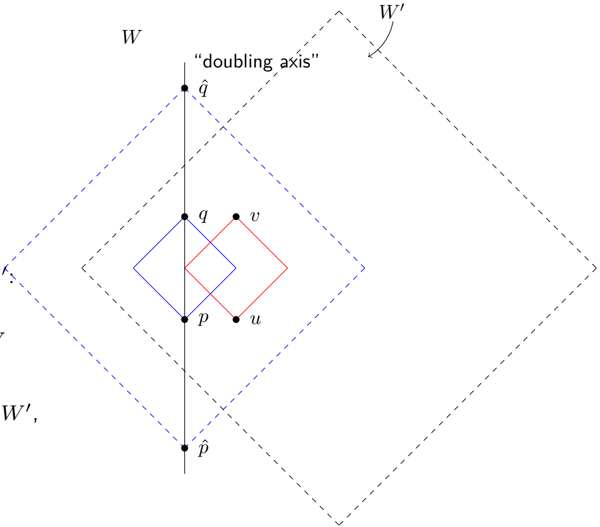
Doubling of causal diamonds in cont. spacetimes (1/2)

1. W open, conn., rel. comp. chart
2. $W = (0, B) \times Z$
3. $\partial_t = \partial_{x^0}$ unif. TL
4. $W' \subseteq W$ open, caus. conv. in W s.t. $p = (t, x), q = (s, x) \in W'$: $\hat{p} = (t - \lambda(s - t), x)$, $\hat{q} = (s + \lambda(s - t), x) \in W$
5. $p = (t, x) \ll q = (s, x)$, $u = (r, y) \ll v = (l, y) \in W'$, $l - r \leq 2(s - t)$, $J(p, q) \cap J(u, v) \neq \emptyset$ $\Rightarrow J(u, v) \subseteq J(\hat{p}, \hat{q}, W)$
6. W arb. small, inside g.h. nhd.



Doubling of causal diamonds in cont. spacetimes (1/2)

1. W open, conn., rel. comp. chart
2. $W = (0, B) \times Z$
3. $\partial_t = \partial_{x^0}$ unif. TL
4. $W' \subseteq W$ open, caus. conv. in W s.t.
   $p = (t, x), q = (s, x) \in W'$:
   $\hat{p} = (t - \lambda(s - t), x)$,
   $\hat{q} = (s + \lambda(s - t), x) \in W$
5. $p = (t, x) \ll q = (s, x)$,
   $u = (r, y) \ll v = (l, y) \in W'$,
   $l - r \leq 2(s - t)$,
   $J(p, q) \cap J(u, v) \neq \emptyset$
   $\Rightarrow J(u, v) \subseteq J(\hat{p}, \hat{q}, W)$
6. W arb. small, inside g.h. nhd.



Doubling of causal diamonds in cont. spacetimes (2/2)

cylindrical nhd. W: $\mathrm{vol}^g(J(\hat{p},\hat{q},W)) \leq L\ \mathrm{vol}^g(J(p,q))$

Definition

Borel measure m on M is *loc. causally doubling* if $\forall$ cyl. nhds. (W', W) $\exists L \geq 1$:

1. $\forall p = (t,x), q = (s,x) \in W'$: $m(J(\hat{p},\hat{q},W)) \leq L\, m(J(p,q))$
2. $m(J(p,q,W)) > 0$ ($p, q \in W$ with $p \ll q$)
3. $m(\overline{W}) < \infty$

Theorem

(M,g) cont., causally plain, strongly causal spacetime; m *loc. causally doubling measure*, *loc. doubling constant* L on all suff. small cyl. nhds $\Rightarrow$

$$\dim(M) = \dim^{\tau}(M) \leq \log_{1+2\lambda}(L)$$

Doubling of causal diamonds in cont. spacetimes (2/2)

cylindrical nhd. W: $\mathrm{vol}^g(J(\hat{p},\hat{q},W)) \leq L\ \mathrm{vol}^g(J(p,q))$

Definition

Borel measure m on M is *loc. causally doubling* if $\forall$ cyl. nhds. (W',W) $\exists L \geq 1$:

1. $\forall p=(t,x), q=(s,x) \in W'$: $m(J(\hat{p},\hat{q},W)) \leq L\, m(J(p,q))$
2. $m(J(p,q,W)) > 0$ ($p,q \in W$ with $p \ll q$)
3. $m(\overline{W}) < \infty$

Theorem

(M,g) cont., causally plain, strongly causal spacetime; m *loc. causally doubling measure*, *loc. doubling constant* L on all suff. small cyl. nhds $\Rightarrow$

$$\dim(M) = \dim^\tau(M) \leq \log_{1+2\lambda}(L)$$

Doubling of causal diamonds in cont. spacetimes (2/2)

cylindrical nhd. W: $\mathrm{vol}^g(J(\hat{p},\hat{q},W)) \leq L\ \mathrm{vol}^g(J(p,q))$

Definition

Borel measure m on M is *loc. causally doubling* if $\forall$ cyl. nhds. (W',W) $\exists L \geq 1$:

1. $\forall p=(t,x), q=(s,x) \in W'$: $m(J(\hat{p},\hat{q},W)) \leq L\, m(J(p,q))$
2. $m(J(p,q,W)) > 0$ ($p,q \in W$ with $p \ll q$)
3. $m(\overline{W}) < \infty$

Theorem

(M,g) cont., causally plain, strongly causal spacetime; m *loc. causally doubling measure*, *loc. doubling constant* L on all suff. small cyl. nhds $\Rightarrow$

$$\dim(M) = \dim^\tau(M) \leq \log_{1+2\lambda}(L)$$

Doubling of causal diamonds in cont. spacetimes (2/2)

cylindrical nhd. W: $\mathrm{vol}^g(J(\hat{p},\hat{q},W)) \leq L\ \mathrm{vol}^g(J(p,q))$

Definition

Borel measure m on M is *loc. causally doubling* if $\forall$ cyl. nhds. (W', W) $\exists L \geq 1$:

1. $\forall p = (t,x), q = (s,x) \in W'$: $m(J(\hat{p},\hat{q},W)) \leq L\, m(J(p,q))$
2. $m(J(p,q,W)) > 0$ ($p, q \in W$ with $p \ll q$)
3. $m(\overline{W}) < \infty$

Theorem

(M,g) cont., causally plain, strongly causal spacetime; m *loc. causally doubling measure*, *loc. doubling constant* L on all suff. small cyl. nhds $\Rightarrow$

$$\dim(M) = \dim^{\tau}(M) \leq \log_{1+2\lambda}(L)$$

Doubling of causal diamonds in cont. spacetimes (2/2)

cylindrical nhd. W: $\mathrm{vol}^g(J(\hat{p},\hat{q},W)) \leq L\ \mathrm{vol}^g(J(p,q))$

Definition

Borel measure m on M is *loc. causally doubling* if $\forall$ cyl. nhds. (W',W) $\exists L \geq 1$:

1. $\forall p=(t,x), q=(s,x) \in W'$: $m(J(\hat{p},\hat{q},W)) \leq L\, m(J(p,q))$
2. $m(J(p,q,W)) > 0$ ($p,q \in W$ with $p \ll q$)
3. $m(\overline{W}) < \infty$

Theorem

(M,g) cont., causally plain, strongly causal spacetime; m *loc. causally doubling measure*, *loc. doubling constant* L on all suff. small cyl. nhds $\Rightarrow$

$$\dim(M) = \dim^\tau(M) \leq \log_{1+2\lambda}(L)$$

Doubling of causal diamonds in cont. spacetimes (3/3)

Sketch of the proof:

Suffices to show $\mathcal{V}^\kappa(J)$ for small CD J ($\kappa := \log_{1+2\lambda}(L)$) in cyl. nhd.

$J_i := J(p_i, q_i)$ ($i \in I_\xi$) *maximally T_ξ-separated*, i.e.,

1. $p_i = (t_i, x_i)$, $q_i = (s_i, x_i) \in \tilde{W}$,
2. $s_i - t_i = T_\xi$,
3. for all $i, j \in I_\xi$, $i \neq j$ one has $p_i \not\leq q_j$ or $|s_j - t_i| > 2T_\xi$ or $p_j \not\leq q_i$ or $|s_i - t_j| > 2T_\xi$, and finally
4. $J_i \cap J \neq \emptyset$ for all $i \in I_\xi$.

Then $\mathrm{diam}(J_i) \leq \xi$, $\mathrm{diam}(J(\hat{p}_i, \hat{q}_i)) \leq \xi$; $\tau(p_i, q_i) \geq C_1 \xi$, $\tau(\hat{p}_i, \hat{q}_i) \leq C_2 \xi$

$(J_i)_{i \in I_\xi}$ disjoint and $J \subseteq \bigcup_{i \in I_\xi} J(\hat{p}_i, \hat{q}_i)$; $m(J_i) \geq \tilde{K}\, \tau(p_i, q_i)^\kappa$

$$\infty > m(\overline{W}) \geq m\Big(\bigcup_{i \in I_\xi} J_i\Big) = \sum_{i \in I_\xi} m(J_i) \geq \tilde{K} \sum_{i \in I_\xi} \tau(p_i, q_i)^\kappa \geq \tilde{K} C_1^\kappa \xi^\kappa |I_\xi|$$

$\rightsquigarrow |I_\xi| \leq C_3 \xi^{-\kappa}$

$$\mathcal{V}^\kappa_\xi(J) \leq \sum_{i \in I_\xi} \rho_\kappa(J(\hat{p}_i, \hat{q}_i)) = \omega_\kappa \sum_{i \in I_\xi} \tau(\hat{p}_i, \hat{q}_i)^\kappa \leq \omega_\kappa |I_\xi| C_2^\kappa \xi^\kappa \leq \omega_\kappa C_3 C_2^\kappa < \infty$$

Doubling of causal diamonds in cont. spacetimes (3/3)

Sketch of the proof:

Suffices to show $\mathcal{V}^\kappa(J)$ for small CD J ($\kappa := \log_{1+2\lambda}(L)$) in cyl. nhd.

$J_i := J(p_i, q_i)$ ($i \in I_\xi$) *maximally T_ξ-separated*, i.e.,

1. $p_i = (t_i, x_i)$, $q_i = (s_i, x_i) \in \tilde{W}$,
2. $s_i - t_i = T_\xi$,
3. for all $i, j \in I_\xi$, $i \neq j$ one has $p_i \not\leq q_j$ or $|s_j - t_i| > 2T_\xi$ or $p_j \not\leq q_i$ or $|s_i - t_j| > 2T_\xi$, and finally
4. $J_i \cap J \neq \emptyset$ for all $i \in I_\xi$.

Then $\operatorname{diam}(J_i) \leq \xi$, $\operatorname{diam}(J(\hat{p}_i, \hat{q}_i)) \leq \xi$; $\tau(p_i, q_i) \geq C_1\xi$, $\tau(\hat{p}_i, \hat{q}_i) \leq C_2\xi$

$(J_i)_{i\in I_\xi}$ disjoint and $J \subseteq \bigcup_{i \in I_\xi} J(\hat{p}_i, \hat{q}_i)$; $m(J_i) \geq \tilde{K}\,\tau(p_i, q_i)^\kappa$

$$\infty > m(\overline{W}) \geq m\Big(\bigcup_{i\in I_\xi} J_i\Big) = \sum_{i\in I_\xi} m(J_i) \geq \tilde{K} \sum_{i\in I_\xi} \tau(p_i, q_i)^\kappa \geq \tilde{K} C_1^\kappa \xi^\kappa |I_\xi|$$

$\rightsquigarrow |I_\xi| \leq C_3 \xi^{-\kappa}$

$$\mathcal{V}^\kappa_\xi(J) \leq \sum_{i\in I_\xi} \rho_\kappa(J(\hat{p}_i, \hat{q}_i)) = \omega_\kappa \sum_{i\in I_\xi} \tau(\hat{p}_i, \hat{q}_i)^\kappa \leq \omega_\kappa |I_\xi| C_2^\kappa \xi^\kappa \leq \omega_\kappa C_3 C_2^\kappa < \infty$$

Doubling of causal diamonds in cont. spacetimes (3/3)

Sketch of the proof:

Suffices to show $\mathcal{V}^\kappa(J)$ for small CD J ($\kappa := \log_{1+2\lambda}(L)$) in cyl. nhd.

$J_i := J(p_i, q_i)$ ($i \in I_\xi$) *maximally T_ξ-separated*, i.e.,

1. $p_i = (t_i, x_i)$, $q_i = (s_i, x_i) \in \tilde{W}$,
2. $s_i - t_i = T_\xi$,
3. for all $i, j \in I_\xi$, $i \neq j$ one has $p_i \not\leq q_j$ or $|s_j - t_i| > 2T_\xi$ or $p_j \not\leq q_i$ or $|s_i - t_j| > 2T_\xi$, and finally
4. $J_i \cap J \neq \emptyset$ for all $i \in I_\xi$.

Then $\operatorname{diam}(J_i) \leq \xi$, $\operatorname{diam}(J(\hat{p}_i, \hat{q}_i)) \leq \xi$; $\tau(p_i, q_i) \geq C_1\xi$, $\tau(\hat{p}_i, \hat{q}_i) \leq C_2\xi$

$(J_i)_{i\in I_\xi}$ disjoint and $J \subseteq \bigcup_{i\in I_\xi} J(\hat{p}_i, \hat{q}_i)$; $m(J_i) \geq \tilde{K}\,\tau(p_i, q_i)^\kappa$

$$\infty > m(\overline{W}) \geq m\Big(\bigcup_{i\in I_\xi} J_i\Big) = \sum_{i\in I_\xi} m(J_i) \geq \tilde{K} \sum_{i\in I_\xi} \tau(p_i, q_i)^\kappa \geq \tilde{K} C_1^\kappa \xi^\kappa |I_\xi|$$

$\rightsquigarrow |I_\xi| \leq C_3 \xi^{-\kappa}$

$$\mathcal{V}^\kappa_\xi(J) \leq \sum_{i\in I_\xi} \rho_\kappa(J(\hat{p}_i, \hat{q}_i)) = \omega_\kappa \sum_{i\in I_\xi} \tau(\hat{p}_i, \hat{q}_i)^\kappa \leq \omega_\kappa |I_\xi| C_2^\kappa \xi^\kappa \leq \omega_\kappa C_3 C_2^\kappa < \infty$$

Doubling of causal diamonds in cont. spacetimes (3/3)

Sketch of the proof:

Suffices to show $\mathcal{V}^\kappa(J)$ for small CD J ($\kappa := \log_{1+2\lambda}(L)$) in cyl. nhd.

$J_i := J(p_i, q_i)$ ($i \in I_\xi$) *maximally T_ξ-separated*, i.e.,

1. $p_i = (t_i, x_i)$, $q_i = (s_i, x_i) \in \tilde{W}$,
2. $s_i - t_i = T_\xi$,
3. for all $i, j \in I_\xi$, $i \neq j$ one has $p_i \not\leq q_j$ or $|s_j - t_i| > 2T_\xi$ or $p_j \not\leq q_i$ or $|s_i - t_j| > 2T_\xi$, and finally
4. $J_i \cap J \neq \emptyset$ for all $i \in I_\xi$.

Then $\operatorname{diam}(J_i) \leq \xi$, $\operatorname{diam}(J(\hat{p}_i, \hat{q}_i)) \leq \xi$; $\tau(p_i, q_i) \geq C_1\xi$, $\tau(\hat{p}_i, \hat{q}_i) \leq C_2\xi$

$(J_i)_{i\in I_\xi}$ disjoint and $J \subseteq \bigcup_{i\in I_\xi} J(\hat{p}_i, \hat{q}_i)$; $m(J_i) \geq \tilde{K}\,\tau(p_i, q_i)^\kappa$

$$\infty > m(\overline{W}) \geq m(\bigcup_{i\in I_\xi} J_i) = \sum_{i\in I_\xi} m(J_i) \geq \tilde{K} \sum_{i\in I_\xi} \tau(p_i, q_i)^\kappa \geq \tilde{K} C_1^\kappa \xi^\kappa |I_\xi|$$

$\leadsto |I_\xi| \leq C_3 \xi^{-\kappa}$

$$\mathcal{V}^\kappa_\xi(J) \leq \sum_{i\in I_\xi} \rho_\kappa(J(\hat{p}_i, \hat{q}_i)) = \omega_\kappa \sum_{i\in I_\xi} \tau(\hat{p}_i, \hat{q}_i)^\kappa \leq \omega_\kappa |I_\xi| C_2^\kappa \xi^\kappa \leq \omega_\kappa C_3 C_2^\kappa < \infty$$

Doubling of causal diamonds in cont. spacetimes (3/3)

Sketch of the proof:

Suffices to show $\mathcal{V}^\kappa(J)$ for small CD J ($\kappa := \log_{1+2\lambda}(L)$) in cyl. nhd.

$J_i := J(p_i, q_i)$ ($i \in I_\xi$) *maximally T_ξ-separated*, i.e.,

1. $p_i = (t_i, x_i)$, $q_i = (s_i, x_i) \in \tilde{W}$,
2. $s_i - t_i = T_\xi$,
3. for all $i, j \in I_\xi$, $i \neq j$ one has $p_i \not\leq q_j$ or $|s_j - t_i| > 2T_\xi$ or $p_j \not\leq q_i$ or $|s_i - t_j| > 2T_\xi$, and finally
4. $J_i \cap J \neq \emptyset$ for all $i \in I_\xi$.

Then $\mathrm{diam}(J_i) \leq \xi$, $\mathrm{diam}(J(\hat{p}_i, \hat{q}_i)) \leq \xi$; $\tau(p_i, q_i) \geq C_1\xi$, $\tau(\hat{p}_i, \hat{q}_i) \leq C_2\xi$
$(J_i)_{i\in I_\xi}$ disjoint and $J \subseteq \bigcup_{i\in I_\xi} J(\hat{p}_i, \hat{q}_i)$; $m(J_i) \geq \tilde{K}\,\tau(p_i, q_i)^\kappa$

$$\infty > m(\overline{W}) \geq m(\bigcup_{i\in I_\xi} J_i) = \sum_{i\in I_\xi} m(J_i) \geq \tilde{K} \sum_{i\in I_\xi} \tau(p_i, q_i)^\kappa \geq \tilde{K} C_1^\kappa \xi^\kappa |I_\xi|$$

$$\rightsquigarrow |I_\xi| \leq C_3 \xi^{-\kappa}$$

$$\mathcal{V}^\kappa_\xi(J) \leq \sum_{i\in I_\xi} \rho_\kappa(J(\hat{p}_i, \hat{q}_i)) = \omega_\kappa \sum_{i\in I_\xi} \tau(\hat{p}_i, \hat{q}_i)^\kappa \leq \omega_\kappa |I_\xi| C_2^\kappa \xi^\kappa \leq \omega_\kappa C_3 C_2^\kappa < \infty$$

Doubling of causal diamonds in cont. spacetimes (3/3)

Sketch of the proof:

Suffices to show $\mathcal{V}^\kappa(J)$ for small CD J ($\kappa := \log_{1+2\lambda}(L)$) in cyl. nhd.

$J_i := J(p_i, q_i)$ ($i \in I_\xi$) *maximally T_ξ-separated*, i.e.,

1. $p_i = (t_i, x_i)$, $q_i = (s_i, x_i) \in \tilde{W}$,
2. $s_i - t_i = T_\xi$,
3. for all $i, j \in I_\xi$, $i \neq j$ one has $p_i \not\leq q_j$ or $|s_j - t_i| > 2T_\xi$ or $p_j \not\leq q_i$ or $|s_i - t_j| > 2T_\xi$, and finally
4. $J_i \cap J \neq \emptyset$ for all $i \in I_\xi$.

Then $\mathrm{diam}(J_i) \leq \xi$, $\mathrm{diam}(J(\hat{p}_i, \hat{q}_i)) \leq \xi$; $\tau(p_i, q_i) \geq C_1 \xi$, $\tau(\hat{p}_i, \hat{q}_i) \leq C_2 \xi$

$(J_i)_{i \in I_\xi}$ disjoint and $J \subseteq \bigcup_{i \in I_\xi} J(\hat{p}_i, \hat{q}_i)$; $m(J_i) \geq \tilde{K}\, \tau(p_i, q_i)^\kappa$

$$\infty > m(\overline{W}) \geq m\Big(\bigcup_{i \in I_\xi} J_i\Big) = \sum_{i \in I_\xi} m(J_i) \geq \tilde{K} \sum_{i \in I_\xi} \tau(p_i, q_i)^\kappa \geq \tilde{K} C_1^\kappa \xi^\kappa |I_\xi|$$

$\rightsquigarrow |I_\xi| \leq C_3 \xi^{-\kappa}$

$$\mathcal{V}^\kappa_\xi(J) \leq \sum_{i \in I_\xi} \rho_\kappa(J(\hat{p}_i, \hat{q}_i)) = \omega_\kappa \sum_{i \in I_\xi} \tau(\hat{p}_i, \hat{q}_i)^\kappa \leq \omega_\kappa |I_\xi| C_2^\kappa \xi^\kappa \leq \omega_\kappa C_3 C_2^\kappa < \infty$$

Doubling of causal diamonds in cont. spacetimes (3/3)

Sketch of the proof:

Suffices to show $\mathcal{V}^\kappa(J)$ for small CD J ($\kappa := \log_{1+2\lambda}(L)$) in cyl. nhd.

$J_i := J(p_i, q_i)$ ($i \in I_\xi$) *maximally T_ξ-separated*, i.e.,

1. $p_i = (t_i, x_i)$, $q_i = (s_i, x_i) \in \tilde{W}$,
2. $s_i - t_i = T_\xi$,
3. for all $i, j \in I_\xi$, $i \neq j$ one has $p_i \not\leq q_j$ or $|s_j - t_i| > 2T_\xi$ or $p_j \not\leq q_i$ or $|s_i - t_j| > 2T_\xi$, and finally
4. $J_i \cap J \neq \emptyset$ for all $i \in I_\xi$.

Then $\mathrm{diam}(J_i) \leq \xi$, $\mathrm{diam}(J(\hat{p}_i, \hat{q}_i)) \leq \xi$; $\tau(p_i, q_i) \geq C_1 \xi$, $\tau(\hat{p}_i, \hat{q}_i) \leq C_2 \xi$

$(J_i)_{i \in I_\xi}$ disjoint and $J \subseteq \bigcup_{i \in I_\xi} J(\hat{p}_i, \hat{q}_i)$; $m(J_i) \geq \tilde{K}\, \tau(p_i, q_i)^\kappa$

$$\infty > m(\overline{W}) \geq m\Big(\bigcup_{i \in I_\xi} J_i\Big) = \sum_{i \in I_\xi} m(J_i) \geq \tilde{K} \sum_{i \in I_\xi} \tau(p_i, q_i)^\kappa \geq \tilde{K} C_1^\kappa \xi^\kappa |I_\xi|$$

$\rightsquigarrow |I_\xi| \leq C_3 \xi^{-\kappa}$

$$\mathcal{V}_\xi^\kappa(J) \leq \sum_{i \in I_\xi} \rho_\kappa(J(\hat{p}_i, \hat{q}_i)) = \omega_\kappa \sum_{i \in I_\xi} \tau(\hat{p}_i, \hat{q}_i)^\kappa \leq \omega_\kappa |I_\xi| C_2^\kappa \xi^\kappa \leq \omega_\kappa C_3 C_2^\kappa < \infty$$

Synthetic TL Ricci curvature bounds and doubling

$B_r^\tau(x) := \{y \in X : \tau(x,y) < r\}$, $E_r := E \cap \overline{B_r^\tau(x)}$

Lemma

glob. hyp. locally causally closed measured Lorentzian length space sat. $\mathsf{wTCD}^e_p(K,N)$ ($K \in \mathbb{R}$, $N \in [1,\infty)$, $\mathsf{p} \in (0,1)$) $\Rightarrow \exists L = L(K,N) \geq 1$: $\forall x_0 \in X$, $E \subseteq I^+(x_0) \cup \{x_0\}$ comp., τ-star-shaped wrt x_0, $r > 0$ small

$$m(E_{2r}) \leq L\, m(E_r)$$

! does *NOT* imply doubling for causal diamonds!

Theorem

(M,g) cont., glob. hyp. TL non-branching, causally plain spacetime sat. $\mathsf{wTCD}^e_p(K,N)$ wrt vol^g ($K \in \mathbb{R}$, $N \in [1,\infty)$, $\mathsf{p} \in (0,1)$) (+causally-reversed) $\Rightarrow$

$$\dim(M) = \dim^\tau(M) \leq N$$

Synthetic TL Ricci curvature bounds and doubling

$B_r^\tau(x) := \{y \in X : \tau(x,y) < r\}$, $E_r := E \cap \overline{B_r^\tau(x)}$

Lemma

glob. hyp. locally causally closed measured Lorentzian length space sat. $\mathsf{wTCD}_{\mathsf{p}}^{\mathsf{e}}(K,N)$ ($K \in \mathbb{R}$, $N \in [1,\infty)$, $\mathsf{p} \in (0,1)$) $\Rightarrow \exists L = L(K,N) \geq 1$: $\forall x_0 \in X$, $E \subseteq I^+(x_0) \cup \{x_0\}$ comp., τ-star-shaped wrt x_0, $r > 0$ small

$$m(E_{2r}) \leq L\, m(E_r)$$

! does *NOT* imply doubling for causal diamonds!

Theorem

(M,g) cont., glob. hyp. TL non-branching, causally plain spacetime sat. $\mathsf{wTCD}_{\mathsf{p}}^{\mathsf{e}}(K,N)$ wrt vol^g ($K \in \mathbb{R}$, $N \in [1,\infty)$, $\mathsf{p} \in (0,1)$) (+causally-reversed) $\Rightarrow$

$$\dim(M) = \dim^\tau(M) \leq N$$

Synthetic TL Ricci curvature bounds and doubling

$B_r^\tau(x) := \{y \in X : \tau(x,y) < r\}$, $E_r := E \cap \overline{B_r^\tau(x)}$

Lemma

glob. hyp. locally causally closed measured Lorentzian length space sat. $\mathsf{wTCD}_{\mathsf{p}}^{\mathsf{e}}(K,N)$ ($K \in \mathbb{R}$, $N \in [1,\infty)$, $\mathsf{p} \in (0,1)$) $\Rightarrow \exists L = L(K,N) \geq 1$: $\forall x_0 \in X$, $E \subseteq I^+(x_0) \cup \{x_0\}$ comp., τ-star-shaped wrt x_0, $r > 0$ small

$$m(E_{2r}) \leq L\, m(E_r)$$

! does ***NOT*** imply doubling for causal diamonds!

Theorem

(M,g) cont., glob. hyp. TL non-branching, causally plain spacetime sat. $\mathsf{wTCD}_{\mathsf{p}}^{\mathsf{e}}(K,N)$ wrt vol^g ($K \in \mathbb{R}$, $N \in [1,\infty)$, $\mathsf{p} \in (0,1)$) (+causally-reversed) $\Rightarrow$

$$\dim(M) = \dim^\tau(M) \leq N$$

Synthetic TL Ricci curvature bounds and doubling

$B_r^\tau(x) := \{y \in X : \tau(x,y) < r\}$, $E_r := E \cap \overline{B_r^\tau(x)}$

Lemma

glob. hyp. locally causally closed measured Lorentzian length space sat. $\mathsf{wTCD}^{\mathsf{e}}_{\mathsf{p}}(K,N)$ ($K \in \mathbb{R}$, $N \in [1,\infty)$, $\mathsf{p} \in (0,1)$) $\Rightarrow \exists L = L(K,N) \geq 1$: $\forall x_0 \in X$, $E \subseteq I^+(x_0) \cup \{x_0\}$ comp., τ-star-shaped wrt x_0, $r > 0$ small

$$m(E_{2r}) \leq L\, m(E_r)$$

! does *NOT* imply doubling for causal diamonds!

Theorem

(M,g) cont., glob. hyp. TL non-branching, causally plain spacetime sat. $\mathsf{wTCD}^{\mathsf{e}}_{\mathsf{p}}(K,N)$ wrt vol^g ($K \in \mathbb{R}$, $N \in [1,\infty)$, $\mathsf{p} \in (0,1)$) (+causally-reversed) $\Rightarrow$

$$\dim(M) = \dim^\tau(M) \leq N$$

Outlook

- synthetic dimension of *semi-Riemannian submanifolds* of spacetimes
- how to *define doubling* of *causal diamonds* in general, i.e., not using coordinates?
- synthetic timelike *Ricci curvature bounds* wrt $\mathcal{V}^N$
- applications to *singularity theorems*
- relation to Hausdorff measure/dimension wrt *Sormani-Vega null distance*
- one-dimensional measures restricted to *acausal sets* / *level sets* of (steep) time functions

Thanks!

Outlook

- synthetic dimension of *semi-Riemannian submanifolds* of spacetimes
- how to *define doubling* of *causal diamonds* in general, i.e., not using coordinates?
- synthetic timelike *Ricci curvature bounds* wrt $\mathcal{V}^N$
- applications to *singularity theorems*
- relation to Hausdorff measure/dimension wrt *Sormani-Vega null distance*
- one-dimensional measures restricted to *acausal sets* / *level sets* of (steep) time functions

Thanks!

Outlook

- synthetic dimension of *semi-Riemannian submanifolds* of spacetimes
- how to *define doubling* of *causal diamonds* in general, i.e., not using coordinates?
- synthetic timelike *Ricci curvature bounds* wrt $\mathcal{V}^N$
- applications to *singularity theorems*
- relation to Hausdorff measure/dimension wrt *Sormani-Vega null distance*
- one-dimensional measures restricted to *acausal sets* / *level sets* of (steep) time functions

Thanks!

Outlook

- synthetic dimension of *semi-Riemannian submanifolds* of spacetimes
- how to *define doubling* of *causal diamonds* in general, i.e., not using coordinates?
- synthetic timelike *Ricci curvature bounds* wrt $\mathcal{V}^N$
- applications to *singularity theorems*
- relation to Hausdorff measure/dimension wrt *Sormani-Vega null distance*
- one-dimensional measures restricted to *acausal sets* / *level sets* of (steep) time functions

Thanks!

Outlook

- synthetic dimension of *semi-Riemannian submanifolds* of spacetimes
- how to *define doubling* of *causal diamonds* in general, i.e., not using coordinates?
- synthetic timelike *Ricci curvature bounds* wrt $\mathcal{V}^N$
- applications to *singularity theorems*
- relation to Hausdorff measure/dimension wrt *Sormani-Vega null distance*
- one-dimensional measures restricted to *acausal sets* / *level sets* of (steep) time functions

Thanks!

Outlook

- synthetic dimension of *semi-Riemannian submanifolds* of spacetimes
- how to *define doubling* of *causal diamonds* in general, i.e., not using coordinates?
- synthetic timelike *Ricci curvature bounds* wrt $\mathcal{V}^N$
- applications to *singularity theorems*
- relation to Hausdorff measure/dimension wrt *Sormani-Vega null distance*
- one-dimensional measures restricted to *acausal sets* / *level sets* of (steep) time functions

Thanks!

Outlook

- synthetic dimension of *semi-Riemannian submanifolds* of spacetimes
- how to *define doubling* of *causal diamonds* in general, i.e., not using coordinates?
- synthetic timelike *Ricci curvature bounds* wrt $\mathcal{V}^N$
- applications to *singularity theorems*
- relation to Hausdorff measure/dimension wrt *Sormani-Vega null distance*
- one-dimensional measures restricted to *acausal sets* / *level sets* of (steep) time functions

Thanks!

References

S. Alexander, M. Graf, M. Kunzinger, C.S.
Generalized cones as Lorentzian length spaces: Causality, curvature, and singularity theorems. Comm. Anal. Geom. to appear, arXiv:1909.09575.

S. B. Alexander and R. L. Bishop.
Lorentz and semi-Riemannian spaces with Alexandrov curvature bounds. Comm. Anal. Geom., 16(2):251–282, 2008.

F. Cavalletti, A. Mondino.
Optimal transport in Lorentzian synthetic spaces, synthetic timelike Ricci curvature lower bounds and applications. preprint, arXiv:2004.08934 [math.MG].

J. D. E. Grant, M. Kunzinger, C.S.
Inextendibility of spacetimes and Lorentzian length spaces. Ann. Global Anal. Geom. 55, no. 1, 133–147, 2019.

M. Kunzinger, C.S.
Lorentzian length spaces. Ann. Global Anal. Geom. 54, no. 3, 399–447, 2018.

R. J. McCann, C.S.
A Lorentzian analog for Hausdorff dimension and measure. Pure and Applied Analysis, Vol. 4, No. 2, 367—400, 2022.